Chapter 6

Banach-space-valued ultradistributions involving the Weinstein transform

6.1 Introduction

Distributions are functions that allow for rigorous treatment of derivatives and other operations on non-smooth or singular objects. Scalar distributions are a specific type of distribution that maps test functions to scalar values. The dirac delta function and the Heaviside step function are examples of scalar distributions.
The theory of Banach-space-valued testing functions to the distributions was originated by Zemanian [89]. This type of space is more general than scalar distributions. He introduced the inductive limit space $D^m(A)$ which is given by

$$D^m(A) = D^m_{\mathbb{R}^n}(A) = \cup_{i=1}^{\infty} D^m_{K_i}(A);$$

where $D^m_{K_i}(A)$ is the linear space of all smooth function f from $\mathbb{R}^n$ into a Banach space A, such that $supp\ \phi \subset K_i$ are the compact subsets of $\mathbb{R}^n$ and $K_i \subset K_{i+1}, \cup_{i=1} K_i = \mathbb{R}^n$. The topology generated of the semi-norm of $D^m_{K_i}(A)$ is defined by

$$\gamma_k(\phi) = \sup_{t \in K_i} \| \; D^k_t \phi(t) \; \|_A, \quad 0 \leq k \leq m.$$

Motivated by the result of Zemanian [89], exploiting the theory of Mellin transform, Tiwari [79] defined Banach space-valued distributions and proved several results, including a Mellin-type convolution theorem. From the result of [79, 89], Koh and Lie [27] proved the subspace ${}_\mu D_I(A)$ is dense in $H_\mu(A)$ and further showed that there is a bijection from $[H_\mu(A); B]$ onto $[H_\mu; [A; B]]$. It is also shown that the Hankel transformation of an arbitrary order on $H_\mu(A)$ is an automorphism on $H_\mu(A)$. Upadhyay [83] investigated the $H^\omega_\mu(A)$ type space, which was defined by the set of all those smooth, complex valued functions $\phi(x)$ on $I = (0, \infty)$ such that

$$\gamma^\mu_{\lambda,k}(\phi) = \left|\left| exp[\lambda\omega(x)] \left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-\frac{1}{2}}\phi(x) \right|\right|_A < \infty, \ \forall\ \lambda, k \in \mathbb{N}_0.$$

and discussed their algebraic and topological properties by using the theory of the Hankel transform. Motivated by the results of [27, 79, 83, 89], in the present chapter we investigated various properties of the Banach-space-valued ultradistributions associated with the Weinstein transform.

6.2 The Space $[H^\beta_\omega(A)]$ and its Properties

In this section, the space $H^\beta_\omega(A)$ is defined, and its various properties are discussed by utilizing the theory of the Weinstein transform.

Definition 6.2.1. The space $H(A)$ is the set of all those smooth, complex-valued function $\phi(x)$ from $\mathbb{R}_+^{n+1}$ into A which satisfy the following norm:

$$\gamma_{m,k}(\phi) = \sup_{x\in\mathbb{R}_+^{n+1}} \| \, x^m (\Delta_{W,\beta}^n)_x^k \phi(x) \, \|_A \;\; < \;\; \infty, \; \forall \; m, k \; \in \mathbb{N}_0. \tag{6.2.1}$$

Theorem 6.2.2. *The Weinstein transform* $(F_w\phi)$ *is an automorphism on* $H(A)$.

Proof: Let $\phi \in H(A)$, then we show that $\mathcal{F}_w(\phi) \in H(A)$. From (6.2.1), we have

$$\gamma_{\alpha,m}[(\mathcal{F}_w\phi)(\xi)] = \sup_{\xi\in\mathbb{R}_+^{n+1}} \left\| \xi^m (\Delta_{W,\beta}^n)_\xi^\alpha (\mathcal{F}_w\phi)(\xi) \right\|_A . \tag{6.2.2}$$

Now, first, we can find

$$\begin{aligned}(\Delta_{W,\beta}^n)_\xi^\alpha (\mathcal{F}_w\phi)(\xi) &= (\Delta_{W,\beta}^n)_\xi^\alpha \int_{\mathbb{R}_+^{n+1}} e^{-i\langle\xi',y'\rangle} J_\beta(\xi_{n+1}y_{n+1})\phi(y)\, d\mu_\beta(y)\\ &= \int_{\mathbb{R}_+^{n+1}} (\Delta_{W,\beta}^n)_\xi^\alpha e^{-i\langle\xi',y'\rangle} J_\beta(\xi_{n+1}y_{n+1})\phi(y)\, d\mu_\beta(y).\end{aligned}$$

In view of (1.4.5), we get

$$(\Delta_{W,\beta}^n)_\xi^\alpha (\mathcal{F}_w\phi)(\xi) = \left(\int_{\mathbb{R}_+^{n+1}} (- \| \, y \, \|^2)^\alpha e^{-i\langle\xi',y'\rangle} J_\beta(\xi_{n+1}y_{n+1})\phi(y)\, d\mu_\beta(y) \right).$$

Set $(- \| \, y \, \|^2)^\alpha \phi(y) = g(y)$. Then above expression yields

$$(\Delta_{W,\beta}^n)_\xi^\alpha (\mathcal{F}_w\phi)(\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle\xi',y'\rangle} J_\beta(\xi_{n+1}y_{n+1}) g(y)\, d\mu_\beta(y).$$

