Chapter 4

Global exponential stability of neural networks with time-varying delays via matrix measure method

4.1 Introduction

In this chapter, the global exponential stability of octonion-valued neural network (OVNN) is studied. OVNN is different from real, complex and quaternion-valued neural network. Octonion numbers are neither commutative nor associative. One neuron of OVNNs can store eight times the information stored by one neuron of RVNNs. OVNNs are frequently used in the processing of color images, time series prediction, signal transmission etc [77, 78, 79]. In [78], the author have applied OVNNs into five real-world forecasting problems. In [6], OVNNs are proposed to estimate runoff rates. The authors in [80] have applied OVNNs in designing control for robot manipulator. Therefore, it gives ample scope for OVNNs to be explored.

The research on OVNNs is gaining rhythm because of its diversified applications. Recent works on OVNNs are mainly due to CA Popa [20, 79]. In the present chapter, the OVNNs with time varying delay is explored. The sufficient conditions for the exponential stability are obtained using the matrix measure.

4.2 Model Description and Preliminaries

Consider the following octonion valued neural network with delay as

$$\dot{\omega}(t) = -D\omega(t) + Af(\omega(t)) + Bg(\omega(t - \varkappa(t))) + I, \tag{4.1}$$

where $\omega(t) = [\omega_1(t), \omega_2(t), ..., \omega_n(t)]^T \in \mathbb{O}^n$ is the state vector of n neurons in the network; $D = \text{diag}\{d_1, d_2, ..., d_n\} \in \mathbb{R}^{n\times n}$ is the self-feedback connection weights matrix with $d_i > 0$; $A = [a_{ij}]_{n\times n} \in \mathbb{O}^{n\times n}$ and $B = [b_{ij}]_{n\times n} \in \mathbb{O}^{n\times n}$ are the connection weights matrices; $f(.) : \mathbb{O}^n \to \mathbb{O}^n$ and $g(.) : \mathbb{O}^n \to \mathbb{O}^n$, are octonion-valued nonlinear activation functions for time t and $t - \varkappa(t)$ respectively; $\varkappa(t)$ is the time-varying delay such that $\varkappa(t) \leq \varkappa$ for some $\varkappa \in \mathbb{R}$; $I = [I_1, I_2, ..., I_n]^T$ is an external input vector. The state vector being octonion is written as

$$\omega(t) = \omega^0(t)\varrho_0 + \omega^1(t)\varrho_1 + \omega^2(t)\varrho_2 + \omega^3(t)\varrho_3 + \omega^4(t)\varrho_4 + \omega^5(t)\varrho_5 + \omega^6(t)\varrho_6 + \omega^7(t)\varrho_7, \tag{4.2}$$

where ϱ_i are the octonion units for $i = 0, ..., 7$.

The addition and scalar multiplication in octonion numbers are defined in equations (1.2) and (1.3). The multiplication of octonion units is given in Table 1, in subsection 1.3.2.2 in the chapter 1.

Following assumptions are be frequently used as and when required.

Assumption 5. The activation functions f and g can be written as

$$\begin{aligned}
f(\omega(t)) =& f^0(\omega^0(t))\varrho_0 + f^1(\omega^1(t))\varrho_1 + f^2(\omega^2(t))\varrho_2 + f^3(\omega^3(t))\varrho_3 \\
&+ f^4(\omega^4(t))\varrho_4 + f^5(\omega^5(t))\varrho_5 + f^6(\omega^6(t))\varrho_6 + f^7(\omega^7(t))\varrho_7, \\
g(\omega(t)) =& g^0(\omega^0(t))\varrho_0 + g^1(\omega^1(t))\varrho_1 + g^2(\omega^2(t))\varrho_2 + g^3(\omega^3(t))\varrho_3 \\
&+ g^4(\omega^4(t))\varrho_4 + g^5(\omega^5(t))\varrho_5 + g^6(\omega^6(t))\varrho_6 + g^7(\omega^7(t))\varrho_7,
\end{aligned}$$

where each of $f^i(.)$ and $g^i(.)$ are functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ for $i = 0, 1, ..., 7$.

Now, the neural network (4.1) with Assumption 5 can be separated into eight real valued systems as

$$\begin{aligned}
\dot{\omega}^0(t) =& - D\omega^0(t) + A^0 f^0(\omega^0(t)) - \sum_{i=1}^{7} A^i f^i(\omega^i(t)) + B^0 g^0(\omega^0(t - \varkappa(t))) \\
&- \sum_{i=1}^{7} B^i g^i(\omega^i(t - \varkappa(t))) + I^0, \\
\dot{\omega}^1(t) =& - D\omega^1(t) + A^0 f^1(\omega^1(t)) + A^1 f^0(\omega^0(t)) + A^2 f^3(\omega^3(t)) - A^3 f^2(\omega^2(t)) \\
&+ A^4 f^5(\omega^5(t)) - A^5 f^4(\omega^4(t)) - A^6 f^7(\omega^7(t)) + A^7 f^6(\omega^6(t)) \\
&+ B^0 g^1(\omega^1(t - \varkappa(t))) + B^1 g^0(\omega^0(t - \varkappa(t))) + B^2 g^3(\omega^3(t - \varkappa(t))) \\
&- B^3 g^2(\omega^2(t - \varkappa(t))) + B^4 g^5(\omega^5(t - \varkappa(t))) - B^5 g^4(\omega^4(t - \varkappa(t))) \\
&- B^6 g^7(\omega^7(t - \varkappa(t))) + B^7 g^6(\omega^6(t - \varkappa(t))) + I^1, \\
\dot{\omega}^2(t) =& - D\omega^2(t) + A^0 f^2(\omega^2(t)) - A^1 f^3(\omega^3(t)) + A^2 f^0(\omega^0(t)) + A^3 f^1(\omega^1(t)) \\
&+ A^4 f^6(\omega^6(t)) + A^5 f^7(\omega^7(t)) - A^6 f^4(\omega^4(t)) - A^7 f^5(\omega^5(t)) \\
&+ B^0 g^2(\omega^2(t - \varkappa(t))) - B^1 g^3(\omega^3(t - \varkappa(t))) + B^2 g^0(\omega^0(t - \varkappa(t))) \\
&+ B^3 g^1(\omega^1(t - \varkappa(t))) + B^4 g^6(\omega^6(t - \varkappa(t))) + B^5 g^7(\omega^7(t - \varkappa(t))) \\
&- B^6 g^4(\omega^4(t - \varkappa(t))) - B^7 g^5(\omega^5(t - \varkappa(t))) + I^2,
\end{aligned}$$

