

# Chapter 4

## Numerical algorithm for the multi-term time-fractional electromagnetic wave model arising in dielectric medium

In this chapter, we have provided a numerical algorithm for solving multi-term time-fractional electromagnetic wave model is provided by using finite difference scheme. We have approximated the Caputo fractional derivative of order by the L3 approximation (in Sect. 4.2), and central difference scheme for the second order spatial derivative in Sect. 4.3. The proposed difference scheme is of second order in space and time direction for all fractional order  $1 < \alpha < \beta < 2$ . In Sect. 4.4, two test problems are given to validate the efficiency and accuracy of the difference scheme. To demonstrate the effectiveness of our scheme, a comparative study is also provided with the existing scheme. The chapter is concluded in Sect. 4.5.

## 4.1 Introduction

Consider the following multi-term time-fractional electromagnetic wave model (MT-TFEMWM) [? ]

$$\begin{cases} \lambda_1 {}_0^C D_t^\alpha \mathfrak{U}(x, t) + \lambda_2 {}_0^C D_t^\beta \mathfrak{U}(x, t) = \mathfrak{U}_{xx}(x, t) + f(x, t), & (x, t) \in \Omega = [x_1, x_r] \times [0, T], \\ I.C. : \mathfrak{U}(x, 0) = \phi_1(x), \quad \partial_t \mathfrak{U}(x, 0) = \phi_2(x). \\ B.C. : \mathfrak{U}(x_1, t) = \psi_1(x), \quad \mathfrak{U}(x_r, t) = \psi_2(x), \end{cases} \quad (4.1)$$

where,  $1 < \beta < \alpha < 2$  and The Caputo fractional derivatives is defined as

$${}_0^C D_t^\alpha \mathfrak{U}(x, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\mathfrak{U}''(\cdot, s)}{(t-s)^{\alpha-1}} ds, \quad \alpha \in (1, 2). \quad (4.2)$$

The primary contribution and advantage of the proposed numerical scheme can be expressed through the following points:

- We have applied numerical approximation for the Caputo derivative of order  $\alpha$  and  $\beta$  by L3 approximation [189] where,  $1 < \beta < \alpha < 2$ .
- A difference scheme is proposed that combines this approximation with central difference approximation for spatial derivatives. To validate the accuracy of our scheme, we provide some test examples of the MT-TFWM.
- In a computational framework, the order of convergence of the proposed difference scheme is  $\mathcal{O}(\tau^2, h^2)$ .
- A comparison of the proposed scheme with the existing schemes [132, 133] is presented to demonstrate it's efficiency and accuracy.

## 4.2 Time discretization

This section includes the L3 approximation of order  $\alpha$  and  $\beta$ , where  $1 < \beta < \alpha < 2$  of the Caputo derivative.

### 4.2.1 L3 approximation of the Caputo derivative of order $\alpha$ and $\beta \in (1, 2)$

The Caputo fractional derivative of order  $\alpha \in (1, 2)$  is defined as

$${}_0^C D_t^\alpha \mathfrak{U}(t)|_{t=t_k} = \frac{1}{\Gamma(2-\alpha)} \int_{t_0}^{t_k} \frac{\mathfrak{U}''(s)}{(t_k-s)^{\alpha-1}} ds.$$

We have applied the newly developed L3 approximation [189] of discretizing the Caputo time fractional derivative of order  $\alpha \in (1, 2)$ . Let  $\{t_k = k\tau, k = 0, 1, 2, \dots, N_t\}$  with step length  $\tau = T/N_t$  where,  $N_t$  denotes the temporal discretization parameter, then the L3 approximation of the Caputo derivative is defined as

$${}_0 \mathbb{D}_t^\alpha \mathfrak{U}(t_k) = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} \mathfrak{c}_{k-j-1}^{(k,\alpha)} \Delta^2 \mathfrak{U}_j. \quad (4.3)$$

The coefficients  $\mathfrak{a}_j^{(\alpha)}$ ,  $\mathfrak{b}_j^{(\alpha)}$  and  $\mathfrak{c}_j^{(k,\alpha)}$  for  $0 \leq j \leq k-1$  is defined as,

$$\mathfrak{a}_j^{(\alpha)} = (j+1)^{2-\alpha} - j^{2-\alpha}, \quad (4.4)$$

$$\mathfrak{b}_j^{(\alpha)} = \frac{1}{(3-\alpha)} \{(j+1)^{3-\alpha} - j^{3-\alpha}\} + (j+1)^{2-\alpha} - 2j^{2-\alpha}, \quad (4.5)$$

for  $k = 1$ ,

$$\mathfrak{c}_0^{(k,\alpha)} = 1,$$

for  $k = 2$ ,

$$c_j^{(k,\alpha)} = \begin{cases} -a_1^{(\alpha)} + b_0^{(\alpha)} + b_1^{(\alpha)}, & j = 0, \\ a_0^{(\alpha)} + 2a_1^{(\alpha)} - b_0^{(\alpha)} - b_1^{(\alpha)}, & j = 1, \end{cases} \quad (4.6)$$

for  $k = 3$ ,

$$c_j^{(k,\alpha)} = \begin{cases} b_0^{(\alpha)}, & j = 0, \\ a_0^{(\alpha)} - a_2^{(\alpha)} - b_0^{(\alpha)} + b_1^{(\alpha)} + b_2^{(\alpha)}, & j = 1, \\ a_1^{(\alpha)} + 2a_2^{(\alpha)} - b_1^{(\alpha)} - b_2^{(\alpha)}, & j = 2, \end{cases} \quad (4.7)$$

for  $k \geq 4$ ,

$$c_j^{(k,\alpha)} = \begin{cases} b_0^{(\alpha)} & j = 0, \\ a_{j-1}^{(\alpha)} - b_{j-1}^{(\alpha)} + b_j^{(\alpha)} & 1 \leq j \leq k-3, \\ a_{k-3}^{(\alpha)} - a_{k-1}^{(\alpha)} - b_{k-3}^{(\alpha)} + b_{k-2}^{(\alpha)} + b_{k-1}^{(\alpha)}, & j = k-2, \\ a_{k-2}^{(\alpha)} + 2a_{k-1}^{(\alpha)} - b_{k-2}^{(\alpha)} - b_{k-1}^{(\alpha)}, & j = k-1. \end{cases} \quad (4.8)$$

**Theorem 4.2.1.** The truncation error in L3 approximation is given by

$${}_0^C D_t^\alpha \mathcal{U}(t)|_{t=t_k} - {}_0 \mathbb{D}_t^\alpha \mathcal{U}(t_k) = \mathcal{O}(\tau^2). \quad (4.9)$$

*Proof.* The proof follows [189]. □

### 4.3 Numerical scheme for multi-term time-fractional electromagnetic wave model (MTTFEMWM)

Let  $\{x_i = x_1 + ih, i = 0, 1, 2, \dots, M_x\}$  and  $\{t_k = k\tau, k = 0, 1, 2, \dots, N_t\}$  be the uniform meshes with step lengths  $h = (x_r - x_l)/M_x$  and  $\tau = T/N_t$  where  $M_x$  and  $N_t$  are the discretization parameters in  $x$  and  $t$  respectively. The MTTFEMWM at each discrete point  $(x_i, t_k)$  can be written as

$$\lambda_{10}^C D_t^\alpha \mathfrak{U}(x_i, t_k) + \lambda_{20}^C D_t^\beta \mathfrak{U}(x_i, t_k) = \lambda_3 \frac{\partial^2 \mathfrak{U}(x_i, t_k)}{\partial x^2} + f(x_i, t_k), \quad (4.10)$$

with initial conditions

$$\mathfrak{U}(x_i, 0) = \phi_1(x_i), \quad \partial_t \mathfrak{U}(x_i, 0) = \phi_2(x_i), \quad (4.11)$$

and boundary conditions

$$\mathfrak{U}(x_1, t_k) = \psi_1(t_k), \quad \mathfrak{U}(x_r, t_k) = \psi_2(t_k). \quad (4.12)$$