From (6.2.2), we find that

$$\begin{aligned}
&\gamma_{\alpha,m}[(\mathcal{F}_w\phi)(\xi)] \\
&\quad = \sup_{\xi\in\mathbb{R}^{n+1}_+}\left|\left|\xi^m\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',y'\rangle}J_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y)\right|\right|_A \\
&\quad \le \sup_{\xi\in\mathbb{R}^{n+1}_+}\left|\left|\int_{\mathbb{R}^{n+1}_+}\xi^{2m}e^{-i\langle\xi',y'\rangle}J_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y)\right|\right|_A \\
&\quad \le \sup_{\xi\in\mathbb{R}^{n+1}_+}\left|\left|\int_{\mathbb{R}^{n+1}_+}(||\,\xi\,||^2)^m e^{-i\langle\xi',y'\rangle}J_\beta(\xi_{n+1}y_{n+1})g(y)\,d\mu_\beta(y)\right|\right|_A \\
&\quad = \sup_{\xi\in\mathbb{R}^{n+1}_+}\left|\left|\int_{\mathbb{R}^{n+1}_+}(-1)^m(\Delta^n_{W,\beta})^m_y\left[e^{-i\langle\xi',y'\rangle}J_\beta(\xi_{n+1}y_{n+1})\right]g(y)\,d\mu_\beta(y)\right|\right|_A.
\end{aligned}$$

By integrating the parts, we obtain

$$\begin{aligned}
&\gamma_{\alpha,m}[(\mathcal{F}_w\phi)(\xi)] \\
&\quad \le \sup_{\xi\in\mathbb{R}^{n+1}_+}\left|\left|\int_{\mathbb{R}^{n+1}_+}e^{-i\langle\xi',y'\rangle}J_\beta(\xi_{n+1}y_{n+1})(\Delta^n_{W,\beta})^m_y g(y)\,d\mu_\beta(y)\right|\right|_A \\
&\quad \le \sup_{\xi\in\mathbb{R}^{n+1}_+}\left|\left|e^{-i\langle\xi',y'\rangle}J_\beta(\xi_{n+1}y_{n+1})\right|\right|_A\cdot\left|\int_{\mathbb{R}^{n+1}_+}(\Delta^n_{W,\beta})^m_y g(y)\,d\mu_\beta(y)\right| \\
&\quad \le \left|\int_{\mathbb{R}^{n+1}_+}(1+y^2)^k(\Delta^n_{W,\beta})^m_y g(y)(1+y^2)^{-k}\,d\mu_\beta(y)\right| \\
&\quad \le \sup_{\xi\in\mathbb{R}^{n+1}_+}\left|\left|(1+y^2)^k(\Delta^n_{W,\beta})^m_y g(y)\right|\right|_A\cdot\left|\int_{\mathbb{R}^{n+1}_+}(1+y^2)^{-k}\,d\mu_\beta(y)\right| \\
&\quad \le \sup_{\xi\in\mathbb{R}^{n+1}_+}\left|\left|\sum_{r=0}^{k}\binom{k}{r}(y^2)^r(\Delta^n_{W,\beta})^m_y g(y)\right|\right|_A\cdot\left|\int_{\mathbb{R}^{n+1}_+}(1+y^2)^{-k}\,d\mu_\beta(y)\right| \\
&\quad < \infty.
\end{aligned}$$

By the inversion formula for the Weinstein transform, we have $\mathcal{F}_w^{-1}\phi(y) = \mathcal{F}_w\phi(-y)$. Therefore, $\mathcal{F}_w$ is one-one. So that $\mathcal{F}_w$ is an automorphism.

From the definition 2.2.7, we restate the following definition:

Definition 6.2.3. Let ω be a continuous real-valued function defined on $\mathbb{R}_+^{n+1}$ possessing the following condition:

$$(i) \quad 0 = \omega(0) = \lim_{x\to 0} \omega(x) \leq \omega(x+y) \leq \omega(x) + \omega(y), \forall\, x,\, y \in \mathbb{R}_+^{n+1}.$$

$$(ii) \quad J_n(\omega) = \int_{|y|\geq 1} \frac{\omega(y)}{1+ \|\, y \,\|^2}\, d\mu_\beta(y) < \infty.$$

$$(iii) \quad a + b\log(1+y) \leq \omega(y),\ \forall\, y \in \mathbb{R}_+^{n+1},\ a \in \mathbb{R},\ \ and\ b > 0.$$

We denote by M the set of all continuous functions satisfying (i), (ii) and (iii).

Definition 6.2.4. Let $\omega \in M$. The space $H_\omega(A)$ is the set of all complex valued infinitely differentiable functions ϕ from $\mathbb{R}_+^{n+1}$ into a Banach space A, satisfying

$$p^{\omega}_{\lambda,k}(\phi) = \sup_{x\in\mathbb{R}_+^{n+1}} \|\, e^{\lambda\omega(x)} D^k \phi(x) \,\|_A < \infty,\ \ \forall\ \lambda, k\ \in \mathbb{N}_0. \tag{6.2.3}$$

Theorem 6.2.5. *Let $\alpha \in \mathbb{N}_0$ and $\phi \in H_\omega(A)$, then we have the following relation:*

$$e^{\lambda\omega(x)} (\Delta^n_{W,\beta})^\alpha_x \phi(x) = \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{\delta_1,\delta_2,...,\delta_n\geq 0} \binom{\alpha}{j} \binom{\alpha-j}{\delta_1, \delta_2, ...\delta_n} E'_{\beta,r} x_{n+1}^{r-\alpha} e^{\lambda\omega(x)} D_x^{2\delta'+r} \phi(x). \tag{6.2.4}$$

Proof : From [13, P.14], for a constant $E'_{\beta,r}$, $r \in \{1, 2, ..., \alpha\}$ depends only on β satisfying

$$(\Delta^n_{W,\beta})^\alpha_x \phi(x) = \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \binom{\alpha}{j} E'_{\beta,r}\ x_{n+1}^{r-\alpha} (\Delta_n)^{\alpha-j}_{x'} \frac{\partial^r}{\partial x^r_{n+1}} \phi(x), \tag{6.2.5}$$

where

$$(\Delta_n)^{\alpha-j}_{x'} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2} \right)^{\alpha-j}.$$