$$
\begin{aligned}
\dot{\omega}^3(t) =& - D\omega^3(t) + A^0 f^3(\omega^3(t)) + A^1 f^2(\omega^2(t)) - A^2 f^1(\omega^1(t)) + A^3 f^0(\omega^0(t)) \\
& + A^4 f^7(\omega^7(t)) - A^5 f^6(\omega^6(t)) + A^6 f^5(\omega^5(t)) - A^7 f^4(\omega^4(t)) \\
& + B^0 g^3(\omega^3(t-\varkappa(t))) + B^1 g^2(\omega^2(t-\varkappa(t))) - B^2 g^1(\omega^1(t-\varkappa(t))) \\
& + B^3 g^0(\omega^0(t-\varkappa(t))) + B^4 g^7(\omega^7(t-\varkappa(t))) - B^5 g^6(\omega^6(t-\varkappa(t))) \\
& + B^6 g^5(\omega^5(t-\varkappa(t))) - B^7 g^4(\omega^4(t-\varkappa(t))) + I^3, \\
\dot{\omega}^4(t) =& - D\omega^4(t) + A^0 f^4(\omega^4(t)) - A^1 f^5(\omega^5(t)) - A^2 f^6(\omega^6(t)) - A^3 f^7(\omega^7(t)) \\
& + A^4 f^0(\omega^0(t)) + A^5 f^1(\omega^1(t)) + A^6 f^2(\omega^2(t)) + A^7 f^3(\omega^3(t)) \\
& + B^0 g^4(\omega^4(t-\varkappa(t))) - B^1 g^5(\omega^5(t-\varkappa(t))) - B^2 g^6(\omega^6(t-\varkappa(t))) \\
& - B^3 g^7(\omega^7(t-\varkappa(t))) + B^4 g^0(\omega^0(t-\varkappa(t))) + B^5 g^1(\omega^1(t-\varkappa(t))) \\
& + B^6 g^2(\omega^2(t-\varkappa(t))) + B^7 g^3(\omega^3(t-\varkappa(t))) + I^4, \\
\dot{\omega}^5(t) =& - D\omega^5(t) + A^0 f^5(\omega^5(t)) + A^1 f^4(\omega^4(t)) - A^2 f^7(\omega^7(t)) + A^3 f^6(\omega^6(t)) \\
& - A^4 f^1(\omega^1(t)) + A^5 f^0(\omega^0(t)) - A^6 f^3(\omega^3(t)) + A^7 f^2(\omega^2(t)) \\
& + B^0 g^5(\omega^5(t-\varkappa(t))) + B^1 g^4(\omega^4(t-\varkappa(t))) - B^2 g^7(\omega^7(t-\varkappa(t))) \\
& + B^3 g^6(\omega^6(t-\varkappa(t))) - B^4 g^1(\omega^1(t-\varkappa(t))) + B^5 g^0(\omega^0(t-\varkappa(t))) \\
& - B^6 g^3(\omega^3(t-\varkappa(t))) + B^7 g^2(\omega^2(t-\varkappa(t))) + I^5, \\
\dot{\omega}^6(t) =& - D\omega^6(t) + A^0 f^6(\omega^6(t)) + A^1 f^7(\omega^7(t)) + A^2 f^4(\omega^4(t)) - A^3 f^5(\omega^5(t)) \\
& - A^4 f^2(\omega^2(t)) + A^5 f^3(\omega^3(t)) + A^6 f^0(\omega^0(t)) - A^7 f^1(\omega^1(t)) \\
& + B^0 g^6(\omega^6(t-\varkappa(t))) + B^1 g^7(\omega^7(t-\varkappa(t))) + B^2 g^4(\omega^4(t-\varkappa(t))) \\
& - B^3 g^5(\omega^5(t-\varkappa(t))) - B^4 g^2(\omega^2(t-\varkappa(t))) + B^5 g^3(\omega^3(t-\varkappa(t))) \\
& + B^6 g^0(\omega^0(t-\varkappa(t))) - B^7 g^1(\omega^1(t-\varkappa(t))) + I^6,
\end{aligned}
$$

$$
\begin{aligned}
\dot{\omega}^7(t) = &- D\omega^7(t) + A^0 f^7(\omega^7(t)) - A^1 f^6(\omega^6(t)) + A^2 f^5(\omega^5(t)) + A^3 f^4(\omega^4(t)) \\
&- A^4 f^3(\omega^3(t)) - A^5 f^2(\omega^2(t)) + A^6 f^1(\omega^1(t)) + A^7 f^0(\omega^0(t)) \\
&+ B^0 g^7(\omega^7(t-\varkappa(t))) - B^1 g^6(\omega^6(t-\varkappa(t))) + B^2 g^5(\omega^5(t-\varkappa(t))) \\
&+ B^3 g^4(\omega^4(t-\varkappa(t))) - B^4 g^3(\omega^3(t-\varkappa(t))) - B^5 g^2(\omega^2(t-\varkappa(t))) \\
&+ B^6 g^1(\omega^1(t-\varkappa(t))) + B^7 g^0(\omega^0(t-\varkappa(t))) + I^7. \qquad (4.3)
\end{aligned}
$$