For the first time level  $t_1$ , applying L3 approximation in time-fractional derivative of order  $\alpha$  and orders  $\beta$  and central difference scheme for space derivative in (4.10), we have

$$\begin{aligned} \lambda_1 \mathbb{D}_t^\alpha \mathfrak{U}(x_i, t_1) + \lambda_2 \mathbb{D}_t^\beta \mathfrak{U}(x_i, t_1) &= \lambda_3 \frac{\partial^2 \mathfrak{U}(x_i, t_1)}{\partial x^2} + f(x_i, t_1), \\ \lambda_1 \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} (\mathfrak{U}_i^1 - 2\mathfrak{U}_i^0 + \mathfrak{U}_i^{-1}) + \lambda_2 \frac{\tau^{-\beta}}{\Gamma(3-\beta)} (\mathfrak{U}_i^1 - 2\mathfrak{U}_i^0 + \mathfrak{U}_i^{-1}) &= \frac{\lambda_3}{h^2} (\mathfrak{U}_{i+1}^1 - 2\mathfrak{U}_i^1 + \mathfrak{U}_{i-1}^1) \\ &\quad + f_i^1. \end{aligned} \quad (4.13)$$

Since,  $\partial_t \mathfrak{U}(x_i, 0) = \phi_2(x_i)$  therefore

$$\frac{\mathfrak{U}_i^1 - \mathfrak{U}_i^{-1}}{2\tau} = \phi_2(x_i) \quad \text{and} \quad \mathfrak{U}_i^{-1} = \mathfrak{U}_i^1 - 2\tau\phi_2(x_i). \quad (4.14)$$

Let  $\mu^{(\alpha)} = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}$  and  $\mu^{(\beta)} = \frac{\tau^{-\beta}}{\Gamma(3-\beta)}$ . Now, rearranging the terms, we get

$$\begin{aligned} \left( \lambda_1 \mu^{(\alpha)} \mathbf{c}_0^{(k,\alpha)} + \lambda_2 \mu^{(\beta)} \mathbf{c}_0^{(k,\beta)} \right) (-2\mathfrak{U}_i^0 + \mathfrak{U}_i^{-1}) - \mathfrak{f}_i^1 &= \frac{\lambda_3}{h^2} \mathfrak{U}_{i+1}^1 - \left( \lambda_1 \mu^{(\alpha)} \mathbf{c}_0^{(k,\alpha)} + \lambda_2 \mu^{(\beta)} \mathbf{c}_0^{(k,\beta)} \right) \\ &\quad \mathfrak{U}_i^1 + \frac{\lambda_3}{h^2} \mathfrak{U}_{i-1}^1. \end{aligned} \quad (4.15)$$

From the boundary conditions, we can get  $\mathfrak{U}_0^1 = \psi_1(\mathbf{t}_1)$  and  $\mathfrak{U}_M^1 = \psi_2(\mathbf{t}_k)$ . Therefore the discrete form of TFWE at time level  $\mathbf{t}_k$  for  $k \geq 2$  is

$$\lambda_1 \mathbb{D}_t^\alpha \mathfrak{U}(x_i, \mathbf{t}_k) + \lambda_2 \mathbb{D}_t^\beta \mathfrak{U}(x_i, \mathbf{t}_k) = \lambda_3 \frac{\partial^2 \mathfrak{U}(x_i, \mathbf{t}_k)}{\partial x^2} + \mathfrak{f}(x_i, \mathbf{t}_k), \quad (4.16)$$

$$\lambda_1 \mu^{(\alpha)} \left[ \sum_{j=0}^{k-1} \mathbf{c}_{k-j-1}^{(k,\alpha)} \Delta^2 \mathfrak{U}_i^{j-1} \right] + \lambda_2 \mu^{(\beta)} \left[ \sum_{j=0}^{k-1} \mathbf{c}_{k-j-1}^{(k,\beta)} \Delta^2 \mathfrak{U}_i^{j-1} \right] = \frac{\lambda_3}{h^2} (\mathfrak{U}_{i+1}^k - 2\mathfrak{U}_i^k + \mathfrak{U}_{i-1}^k) + \mathfrak{f}_i^k, \quad (4.17)$$

on simplifying, we have

$$\begin{aligned} &\left[ \lambda_1 \mu^{(\alpha)} \mathbf{c}_{k-1}^{(k,\alpha)} + \lambda_2 \mu^{(\beta)} \mathbf{c}_{k-1}^{(k,\beta)} \right] \mathfrak{U}_i^{-1} + \left[ \lambda_1 \mu^{(\alpha)} (\mathbf{c}_{k-2}^{(k,\alpha)} - 2\mathbf{c}_{k-1}^{(k,\alpha)}) + \lambda_2 \mu^{(\beta)} (\mathbf{c}_{k-2}^{(k,\beta)} - 2\mathbf{c}_{k-1}^{(k,\beta)}) \right] \mathfrak{U}_i^0 \\ &+ \left[ \lambda_1 \mu^{(\alpha)} \sum_{j=2}^{k-1} (\mathbf{c}_{k-j+1}^{(k,\alpha)} - 2\mathbf{c}_{k-j}^{(k,\alpha)} + \mathbf{c}_{k-j-1}^{(k,\alpha)}) + \lambda_2 \mu^{(\beta)} \sum_{j=2}^{k-1} (\mathbf{c}_{k-j+1}^{(k,\beta)} - 2\mathbf{c}_{k-j}^{(k,\beta)} + \mathbf{c}_{k-j-1}^{(k,\beta)}) \right] \mathfrak{U}_i^{j-1} + \\ &\left[ \lambda_1 \mu^{(\alpha)} (\mathbf{c}_1^{(k,\alpha)} - 2\mathbf{c}_0^{(k,\alpha)}) + \lambda_2 \mu^{(\beta)} (\mathbf{c}_1^{(k,\beta)} - 2\mathbf{c}_0^{(k,\beta)}) \right] \mathfrak{U}_i^{k-1} \\ &= \frac{\lambda_3}{h^2} \mathfrak{U}_{i+1}^k - \left( \frac{2\lambda_3}{h^2} + \lambda_1 \mu^{(\alpha)} \mathbf{c}_0^{(k,\alpha)} + \lambda_2 \mu^{(\beta)} \mathbf{c}_0^{(k,\beta)} \right) \mathfrak{U}_i^k + \frac{\lambda_3}{h^2} \mathfrak{U}_{i-1}^k + \mathfrak{f}_i^k. \end{aligned} \quad (4.18)$$