Using multi-index theorem for $\alpha - j \in \mathbb{N}_0$ we get

$$(\Delta_n)_{x'}^{\alpha-j} = \sum_{\delta_1,\delta_2,...,\delta_n \geq 0} \binom{\alpha - j}{\delta_1, \delta_2, ..., \delta_n} \frac{\partial^{2\delta_1}}{\partial x_1^{2\delta_1}} \cdots \frac{\partial^{2\delta_n}}{\partial x_n^{2\delta_n}}. \tag{6.2.6}$$

From (6.2.5) and (6.2.6), we find

$$(\Delta_{W,\beta}^n)_x^\alpha \phi(x) = \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \sum_{\delta_1,\delta_2,...,\delta_n \geq 0} \binom{\alpha}{j} \binom{\alpha - j}{\delta_1, \delta_2, ..., \delta_n} E'_{\beta,r}\; x_{n+1}^{r-\alpha} \left[\frac{\partial^{2|\delta'|+r}}{\partial x_1^{2\delta_1} \cdots \partial x_n^{2\delta_n}\, \partial x_{n+1}^r} \phi(x) \right],$$

where $\delta' = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{N}_0^n$ and $\mid \delta' \mid = \delta_1 + \delta_2 + ... + \delta_n$.

This implies that

$$e^{\lambda\omega(x)} (\Delta_{W,\beta}^n)_x^\alpha \phi(x) = \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \binom{\alpha}{j} \binom{\alpha - j}{\delta_1, \delta_2, ..., \delta_n} E'_{\beta,r}\; x_{n+1}^{r-\alpha} e^{\lambda\omega(x)} D_x^{2|\delta'|+r} \phi(x).$$

Lemma 6.2.6. *Let $\phi \in H_\omega(A)$. Then for any non-negative real number λ, multi-index α and positive constants $E'_{\beta,\alpha}$ we find the following relation:*

$$\left| e^{\lambda\omega(x)} (\Delta_{W,\beta}^n)_x^\alpha \phi(x) \right| \leq E'_{\beta,\alpha} \left| x_{n+1}^\alpha e^{\lambda\omega(x)} D_x^{2\alpha} \phi(x) \right|.$$

Proof: Let $\phi \in H_\omega(A)$. Then from (6.2.4) we have

$$e^{\lambda\omega(x)} (\Delta_{W,\beta}^n)_x^\alpha \phi(x) = \sum_{j=0}^{\alpha} \sum_{r=1}^{2j} \binom{\alpha}{j} \binom{\alpha - j}{\delta_1, \delta_2, ..., \delta_n} E'_{\beta,r}\; e^{\lambda\omega(x)} x_{n+1}^{r-\alpha} D_x^{2|\delta'|+r} \phi(x).$$

Therefore

$$\begin{aligned}
\left|e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^{\alpha}_x\phi(x)\right| =& \left|\sum_{j=0}^{\alpha}\sum_{r=1}^{2j}\binom{\alpha}{j}\binom{\alpha-j}{\delta_1,\delta_2,...,\delta_n}E'_{\beta,r}\ e^{\lambda\omega(x)}x_{n+1}^{r-\alpha}D_x^{2|\delta'|+r}\phi(x)\right| \\
\leq& \sum_{j=0}^{\alpha}\sum_{r=1}^{2j}\binom{\alpha}{j}\binom{\alpha-j}{\delta_1,\delta_2,...,\delta_n}E'_{\beta,r}\ \left|e^{\lambda\omega(x)}x_{n+1}^{r-\alpha}D_x^{2|\delta'|+r}\phi(x)\right| \\
\leq& \sum_{j=0}^{\alpha}\sum_{r=1}^{2j}\binom{\alpha}{j}E'_{\beta,r}\ \left|e^{\lambda\omega(x)}x_{n+1}^{r-\alpha}D_x^{r}\phi(x)\right| \\
\leq& \sum_{j=0}^{\alpha}\binom{\alpha}{j}E'_{\beta,j}\ \left|x_{n+1}^{2j-\alpha}e^{\lambda\omega(x)}D_x^{2j}\phi(x)\right| \\
\leq& E'_{\beta,\alpha}\ \left|e^{\lambda\omega(x)}x_{n+1}^{\alpha}D_x^{2\alpha}\phi(x)\right|.
\end{aligned}$$

Hence

$$\left|e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^{\alpha}_x\phi(x)\right| \leq E'_{\beta,\alpha}\ \left|x_{n+1}^{\alpha}e^{\lambda\omega(x)}D_x^{2\alpha}\phi(x)\right|.$$

From the above concepts, we are able to define the space $H^{\beta}_{\omega}(A)$ which are given below:

Definition 6.2.7. The space $H^{\beta}_{\omega}(A)$ is defined as the set of all functions $\phi \in H_{\omega}(A)$ such that

$$\gamma^{\omega}_{\lambda,k}(\phi) = \sup_{x\in\mathbb{R}^{n+1}_{+}} ||\ e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\ ||_A < \infty,\ \forall\ \lambda, k\ \in \mathbb{N}_0. \tag{6.2.7}$$

Theorem 6.2.8. *For $\omega \in M$, the space $H^{\beta}_{\omega}(A)$ is a subspace of $H(A)$.*

Proof: Let $\phi \in H^{\beta}_{\omega}(A)$. Then we have to show that $\gamma_{m,k}(\phi) < \infty$.

From (6.2.1), we have

$$\sup_{x\in\mathbb{R}^{n+1}_{+}} \big|\big|x^m(\Delta^n_{W,\beta})^k_x\phi(x)\big|\big|_A = \sup_{x\in\mathbb{R}^{n+1}_{+}} \big|\big|x^m e^{\lambda\omega(x)}e^{-\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\big|\big|_A.$$

Since $x^m e^{-\lambda\omega(x)} \leq 1$ for some $\lambda > 0$.