Remark 4.1. The commutativity and associativity do not hold in OVNNs. To reduce the computation complexity, the OVNNs are separated into eight RVNNs. The separation of OVNNs into constituent RVNNs is based on the multiplication defined in Table 1.

This system of eight equations can be written in one equation as

$$
\dot{\alpha}(t) = - D_1\alpha(t) + \sum_{i=0}^{7} A_i f_i(\alpha(t)) + \sum_{i=0}^{7} B_i g_i(\alpha(t-\varkappa(t))) + \mathcal{I}, \qquad (4.4)
$$

where $\alpha(t) = [(\omega^0(t))^T, (\omega^1(t))^T, ..., (\omega^7(t))^T]^T$; $D_1 = diag[D, D, ..., D]_{8n\times 8n}$; $\sqsupset_0 = diag[\sqsupset^0, \sqsupset^1, \sqsupset^2, \sqsupset^3, \sqsupset^4, \sqsupset^5, \sqsupset^6, \sqsupset^7]$, $\sqsupset_1 = diag[-\sqsupset^1, \sqsupset^0, \sqsupset^3, \sqsupset^2, \sqsupset^5, \sqsupset^4, \sqsupset^7, \sqsupset^6]$, $\sqsupset_2 = diag[-\sqsupset^2, -\sqsupset^3, \sqsupset^0, \sqsupset^1, \sqsupset^6, \sqsupset^7, -\sqsupset^4, -\sqsupset^5]$, $\sqsupset_3 = diag[-\sqsupset^3, \sqsupset^2, -\sqsupset^1, \sqsupset^0, \sqsupset^7, -\sqsupset^6, \sqsupset^5, -\sqsupset^4]$, $\sqsupset_4 = diag[-\sqsupset^4, -\sqsupset^5, -\sqsupset^6, -\sqsupset^7, \sqsupset^0, \sqsupset^1, \sqsupset^2, \sqsupset^3]$, $\sqsupset_5 = diag[-\sqsupset^5, \sqsupset^4, -\sqsupset^7, \sqsupset^6, -\sqsupset^1, \sqsupset^0, -\sqsupset^3, \sqsupset^2]$, $\sqsupset_6 = diag[-\sqsupset^6, \sqsupset^7, \sqsupset^4, -\sqsupset^5, -\sqsupset^2, \sqsupset^3, \sqsupset^0, -\sqsupset^1]$, $\sqsupset_7 = diag[-\sqsupset^7, -\sqsupset^6, \sqsupset^5, \sqsupset^4, -\sqsupset^3, -\sqsupset^2, \sqsupset^1, \sqsupset^0]$ for $\sqsupset = A, B$; $f_i(\alpha(t)) = [(f^i(\omega^i(t)))^T, (f^i(\omega^i(t)))^T, ..., (f^i(\omega^i(t)))^T]^T_{8n\times 1}$, $g_i(\alpha(t-\varkappa(t))) = [(g^i(\omega^i(t-\varkappa(t))))^T, (g^i(\omega^i(t-\varkappa(t))))^T, ..., (g^i(\omega^i(t-\varkappa(t))))^T]^T_{8n\times 1}$ for $i = 0, 1, ..., 7$ and $\mathcal{I} = [(I^0)^T, (I^1)^T, ..., (I^7)^T]^T$.

Remark 4.2. The OVNNs (4.1) are separated into eight real valued components. Then it is merged into a single equation (4.4). Now, the global exponential stability of this system will be shown by using the Matrix measure method and Halanay inequality. Some variants of the Halanay inequality are discussed in [81].

Let, $\epsilon(t) = [(\epsilon_0(t))^T, (\epsilon_1(t))^T, ..., (\epsilon_7(t))^T]^T = \alpha(t) - \alpha^\star = [(\alpha_0(t) - \alpha_0^\star)^T, (\alpha_1(t) - \alpha_1^\star)^T, ..., (\alpha_7(t) - \alpha_7^\star)^T]^T$.

Then,

$$\dot{\epsilon}(t) = -D_1\epsilon(t) + \sum_{i=0}^{7} A_i F_i(\epsilon(t)) + \sum_{i=0}^{7} B_i G_i(\epsilon(t - \varkappa(t))),$$

where $F_i(\epsilon(t)) = f_i(\alpha(t)) - f_i(\alpha^\star)$ and $G_i(\epsilon(t - \varkappa(t))) = g_i(\alpha(t - \varkappa(t))) - g_i(\alpha^\star)$ for $i = 0, 1, ..., 7$.

Assumption 6. The eight constituents of each of the octonion valued activation functions f^i and g^i satisfy the following inequalities:

$$|f_j^i(x) - f_j^i(y)| \leq r_i|x - y|,$$
$$|g_j^i(x) - g_j^i(y)| \leq l_i|x - y|,$$

where r_i and l_i are real constants for $j = 0, 1, ..., 7$.