The scheme (4.18) can be expressed in matrix form as follows:

$$\left\{ \begin{array}{l} \mathbb{A}_1 \mathbb{U}^1 = -2\mathbb{U}^0 + \mathbb{U}^{-1} - \mathbb{F}^1, \quad \text{for } k = 1. \\ \mathbb{A}_2 \mathbb{U}^2 = \left[ \lambda_1 \mu^{(\alpha)} \mathbf{c}_1^{(k,\alpha)} + \lambda_2 \mu^{(\beta)} \mathbf{c}_1^{(k,\beta)} \right] \mathbb{U}_i^{-1} + \left[ \lambda_1 \mu^{(\alpha)} (\mathbf{c}_0^{(k,\alpha)} - 2\mathbf{c}_1^{(k,\alpha)}) + \lambda_2 \mu^{(\beta)} (\mathbf{c}_0^{(k,\beta)} - 2\mathbf{c}_1^{(k,\beta)}) \right] \mathbb{U}_i^0 + \left[ \lambda_1 \mu^{(\alpha)} (\mathbf{c}_1^{(k,\alpha)} - 2\mathbf{c}_0^{(k,\alpha)}) + \lambda_2 \mu^{(\beta)} (\mathbf{c}_1^{(k,\beta)} - 2\mathbf{c}_0^{(k,\beta)}) \right] \mathbb{U}_i^1 - \mathbb{F}^k, \\ \text{for } k = 2. \\ \mathbb{A}_k \mathbb{U}^k = \left[ \lambda_1 \mu^{(\alpha)} \mathbf{c}_{k-1}^{(k,\alpha)} + \lambda_2 \mu^{(\beta)} \mathbf{c}_{k-1}^{(k,\beta)} \right] \mathbb{U}_i^{-1} + \left[ \lambda_1 \mu^{(\alpha)} (\mathbf{c}_{k-2}^{(k,\alpha)} - 2\mathbf{c}_{k-1}^{(k,\alpha)}) + \lambda_2 \mu^{(\beta)} (\mathbf{c}_{k-2}^{(k,\beta)} - 2\mathbf{c}_{k-1}^{(k,\beta)}) \right] \mathbb{U}_i^0 + \left[ \lambda_1 \mu^{(\alpha)} \sum_{j=2}^{k-1} (\mathbf{c}_{k-j+1}^{(k,\alpha)} - 2\mathbf{c}_{k-j}^{(k,\alpha)} + \mathbf{c}_{k-j-1}^{(k,\alpha)}) + \lambda_2 \mu^{(\beta)} \sum_{j=2}^{k-1} (\mathbf{c}_{k-j+1}^{(k,\beta)} - 2\mathbf{c}_{k-j}^{(k,\beta)} + \mathbf{c}_{k-j-1}^{(k,\beta)}) \right] \mathbb{U}_i^{j-1} + \left[ \lambda_1 \mu^{(\alpha)} (\mathbf{c}_1^{(k,\alpha)} - 2\mathbf{c}_0^{(k,\alpha)}) + \lambda_2 \mu^{(\beta)} (\mathbf{c}_1^{(k,\beta)} - 2\mathbf{c}_0^{(k,\beta)}) \right] \mathbb{U}_i^{k-1} - \mathbb{F}^k, \text{ for } k \geq 3. \end{array} \right.$$

Where the matrix  $\mathbb{A}_k$ ,  $k = 1, 2, \dots, (N_t - 1)$  is a tridiagonal matrix of order  $(N_t - 1)$

$$\mathbb{A}_k = \text{tridiag} \left[ \frac{\lambda_3}{h^2}, - \left( \frac{2\lambda_3}{h^2} + \lambda_1 \mu^{(\alpha)} \mathbf{c}_0^{(k,\alpha)} + \lambda_2 \mu^{(\beta)} \mathbf{c}_0^{(k,\beta)} \right), \frac{\lambda_3}{h^2} \right], \text{ for } k \geq 1, \quad (4.19)$$

and  $\mathbb{F}^k = [f_1^k, f_2^k, \dots, f_{N_t-1}^k]^T$ . The vector  $\mathbb{U}^k = [\mathfrak{U}_1^k, \mathfrak{U}_2^k, \dots, \mathfrak{U}_{N_t-1}^k]^T$  be the numerical solution at  $t_k$ .

The algorithm for solving by the proposed numerical scheme (4.18) is given below.

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**Algorithm 3:** To evaluate the numerical solution of MTTFEMWM (4.1)

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**Input:** The rectangular domain  $\Omega = [0, L] \times [0, T]$ ,  $M_x$ ,  $h = L/M_x$ ,  $N_t$ ,  $\tau = T/N_t$ ,

initial conditions  $\phi_1(x)$ ,  $\phi_2(x)$ , boundary conditions  $\psi_1(x)$ ,  $\psi_2(x)$  and

$1 < \beta < \alpha < 2$ .

**Output:** The discrete solutions at each grid point  $(x_i, t_k)$

**for** Numerical solution of MTTFEMWM by the difference scheme **do**

**Step-1.1** Discretize the rectangular domain in uniform step size

$x_i = ih$ ,  $i = 0, 1, \dots, M_x$  and  $t_k = k\tau$ ,  $k = 0, 1, \dots, N_t$ .

**Step-1.2** Apply L3 approximation (4.3) to discretize the Caputo fractional

derivative in time of order (1,2) and central difference scheme to discretise the spatial derivatives term.

**Step-1.3** Use initial condition  $\mathfrak{U}(x_i, 0) = \phi_1(x_i)$  and  $\partial_t \mathfrak{U}(x_i, 0) = \phi_2(x_i)$  to

calculate  $\mathfrak{U}_i^0$  and  $\mathfrak{U}_i^{-1}$ , respectively.

**Step-1.4** Use the boundary conditions  $\mathfrak{U}(0, t) = \psi_1(x_i)$  and  $\mathfrak{U}(L, t) = \psi_1(x_i)$  to

calculate the values of  $\mathfrak{U}_0^k$  and  $\mathfrak{U}_{M_x}^k$ , respectively.

**Step-1.5** The difference scheme for the numerical solution at first time level  $\mathfrak{U}_i^1$  is

given in (4.15).

**Step-1.6** Use this solution  $\mathfrak{U}_i^1$  to get the numerical solution  $\mathfrak{U}_i^k$  at time level  $t_k$  for

$k \geq 2$  by the difference scheme (4.18).

**Step-1.7** Repeat Step 1.6 and use all the values of  $\mathfrak{U}_i^k$  at each previous time levels

till we get the discrete solution of MT-TFWE at each discretised points  $(x_i, t_k)$ .

**end**

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## 4.4 Numerical Results and discussion

### 4.4.1 Numerical examples for the MTTFEMWM

In this section, we give the numerical example of the multi-term time-fractional electromagnetic wave model to demonstrate the effectiveness of the numerical scheme.

**Example 4.1.** Consider the following MTTFEMWM with homogeneous boundary conditions [132]

$$\begin{cases} {}^C D_t^\alpha \mathfrak{U}(x, t) + {}^C D_t^\beta \mathfrak{U}(x, t) - \frac{\partial^2 \mathfrak{U}(x, t)}{\partial x^2} = \text{curl } j(x, t), & (x, t) \in [0, 1] \times [0, 1] \\ \text{Initial condition : } \mathfrak{U}(x, 0) = \partial_t \mathfrak{U}(x, 0) = 0, & x \in [0, 1], \\ \text{Boundary condition : } \mathfrak{U}(l, t) = 0, & l = 0, 1. \end{cases} \quad (4.20)$$

The value of source term  $\text{curl } j(x, t)$  is different for different choices of  $\beta$  and  $\alpha$ . Exact solution of Ex. 4.1 is  $\mathfrak{U}(x, t) = t^3 x^{(1+\alpha+\beta)}(1-x)$ .

Let  $N_t = 100$  and  $M_x = 100$  then the findings of Ex. 4.1 are given as follows:

- Fig. 4.1-4.4 shows the maximum absolute error (MAE) for different values of  $(\beta, \alpha)$  such as (1.2,1.7), (1.3,1.8), (1.4,1.9) and (1.5,1.99) respectively, when  $\tau = 1/100$  and  $h = 1/100$ . It shows that, the numerical solutions attain good accuracy at every time step for all values of  $\beta, \alpha$ .
- Fig. 4.5 depicts the MAE at  $T = 1$  for various  $\beta$  and  $\alpha$ , when  $M_x = 100$  and  $N_t = 100$ . It confirms that the proposed method can be credible in the large scale problems.