Therefore, above expression yields

$$\begin{aligned}\sup_{x\in\mathbb{R}^{n+1}_{+}} ||\ x^m(\Delta^n_{W,\beta})^k_x\phi(x)\ ||_A &= \sup_{x\in\mathbb{R}^{n+1}_{+}} ||\ x^m e^{\lambda\omega(x)}e^{-\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\ ||_A\\ &\leq \sup_{x\in\mathbb{R}^{n+1}_{+}} ||\ e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\ ||_A\\ &< \infty.\end{aligned}$$

Therefore, we get:

$$\gamma_{m,k}(\phi) < \infty.$$

The above implies that $H^{\beta}_{\omega}(A) \subset H(A)$.

Definition 6.2.9. $\phi(x)$ belongs to $D_{\mathbb{R}^{n+1}_{+}}(A)$ iff ϕ is defined on $\mathbb{R}^{n+1}_{+}$, takes its value in A and there exist $b \in \mathbb{R}^{n+1}_{+}$ such that $\phi(x) = 0$ for $|\ x\ |\geq b$.

Let

$$\omega D_{\mathbb{R}^{n+1}_{+}}(A) = D_{\mathbb{R}^{n+1}_{+}}(A) \cap H_{\omega}(A). \tag{6.2.8}$$

Theorem 6.2.10. *The subspace* $\omega D_{\mathbb{R}^{n+1}_{+}}(A)$ *is dense in* $H^{\beta}_{\omega}(A)$.

Proof: Let $\theta(x) \in D_{\mathbb{R}^{n+1}_{+}}(A)$ such that

$$\theta(x) = 1,\ \ for\ \ 0 <|\ x\ |\leq 1$$

$$and\ \theta(x) = 0,\ \ for\ \ \ |\ x\ |\geq 2.$$

Now, we take $\phi(x) \in H_\omega(A)$ and $\lambda, k \in \mathbb{N}_0^{n+1}$. Then

$$e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x \phi(x)\left[\theta\left(\frac{x}{\eta}\right)\phi(x) - \phi(x)\right] = e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x \phi(x)\left[\phi(x)\left(\theta\left(\frac{x}{\eta}\right) - 1\right)\right].$$

From (1.4.15), we have

$$\begin{aligned} & e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x \phi(x)\left[\theta\left(\frac{x}{\eta}\right)\phi(x) - \phi(x)\right] \\ & \quad = e^{\lambda\omega(x)} \sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,\ldots,\delta_n\ge 0} \binom{k}{j}\binom{m}{q}\binom{k-j}{\delta_1,\delta_2,\ldots,\delta_n}\frac{1}{\rho'!}E'_{\beta,m} \\ & \quad \times x_{n+1}^{m-k} D_x^{\rho+q}\phi(x)\cdot D_x^{\rho'+2\delta'+m-q}\left[\theta\left(\frac{x}{\eta}\right) - 1\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{x\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x \phi(x)\left[\theta\left(\frac{x}{\eta}\right)\phi(x) - \phi(x)\right]\right\|_A \\ & \quad = \sup_{x\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda\omega(x)} \sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\le 2(k-j)}^{max}\sum_{\delta_1,\delta_2,\ldots,\delta_n\ge 0} \binom{k}{j}\binom{m}{q}\binom{k-j}{\delta_1,\delta_2,\ldots,\delta_n}\right. \\ & \quad \left.\times \frac{1}{\rho'!}E'_{\beta,m} x_{n+1}^{m-k} D_x^{\rho+q}\phi(x)\cdot D_x^{\rho'+2\delta'+m-q}\left[\theta\left(\frac{x}{\eta}\right) - 1\right]\right\|_A. \end{aligned}$$

This above yields

$$\begin{aligned} & \sup_{x\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x \phi(x)\left[\theta\left(\frac{x}{\eta}\right)\phi(x) - \phi(x)\right]\right\|_A \\ & \quad \le \sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\le 2(k-j)}^{max}\sum_{\delta_1,\delta_2,\ldots,\delta_n\ge 0} \binom{k}{j}\binom{m}{q}\binom{k-j}{\delta_1,\delta_2,\ldots,\delta_n}\frac{1}{\rho'!}E'_{\beta,m} \\ & \quad \times \sup_{x\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda\omega(x)} D_x^{\rho+q}\phi(x)\right\|_A \cdot \sup_{x\ge\eta}\left|\frac{D_x^{\rho'+2\delta'+m-q}\left[\theta\left(\frac{x}{\eta}\right) - 1\right]}{x_{n+1}^{k-m}}\right|. \end{aligned}$$

From (6.2.3), $\sup_{x\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda\omega(x)} D_x^{\rho+q}\phi(x)\right\|_A < \infty$ and since $\theta(x)$ and its derivatives are bounded, it follows that

$\sup_{x\geq\eta}\left|\frac{D_x^{\rho'+2\delta'+m-q}\left[\theta\left(\frac{x}{\eta}\right)-1\right]}{x_{n+1}^{k-m}}\right|\longrightarrow 0\ as\ \mid\eta\mid\longrightarrow\infty,$

for fixed k and $0\leq\rho'+2\delta'+m-q\leq k$.

Hence, this imply that

$$\theta\left(\frac{x}{\eta}\right)\phi(x)\longrightarrow\phi(x)\ in\ H_\omega^\beta(A).$$

This prove that the subspace $\omega D_{\mathbb{R}_+^{n+1}}(A)$ is dense in $H_\omega^\beta(A)$.

