Chapter 4

Compactly supported distributions

In this chapter, we will prove that the Weyl transform of a compactly supported distribution on $\mathbb{R}^{2n}$ is in $S^p(\mathcal{H})$ if and only if the Fourier transform of the distribution is in $L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$. Moreover, we will prove that the Weyl transform of a compactly supported distribution on $\mathbb{R}^{2n}$ is a compact operator if and only if the Fourier transform of the distribution vanishes at infinity.

4.1 Introduction

In Chapter 2, we considered smooth measures supported on a compact smooth hypersurface in $\mathbb{R}^{2n}$ of positive Gaussian curvature, and proved that the Weyl transform of such a measure is compact and belongs to $S^p(\mathcal{H})$ if $p > n \geq 6$ (see Theorem 2.1.1). We also conjectured that the Weyl transform of a smooth measure supported on a compact smooth hypersurface in $\mathbb{R}^{2n}$ of positive Gaussian curvature is in $S^p(\mathcal{H})$ if $p > 4n/(2n-1)$, with no restriction on n (see Conjecture 2.5.1).

This conjecture was recently settled by Luef and Samuelsen in [29, Theorem 1.4]. They settled the conjecture in the affirmative by proving the following more general result.

Theorem 4.1.1. *Let μ be a compactly supported Radon measure on $\mathbb{R}^{2n}$. Let $\breve{\mu}$ denote the symplectic Fourier transform of μ. Then $W(\mu) \in S^p(\mathcal{H})$ if and only if $\breve{\mu} \in L^p(\mathbb{R}^{2n})$ for $1 \leq p \leq \infty$. Moreover, $W(\mu)$ is compact if and only if $\breve{\mu} \in \mathcal{C}_0(\mathbb{R}^{2n})$.*

The main result of this chapter is the following theorem, which is a generalization of Theorem 4.1.1, and is proved in Section 4.4.

Theorem 4.1.2. *Let T be a compactly supported distribution on $\mathbb{R}^{2n}$. Let $\widehat{T}$ denote the Fourier transform of T.*

(a) For $1 \leq p \leq \infty$, $W(T) \in S^p(\mathcal{H})$ if and only if $\widehat{T} \in L^p(\mathbb{R}^{2n})$. Moreover, if K is a compact set in $\mathbb{R}^{2n}$, then there exists a constant C_K such that

$$C_K^{-1} \left\| \widehat{T} \right\|_p \leq \| W(T) \|_{S^p} \leq C_K \left\| \widehat{T} \right\|_p ,$$

whenever $\operatorname{supp}(T) \subseteq K$.

(b) Furthermore, $W(T)$ is compact if and only if $\widehat{T} \in \mathcal{C}_0(\mathbb{R}^{2n})$.

4.2 Weyl transform of a tempered distribution

In this section, we study the Weyl transform of a tempered distribution. We only give a brief introduction to the topic. For a detailed discussion, we refer to [26, 30].

Definition 4.2.1. An operator $X \in \mathcal{B}(\mathcal{H})$ is called a *Schwartz operator* if there exists $k \in \mathcal{S}(\mathbb{R}^{2n})$ such that for every $\varphi \in \mathcal{H}$,

$$(X\varphi)(y) = \int_{\mathbb{R}^n} k(x,y)\varphi(x)\, dx, \quad y \in \mathbb{R}^n.$$

Let $\mathcal{S}(\mathcal{H})$ denote the set of Schwartz operators on $\mathcal{H}$. We define a family of seminorms on $\mathcal{S}(\mathcal{H})$ as follows. Let Q_i and P_i be the operators defined by

$$\begin{aligned}(Q_i f)(x) &= x_i f(x), \quad \text{and} \\ (P_i f)(x) &= -i\frac{\partial}{\partial x_i} f(x).\end{aligned}$$

For multi-indices $\alpha, \beta, \alpha', \beta' \in \mathbb{N}_0^n$ and $X \in \mathcal{S}(\mathcal{H})$, define

$$\|X\|_{\alpha,\beta,\alpha',\beta'} = \left\| Q^\alpha P^\beta X P^{\beta'} Q^{\alpha'} \right\|_{\text{op}},$$

where $Q^\alpha = Q_1^{\alpha_1} \dots Q_n^{\alpha_n}$ and $P^\beta = P_1^{\beta_1} \dots P_n^{\beta_n}$. Then $X \to \|X\|_{\alpha,\beta,\alpha',\beta'}$ is a seminorm. These seminorms turn $\mathcal{S}(\mathcal{H})$ into a Fréchet space. Let $\mathcal{S}'(\mathcal{H})$ denote the topological dual of $\mathcal{S}(\mathcal{H})$. Observe that if $X \in \mathcal{S}(\mathcal{H})$, then $X^* \in \mathcal{S}(\mathcal{H})$, where X^* is the adjoint of X.