Assumption 7. The above inequalities can be re-written for $x \neq y$ as

$$\frac{|f_j^i(x) - f_j^i(y)|}{|x - y|} \leq r_i, \qquad \frac{|g_j^i(x) - g_j^i(y)|}{|x - y|} \leq l_i, \quad j = 0, 1, ...7.$$

Lemma 4.3. *[76] Let $y(t)$ is a non negative continuous function defined on $[t_0 - \varkappa, +\infty)$, and k_1 and k_2 be constants with $k_1 > k_2 > 0$ satisfying the following inequality for $t \geq t_0$,*

$$D^+(y(t)) \leq -k_1 y(t) + k_2 \sup_{t-\varkappa \leq s \leq t} y(s),$$

then the following inequality stands true for $t \geq t_0$,

$$y(t) \leq \sup_{t_0 - \varkappa \leq s \leq t_0} y(s) e^{-r(t-t_0)}, \tag{4.5}$$

where r *is a bound on the exponential convergence rate and is the unique positive solution of* $r = k_1 - k_2 e^{r\varkappa}$ *and* $D^+(y(t))$ *is the upper right Dinni derivative.*

Lemma 4.4. *[76] Let the Assumption 7 holds and let* $\|.\|_q$ *is an induced norm on* $\mathbb{R}^{n\times n}$. *Let* μ_q *be the corresponding matrix measure. Then* $\mu_q(AG(e(t))) \leq \mu_q(A^\star C)$, *where* $G(e(t)) = diag\left(\frac{f_1(\varrho_1(t))}{\varrho_1(t)}, \frac{f_1(\varrho_1(t))}{\varrho_2(t)}, ..., \frac{f_n(\varrho_n(t))}{\varrho_n(t)}\right)$, $C = diag(c_1, c_2, ..., c_n)$, *p=1,∞,* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $A^\star = (a^\star_{ij}) = \begin{cases} max(0, a_{ii}), & i = j, \\ a_{ij}, & i \neq j. \end{cases}$

4.3 Main Results

In this section, the matrix measure method is applied to obtain exponential stability criterion for the delayed OVNNs.

Theorem 4.5. *The OVNNs with the help of Assumptions 5 and 6 converge globally exponentially to the equilibrium point if there exist*
$k_1 = -\mu_q(-D_1) - \sum_{i=0}^{7} \|A_i\|_q (8)^{\frac{1}{q}} r_i$,
$k_2 = \sum_{i=0}^{7} \|B_i\|_q (8)^{\frac{1}{q}} l_i$ *for q = 1,2,∞ such that* $k_1 > k_2 > 0$.

Proof. By the definition of dinni derivative,

$$\begin{aligned} D^+\big(\epsilon(t)\big) &= \lim_{h\to 0^+} \frac{\|\epsilon(t+h)\|_q - \|\epsilon(t)\|_q}{h} \\ &= \lim_{h\to 0^+} \frac{\|\epsilon(t) + h\dot{\epsilon}(t) + o(h^2)\|_q - \|\epsilon(t)\|_q}{h}. \end{aligned} \tag{4.6}$$

Now,

$$\begin{aligned}\|\epsilon(t)+h\dot{\epsilon}(t)+o(h^2)\|_q =&\|\epsilon(t)+h\big(-D_1\epsilon(t)+\sum_{i=0}^{7}A_iF_i(\epsilon(t))\\&+\sum_{i=0}^{7}B_iG_i(\epsilon(t-\varkappa(t)))\big)+o(h^2)\|_q \qquad (4.7)\\ \le&\|I-hD_1\|_q\|\epsilon(t)\|_q+h\big(\sum_{i=0}^{7}\|A_i\|_q\|F_i(\epsilon(t))\|_q\\&+\sum_{i=0}^{7}\|B_i\|_q\|G_i(\epsilon(t-\varkappa(t)))\|_q\big)+\|o(h^2)\|_q, \qquad (4.8)\end{aligned}$$

where

$$\begin{aligned}\|F_0(\epsilon(t))\|_q^q =&\|f_0(\alpha(t))-f_0(\alpha^\star)\|_q^q\\ =&\left\|\begin{matrix}f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\\f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\\f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\\f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\\f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\\f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\\f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\\f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\end{matrix}\right\|_q^q = 8\|f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)\|_q^q.\end{aligned}$$

Therefore, by using assumption 6, $\|F_0(\epsilon(t))\|_q \le (8)^{\frac{1}{q}}r_0\|\epsilon(t)\|_q$.

Similarly, $\|F_i(\epsilon(t))\|_q \le (8)^{\frac{1}{q}}r_i\|\epsilon(t)\|_q$ for $i=1,2,...,7$ and $\|G_i(\epsilon(t-\varkappa(t)))\|_q \le (8)^{\frac{1}{q}}l_i\|\epsilon(t-\varkappa(t))\|_q$ for $i=0,1,...,7$.