Theorem 6.2.11. *If $\phi\in H_\omega^\beta(A)$ and $\omega\in M$, then the Weinstein transform $\mathcal{F}_w\phi$ is an automorphism on $H_\omega^\beta(A)$.*

Proof: Let $\phi\in H_\omega^\beta(A)$, then we have to show that $\mathcal{F}_w(\phi)\in H_\omega^\beta(A)$. From (6.2.7) we have

$$\begin{aligned}&\gamma_{\lambda,\alpha}^\omega[\mathcal{F}_w\phi(\xi)]\\&\quad=\sup_{\xi\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(\xi)}(\Delta_{W,\beta}^n)_\xi^\alpha(\mathcal{F}_w\phi)(\xi)\right\|_A\\&\quad=\sup_{\xi\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(\xi)}(\Delta_{W,\beta}^n)_\xi^\alpha\int_{\mathbb{R}_+^{n+1}}e^{-i\langle\xi',y'\rangle}J_\beta(\xi_{n+1}y_{n+1})\phi(y)\,d\mu_\beta(y)\right\|_A.\end{aligned}$$

Using (1.4.5), we get

$$\begin{aligned}&\gamma_{\lambda,\alpha}^\omega[\mathcal{F}_w\phi(\xi)]\\&\quad=\sup_{\xi\in\mathbb{R}_+^{n+1}}\left\|e^{\lambda\omega(\xi)}\int_{\mathbb{R}_+^{n+1}}e^{-i\langle\xi',y'\rangle}J_\beta(\xi_{n+1}y_{n+1})(-\|y\|^2)^\alpha\phi(y)\,d\mu_\beta(y)\right\|_A.\end{aligned}$$

Now, we take $f(y) = (-\|y\|^2)^\alpha \phi(y)$ then above expression yields

$$\begin{aligned}\gamma^{\omega}_{\lambda,\alpha}[\mathcal{F}_w\phi(\xi)] &= \sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda\omega(\xi)}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle}J_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A\\ &\le \sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda|\omega(\xi)|}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle}J_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A.\end{aligned}$$

Form [46], we find for every $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that

$$|\omega(\xi)| \le \epsilon\|\xi\|^2 + C(\epsilon).$$

Therefore, we have

$$\begin{aligned}&\gamma^{\omega}_{\lambda,\alpha}[\mathcal{F}_w\phi(\xi)]\\ &\le \sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda(\epsilon\|\xi\|^2+C(\epsilon))}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle}J_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A\\ &\le e^{\lambda C(\epsilon)}\sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|e^{\lambda\epsilon\|\xi\|^2}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle}J_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A\\ &\le e^{\lambda C(\epsilon)}\sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\|\xi\|^{2m}\int_{\mathbb{R}^{n+1}_+} e^{-i\langle\xi',\,y'\rangle}J_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A\\ &= e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|\int_{\mathbb{R}^{n+1}_+}\|\xi\|^{2m}e^{-i\langle\xi',\,y'\rangle}J_\beta(\xi_{n+1}y_{n+1})f(y)\,d\mu_\beta(y)\right\|_A\\ &= e^{\lambda C(\epsilon)}\sum_{m=0}^{\infty}\frac{(\lambda\epsilon)^m}{m!}\sup_{\xi\in\mathbb{R}^{n+1}_+}\left\|\int_{\mathbb{R}^{n+1}_+}(\Delta^n_{W,\beta})^m_y(e^{-i\langle\xi',\,y'\rangle}J_\beta(\xi_{n+1}y_{n+1}))f(y)\,d\mu_\beta(y)\right\|_A.\end{aligned}$$

By integrating by parts, we get

$$\begin{aligned}
&\gamma^{\omega}_{\lambda,\alpha}[\mathcal{F}_w\phi(\xi)] \\
&\quad \leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{\xi\in\mathbb{R}^{n+1}_+} \left\| (-1)^m \int_{\mathbb{R}^{n+1}_+} e^{-i\langle \xi', y'\rangle} J_\beta(\xi_{n+1}y_{n+1})(\Delta^n_{W,\beta})^m_y f(y)\, d\mu_\beta(y) \right\|_A \\
&\quad \leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \left(\sup_{\xi\in\mathbb{R}^{n+1}_+} \left\| e^{-i\langle \xi', y'\rangle} J_\beta(\xi_{n+1}y_{n+1}) \right\|_A \right) \left| \int_{\mathbb{R}^{n+1}_+} (\Delta^n_{W,\beta})^m_y f(y)\, d\mu_\beta(y) \right| \\
&\quad \leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \left| \int_{\mathbb{R}^{n+1}_+} e^{\lambda\omega(y)} (\Delta^n_{W,\beta})^m_y f(y) e^{-\lambda\omega(y)}\, d\mu_\beta(y) \right| \\
&\quad \leq e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} \sup_{y\in\mathbb{R}^{n+1}_+} \left\| e^{\lambda\omega(x)} (\Delta^n_{W,\beta})^m_y f(y) \right\|_A \cdot \left| \int_{\mathbb{R}^{n+1}_+} e^{-\lambda\omega(y)}\, d\mu_\beta(y) \right| \\
&\quad = e^{\lambda C(\epsilon)} \sum_{m=0}^{\infty} \frac{(\lambda\epsilon)^m}{m!} p_{m,\lambda}(f(y)) \left| \int_{\mathbb{R}^{n+1}_+} e^{-\lambda\omega(y)}\, d\mu_\beta(y) \right| \\
&\quad < \infty.
\end{aligned}$$

This shows that $\mathcal{F}_w(\phi) \in H^\beta_\omega(A)$.

Then by the inversion formula for the Weinstein transform, we have $\mathcal{F}_w^{-1}\phi(y) = \mathcal{F}_w\phi(-y)$. Therefore $\mathcal{F}_w$ is one-one. So that $\mathcal{F}_w$ is an automorphism.