Theorem 4.2.2. *Let $\mathcal{S}_0(\mathcal{H}) = \{X \in \mathcal{S}(\mathcal{H}) \mid X$ has finite rank$\}$. Then $\mathcal{S}_0(\mathcal{H})$ is dense in $\mathcal{S}(\mathcal{H})$.*

Theorem 4.2.3. *For $1 \le p \le \infty$, $\mathcal{S}(\mathcal{H}) \subseteq S^p(\mathcal{H})$. Moreover, if $1 \le p < \infty$, then $S(\mathcal{H})$ is dense in $S^p(\mathcal{H})$.*

The proof of these results can be found in [26, 61].

Recall that the Fourier-Wigner transform of $X \in S^1(\mathcal{H})$ is the function $\alpha(X) : \mathbb{R}^{2n} \to \mathbb{C}$ defined by

$$\alpha(X)(x,y) = \mathrm{tr}(X\rho(x,y,1)^{-1}).$$

Observe that if $X \in \mathcal{S}(\mathcal{H})$, then $\alpha(X) \in \mathcal{S}(\mathbb{R}^{2n})$. In fact, we have the following fundamental fact.

Lemma 4.2.4. *The map $\alpha : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathbb{R}^{2n})$ is a topological isomorphism.*

Proof. Let $X \in \mathcal{S}(\mathcal{H})$. Observe that (cf. [61, Lemma 11.2])

$$\begin{aligned}
\alpha(P_jX) &= i\frac{\partial}{\partial x_j}\alpha(X) + \pi y_j\alpha(X),\\
\alpha(Q_jX) &= -\frac{1}{2\pi i}\frac{\partial}{\partial y_j}\alpha(X) - x_j\alpha(X)/2,\\
\alpha(XP_j) &= i\frac{\partial}{\partial x_j}\alpha(X) - \pi y_j\alpha(X), \quad \text{and}\\
\alpha(XQ_j) &= -\frac{1}{2\pi i}\frac{\partial}{\partial y_j}\alpha(X) + x_j\alpha(X)/2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial}{\partial x_j}\alpha(X) &= -\frac{i}{2}\left(\alpha(P_jX) + \alpha(XP_j)\right),\\
\frac{\partial}{\partial y_j}\alpha(X) &= -\pi i\left(\alpha(Q_jX) + \alpha(XQ_j)\right),\\
x_j\alpha(X) &= \alpha(XQ_j) - \alpha(Q_jX), \quad \text{and}\\
y_j\alpha(X) &= \frac{1}{2\pi}\left(\alpha(P_jX) - \alpha(XP_j)\right).
\end{aligned}$$

Therefore α is continuous on $\mathcal{S}(\mathcal{H})$. Observe that the Weyl transform maps $\mathcal{S}(\mathbb{R}^{2n})$ to $\mathcal{S}(\mathcal{H})$. By the open mapping theorem, it follows that $\alpha : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathbb{R}^{2n})$ is an isomorphism. □

For a function f on $\mathbb{R}^n$, let $f^\sim$ denote the function defined by $f^\sim(x) = f(-x)$, $x \in \mathbb{R}^n$.

We now define the Weyl transform of a tempered distribution.

Definition 4.2.5. Let $T \in \mathcal{S}'(\mathbb{R}^{2n})$. Define $W(T) : \mathcal{S}(\mathcal{H}) \to \mathbb{C}$ by

$$(W(T))(X) = T(\alpha(X)^{\sim}), \quad X \in \mathcal{S}(\mathcal{H}).$$

It follows from Lemma 4.2.4 that $W(T) \in \mathcal{S}'(\mathcal{H})$; $W(T)$ is called the *Weyl transform* of the tempered distribution T. We now prove that the two definitions of Weyl transform coincide for integrable functions.

For $A \in \mathcal{B}(\mathcal{H})$, define $T_A : \mathcal{S}(\mathcal{H}) \to \mathbb{C}$ by

$$T_A(X) = \operatorname{tr}(AX), \quad X \in \mathcal{S}(\mathcal{H}).$$

It can be shown that $T_A \in \mathcal{S}'(\mathcal{H})$.

For $f \in L^1(\mathbb{R}^{2n})$, let E_f be the tempered distribution defined by

$$E_f(\psi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x,y)\psi(x,y)\, dx\, dy, \quad \psi \in \mathcal{S}(\mathbb{R}^{2n}).$$

Let $X \in \mathcal{S}(\mathcal{H})$. By Definition 4.2.5 and the dominated convergence theorem, it follows that

$$\begin{aligned}(W(E_f))(X) &= E_f(\alpha(X)^{\sim}) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x,y)\alpha(X)(-x,-y)\, dx\, dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x,y)\operatorname{tr}(X\rho(x,y,1))\, dx\, dy \\ &= \operatorname{tr}(XW(f)) \\ &= T_{W(f)}(X).\end{aligned}$$

We now define the Fourier-Wigner transform of an element in $\mathcal{S}'(\mathcal{H})$.