Thus, inequality (4.8) becomes

$$\|\epsilon(t)+h\dot{\epsilon}(t)+o(h)\|_q \leq \|I-hD_1\|_q\|\epsilon(t)\|_q + h\big(\sum_{i=0}^{7}\|A_i\|_q(8)^{\frac{1}{q}}r_i\|\epsilon(t)\|_q$$
$$+\sum_{i=0}^{7}\|B_i\|_q(8)^{\frac{1}{q}}l_i\|\epsilon(t-\varkappa(t))\|_q\big)+\|o(h^2)\|_q.$$

Therefore, by using lemma (4.3), we get

$$D^+\big(\epsilon(t)\big) \leq \lim_{h\to 0^+}\left(\frac{\|I-hD_1\|_q-1}{h}\right)\|\epsilon(t)\|_q + (\sum_{i=0}^{7}\|A_i\|_q(8)^{\frac{1}{q}}r_i)\|\epsilon(t)\|_q$$
$$+(\sum_{i=0}^{7}\|B_i\|_q(8)^{\frac{1}{q}}l_i)\|\epsilon(t-\varkappa(t))\|_q$$
$$\leq -\big(-\mu_q(-D_1)-\sum_{i=0}^{7}\|A_i\|_q(8)^{\frac{1}{q}}r_i\big)\|\epsilon(t)\|_q + \big(\sum_{i=0}^{7}\|B_i\|_q(8)^{\frac{1}{q}}l_i\big)\sup_{t-\varkappa\leq s\leq t}\|\epsilon(s)\|_q$$
$$=-k_1\|e(t)\|_q + k_2\sup_{t-\varkappa\leq s\leq t}\|\epsilon(s)\|_q.$$

Remark 4.6. The above theorem is an extension of the theorem proved in [64]. By taking constants $r_i=0$ for $i=2,3,...,7$, we get $-\mu_q(-D_1)-\|A_0\|_q(2)^{\frac{1}{q}}r_0-\|A_1\|_q(2)^{\frac{1}{q}}r_1 \geq -\mu_q(-D_1)-\|A_0\|_q(8)^{\frac{1}{q}}r_0-\|A_1\|_q(8)^{\frac{1}{q}}r_1 > \|B_0\|_q(8)^{\frac{1}{q}}l_0+\|B_1\|_q(8)^{\frac{1}{q}}l_1 > \|B_0\|_q(2)^{\frac{1}{q}}l_0+\|B_1\|_q(2)^{\frac{1}{q}}l_1 > 0.$

Theorem 4.7. *The OVNNs (4.1) with the use of Assumptions (5) and (7) converge globally exponentially to the equilibrium point if*

$$-\mu_q(-D_1)-\mu_q(\mathcal{A}^{\star}C)-\sum_{i=0}^{7}\|A_i\|_q(r+7^{\frac{1}{q}}r_i) > \sum_{i=0}^{7}\|B_i\|_q(8)^{\frac{1}{q}}l_i > 0,$$

where $q=1,\infty$*;* $r=\max_{1\leq i\leq n} r_i,\ \mathcal{A}=\sum_{i=0}^{7}A_i,\ C=[r_0,r_0,...,r_0,r_1,r_1,...,r_1,...,r_7,r_7,...,r_7]^T_{8n\times 1}$ *and rest of the things are same as in previous theorem.*

Proof. After proceeding similar to the proof previous theorem up to equation (4.7), we get

$$\begin{pmatrix} f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \end{pmatrix} = \begin{pmatrix} f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^1(\epsilon_1(t)+\alpha_1^\star)-f^1(\alpha_1^\star) \\ f^2(\epsilon_2(t)+\alpha_2^\star)-f^2(\alpha_2^\star) \\ f^3(\epsilon_3(t)+\alpha_3^\star)-f^3(\alpha_3^\star) \\ f^4(\epsilon_4(t)+\alpha_4^\star)-f^4(\alpha_4^\star) \\ f^5(\epsilon_5(t)+\alpha_5^\star)-f^5(\alpha_5^\star) \\ f^6(\epsilon_6(t)+\alpha_6^\star)-f^6(\alpha_6^\star) \\ f^7(\epsilon_7(t)+\alpha_7^\star)-f^7(\alpha_7^\star) \end{pmatrix} - \begin{pmatrix} 0 \\ f^1(\epsilon_1(t)+\alpha_1^\star)-f^1(\alpha_1^\star) \\ f^2(\epsilon_2(t)+\alpha_2^\star)-f^2(\alpha_2^\star) \\ f^3(\epsilon_3(t)+\alpha_3^\star)-f^3(\alpha_3^\star) \\ f^4(\epsilon_4(t)+\alpha_4^\star)-f^4(\alpha_4^\star) \\ f^5(\epsilon_5(t)+\alpha_5^\star)-f^5(\alpha_5^\star) \\ f^6(\epsilon_6(t)+\alpha_6^\star)-f^6(\alpha_6^\star) \\ f^7(\epsilon_7(t)+\alpha_7^\star)-f^7(\alpha_7^\star) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \\ f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star) \end{pmatrix}$$

$$= \mathcal{G}(\epsilon(t))\epsilon(t) - \eta_0^1(\epsilon(t)) + \eta_0^2(\epsilon_0(t)),$$

where $\mathcal{G}(\epsilon(t)) = diag\left\{\frac{f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star)}{\epsilon_0(t)}, \frac{f^1(\epsilon_1(t)+\alpha_1^\star)-f^1(\alpha_1^\star)}{\epsilon_1(t)}, ..., \frac{f^7(\epsilon_7(t)+\alpha_7^\star)-f^7(\alpha_7^\star)}{\epsilon_7(t)}\right\}$,
$\eta_0^1(\epsilon(t)) = [0_{1\times n}, (f^1(\epsilon_1(t)+\alpha_1^\star)-f^1(\alpha_1^\star))^T, (f^2(\epsilon_2(t)+\alpha_2^\star)-f^2(\alpha_2^\star))^T, ..., (f^7(\epsilon_7(t)+\alpha_7^\star)-f^7(\alpha_7^\star))^T]^T$, $\eta_0^2(\epsilon_0(t)) = [0_{1\times n}, (f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star))^T, (f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star))^T, ..., (f^0(\epsilon_0(t)+\alpha_0^\star)-f^0(\alpha_0^\star))^T]^T$.
Similarly, $F_i(\epsilon(t)) = \mathcal{G}(\epsilon(t))\epsilon(t) - \eta_i^1(\epsilon(t)) + \eta_i^2(\epsilon_i(t))$, for $i = 1, 2, ..., 7$.