Definition 6.2.12. We denote $\omega D_{\mathbb{R}^{n+1}_+}(A) \oplus (A)$ is the linear space of all $\phi \in \omega D_{\mathbb{R}^{n+1}_+}(A)$ having representation of the form $\phi = \sum_k a_k h_k$ where $a_k \in \omega D_{\mathbb{R}^{n+1}_+}(A), h_k \in A$ and the summation is over a finite number of terms.

Theorem 6.2.13. *The space $\omega D_{\mathbb{R}^{n+1}_+}(A) \oplus (A)$ is dense in $H^\beta_\omega(A)$.*

Proof: Let $\theta(x)$ be the function which is same as in Theorem 6.2.10. Then for $\phi \in \omega D_{\mathbb{R}^{n+1}_+}(A)$, we have to show that

$$\theta\left(\frac{x}{\eta}\right)\mathcal{F}_w(\phi) \longrightarrow \mathcal{F}_w(\phi) \text{ as } \eta \longrightarrow \infty \text{ in } H^\beta_\omega(A).$$

Now, we take $\phi(x) \in H^{\beta}_{\omega}(A)$ and $\lambda, k \in \mathbb{N}_0^{n+1}$, then

$$e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\Big[\theta\Big(\frac{x}{\eta}\Big)(\mathcal{F}_w\phi)(x)-(\mathcal{F}_w\phi)(x)\Big] = e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\Big[(\mathcal{F}_w\phi)(x)\Big(\theta\Big(\frac{x}{\eta}\Big)-1\Big)\Big].$$

Using (1.4.15), we get

$$\begin{aligned}&e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\Big[\theta\Big(\frac{x}{\eta}\Big)(\mathcal{F}_w\phi)(x) - (\mathcal{F}_w\phi)(x)\Big]\\ &\quad= e^{\lambda\omega(x)}\sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,\dots,\delta_n\ge 0}\binom{k}{j}\binom{m}{q}\binom{k-j}{\delta_1,\delta_2,\dots,\delta_n}\frac{1}{\rho'!}E'_{\beta,m}\\ &\quad\times x_{n+1}^{m-k}D_x^{\rho+q}(\mathcal{F}_w\phi)(x)\cdot D_x^{\rho'+2\delta'+m-q}\Big[\theta\Big(\frac{x}{\eta}\Big)-1\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_{x\in\mathbb{R}^{n+1}_+}\Big\|e^{\lambda\omega(x)}(\Delta^n_{W,\beta})^k_x\phi(x)\Big[\theta\Big(\frac{x}{\eta}\Big)(\mathcal{F}_w\phi)(x)-(\mathcal{F}_w\phi)(x)\Big]\Big\|_A\\ &\quad=\sup_{x\in\mathbb{R}^{n+1}_+}\Big\|e^{\lambda\omega(x)}\sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,\dots,\delta_n\ge 0}\binom{k}{j}\binom{m}{q}\binom{k-j}{\delta_1,\delta_2,\dots,\delta_n}\\ &\quad\times\frac{1}{\rho'!}E'_{\beta,m}x_{n+1}^{m-k}D_x^{\rho+q}(\mathcal{F}_w\phi)(x)\cdot D_x^{\rho'+2\delta'+m-q}\Big[\theta\Big(\frac{x}{\eta}\Big)-1\Big]\Big\|_A\\ &\quad\le\sum_{j=1}^{k}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,\dots,\delta_n\ge 0}\binom{k}{j}\binom{m}{q}\binom{k-j}{\delta_1,\delta_2,\dots,\delta_n}\frac{1}{\rho'!}E'_{\beta,m}\\ &\quad\times\sup_{x\in\mathbb{R}^{n+1}_+}e^{\lambda\omega(x)}\Big\|D_x^{\rho+q}(\mathcal{F}_w\phi)(x)\Big\|_A\cdot\sup_{x\ge\eta}\left|\frac{D_x^{\rho'+2\delta'+m-q}\big[\theta\big(\frac{x}{\eta}\big)-1\big]}{x_{n+1}^{k-m}}\right|.\end{aligned}\qquad(6.2.9)$$

Hence,
$\sup_{x\ge\eta}\left|\frac{D_x^{\rho'+2\delta'+m-q}\left[\theta\left(\frac{x}{\eta}\right)-1\right]}{x_{n+1}^{k-m}}\right| \longrightarrow 0 \ as \ \eta\longrightarrow\infty,$
for fixed k and $0\le\rho'+2\delta'+m-q\le k$.

In view of (6.2.3), we have

$$\sup_{x\in\mathbb{R}_+^{n+1}} e^{\lambda\omega(x)}\left|\left|D_x^{\rho+q}(\mathcal{F}_w\phi)(x)\right|\right|_A < \infty.$$

Then we have

$$\left|\left|e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^k\phi(x)\left[\theta\left(\frac{x}{\eta}\right)(\mathcal{F}_w\phi)(x) - (\mathcal{F}_w\phi)\right]\right|\right|_A \longrightarrow 0 \;\; as \;\; \eta \longrightarrow \infty.$$

Therefore,

$$\theta\left(\frac{x}{\eta}\right)\mathcal{F}_w(\phi) \longrightarrow \mathcal{F}_w(\phi) \; is \; in \; H_\omega^\beta(A) \; as \; \eta \longrightarrow \infty.$$

This proves the theorem.