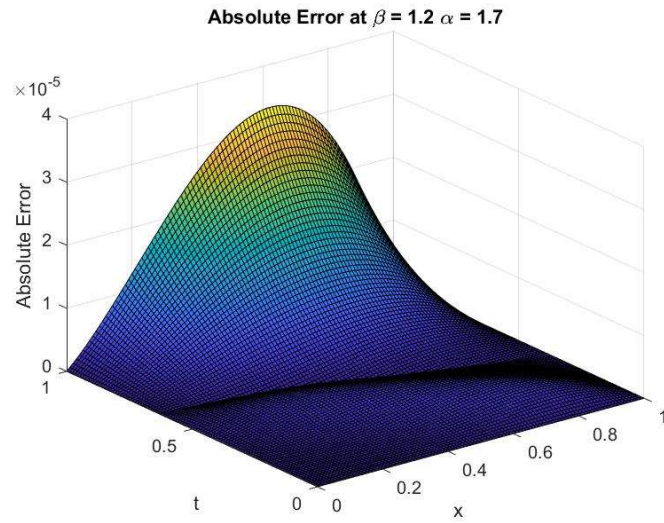


FIGURE 4.1: Absolute Error of Ex. 4.1 at  $\beta = 1.2$  and  $\alpha = 1.7$ ,  $N_t = 100$ ,  $M_x = 100$ .

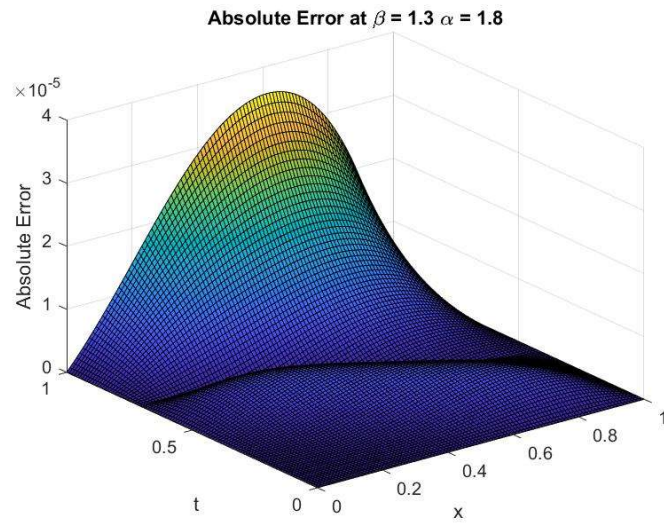


FIGURE 4.2: Absolute Error of Ex. 4.1 at  $\beta = 1.3$  and  $\alpha = 1.8$ ,  $N_t = 100$ ,  $M_x = 100$ .

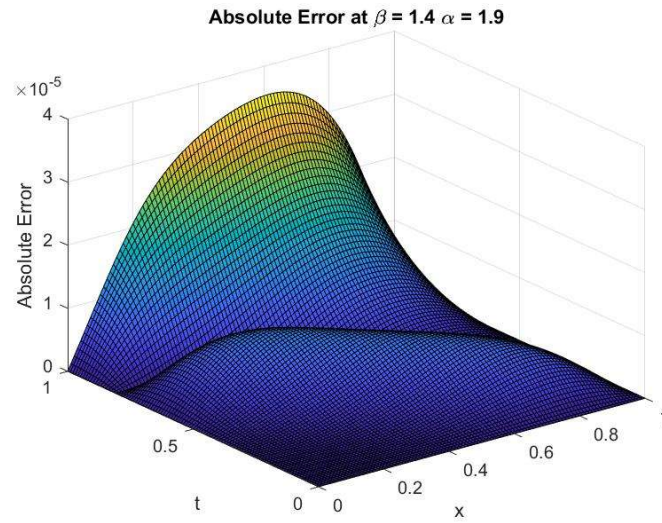


FIGURE 4.3: Absolute Error of Ex. 4.1 at  $\beta = 1.4$  and  $\alpha = 1.9$ ,  $N_t = 100$ ,  $M_x = 100$ .

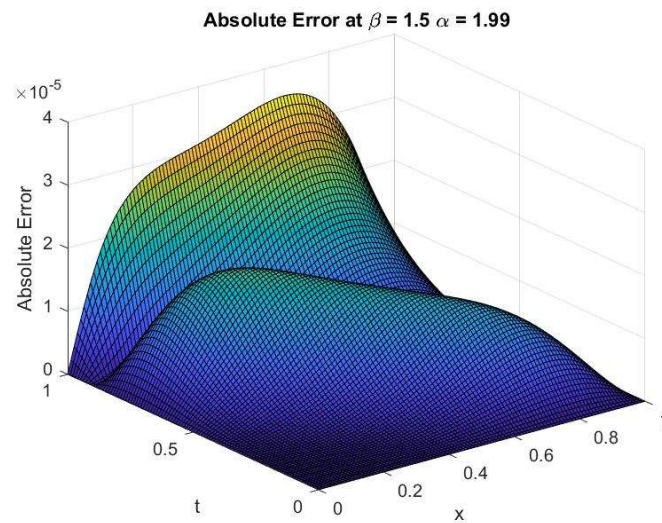
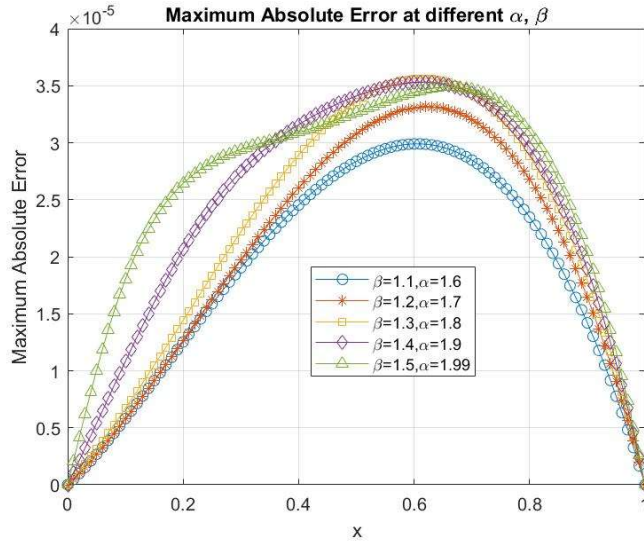


FIGURE 4.4: Absolute Error of Ex. 4.1 at  $\beta = 1.5$  and  $\alpha = 1.99$ ,  $N_t = 100$ ,  $M_x = 100$ .