Therefore,

$$\left\| \begin{matrix} f^0(\epsilon(t)+\alpha^\star)-f^0(\alpha^\star) \\ f^0(\epsilon(t)+\alpha^\star)-f^0(\alpha^\star) \\ f^0(\epsilon(t)+\alpha^\star)-f^0(\alpha^\star) \\ f^0(\epsilon(t)+\alpha^\star)-f^0(\alpha^\star) \\ f^0(\epsilon(t)+\alpha^\star)-f^0(\alpha^\star) \\ f^0(\epsilon(t)+\alpha^\star)-f^0(\alpha^\star) \\ f^0(\epsilon(t)+\alpha^\star)-f^0(\alpha^\star) \\ f^0(\epsilon(t)+\alpha^\star)-f^0(\alpha^\star) \end{matrix} \right\|_q^q \leq \|\mathcal{G}(\epsilon(t))\|_q^q \|\epsilon(t)\|_q^q + r^q \|\epsilon(t)\|_q^q + 7^{\frac{1}{q}} r_0^q \|\epsilon(t)\|_q^q.$$

Similarly,

$$\|F_i(\epsilon(t))\| \leq \|\mathcal{G}(\epsilon(t))\|_q^q \|\epsilon(t)\|_q^q + r^q \|\epsilon(t)\|_q^q + 7^{\frac{1}{q}} r_i^q \|\epsilon(t)\|_q^q,$$

for $i = 1, 2, ..., 7$.

Therefore, the equation (4.7) becomes,

$$\begin{aligned} \|\epsilon(t)+h\dot{\epsilon(t)}+o(h)\|_q =& \|\epsilon(t)+h\big(-D_1\epsilon(t)+\sum_{i=0}^{7} A_i F_i(\epsilon(t))+\sum_{i=0}^{7} B_i G_i(\epsilon(t-\varkappa(t)))\big) \\ &+o(h^2)\|_q \\ \leq& \|\epsilon(t)+h\big(-D_1\epsilon(t)+\sum_{i=0}^{7}(A_i(\mathcal{G}(\epsilon(t))\epsilon(t)-\eta_i^1(\epsilon(t))+\eta_i^2(\epsilon_i(t)))) \\ &+\sum_{i=0}^{7} B_i g_i(\epsilon(t-\varkappa(t)))+o(h^2)\|_q \\ \leq& \|\Big(I-h\big(D_1-\mathcal{A}\mathcal{G}(\epsilon(t))\big)\Big)\epsilon(t)\|_q + h(\sum_{i=0}^{7}\|A_i\|_q(\|\eta_i^1(\epsilon(t))\|_q \\ &+\|\eta_i^2(\epsilon_i(t))\|_q)) + h(\sum_{i=0}^{7}\|B_i\|_q (8)^{\frac{1}{q}} l_i \|\epsilon(t-\varkappa(t))\|_q) + \|o(h^2)\|_q. \end{aligned}$$

From the definition given in (4.6) and by using lemma 4.4 with some properties of matrix measure, we have

$$\begin{aligned} D^{+}\big(\epsilon(t)\big) \leq \lim_{h\to 0^{+}} &\bigg(\frac{\|I-h\big(D_1-\mathcal{A}\mathcal{G}(\epsilon(t))\big)\|_q-1}{h}\bigg)\|\epsilon(t)\|_q \\ &+\sum_{i=0}^{7}\|A_i\|_q(\|\eta_i^1(\epsilon(t))\|_q+\|\eta_i^2(\epsilon_i(t))\|_q)+\sum_{i=0}^{7}\|B_i\|_q(8)^{\frac{1}{q}}l_i\|\epsilon(t-\varkappa(t))\|_q \\ \leq -\big(-\mu_q(-D_1)&-\mu_q(\mathcal{A}\mathcal{G}(\epsilon(t)))-\sum_{i=0}^{7}\|A_i\|_q(r^q+7^{\frac{1}{q}}r_i^q)\big)\|\epsilon(t)\|_q \\ &+\big(\sum_{i=0}^{7}\|B_i\|_q(8)^{\frac{1}{q}}l_i\big)\sup_{t-\varkappa\leq s\leq t}\|\epsilon(s)\|_q \\ \leq -\big(-\mu_q(-D_1)&-\mu_q(\mathcal{A}^{\star}C)-\sum_{i=0}^{7}\|A_i\|_q(r^q+7^{\frac{1}{q}}r_i^q)\big)\|\epsilon(t)\|_q \\ &+\big(\sum_{i=0}^{7}\|B_i\|_q(8)^{\frac{1}{q}}l_i\big)\sup_{t-\varkappa\leq s\leq t}\|\epsilon(s)\|_q. \end{aligned}$$

□

Remark 4.8. A matrix measure is used to prove the above results, which is more general than the matrix norm. The matrix norm is always positive, whereas the matrix measure can be positive or negative. Also, there is no need to construct the Lyapunov function while obtaining sufficient conditions for global exponential stability using the matrix measure method.

Remark 4.9. The assumption used to prove Theorem 4.7 is more general than the assumption used in Theorem 4.5. In particular, Assumption 6 is a special case of Assumption 5. Therefore, the Theorem 4.7 gives more general results than Theorem 4.5.