Theorem 6.2.14. *Let $\omega \in M$, then $H_\omega^\beta(A)$ is topological algebra under point-wise multiplication .*

Proof: Let $\phi, \psi \in H_\omega^\beta(A)$, then we have

$$\gamma_{\lambda,k}^\omega(\phi\psi) = \sup_{x\in\mathbb{R}_+^{n+1}} || \, e^{\lambda\omega(x)}(\Delta_{W,\beta}^n)_x^k\big(\phi\psi\big)(x) \, ||_A \, . \tag{6.2.10}$$

Using (1.4.15), we have

$$(\Delta_{W,\beta}^n)_x^k[\phi(x)\psi(x)] = \sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,..\delta_n\ge 0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,..\delta_n} \times \frac{1}{\rho'!}E'_{\beta,r}x_{n+1}^{r-k}\big(D_x^{\rho'+q}\phi(x)\big)\cdot\big(D_x^{\rho'+2\delta'+r-q}\psi(x)\big) \, .$$

Then from (6.2.10), we find

$$
\begin{aligned}
&\gamma^{\omega}_{\lambda,k}(\phi\psi)\\
&= \sup_{x\in\mathbb{R}^{n+1}_{+}} e^{\lambda\omega(x)}\Bigg|\Bigg|\sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,..\delta_n\ge 0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,..\delta_n}\\
&\times\frac{1}{\rho'!}E'_{\beta,r}x_{n+1}^{r-k}\big(D_x^{\rho'+q}\phi(x)\big)\cdot\big(D_x^{\rho'+2\delta'+r-q}\psi(x)\big)\Bigg|\Bigg|_A\\
&\le \sup_{x\in\mathbb{R}^{n+1}_{+}} e^{\lambda\omega(x)}\Bigg|\Bigg|\sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,..\delta_n\ge 0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,..\delta_n}\\
&\times\frac{1}{\rho'!}E'_{\beta,r}\big(D_x^{\rho'+q}\phi(x)\big)e^{k\omega(x)}\big(D_x^{\rho'+2\delta'+r-q}\psi(x)\big)\Bigg|\Bigg|_A,\ \text{ for some } k\in\mathbb{N}\\
&\le \sup_{x\in\mathbb{R}^{n+1}_{+}}\sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,..\delta_n\ge 0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,..\delta_n}\\
&\times\frac{1}{\rho'!}E'_{\beta,r}e^{\frac{\lambda\omega(x)}{2}}\big|\big|\big(D_x^{\rho'+q}\phi(x)\big)\big|\big|_A\cdot e^{\frac{\lambda+2k}{2}\omega(x)}\big|\big|\big(D_x^{\rho'+2\delta'+r-q}\psi(x)\big)\big|\big|_A\\
&\le\sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,..\delta_n\ge 0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,..\delta_n}\frac{1}{\rho'!}E'_{\beta,r}\\
&\times\sup_{x\in\mathbb{R}^{n+1}_{+}} e^{\frac{\lambda\omega(x)}{2}}\big|\big|\big(D_x^{\rho'+q}\phi(x)\big)\big|\big|_A\sup_{x\in\mathbb{R}^{n+1}_{+}} e^{\frac{\lambda+2k}{2}\omega(x)}\big|\big|\big(D_x^{\rho'+2\delta'+r-q}\psi(x)\big)\big|\big|_A.
\end{aligned}
$$

In veiw of (6.2.3), we get

$$
\begin{aligned}
\gamma^{\omega}_{\lambda,k}(\phi\psi)\le&\sum_{j=0}^{k}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\le 2(k-j)}\sum_{\delta_1,\delta_2,..\delta_n\ge 0}\binom{k}{j}\binom{r}{q}\binom{k-j}{\delta_1,\delta_2,..\delta_n}\\
&\times p^{\omega}_{\rho'+q,\frac{\lambda}{2}}(\phi)\cdot p^{\omega}_{\rho'+2\delta'+r-q,\frac{\lambda+2k}{2}}(\psi)<\infty.
\end{aligned}
$$

This implies that

$$\gamma^{\omega}_{\lambda,k}(\phi\psi)<\infty.$$

Theorem 6.2.15. *Let* $\omega\in M$*, then* $H^{\beta}_{\omega}(A)$ *is topological algebra under convolution.*

Proof: Let ϕ and $\psi \in H^{\beta}_{\omega}(A)$ then

$$\gamma^{\omega}_{\lambda,k}(\phi\#_{\beta}\psi) = \sup_{x\in\mathbb{R}^{n+1}_{+}} e^{\lambda\omega(x)}\big|\big|(\Delta^{n}_{W,\beta})^{k}_{x}\big(\phi\#_{\beta}\psi\big)(x)\big|\big|_{A}. \tag{6.2.11}$$

From [78, P.21], we get

$$\gamma^{\omega}_{\lambda,k}(\phi\#_{\beta}\psi) = \sup_{x\in\mathbb{R}^{n+1}_{+}} e^{\lambda\omega(x)}\big|\big|\big(\phi\#_{\beta}(\Delta^{n}_{W,\beta})^{k}_{x}\psi\big)(x)\big|\big|_{A}. \tag{6.2.12}$$

Using (1.4.12) we have

$$\big|\big|(\phi\#_{\beta}(\Delta^{n}_{W,\beta})^{k}_{x}\psi\big)(x)\big|\big|_{A} = \big|\big|\int_{\mathbb{R}^{n+1}_{+}} \phi(y)(\Delta^{n}_{W,\beta})^{k}_{x}\psi(x,y)\, d\mu_{\beta}(y)\big|\big|_{A}.$$

In veiw of (1.4.7), we obtain

$$\big|\big|(\phi\#_{\beta}(\Delta^{n}_{W,\beta})^{k}_{x}\psi\big)(x)\big|\big| = \big|\big|\int_{\mathbb{R}^{n+1}_{+}} \phi(y)\bigg(\int_{\mathbb{R}^{n+1}_{+}} \psi(z)(\Delta^{n}_{W,\beta})^{k}_{x}D_{\beta}(x,y,z)\, d\mu_{\beta}(z)\bigg)d\mu_{\beta}(y)\big|\big|_{A}.$$