FIGURE 4.5: Absolute Error of Ex. 4.1 at different  $\beta$  and  $\alpha$ ,  $N_t = 100$ ,  $M_x = 100$ .TABLE 4.1: Comparative study of  $L_2$  error for Ex. 4.1 different values of  $\beta$  and  $\alpha$  at  $T = 1$  when  $M_x = 1000$ .

| $\tau$       | $\beta = 1.1, \alpha = 1.5$ |            |            | $\beta = 1.5, \alpha = 1.7$ |            |            | $\beta = 1.3, \alpha = 1.9$ |            |            |
|--------------|-----------------------------|------------|------------|-----------------------------|------------|------------|-----------------------------|------------|------------|
|              | $L_2$ error                 | Error[133] | Error[132] | $L_2$ error                 | Error[133] | Error[132] | $L_2$ error                 | Error[133] | Error[132] |
| <b>1/10</b>  | 1.008e-04                   | 7.489e-03  | 4.063e-04  | 1.721e-04                   | 1.508e-03  | 2.773e-04  | 8.059e-05                   | 2.105e-03  | 1.801e-03  |
| <b>1/20</b>  | 1.766e-05                   | 2.605e-04  | 1.337e-04  | 2.363e-5                    | 6.159e-04  | 8.220e-05  | 8.568e-05                   | 9.941e-04  | 1.038e-03  |
| <b>1/40</b>  | 2.816e-06                   | 9.064e-05  | 4.477e-05  | 7.992e-07                   | 2.487e-04  | 1.887e-05  | 3.361e-05                   | 4.639e-04  | 5.562e-04  |
| <b>1/80</b>  | 6.6613e-07                  | 3.168e-05  | 1.544e-05  | 7.997e-07                   | 1.063e-04  | 6.139e-06  | 9.451e-06                   | 2.182e-04  | 2.804e-04  |
| <b>1/160</b> | 2.968e-07                   | -          | -          | 8.595e-08                   | -          | -          | 2.459e-06                   | -          | -          |

TABLE 4.2: Comparative study of  $L_\infty$  error for Ex. 4.1 different values of  $\beta$  and  $\alpha$  at  $T = 1$  when  $M_x = 1000$ .

| $\tau$       | $\beta = 1.1, \alpha = 1.5$ |            | $\beta = 1.5, \alpha = 1.7$ |            | $\beta = 1.3, \alpha = 1.9$ |            |
|--------------|-----------------------------|------------|-----------------------------|------------|-----------------------------|------------|
|              | $L_\infty$ error            | Error[132] | $L_\infty$ error            | Error[132] | $L_\infty$ error            | Error[132] |
| <b>1/10</b>  | 1.421e-04                   | 5.892e-04  | 2.426e-04                   | 4.169e-04  | 1.463e-04                   | 2.535e-03  |
| <b>1/20</b>  | 2.504e-05                   | 1.966e-04  | 3.803e-05                   | 1.337e-04  | 1.390e-04                   | 1.454e-03  |
| <b>1/40</b>  | 3.967e-06                   | 6.646e-05  | 1.328e-06                   | 3.510e-05  | 5.797e-05                   | 7.795e-04  |
| <b>1/80</b>  | 9.284e-07                   | 2.307e-05  | 1.406e-06                   | 8.778e-06  | 1.647e-05                   | 3.934e-04  |
| <b>1/160</b> | 4.175e-07                   | 8.036e-06  | 1.314e-07                   | 5.513e-06  | 4.226e-06                   | 1.914e-04  |

- Table 4.1 provides the comparative study of the  $L_2$  error for different  $\alpha$ , and  $\beta$  at  $T = 1$  by taking  $M_x = 1000$  grid points. It is found that the scheme are highly accurate for all  $\alpha, \beta$  and in comparison of [133], [132].
- Table 4.2 provides the comparative study of the  $L_\infty$  error for  $\alpha, \beta$  at  $T = 1$  by taking  $M_x = 1000$ . It is found that the scheme are highly accurate for all  $\alpha, \beta$  in comparison to [132].
- It can be observed from Tables 4.1 and 4.2 that the scheme provide high accuracy as compared to the existing schemes.
- Table 4.3 provides the temporal order of convergence in time w.r.t.  $L_2$  error and  $L_\infty$  error for different  $\alpha, \beta$  at  $T = 1$  by taking  $N_t = 10 \times 2^m, m = 0, 1, \dots, 7$  and  $M_x = 2000$  grid points.
- Table 4.4 provides the spatial order of convergence in time w.r.t.  $L_2$  error and  $L_\infty$  error for different  $\alpha, \beta$  at  $T = 1$  by taking  $M_x = 5 \times 2^m, m = 0, 1, \dots, 7$  and  $N_t = 1000$  grid points.
- It can be observed from Table 4.3-4.4 that the scheme provide second order of convergence in space and time direction.

TABLE 4.3: At time  $T=1$ , when  $h = 1/2000$ ,  $L_2$  error,  $L_\infty$  error and order in Ex. 4.1.

| $\beta, \alpha$   | $\tau$ | Our Scheme  |       | Our Scheme       |       |
|-------------------|--------|-------------|-------|------------------|-------|
|                   |        | $L_2$ Error | Order | $L_\infty$ Error | Order |
| <b>(1.2, 1.7)</b> | 1/10   | 2.169e-04   | 0     | 3.102e-04        | 0     |
|                   | 1/20   | 7.765e-05   | 1.482 | 1.116e-04        | 1.475 |
|                   | 1/40   | 1.855e-05   | 2.066 | 2.662e-05        | 2.067 |
|                   | 1/80   | 4.110e-06   | 2.174 | 5.883e-06        | 2.178 |
|                   | 1/160  | 9.261e-07   | 2.150 | 1.324e-06        | 2.152 |
|                   | 1/320  | 2.394e-07   | 1.952 | 3.417e-07        | 1.954 |
|                   | 1/640  | 9.196e-08   | 1.380 | 1.314e-07        | 1.379 |
| <b>(1.3, 1.8)</b> | 1/10   | 2.371e-04   | 0     | 3.427e-04        | 0     |
|                   | 1/20   | 1.125e-04   | 1.075 | 1.725e-04        | 0.990 |
|                   | 1/40   | 3.106e-05   | 1.857 | 4.786e-05        | 1.850 |
|                   | 1/80   | 7.422e-06   | 2.065 | 1.139e-05        | 2.071 |
|                   | 1/160  | 1.735e-06   | 2.097 | 2.639e-06        | 2.110 |
|                   | 1/320  | 4.288e-07   | 2.017 | 6.402e-07        | 2.043 |
|                   | 1/640  | 1.347e-07   | 1.671 | 1.928e-07        | 1.732 |
| <b>(1.4,1.9)</b>  | 1/10   | 1.895e-04   | 0     | 2.863e-04        | 0     |
|                   | 1/20   | 1.312e-04   | 0.530 | 2.143e-04        | 0.418 |
|                   | 1/40   | 4.599e-05   | 1.513 | 7.882e-05        | 1.443 |
|                   | 1/80   | 1.238e-05   | 1.893 | 2.149e-05        | 1.875 |
|                   | 1/160  | 3.088e-06   | 2.003 | 5.367e-06        | 2.002 |
|                   | 1/320  | 7.695e-07   | 2.005 | 1.317e-06        | 2.027 |
|                   | 1/640  | 2.121e-07   | 1.859 | 3.392e-07        | 1.957 |
| <b>(1.5,1.99)</b> | 1/10   | 1.334e-04   | 0     | 2.211e-04        | 0     |
|                   | 1/20   | 1.075e-04   | 0.312 | 1.771e-04        | 0.320 |
|                   | 1/40   | 5.049e-05   | 1.090 | 9.122e-05        | 0.957 |
|                   | 1/80   | 1.637e-05   | 1.625 | 3.020e-05        | 1.595 |
|                   | 1/160  | 4.545e-06   | 1.849 | 8.396e-06        | 1.847 |
|                   | 1/320  | 1.201e-06   | 1.920 | 2.203e-06        | 1.930 |
|                   | 1/640  | 3.250e-07   | 1.886 | 5.722e-07        | 1.945 |

TABLE 4.4:  $L_2$  error,  $L_\infty$  error and order in Ex. 4.1 at time  $t = 1$  when  $\tau = 1/1000$ .