4.4 Numerical Examples

Two examples are discussed in this section to show the effectiveness of the two theorems stated in the previous section.

Example 4.1. *Consider OVNNs (4.1) as*

$$\dot{\omega}(t) = -D\omega(t) + Af(\omega(t)) + Bg(\omega(t - \varkappa(t))) + I, \tag{4.9}$$

where $D = \begin{pmatrix} 40 & 0 \\ 0 & 50 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, $I = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$ *with*

$a_{11} = 0.2\varrho_0 + 0.1\varrho_1 - 0.4\varrho_2 + 0.3\varrho_3 + 0.2\varrho_4 - 0.2\varrho_5 - 0.2\varrho_6 - 0.1\varrho_7,$

$a_{12} = 0.5\varrho_0 + 0.2\varrho_1 + 0.3\varrho_2 - 0.2\varrho_3 - 0.1\varrho_4 + 0.2\varrho_5 + 0.1\varrho_6 + 0.4\varrho_7,$

$a_{21} = -0.1\varrho_0 - 0.3\varrho_1 - 0.2\varrho_2 + 0.2\varrho_3 + 0.6\varrho_4 - 0.3\varrho_5 + 0.4\varrho_6 + 0.2\varrho_7,$

$a_{22} = -0.2\varrho_0 - 0.1\varrho_1 + 0.4\varrho_2 + 0.3\varrho_3 + 0.4\varrho_4 + 0.1\varrho_5 - 0.1\varrho_6 - 0.3\varrho_7,$

$b_{11} = 0.5\varrho_0 - 0.1\varrho_1 + 0.2\varrho_2 + 0.2\varrho_3 + 0.1\varrho_4 - 0.4\varrho_5 + 0.2\varrho_6 + 0.1\varrho_7,$

$b_{12} = 0.2\varrho_0 - 0.3\varrho_1 - 0.1\varrho_2 - 0.3\varrho_3 + 0.1\varrho_4 - 0.5\varrho_5 - 0.2\varrho_6 + 0.3\varrho_7,$

$b_{21} = 0.1\varrho_0 - 0.4\varrho_1 - 0.2\varrho_2 + 0.2\varrho_3 - 0.1\varrho_4 - 0.4\varrho_5 - 0.2\varrho_6 + 0.1\varrho_7,$

$b_{22} = 0.6\varrho_0 + 0.4\varrho_1 + 0.2\varrho_2 - 0.1\varrho_3 - 0.2\varrho_4 + 0.1\varrho_5 + 0.4\varrho_6 - 0.5\varrho_7,$

$i_1 = 18\varrho_0 + 19\varrho_1 + 20\varrho_2 + 21\varrho_3 + 22\varrho_4 + 23\varrho_5 + 24\varrho_6 + 25\varrho_7,$

$i_2 = 25\varrho_0 + 24\varrho_1 + 23\varrho_2 + 22\varrho_3 + 21\varrho_4 + 20\varrho_5 + 19\varrho_6 + 18\varrho_7.$

Let the activation functions are given by $f_j^i(\omega(t)) = tanh(\omega_j^i(t))\varrho_i$, $g_j^i(\omega(t)) = tanh(\omega_j^i(t - \varkappa(t)))\varrho_i$ *for* $i = 0, 1, ..., 7$ *and* $j = 1, 2$. *Take* $q = 2$, $r_i = 1 = l_i$ *for* $i = 0, 1, ..., 7$ *and* $\varkappa(t) = 0.45\sin(t)$.

Then, all the conditions of Theorem 4.5 are satisfied with $k_1 = 23.4780 > 16.0745 = k_2 > 0$. *Hence, the trajectories of the system converge globally exponentially to the equilibrium point. The trajectories of the system (4.9) are shown in Figure 4.1.*

Example 4.2. *Consider the OVNNs (1) as*

$$\dot{\omega}(t) = -D\omega(t) + Af(\omega(t)) + Bg(\omega(t - \varkappa(t))) + I(t), \tag{4.10}$$

where $D = \begin{pmatrix} 120 & 0 \\ 0 & 130 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, $I = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$ *with*

$a_{11} = 0.3\varrho_0 + 0.2\varrho_1 - 0.2\varrho_2 + 0.4\varrho_3 + 0.1\varrho_4 - 0.1\varrho_5 - 0.3\varrho_6 - 0.3\varrho_7,$

$a_{12} = -0.3\varrho_0 + 0.3\varrho_1 + 0.1\varrho_2 + 0.4\varrho_3 + 0.3\varrho_4 - 0.1\varrho_5 + 0.2\varrho_6 + 0.2\varrho_7,$

$a_{21} = 0.2\varrho_0 - 0.5\varrho_1 - 0.2\varrho_2 + 0.1\varrho_3 + 0.1\varrho_4 + 0.3\varrho_5 - 0.4\varrho_6 - 0.1\varrho_7,$

$a_{22} = 0.1\varrho_0 + 0.3\varrho_1 + 0.1\varrho_2 + 0.2\varrho_3 - 0.2\varrho_4 - 0.1\varrho_5 + 0.4\varrho_6 + 0.3\varrho_7,$

$b_{11} = 0.1\varrho_0 + 0.2\varrho_1 + 0.5\varrho_2 + 0.3\varrho_3 + 0.2\varrho_4 - 0.2\varrho_5 - 0.4\varrho_6 - 0.3\varrho_7,$