Definition 4.2.6. Let $\Psi \in \mathcal{S}'(\mathcal{H})$. Define $\alpha(\Psi) : \mathcal{S}(\mathbb{R}^{2n}) \to \mathbb{C}$ by

$$\alpha(\Psi)(\varphi) = \Psi(W(\varphi^{\sim})), \quad \varphi \in \mathcal{S}(\mathbb{R}^{2n}).$$

It follows from Lemma 4.2.4 that $\alpha(\Psi) \in \mathcal{S}'(\mathbb{R}^{2n})$; $\alpha(\Psi)$ is called the *Fourier-Wigner* transform of Ψ. We now prove that the two definitions of Fourier-Wigner transform coincide for trace class operators.

Let $A \in S^1(\mathcal{H})$, and $\psi \in \mathcal{S}(\mathbb{R}^{2n})$. By Definition 4.2.6 and the dominated convergence theorem, it follows that

$$\begin{aligned}
\alpha(T_A)(\psi) &= T_A(W(\psi^{\sim})) \\
&= \operatorname{tr}(AW(\psi^{\sim})) \\
&= \operatorname{tr}\left(A \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(-x,-y)\rho(x,y,1)\,dx\,dy\right) \\
&= \operatorname{tr}\left(A \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x,y)\rho(x,y,1)^{-1}\,dx\,dy\right) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x,y)\operatorname{tr}(A\rho(x,y,1)^{-1})\,dx\,dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi(x,y)\alpha(A)(x,y)\,dx\,dy \\
&= E_{\alpha(A)}(\psi).
\end{aligned}$$

If $T \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, then

$$\alpha(W(T))(\varphi) = W(T)(W(\varphi^{\sim})) = T(\alpha(W(\varphi^{\sim}))^{\sim}) = T(\varphi).$$

Therefore if $T \in \mathcal{S}'(\mathbb{R}^{2n})$, then

$$\alpha(W(T)) = T. \tag{4.1}$$

Similarly, we conclude that $W(\alpha(\Psi)) = \Psi$ for all $\Psi \in \mathcal{S}'(\mathcal{H})$.

4.3 An observation of Tim Steger

In this section, we prove that Fourier-Wigner transforms of trace class operators are locally the same as symplectic Fourier transforms of integrable functions. The results in this section are from [59].

Definition 4.3.1. Let $(x, y) \in \mathbb{R}^{2n}$ and $X \in S^p(\mathcal{H})$, $1 \leq p \leq \infty$. Define $(x, y) \cdot X$ to be the operator

$$(x, y) \cdot X = \rho(x, y, 1) X \rho(x, y, 1)^{-1}.$$

Since ρ is unitary, it follows from Theorem 2.2.9 that if $(x, y) \in \mathbb{R}^{2n}$ and $X \in S^p(\mathcal{H})$, then $(x, y) \cdot X \in S^p(\mathcal{H})$, and $\|(x, y) \cdot X\|_{S^p} = \|X\|_{S^p}$. This action of $\mathbb{R}^{2n}$ on $S^p(\mathcal{H})$ is called *quantum translation*. The motivation for this definition may be found in [56, p28]. This action plays an important role in the works [56, 60, 62].

Definition 4.3.2. Let $q \in L^1(\mathbb{R}^{2n})$ and $X \in S^p(\mathcal{H})$, $1 \leq p \leq \infty$. Define $q \cdot X$ to be the operator given by

$$q \cdot X = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} q(x, y) \left((x, y) \cdot X\right) \, dx \, dy.$$

The following result was proved in [60, Theorem 3.7] for $p = 1$.

Lemma 4.3.3. *If $q \in L^1(\mathbb{R}^{2n})$ and $X \in S^p(\mathcal{H})$, $1 \leq p \leq \infty$, then $q \cdot X \in S^p(\mathcal{H})$. Moreover,*

$$\|q \cdot X\|_{S^p} \leq \|q\|_1 \|X\|_{S^p}.$$

Proof. Let $q \in L^1(\mathbb{R}^{2n})$ and $X \in S^p(\mathcal{H})$. Then

$$\begin{aligned} \|q \cdot X\|_{S^p} &= \left\| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} q(x,y)\,((x,y)\cdot X)\; dx\,dy \right\|_{S^p} \\ &\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |q(x,y)|\, \|(x,y)\cdot X\|_{S^p}\; dx\,dy \\ &\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |q(x,y)|\, \|X\|_{S^p}\; dx\,dy \\ &= \|q\|_1 \, \|X\|_{S^p}\,. \end{aligned}$$

Therefore $q \cdot X \in S^p(\mathcal{H})$. □

Let $e : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{C}$ be the function defined by

$$e((x,y),(\xi,\eta)) = e^{2\pi i(x\cdot\eta - y\cdot\xi)}.$$

The *symplectic Fourier transform* of a function $f \in L^1(\mathbb{R}^{2n})$ is the function on $\mathbb{R}^{2n}$ defined by

$$\breve{f}(\xi,\eta) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e((x,y),(\xi,\eta)) f(x,y)\, dx\, dy.$$