| $\beta, \alpha$    | $h$   | Our Scheme  |       | Our Scheme       |       |
|--------------------|-------|-------------|-------|------------------|-------|
|                    |       | $L_2$ Error | Order | $L_\infty$ Error | Order |
| <b>(1.2, 1.7)</b>  | 1/5   | 8.141e-03   | 0     | 1.171e-02        | 0     |
|                    | 1/10  | 2.057e-03   | 1.985 | 2.953e-03        | 1.988 |
|                    | 1/20  | 5.153e-04   | 1.997 | 7.433e-04        | 1.990 |
|                    | 1/40  | 1.289e-04   | 1.999 | 1.861e-04        | 1.998 |
|                    | 1/80  | 3.224e-05   | 1.999 | 4.657e-05        | 1.999 |
|                    | 1/160 | 8.071e-06   | 1.998 | 1.166e-05        | 1.998 |
|                    | 1/320 | 2.029e-06   | 1.992 | 2.930e-06        | 1.992 |
| <b>(1.3, 1.8)</b>  | 1/5   | 8.233e-03   | 0     | 1.193e-02        | 0     |
|                    | 1/10  | 2.092e-03   | 1.977 | 3.050e-03        | 1.968 |
|                    | 1/20  | 5.250e-04   | 1.995 | 7.726e-04        | 1.981 |
|                    | 1/40  | 1.314e-04   | 1.998 | 1.933e-04        | 1.999 |
|                    | 1/80  | 3.287e-05   | 1.999 | 4.838e-05        | 1.998 |
|                    | 1/160 | 8.287e-06   | 1.996 | 1.212e-05        | 1.997 |
|                    | 1/320 | 2.082e-07   | 1.985 | 3.056e-06        | 1.988 |
| <b>(1.4, 1.9)</b>  | 1/5   | 8.208e-03   | 0     | 1.193e-02        | 0     |
|                    | 1/10  | 2.098e-03   | 1.968 | 3.168e-03        | 1.913 |
|                    | 1/20  | 5.270e-04   | 1.993 | 7.948e-04        | 1.995 |
|                    | 1/40  | 1.319e-04   | 1.998 | 1.992e-04        | 1.997 |
|                    | 1/80  | 3.303e-05   | 1.998 | 4.984e-05        | 1.999 |
|                    | 1/160 | 8.290e-06   | 1.994 | 1.248e-05        | 1.997 |
|                    | 1/320 | 2.106e-06   | 1.977 | 3.141e-06        | 1.991 |
| <b>(1.5, 1.99)</b> | 1/5   | 8.203e-03   | 0     | 1.190e-02        | 0     |
|                    | 1/10  | 2.107e-03   | 1.961 | 3.281e-03        | 1.858 |
|                    | 1/20  | 5.299e-04   | 1.991 | 8.238e-04        | 1.994 |
|                    | 1/40  | 1.327e-04   | 1.998 | 2.062e-04        | 1.998 |
|                    | 1/80  | 3.323e-05   | 1.998 | 5.157e-05        | 1.999 |
|                    | 1/160 | 8.346e-06   | 1.993 | 1.291e-05        | 1.998 |
|                    | 1/320 | 2.127e-06   | 1.972 | 3.239e-06        | 1.994 |

**Example 4.2.** [132] Consider the example

$$\begin{cases} {}^C_0D_t^\alpha \mathfrak{U}(x, t) + {}^C_0D_t^\beta \mathfrak{U}(x, t) = \frac{\partial^2 \mathfrak{U}(x, t)}{\partial x^2} + \text{curl } j(x, t), & (x, t) \in [0, 1] \times [0, 1] \\ I.C. : \mathfrak{U}(x, 0) = \partial_t \mathfrak{U}(x, 0) = 0, & x \in [0, 1], \\ B.C. : \mathfrak{U}(l, t) = 0, & l = 0, 1, \end{cases} \quad (4.21)$$

where  $\text{curl } j(x, t)$  is the source function. The exact solution of Ex. 4.2 is  $\mathfrak{U}(x, t) = t^3 \sin \pi x$ .

Let  $N_t = 100$  and  $M_x = 100$  then the findings of Ex. 4.2 are given as follows:

- Figs. 4.6-4.9 shows the MAE for different value of  $(\beta, \alpha)$  such as (1.2,1.7), (1.3,1.8), (1.4,1.9) and (1.5,1.99) respectively, when  $\tau = 1/100$  and  $h = 1/100$ . It shows that, the numerical solutions attains good accuracy at every time step for all  $\alpha, \beta$ .
- Fig. 4.10 depicts the MAE at  $T = 1$  for various  $\alpha, \beta$  when  $M_x = 100$  and  $N_t = 100$ . It confirms that the proposed method can be trusted in the large scale problems.

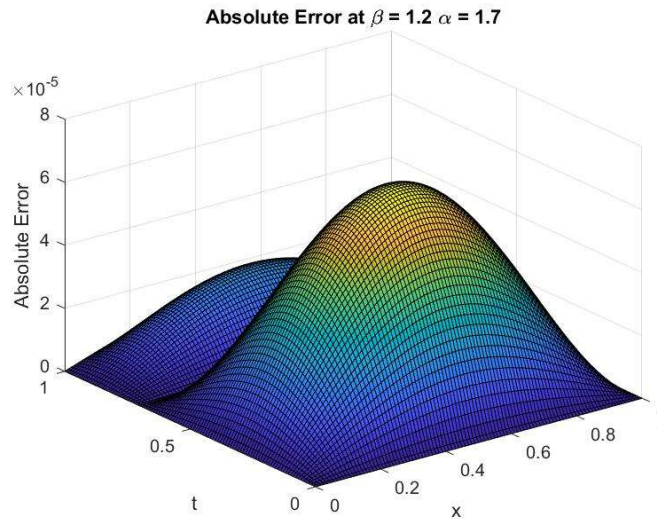


FIGURE 4.6: Absolute Error of Ex. 4.2 at  $\beta = 1.2$  and  $\alpha = 1.7$ ,  $N_t = 100$ ,  $M_x = 100$ .

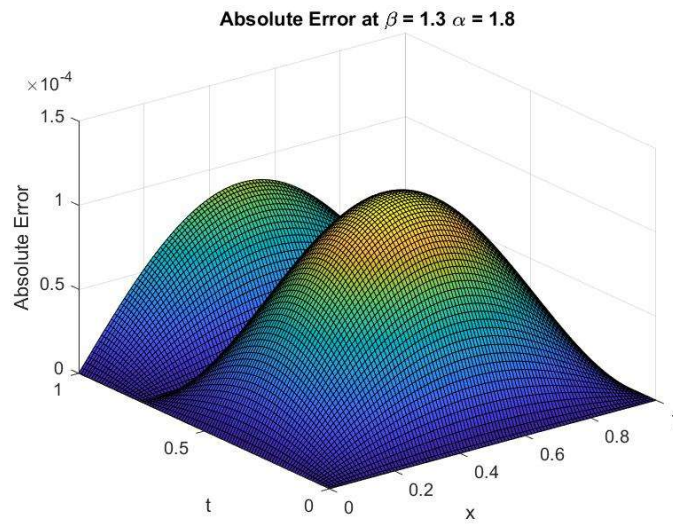


FIGURE 4.7: Absolute Error of Ex. 4.2 at  $\beta = 1.3$  and  $\alpha = 1.8$ ,  $N_t = 100$ ,  $M_x = 100$ .

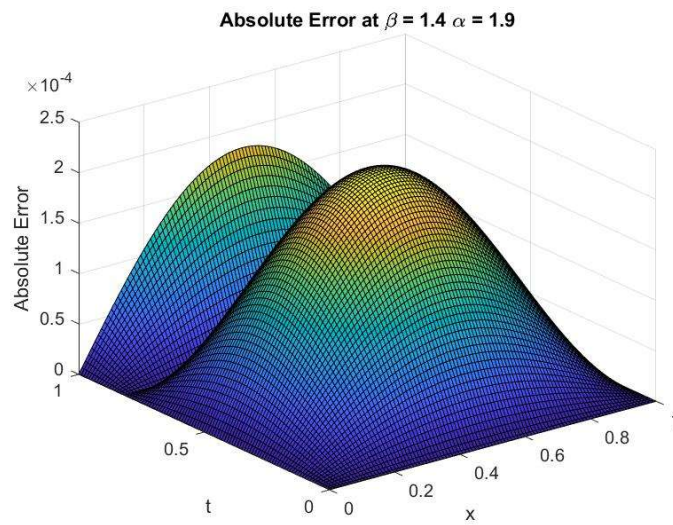


FIGURE 4.8: Absolute Error of Ex. 4.2 at  $\beta = 1.4$  and  $\alpha = 1.9$ ,  $N_t = 100$ ,  $M_x = 100$ .