$b_{12} = 0.4\varrho_0 - 0.3\varrho_1 - 0.7\varrho_2 + 0.1\varrho_3 + 0.1\varrho_4 + 0.4\varrho_5 + 0.3\varrho_6 - 0.5\varrho_7,$

$b_{21} = 0.3\varrho_0 - 0.2\varrho_1 - 0.1\varrho_2 - 0.1\varrho_3 - 0.3\varrho_4 + 0.2\varrho_5 + 0.3\varrho_6 + 0.2\varrho_7,$

$b_{22} = -0.4\varrho_0 - 0.2\varrho_1 - 0.3\varrho_2 + 0.1\varrho_3 + 0.1\varrho_4 + 0.2\varrho_5 + 0.4\varrho_6 - 0.3\varrho_7,$

$i_1 = -50\varrho_0 - 49\varrho_1 - 48\varrho_2 - 47\varrho_3 - 46\varrho_4 - 45\varrho_5 - 44\varrho_6 - 43\varrho_7,$

$i_2 = 40\varrho_0 + 42\varrho_1 + 44\varrho_2 + 46\varrho_3 + 48\varrho_4 + 50\varrho_5 + 52\varrho_6 + 54\varrho_7.$

Let the activation functions are given by $f_j^i(\omega(t)) = tanh(\omega_j^i(t))\varrho_i$, $g_j^i(\omega(t)) = tanh(\omega_j^i(t - \varkappa(t)))\varrho_i$ *for* $i = 0, 1, ..., 7$ *and* $j = 1, 2$. *Take* $q = 1$, $r_i = 1 = l_i$ *for* $i = 0, 1, ..., 7$ *and* $\varkappa(t) = \cos(t)$.

Then all the sufficient conditions of Theorem 4.7 are satisfied. Hence, the trajectories of neural network (4.10) converge globally exponentially to equilibrium point. The trajectories of this system are shown in Figure 4.2.

4.5 Conclusion

Sufficient criteria for global exponential stability of OVNNs have been derived using the matrix measure method and Halanay inequality. Matrix measure is a better

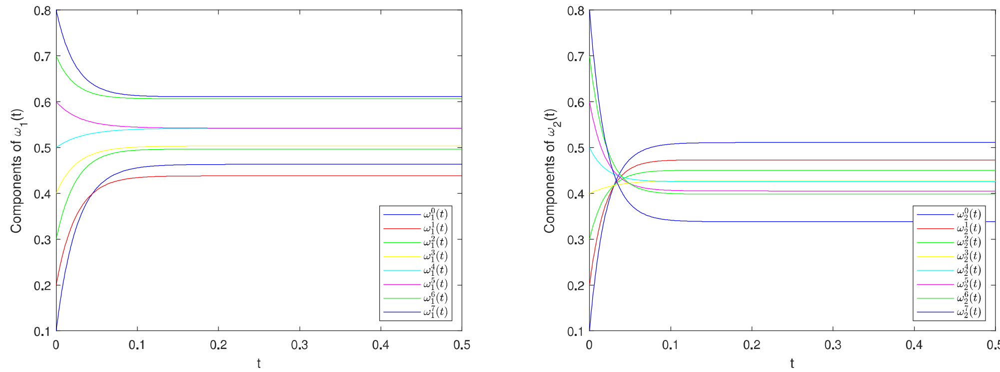

FIGURE 4.1: Trajectories of $\omega^0(t)$,$\omega^1(t)$,$\omega^2(t)$,$\omega^3(t)$,$\omega^4(t)$,$\omega^5(t)$,$\omega^6(t)$,$\omega^7(t)$ for $n = 2$.

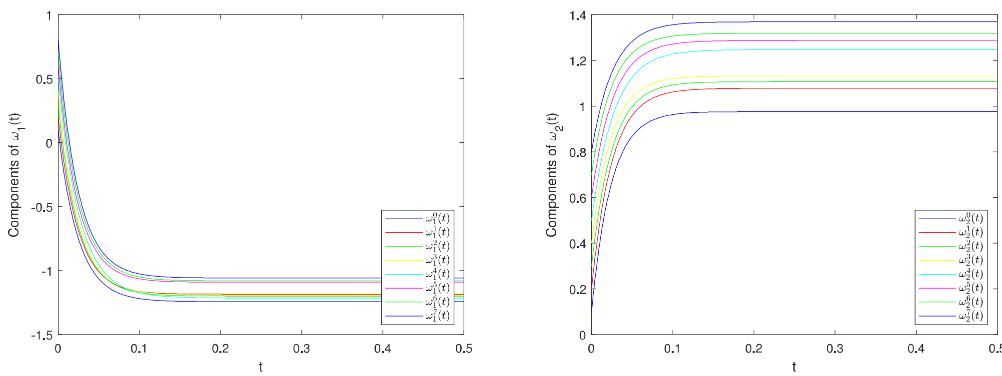

FIGURE 4.2: Trajectories of all the components of $\omega(t)$ for $n = 2$.

choice than matrix norm, as it can be positive as well as negative, whereas a matrix norm is always positive. Furthermore, two kinds of assumptions are used to obtain two sets of different sufficient conditions for the required global exponential stability. The results on RVNNs, CVNNs and QVNNs are special cases of the results obtained here. By using Halanay inequality, the results have been easily obtained. Also, OVNNs have been researched very little in the past. So, there is a plenty scope of research in OVNNs, in the years ahead. As an extension of the present work, global exponential stability of OVNNs with mixed time-varying delay could be done. Furthermore, an attempt to find the fixed-time stability and finite-time stability of OVNNs with different delays will be an exciting challenge for future research.

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