More generally (see e.g., [30]), if T is a tempered distribution on $\mathbb{R}^{2n}$, the *symplectic Fourier transform* of T is the tempered distribution $\breve{T}$ given by

$$\breve{T}(\varphi) = T((\breve{\varphi})^{\sim}), \quad \varphi \in \mathcal{S}(\mathbb{R}^{2n}).$$

It is well known that if T is a compactly supported distribution on $\mathbb{R}^{2n}$, then $\breve{T}$ is a function on $\mathbb{R}^{2n}$ given by the formula

$$\breve{T}(x,y) = T(e(\cdot,(x,y))), \quad (x,y) \in \mathbb{R}^{2n}.$$

Since the symplectic Fourier transform is just a rotation of the Fourier transform, it follows that $\widehat{T} \in L^p(\mathbb{R}^{2n})$ if and only if $\check{T} \in L^p(\mathbb{R}^{2n})$, and $\left\|\check{T}\right\|_p = \left\|\widehat{T}\right\|_p$, $1 \leq p \leq \infty$. Moreover $\widehat{T} \in \mathcal{C}_0(\mathbb{R}^{2n})$ if and only if $\check{T} \in \mathcal{C}_0(\mathbb{R}^{2n})$.

Let $q \in L^1(\mathbb{R}^{2n})$ and $X \in S^1(\mathcal{H})$, then $q \cdot X \in S^1(\mathcal{H})$. The following lemma describes the relation between $\alpha(q \cdot X)$ and $\alpha(X)$ ([60, Lemma 3.10]).

Lemma 4.3.4. *If $q \in L^1(\mathbb{R}^{2n})$ and $X \in S^1(\mathcal{H})$, then*

$$\alpha(q \cdot X) = \check{q}\alpha(X).$$

Proof. Let $(x, y), (\xi, \eta) \in \mathbb{R}^{2n}$ and $X \in S^1(\mathcal{H})$. Then

$$\begin{aligned}
\alpha((x,y) \cdot X)(\xi,\eta) &= \alpha(\rho(x,y,1)X\rho(x,y,1)^{-1})(\xi,\eta) \\
&= \operatorname{tr}(\rho(x,y,1)X\rho(x,y,1)^{-1}\rho(\xi,\eta,1)^{-1}) \\
&= \operatorname{tr}(\rho(x,y,1)X\rho((-x,-y,1)(-\xi,-\eta,1)) \\
&= e((x,y),(\xi,\eta))\operatorname{tr}(\rho(x,y,1)X\rho((-\xi,-\eta,1)(-x,-y,1)) \\
&= e((x,y),(\xi,\eta))\operatorname{tr}(\rho(x,y,1)X\rho(\xi,\eta,1)^{-1}\rho(x,y,1)^{-1}) \\
&= e((x,y),(\xi,\eta))\operatorname{tr}(X\rho(\xi,\eta,1)^{-1}) \\
&= e((x,y),(\xi,\eta))\alpha(X)(\xi,\eta).
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha(q \cdot X)(\xi,\eta) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} q(x,y)\alpha((x,y) \cdot X)(\xi,\eta)\,dx\,dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} q(x,y)e((x,y),(\xi,\eta))\alpha(X)(\xi,\eta)\,dx\,dy \\
&= \check{q}(\xi,\eta)\alpha(X)(\xi,\eta).
\end{aligned}$$

□

Observe that if $q \in \mathcal{S}(\mathbb{R}^{2n})$ and $X \in \mathcal{S}(\mathcal{H})$, then $q \cdot X \in \mathcal{S}(\mathcal{H})$. More generally, if $q \in \mathcal{S}(\mathbb{R}^{2n})$ and $T \in \mathcal{S}'(\mathcal{H})$, then $q \cdot T : \mathcal{S}(\mathcal{H}) \to \mathbb{C}$ is defined by

$$(q \cdot T)(X) = T(q^{\sim} \cdot X), \quad X \in \mathcal{S}(\mathcal{H}).$$

It follows that $q \cdot T \in \mathcal{S}'(\mathcal{H})$.

Lemma 4.3.5. *Let $q \in \mathcal{S}(\mathbb{R}^{2n})$ and $T \in \mathcal{S}'(\mathcal{H})$. Then*

$$\alpha(q \cdot T) = \breve{q}\alpha(T).$$

Proof. If $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, then

$$\begin{aligned}(\alpha(q \cdot T))(\varphi) &= (q \cdot T)(W(\varphi^{\sim})) \\ &= T(q^{\sim} \cdot W(\varphi^{\sim})) \\ &= \alpha(T)(\breve{q}\varphi) \\ &= \left(\breve{q}\alpha(T)\right)(\varphi).\end{aligned}$$

□

Let $g(x, y) = e^{-\frac{\pi}{2}(|x|^2+|y|^2)}$, $(x, y) \in \mathbb{R}^{2n}$. Then $g \in L^{\infty}(\mathbb{R}^{2n})$, and $\|g\|_{\infty} = 1$.