Taking (1.4.8), we get

$$
\begin{aligned}
&\big|\big|(\phi\#_\beta(\Delta^n_{W,\beta})^k_x\psi)(x)\big|\big|_A\\
&= \big|\big|\int_{\mathbb{R}^{n+1}_+}\phi(y)\bigg(\int_{\mathbb{R}^{n+1}_+}\psi(z)(\Delta^n_{W,\beta})^k_x\bigg(\int_{\mathbb{R}^{n+1}_+}e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})\\
&\times e^{i\langle y',\xi'\rangle}J_\beta(y_{n+1}\xi_{n+1})e^{-i\langle z',\xi'\rangle}J_\beta(z_{n+1}\xi_{n+1})\,d\mu_\beta(\xi)\bigg)\,d\mu_\beta(z)\bigg)\,d\mu_\beta(y)\big|\big|_A\\
&= \big|\big|\int_{\mathbb{R}^{n+1}_+}\phi(y)\bigg(\int_{\mathbb{R}^{n+1}_+}\psi(z)\bigg(\int_{\mathbb{R}^{n+1}_+}\big(-||\xi||^2\big)^k e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})\\
&\times e^{i\langle y',\xi'\rangle}J_\beta(y_{n+1}\xi_{n+1})e^{-i\langle z',\xi'\rangle}J_\beta(z_{n+1}\xi_{n+1})\,d\mu_\beta(\xi)\bigg)\,d\mu_\beta(z)\bigg)\,d\mu_\beta(y)\big|\big|_A\\
&= \big|\big|\int_{\mathbb{R}^{n+1}_+}\phi(y)\bigg(\int_{\mathbb{R}^{n+1}_+}\psi(z)\bigg(\int_{\mathbb{R}^{n+1}_+}e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})e^{i\langle y',\xi'\rangle}J_\beta(y_{n+1}\xi_{n+1})\\
&\times\big(-||\xi||^2\big)^k e^{-i\langle z',\xi'\rangle}J_\beta(z_{n+1}\xi_{n+1})\,d\mu_\beta(\xi)\bigg)\,d\mu_\beta(z)\bigg)\,d\mu_\beta(y)\big|\big|_A\\
&= \big|\big|\int_{\mathbb{R}^{n+1}_+}\phi(y)\bigg(\int_{\mathbb{R}^{n+1}_+}\psi(z)(\Delta^n_{W,\beta})^k_z D_\beta(x,y,z)\,d\mu_\beta(z)\bigg)d\mu_\beta(y)\big|\big|_A\\
&= \big|\big|\int_{\mathbb{R}^{n+1}_+}\phi(y)\bigg(\int_{\mathbb{R}^{n+1}_+}(-1)^k(\Delta^n_{W,\beta})^\alpha_z\psi(z)D_\beta(x,y,z)\bigg)\,d\mu_\beta(y)\big|\big|_A\\
&\leq \sup_{z\in\mathbb{R}^{n+1}_+}\big|\big|(\Delta^n_{W,\beta})^k_z\psi(z)\big|\big|_A\cdot\big|\big|\int_{\mathbb{R}^{n+1}_+}\phi(y)\big(\int_{\mathbb{R}^{n+1}_+}D_\beta(x,y,z)\,d\mu_\beta(z)\big)d\mu_\beta(y)\big|\big|_A\,.
\end{aligned}
$$

Therefore,

$$
\big|\big|(\phi\#_\beta(\Delta^n_{W,\beta})^k_x\psi)(x)\big|\big|_A\leq\sup_{z\in\mathbb{R}^{n+1}_+}\big|\big|(\Delta^n_{W,\beta})^k_z\psi(z)\big|\big|_A\cdot\big|\big|\int_{\mathbb{R}^{n+1}_+}\phi(y)d\mu_\beta(y)\big|\big|_A\,.
$$

Using above expression in (6.2.12), we get

$$\begin{aligned}
\gamma^{\omega}_{\lambda,k}(\phi\#_{\beta}\psi) \leq& \sup_{z\in\mathbb{R}^{n+1}_{+}} e^{\lambda\omega(z)}\big|\big|(\Delta^{n}_{W,\beta})^{k}_{z}\psi(z)\big|\big|_{A}\cdot\big|\big|\int_{\mathbb{R}^{n+1}_{+}}\phi(y)d\mu_{\beta}(y)\big|\big|_{A}\\
=&\gamma^{\omega}_{\lambda,k}(\psi)\big|\big|\int_{\mathbb{R}^{n+1}_{+}} e^{l\omega(y)}\phi(y)e^{-l\omega(y)}d\mu_{\beta}(y)\big|\big|_{A}\\
\leq&\gamma^{\omega}_{\lambda,k}(\psi)\sup_{y\in\mathbb{R}^{n+1}_{+}}\big|\big|e^{l\omega(y)}\phi(y)\big|\big|_{A}\cdot\big|\int_{\mathbb{R}^{n+1}_{+}} e^{-l\omega(y)}\,d\mu_{\beta}(y)\big|\\
\leq&\gamma^{\omega}_{\lambda,k}(\psi)\sup_{y\in\mathbb{R}^{n+1}_{+}} e^{l\omega(y)}\big|\big|\phi(y)\big|\big|_{A}\cdot\big|\int_{\mathbb{R}^{n+1}_{+}} e^{-l\omega(y)}\,d\mu_{\beta}(y)\big|\\
=&\gamma^{\omega}_{\lambda,k}(\psi)\gamma^{\omega}_{l,0}(\phi)\big|\int_{\mathbb{R}^{n+1}_{+}} e^{-l\omega(y)}\,d\mu_{\beta}(y)\big| < \infty\,.
\end{aligned}$$

This proves the present theorem.

***