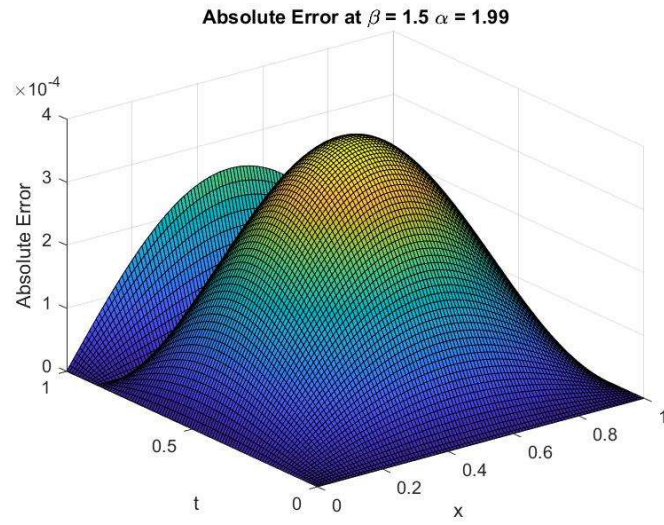


FIGURE 4.9: Absolute Error of Ex. 4.2 at  $\beta = 1.5$  and  $\alpha = 1.99$ ,  $N_t = 100$ ,  $M_x = 100$ .

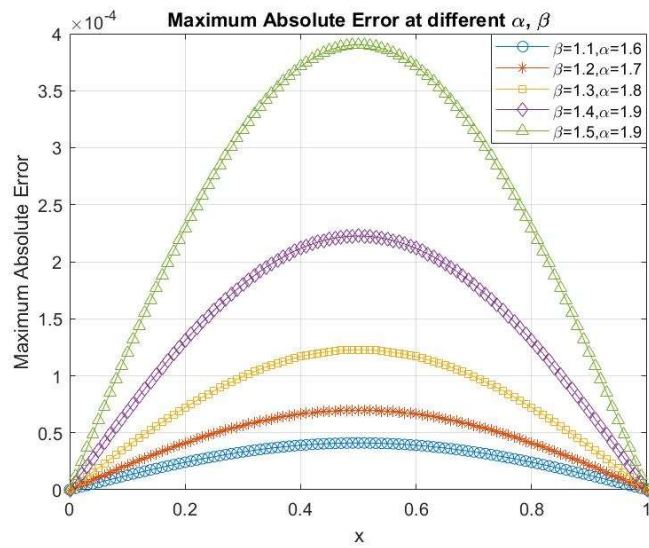


FIGURE 4.10: MAE of Ex. 4.2 at different  $\beta$  and  $\alpha$ ,  $N_t = 100$ ,  $M_x = 100$ .

TABLE 4.5: Comparative study of  $L_2$  error for Ex. 4.2 different values of  $\beta$  and  $\alpha$  at  $T = 1$  when  $M_x = 1000$ .

| $\tau$       | $\beta = 1.1, \alpha = 1.5$ |            |            | $\beta = 1.5, \alpha = 1.7$ |            |            | $\beta = 1.3, \alpha = 1.9$ |            |            |
|--------------|-----------------------------|------------|------------|-----------------------------|------------|------------|-----------------------------|------------|------------|
|              | $L_2$ error                 | Error[133] | Error[132] | $L_2$ error                 | Error[133] | Error[132] | $L_2$ error                 | Error[133] | Error[132] |
| <b>1/10</b>  | 1.842e-03                   | 9.436e-04  | 7.298e-03  | 4.122e-03                   | 8.233e-04  | 6.365e-03  | 1.482e-03                   | 5.533e-03  | 4.286e-02  |
| <b>1/20</b>  | 3.187e-04                   | 3.127e-04  | 2.387e-03  | 5.136e-04                   | 2.339e-04  | 1.765e-03  | 1.876e-03                   | 3.198e-03  | 2.477e-02  |
| <b>1/40</b>  | 4.732e-05                   | 1.157e-04  | 7.970e-04  | 2.596e-05                   | 6.405e-05  | 2.762e-04  | 6.664e-04                   | 1.717e-03  | 1.329e-02  |
| <b>1/80</b>  | 7.860e-06                   | 6.408e-05  | 2.766e-04  | 2.269e-05                   | 5.346e-05  | 3.243e-05  | 1.830e-04                   | 8.683e-04  | 6.714e-03  |
| <b>1/160</b> | 1.172e-06                   | 1.264e-05  | 9.8344e-05 | 7.021e-06                   | 6.513e-6   | 4.967e-05  | 4.584e-05                   | 4.220e-04  | 3.269e-03  |

TABLE 4.6: Comparative study of  $L_\infty$  error for Ex. 4.2 for different values of  $\beta$  and  $\alpha$  at  $T = 1$  when  $M_x = 1000$ .

| $\tau$       | $\beta = 1.1, \alpha = 1.5$ |           | $\beta = 1.5, \alpha = 1.7$ |           | $\beta = 1.3, \alpha = 1.9$ |           |
|--------------|-----------------------------|-----------|-----------------------------|-----------|-----------------------------|-----------|
|              | $L_\infty$ error            | In[132]   | $L_\infty$ error            | In[132]   | $L_\infty$ error            | [132]     |
| <b>1/10</b>  | 2.605e-03                   | 1.032e-02 | 5.829e-03                   | 9.001e-03 | 2.096e-03                   | 6.062e-02 |
| <b>1/20</b>  | 4.506e-04                   | 3.376e-03 | 7.264e-04                   | 2.496e-03 | 2.654e-03                   | 3.503e-02 |
| <b>1/40</b>  | 6.692e-05                   | 1.127e-03 | 3.671e-05                   | 3.906e-04 | 9.425e-04                   | 1.880e-02 |
| <b>1/80</b>  | 1.112e-05                   | 3.912e-04 | 3.209e-05                   | 4.587e-05 | 2.588e-04                   | 9.495e-03 |
| <b>1/160</b> | 1.658e-06                   | 1.390e-04 | 9.929e-06                   | 7.024e-05 | 6.483e-05                   | 4.623e-03 |

- Table 4.5 provide the comparative study of  $L_2$  error with the scheme given in [133],[132] at time  $T = 1$ , for different value of  $\beta$  and  $\alpha$ , taking  $h = 1/1000$ .
- Table 4.6 provides the comparative study of  $L_\infty$  error with existing scheme [132] at time  $T = 1$  for various value of  $\alpha$  and  $\beta$ .
- It can be observed from Tables 4.5-4.6 that our scheme provide high accuracy as compared to the existing schemes.
- Table 4.7 provides the temporal order of convergence in time w.r.t.  $L_2$  error and  $L_\infty$  error for different  $\alpha, \beta$  at  $T = 1$  by taking  $N_t = 10 \times 2^m, m = 0, 1, \dots, 7$  and  $M_x = 1000$  grid points.