Definition 4.3.6. For $X \in S^1(\mathcal{H})$, let $\beta(X)$ be the function on $\mathbb{R}^{2n}$ defined by

$$\beta(X)(x, y) = g(x, y)\alpha(X)(x, y).$$

Observe that if $X \in S^1(\mathcal{H})$, then $\beta(X) \in L^1(\mathbb{R}^{2n})$. For $X \in S^1(\mathcal{H})$, let $\breve{\beta}(X)$ denote the symplectic Fourier transform of $\beta(X)$.

Theorem 4.3.7. *If $X \in S^1(\mathcal{H})$, then $\breve{\beta}(X) \in L^1(\mathbb{R}^{2n})$. Moreover,*

$$\left\|\breve{\beta}(X)\right\|_1 \leq \|X\|_{S^1}.$$

Corollary 4.3.8. *If $X \in S^1(\mathcal{H})$ and $\alpha(X)$ is compactly supported, then $\alpha(X)$ is the symplectic Fourier transform of an L^1 function.*

Theorem 4.3.7 and Corollary 4.3.8 are due to Tim Steger. The detailed proofs were first given in [59] (see also [60]).

4.4 Proof of the main result

We will prove Theorem 4.1.2 in this section.

By Theorem 4.3.7, if $X \in S^1(\mathcal{H})$, then $\breve{\beta}(X) \in L^1(\mathbb{R}^{2n})$ and

$$\left\|\breve{\beta}(X)\right\|_1 \leq \|X\|_{S^1}.$$

Observe that $\breve{\beta} : S^1(\mathcal{H}) \to L^1(\mathbb{R}^{2n})$ is bounded.

Lemma 4.4.1. *Let $1 < p \leq 2$. Then $\breve{\beta}$ extends to a bounded linear map from $S^p(\mathcal{H})$ to $L^p(\mathbb{R}^{2n})$. Moreover if $X \in S^p(\mathcal{H})$, then*

$$\left\|\breve{\beta}(X)\right\|_p \leq \|X\|_{S^p}.$$

Proof. Since α extends to an isometric isomorphism from $S^2(\mathcal{H})$ to $L^2(\mathbb{R}^{2n})$, and multiplication by an L^∞ function defines a bounded linear operator on $L^2(\mathbb{R}^{2n})$, it follows that β extends to a bounded linear map from $S^2(\mathcal{H})$ to $L^2(\mathbb{R}^{2n})$. By the Plancherel theorem, it follows that $\breve{\beta}$ extends to a bounded linear map from $S^2(\mathcal{H})$

to $L^2(\mathbb{R}^{2n})$, and if $X \in S^2(\mathcal{H})$ then

$$\left\|\breve{\beta}(X)\right\|_2 = \|\beta(X)\|_2 \leq \|g\|_\infty \|\alpha(X)\|_2 = \|\alpha(X)\|_2 = \|X\|_{S^2}.$$

By the Calderon-Lions interpolation theorem ([41, Theorem 9.20, Example 1, Proposition 8]), it follows that if $1 \leq p \leq 2$, then $\breve{\beta}$ extends to a bounded linear map from $S^p(\mathcal{H})$ to $L^p(\mathbb{R}^{2n})$. Moreover if $X \in S^p(\mathcal{H})$, then

$$\left\|\breve{\beta}(X)\right\|_p \leq \|X\|_{S^p}.$$

□

For $f \in L^1(\mathbb{R}^{2n})$, let $\Gamma(f) = W(\check{f}g)$, where $g(x,y) = e^{-\frac{\pi}{2}(|x|^2+|y|^2)}$, $(x,y) \in \mathbb{R}^{2n}$. By Lemma 4.3.4,

$$\Gamma(f) = f \cdot W(g).$$

Observe that $W(g) = \varphi \otimes \varphi$, where $\varphi(t) = 2^{-1/4}e^{-\pi|t|^2}$, $t \in \mathbb{R}^n$. It follows that $\|W(g)\|_{S^1} = 1$. Therefore $\Gamma : L^1(\mathbb{R}^{2n}) \to S^1(\mathcal{H})$ is bounded, and if $f \in L^1(\mathbb{R}^{2n})$, then

$$\|\Gamma f\|_{S^1} \leq \|f\|_1.$$

Lemma 4.4.2. *Let $1 < p \leq 2$. Then Γ extends to a bounded linear map from $L^p(\mathbb{R}^{2n})$ to $S^p(\mathcal{H})$. Moreover if $f \in L^p(\mathbb{R}^{2n})$, then*

$$\|\Gamma f\|_{S^p} \leq \|f\|_p.$$

Proof. By the Plancherel theorem and the fact that W extends to an isometric isomorphism from $L^2(\mathbb{R}^{2n})$ onto $S^2(\mathcal{H})$, it follows that Γ extends to a bounded linear map from $L^2(\mathbb{R}^{2n})$ to $S^2(\mathcal{H})$. Moreover if $f \in L^2(\mathbb{R}^{2n})$, then

$$\|\Gamma f\|_{S^2} = \left\|g\check{f}\right\|_2 \leq \|g\|_\infty \left\|\check{f}\right\|_2 = \left\|\check{f}\right\|_2 = \|f\|_2 .$$

By the Calderon-Lions interpolation theorem, it follows that if $1 \leq p \leq 2$, then Γ extends to a bounded linear map from $L^p(\mathbb{R}^{2n})$ to $S^p(\mathcal{H})$, and if $f \in L^p(\mathbb{R}^{2n})$, then

$$\|\Gamma f\|_{S^p} \leq \|f\|_p .$$

□

We will now prove that if $2 < p \leq \infty$, then $\breve{\beta}$ extends to a bounded linear map from $S^p(\mathcal{H})$ to $L^p(\mathbb{R}^{2n})$ and Γ extends to a bounded linear map from $L^p(\mathbb{R}^{2n})$ to $S^p(\mathcal{H})$.

Observe that $\breve{\beta}$ is the formal adjoint of the operator Γ. Indeed if $X \in \mathcal{S}(\mathcal{H})$ and $f \in \mathcal{S}(\mathbb{R}^{2n})$, then

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \breve{\beta}(X)(x,y)f(x,y)\,dx\,dy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta(X)(x,y)\check{f}(-x,-y)\,dx\,dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x,y)\alpha(X)(x,y)\check{f}(-x,-y)\,dx\,dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha(X)(x,y)g(-x,-y)\check{f}(-x,-y)\,dx\,dy \\
&= \operatorname{tr}\left(XW(g\check{f})\right) \\
&= \operatorname{tr}\left(X\Gamma(f)\right).
\end{aligned}$$

Let $2 < p \leq \infty$, and $X \in S^p(\mathcal{H})$. Let p' be the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Then $1 \leq p' < 2$. Define $M_X : L^{p'}(\mathbb{R}^{2n}) \to \mathbb{C}$ by

$$M_X(\varphi) = \operatorname{tr}(\Gamma(\varphi)X), \quad \varphi \in L^{p'}(\mathbb{R}^{2n}).$$

Then, by Lemma 4.4.2,

$$|M_X(\varphi)| \leq \|\Gamma(\varphi)\|_{S^{p'}} \|X\|_{S^p} \leq \|\varphi\|_{p'} \|X\|_{S^p}.$$

Therefore M_X is a bounded linear functional on $L^{p'}(\mathbb{R}^{2n})$, and $\|M_X\|_{\text{op}} \leq \|X\|_{S^p}$. Therefore, there exists $\psi \in L^p(\mathbb{R}^{2n})$ with $\|\psi\|_p \leq \|X\|_{S^p}$ such that $M_X(\varphi) = \int \varphi\psi$. Then $\breve{\beta}(X) = \psi$. Indeed, since $\breve{\beta}$ is the formal adjoint of Γ, it follows that if $X \in \mathcal{S}(\mathcal{H})$ and $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$, then

$$\int \varphi\psi = M_X(\varphi) = \operatorname{tr}(\Gamma(\varphi)X) = \int \varphi\breve{\beta}(X).$$

Therefore $\breve{\beta}(X) \in L^p(\mathbb{R}^{2n})$, and

$$\left\|\breve{\beta}(X)\right\|_p \leq \|X\|_{S^p}. \tag{4.2}$$

Let $2 < p \leq \infty$, and $f \in L^p(\mathbb{R}^{2n})$. Let p' be the conjugate of p. Then $1 \leq p' < 2$. Define $L_f : S^{p'}(\mathcal{H}) \to \mathbb{C}$ by

$$L_f(X) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x,y)\breve{\beta}(X)(x,y)\, dx\, dy, \quad X \in S^{p'}(\mathcal{H}).$$

Then, by Lemma 4.4.1,

$$|L_f(X)| \leq \left\|\breve{\beta}(X)\right\|_{p'} \|f\|_p \leq \|X\|_{S^{p'}} \|f\|_p.$$

Therefore L_f is a bounded linear functional on $S^{p'}(\mathcal{H})$, and $\|L_f\|_{\text{op}} \leq \|f\|_p$. By Theorem 2.2.11, there exists $Y \in S^p(\mathcal{H})$ with $\|Y\|_{S^p} \leq \|f\|_p$ such that $L_f(X) = \operatorname{tr}(YX)$. Since Γ is the formal adjoint of $\breve{\beta}$, it follows that $\Gamma(f) = Y$. Therefore