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- Table 4.8 provides the spatial order of convergence in time w.r.t.  $L_2$  error and  $L_\infty$  error for different  $\alpha, \beta$  at  $T = 1$  by taking  $M_x = 5 \times 2^m$ ,  $m = 0, 1, \dots, 7$  and  $N_t = 1000$  grid points.
  - It can be observed from Tables 4.7-4.8 that the scheme provide second order of convergence in space and time direction.

TABLE 4.7:  $L_2$  error,  $L_\infty$  error and order in Ex. 4.2 at time  $t = 1$  when  $h = 1/1000$ .

| $\beta, \alpha$    | $\tau$ | Our Scheme  |        | Our Scheme       |        |
|--------------------|--------|-------------|--------|------------------|--------|
|                    |        | $L_2$ Error | Order  | $L_\infty$ Error | Order  |
| <b>(1.2, 1.7)</b>  | 1/10   | 4.541e-03   | 0      | 6.422e-03        | 0      |
|                    | 1/20   | 1.620e-03   | 1.487  | 2.291e-03        | 1.487  |
|                    | 1/40   | 3.886e-04   | 2.060  | 5.495e-04        | 2.060  |
|                    | 1/80   | 8.487e-05   | 2.195  | 1.200e-04        | 2.195  |
|                    | 1/160  | 1.794e-05   | 2.242  | 2.537e-05        | 2.242  |
|                    | 1/320  | 3.516e-06   | 2.351  | 4.973e-06        | 2.351  |
|                    | 1/640  | 4.160e-07   | 3.080  | 5.883e-07        | 3.080  |
| <b>(1.3, 1.8)</b>  | 1/10   | 5.428e-03   | 0      | 7.676e-03        | 0      |
|                    | 1/20   | 2.462e-03   | 1.141  | 3.482e-03        | 1.141  |
|                    | 1/40   | 6.840e-04   | 1.848  | 9.673e-04        | 1.848  |
|                    | 1/80   | 1.645e-04   | 2.056  | 2.327e-04        | 2.056  |
|                    | 1/160  | 3.759e-05   | 2.130  | 5.316e-05        | 2.130  |
|                    | 1/320  | 8.201e-06   | 2.197  | 1.160e-05        | 2.197  |
|                    | 1/640  | 1.527e-06   | 2.425  | 2.159e-06        | 2.425  |
| <b>(1.4, 1.9)</b>  | 1/10   | 4.716e-03   | 0      | 6.670e-03        | 0      |
|                    | 1/20   | 2.966e-03   | 0.669  | 4.194e-03        | 0.669  |
|                    | 1/40   | 9.660e-04   | 1.618  | 1.366e-03        | 1.618  |
|                    | 1/80   | 2.582e-04   | 1.903  | 3.652e-04        | 1.903  |
|                    | 1/160  | 6.408e-05   | 2.011  | 9.062e-05        | 2.011  |
|                    | 1/320  | 1.522e-05   | 2.073  | 2.153e-05        | 2.073  |
|                    | 1/640  | 3.346e-06   | 2.186  | 4.732e-06        | 2.186  |
| <b>(1.5, 1.99)</b> | 1/10   | 2.044e-03   | 0      | 2.891e-03        | 0      |
|                    | 1/20   | 2.728e-03   | -0.417 | 3.858e-03        | -0.417 |
|                    | 1/40   | 1.078e-03   | 1.339  | 1.525e-03        | 1.339  |
|                    | 1/80   | 3.229e-04   | 1.740  | 4.566e-04        | 1.740  |
|                    | 1/160  | 8.713e-05   | 1.890  | 1.232e-04        | 1.890  |
|                    | 1/320  | 2.232e-05   | 1.965  | 3.157e-05        | 1.965  |
|                    | 1/640  | 5.407e-06   | 2.046  | 7.647e-06        | 2.046  |

TABLE 4.8:  $L_2$  error,  $L_\infty$  error and Order in Ex. 4.2 at time  $T = 1$  when  $\tau = 1/1000$ .

| $\beta, \alpha$    | $h$   | Our Scheme  |       | Our Scheme       |       |
|--------------------|-------|-------------|-------|------------------|-------|
|                    |       | $L_2$ Error | Order | $L_\infty$ Error | Order |
| <b>(1.2, 1.7)</b>  | 1/5   | 1.765e-02   | 0     | 2.374e-02        | 0     |
|                    | 1/10  | 4.395e-03   | 2.006 | 6.216e-03        | 1.933 |
|                    | 1/20  | 1.097e-03   | 2.002 | 1.552e-03        | 2.002 |
|                    | 1/40  | 2.741e-04   | 2.002 | 3.876e-04        | 2.002 |
|                    | 1/80  | 6.827e-05   | 2.005 | 9.655e-05        | 2.005 |
|                    | 1/160 | 1.683e-05   | 2.020 | 2.380e-05        | 2.020 |
|                    | 1/320 | 3.967e-06   | 2.085 | 5.610e-06        | 2.085 |
| <b>(1.3, 1.8)</b>  | 1/5   | 1.658e-02   | 0     | 2.230e-02        | 0     |
|                    | 1/10  | 4.138e-03   | 2.003 | 5.852e-03        | 1.930 |
|                    | 1/20  | 1.034e-04   | 2.001 | 1.462e-03        | 2.001 |
|                    | 1/40  | 2.578e-04   | 2.003 | 3.646e-04        | 2.003 |
|                    | 1/80  | 6.389e-05   | 2.013 | 9.036e-05        | 2.013 |
|                    | 1/160 | 1.542e-05   | 2.051 | 2.180e-05        | 2.051 |
|                    | 1/320 | 3.297e-06   | 2.225 | 4.663e-06        | 2.225 |
| <b>(1.4, 1.9)</b>  | 1/5   | 1.532e-02   | 0     | 2.061e-02        | 0     |
|                    | 1/10  | 3.830e-03   | 2.000 | 5.417e-03        | 1.928 |
|                    | 1/20  | 9.565e-04   | 2.002 | 1.353e-03        | 2.002 |
|                    | 1/40  | 2.380e-04   | 2.007 | 3.366e-04        | 2.007 |
|                    | 1/80  | 5.840e-05   | 2.027 | 8.259e-05        | 2.027 |
|                    | 1/160 | 1.349e-05   | 2.114 | 1.908e-05        | 2.114 |
|                    | 1/320 | 2.264e-06   | 2.575 | 3.202e-06        | 2.575 |
| <b>(1.5, 1.99)</b> | 1/5   | 1.419e-02   | 0     | 1.909e-02        | 0     |
|                    | 1/10  | 3.553e-03   | 1.998 | 5.025e-03        | 1.926 |
|                    | 1/20  | 8.869e-04   | 2.002 | 1.254e-03        | 2.002 |
|                    | 1/40  | 2.200e-04   | 2.011 | 3.111e-04        | 2.011 |
|                    | 1/80  | 5.322e-05   | 2.047 | 7.526e-05        | 2.047 |
|                    | 1/160 | 1.152e-05   | 2.207 | 1.630e-05        | 2.207 |
|                    | 1/320 | 1.102e-06   | 3.386 | 1.559e-06        | 3.386 |

## 4.5 Conclusion

In this chapter, we have presented a new difference scheme for multi-term time fractional electromagnetic wave model. The Caputo fractional derivatives of order  $\alpha$  and  $\beta$ , where  $1 < \beta < \alpha < 2$  have been approximated by L3 approximation, whereas spatial derivatives have been approximated by central difference scheme. The numerical scheme is validated on two numerical examples and the obtained results are highly accurate and of second order convergence with respect to space and time. The comparative study of the numerical results with [133] and [132] in Tables 4.1, 4.2, 4.5 and 4.6 prove the effectiveness of our scheme. The proposed approach can be extended for the nonlinear and higher dimensional time-fractional mathematical models.

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