$\Gamma(f) \in S^p(\mathcal{H})$, and

$$\|\Gamma f\|_{S^p} \leq \|f\|_p \, . \tag{4.3}$$

We are now in a position to prove part (a) of Theorem 4.1.2. Let K be a compact set in $\mathbb{R}^{2n}$. Suppose T is supported in K, and $\widehat{T} \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$. Then $\check{T} \in L^p(\mathbb{R}^{2n})$. Let f be a compactly supported smooth function which is identically one on K, and put $h(x,y) = e^{\frac{\pi}{2}(|x|^2+|y|^2)} f(x,y)$. Then $\check{h}$ is integrable; put $C_K = \left\|\check{h}\right\|_1$. Let $T_1 = hT$. Then T_1 is a compactly supported distribution. By Young's inequality, $\check{T}_1 = \check{h} * \check{T} \in L^p(\mathbb{R}^{2n})$. Therefore, by Lemma 4.4.2 and equation (4.3), $\Gamma(\check{T}_1) \in S^p(\mathcal{H})$ and $\left\|\Gamma(\check{T}_1)\right\|_{S^p} \leq \left\|\check{T}_1\right\|_p$. However

$$\Gamma(\check{T}_1) = W(gT_1) = W(ghT) = W(T).$$

Therefore $W(T) \in S^p(\mathcal{H})$, and

$$\|W(T)\|_{S^p} \leq \left\|\check{T}_1\right\|_p \leq C_K \left\|\check{T}\right\|_p = C_K \left\|\widehat{T}\right\|_p .$$

Conversely, assume that T is supported in K, and $W(T) \in S^p(\mathcal{H})$, $1 \leq p \leq \infty$. Let $Z = \check{h} \cdot W(T)$, where $h(x,y) = e^{\frac{\pi}{2}(|x|^2+|y|^2)} f(x,y)$ and f is a compactly supported smooth function which is identically one on K. Then, by Lemma 4.3.3, $Z \in S^p(\mathcal{H})$ and $\|Z\|_{S^p} \leq C_K \|W(T)\|_{S^p}$. Therefore, by Lemma 4.4.1 and equation (4.2), $\check{\beta}(Z) \in L^p(\mathbb{R}^{2n})$ and $\left\|\check{\beta}(Z)\right\|_p \leq \|Z\|_{S^p} \leq C_K \|W(T)\|_{S^p}$. However, by Lemma 4.3.5,

$$\check{\beta}(Z) = (g\alpha(Z))\check{} = (ghT)\check{} = \check{T}.$$

Therefore $\check{T} \in L^p(\mathbb{R}^{2n})$, and hence $\widehat{T} \in L^p(\mathbb{R}^{2n})$. Moreover,

$$\left\|\widehat{T}\right\|_p = \left\|\check{T}\right\|_p \leq C_K \|W(T)\|_{S^p} .$$

This proves part (a) of Theorem 4.1.2. We will now prove part (b) of Theorem 4.1.2.

Let T be a compactly supported distribution on $\mathbb{R}^{2n}$ such that $\widehat{T} \in \mathcal{C}_0(\mathbb{R}^{2n})$. Then $\breve{T} \in \mathcal{C}_0(\mathbb{R}^{2n})$. Let $\varepsilon > 0$. Let B denote the closed unit ball centered at 0 in $\mathbb{R}^{2n}$, and put $K = \operatorname{supp}(T) + B$ (Minkowski sum). Then K is compact. Let ρ be a non-negative smooth function supported in B with $\int \rho = 1$. For each $r > 1$, let ρ_r be defined by $\rho_r(x, y) = r^{2n}\rho(rx, ry)$. Observe that for each $r > 0$, $\breve{\rho}_r \in L^1(\mathbb{R}^{2n})$, $\|\breve{\rho}_r\|_\infty = 1$, and $\lim_{r\to\infty} \breve{\rho}_r = 1$ uniformly on compact sets. It follows that for sufficiently large r, $\left\|\breve{\rho}_r\breve{T} - \breve{T}\right\|_\infty < \varepsilon/C_K$. Put $Y = W(\rho_r * T)$. Then $Y \in S^1(\mathcal{H})$ by part (a) of Theorem 4.1.2. Since $\rho_r * T - T$ is supported in K, it follows from part (a) of Theorem 4.1.2 that

$$\begin{aligned}
\|Y - W(T)\|_{\mathrm{op}} &= \|W(\rho_r * T - T)\|_{\mathrm{op}} \\
&\le C_K \left\|(\rho_r * T)\breve{\ } - \breve{T}\right\|_\infty \\
&= C_K \left\|\breve{\rho}_r\breve{T} - \breve{T}\right\|_\infty < \varepsilon.
\end{aligned}$$

Therefore $W(T)$ is compact.

Conversely, let T be a compactly supported distribution on $\mathbb{R}^{2n}$ such that $W(T)$ is compact. Let $\varepsilon > 0$. Observe that for each $r > 0$, $W(\rho_r) \in S^1(\mathcal{H})$, $\|W(\rho_r)\|_{\mathrm{op}} \le 1$, and $W(\rho_r)$ converges strongly to the identity operator as $r \to \infty$.

Since $W(T)$ is compact and the finite rank operators are dense in $\mathcal{K}(\mathcal{H})$, there exists a finite-rank operator X such that for sufficiently large r,

$$\|W(T) - X\|_{\mathrm{op}} < \frac{\varepsilon}{3C_K}. \tag{4.4}$$

Let $\varphi, \psi \in \mathcal{H}$. Then $(W(\rho_r))(\varphi)$ converges to φ in $\mathcal{H}$, i.e., for sufficiently large r,

$$\|W(\rho_r)\varphi - \varphi\|_2 \leq \frac{\varepsilon}{3C_K \|\psi\|_2}.$$

Therefore, for sufficiently large r,

$$\begin{aligned} \left\|W(\rho_r)\varphi \otimes \overline{\psi} - \varphi \otimes \overline{\psi}\right\|_{\mathrm{op}} &= \left\|(W(\rho_r)\varphi - \varphi) \otimes \overline{\psi}\right\|_{\mathrm{op}} \\ &= \|W(\rho_r)\varphi - \varphi\|_2 \|\psi\|_2 \\ &\leq \frac{\varepsilon}{3C_K}. \end{aligned}$$

It follows that for sufficiently large r,

$$\|W(\rho_r)X - X\|_{\mathrm{op}} \leq \varepsilon/(3C_K). \tag{4.5}$$

By equations (4.4) and (4.5), for sufficiently large r,

$$\begin{aligned} &\|W(\rho_r)W(T) - W(T)\|_{\mathrm{op}} \\ \leq\ &\|W(\rho_r)W(T) - W(\rho_r)X\|_{\mathrm{op}} + \|W(\rho_r)X - X\|_{\mathrm{op}} + \|X - W(T)\|_{\mathrm{op}} \\ \leq\ &\frac{\varepsilon}{3C_K} \|W(\rho_r)\|_{\mathrm{op}} + \frac{2\varepsilon}{3C_K} \\ <\ &\frac{\varepsilon}{C_K}. \end{aligned}$$

Put $f = \rho_r \natural T$. Then $\breve{f} \in \mathcal{S}(\mathbb{R}^{2n})$. Since $\rho_r \natural T - T$ is supported in K, it follows from part (a) of Theorem 4.1.2 that

$$\begin{aligned} \left\| \breve{f} - \breve{T} \right\|_\infty &\leq C_K \left\| W(\rho_r \natural T - T) \right\|_{\mathrm{op}} \\ &= C_K \left\| W(\rho_r \natural T) - W(T) \right\|_{\mathrm{op}} \\ &= C_K \left\| W(\rho_r) W(T) - W(T) \right\|_{\mathrm{op}} \\ &< \varepsilon. \end{aligned}$$

Therefore $\breve{T} \in \mathcal{C}_0(\mathbb{R}^{2n})$, and so $\widehat{T} \in \mathcal{C}_0(\mathbb{R}^{2n})$.

4.5 Conclusion

We conclude that the Weyl transform of a compactly supported distribution on $\mathbb{R}^{2n}$ is in $S^p(\mathcal{H})$ if and only if the Fourier transform of the distribution is in $L^p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$. Moreover, the Weyl transform of a compactly supported distribution on $\mathbb{R}^{2n}$ is compact if and only if the Fourier transform of the distribution vanishes at infinity. As a consequence of this, we conclude the following results.

In Theorem 2.1.1, we proved that the Weyl transform of a smooth measure supported on a smooth compact hypersurface in $\mathbb{R}^{2n}$, $n \geq 2$, is compact, provided that the hypersurface has positive Gaussian curvature. Moreover, we proved that the Weyl transform of such a measure belongs to $S^p(\mathcal{H})$ if $p > n \geq 6$. In Theorem 3.4.1, we proved that the Weyl transform of a smooth measure supported on a smooth compact hypersurface in $\mathbb{R}^2$ is compact, provided that the hypersurface has non-zero curvature. By Theorem 4.1.2 and Theorem 1.4.3, we get the following result.

Theorem 4.5.1. *Suppose S is a compact connected smooth hypersurface in $\mathbb{R}^{2n}$, whose Gaussian curvature is nonzero everywhere. Let μ be a smooth measure on*

S. Then $W(\mu)$ is a compact operator. Moreover, $W(\mu) \in S^p(\mathcal{H})$ if and only if $p > 4n/(2n-1)$.

In Theorem 3.1.1, we proved that the Weyl transform of a smooth measure supported on a finite type real-analytic submanifold of $\mathbb{R}^{2n}$ is compact. By Theorem 4.1.2, Theorem 1.4.5, and Theorem 1.4.6, we get the following result.

Theorem 4.5.2. *Suppose M is a finite type smooth submanifold of $\mathbb{R}^{2n}$. Let $\mu = \psi\sigma$ be a smooth measure on M. Then $W(\mu)$ is a compact operator. Moreover, $W(\mu) \in S^p(\mathcal{H})$ if and only if $p > 2nk$, where k is the type of M inside the support of ψ.*