Chapter 3

Pure C3 Modules

In 1976, W.K. Nicholson [47], defined an R-module M to be direct injective if given a direct summand N of M with injection $i : N \to M$ and any monomorphism $g : N \to M$ there exists $f \in End_R(M)$ such that $f \circ g = i$. In 2003, Nicholson and Yousif [48] proved that the class of direct injective modules is equivalent to the class of $C2$ modules. An R-module is called a $C2$ module if every submodule of N which is isomorphic to a direct summand of N is itself a direct summand of N. In 2022, Maurya et al. [38] generalized the class of $C2$ modules to the class of pure $C2$ modules. Here, we introduced the new notion of pure $C3$ modules, which is a generalization of the concept of pure $C2$ modules and $C3$ modules. We give new characterizations of many rings in terms of pure $C3$ modules, namely semisimple rings, pure-semisimple rings, von Neumann regular rings, Noetherian rings, pure-hereditary rings, pure-V-rings, etc. Moreover, we also discuss the pure $C3$ envelope and pure $C3$ cover of a module and introduce the pure quasi-continuous modules as the generalization of the quasi-continuous modules.

3.1 Pure C3 Modules

We first define pure $C3$ modules with some of their examples and study their various properties.

Definition 3.1.1. *An R-module N is called pure $C3$ if $A \oplus B$ is a pure submodule of N for submodules A and B of N with $A \leq^{\oplus} N$, $B \leq^{\oplus} N$ and $A \cap B = 0$, then $A \oplus B \leq^{\oplus} N$.*

Example 3.1.2.

1. *Any $C2$ module or a pure $C2$ module is a pure $C3$ module.*
2. *Any quasi-pure-injective module is a pure $C3$ module [38].*
3. *$\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module is a quasi-pure-projective module because there are no proper pure submodules. Hence, $\mathbb{Z}_{p^\infty}$ is pure $C3$ module.*
4. *$\mathbb{Z}$ as $\mathbb{Z}$-module is pure $C3$ because there are no proper pure submodules of $\mathbb{Z}$ [Corollary 4.92, [32]].*
5. *Let $N = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ as $\mathbb{Z}$-module, where p is any prime is pure $C3$ module since N is finitely generated module over the Noetherian ring, so the pure submodules of N are only direct summands [Corollary 4.91, [32]].*

Proposition 3.1.3. *Let N be a pure $C3$ module, where $N = N_1 \oplus N_2$ for submodules N_1 and N_2 of module N. Let $f : N_1 \to N_2$ be a homomorphism with $Ker(f) \leq^{\oplus} N_1$ and $Im(f)$ is pure in N_2, then $Im(f) \leq^{\oplus} N_2$.*

Proof. It easily follows from [Proposition 2.3,[2]]. □

Corollary 3.1.4. *Let $N = N_1 \oplus N_2$ for some submodules N_1 and N_2 of N. If N is a pure $C3$ module with $f : N_1 \to N_2$ a monomorphism and $Im(f)$ pure in N_2, then $Im(f) \leq^{\oplus} N_2$.*

Corollary 3.1.5. *Let N be an R-module. If $N \oplus PE(N)$ is a pure $C3$ module, then N is pure-injective.*

Proof. Let $i : N \to PE(N)$ be the inclusion map. By Corollary 3.1.4, $i(N) = N \leq^{\oplus} PE(N)$ it implies $N = PE(N)$. Hence N is pure-injective □

Proposition 3.1.6. *For a pure $C3$ module $N = N_1 \oplus N_2$, the following conditions are equivalent:*

1. *Let a homomorphism $f : N_1 \to N_2$ with $Ker(f) \leq^{\oplus} N_1$ and $Im(f)$ is pure in N_2. Then $Im(f) \leq^{\oplus} N_2$;*
2. *Let a monomorphism $f : N_1 \to N_2$ such that $Im(f)$ is pure in N_2. Then $Im(f) \leq^{\oplus} N_2$.*

Proof. (2) ⇒ (1). Obvious by Corollary 3.1.5.

(1) ⇒ (2). By the given hypothesis, $Ker(f) \leq^{\oplus} N_1$ then $N_1 = Ker(f) \oplus X$ for some submodule $X \leq N_1$. So, $N = X \oplus (Ker(f) \oplus N_2)$. Now, a monomorphism $f|_X : X \to N_2$ and $Im(f|_X) = Im(f)$ (since $N_1 = Ker(f) \oplus X$), it implies $Im(f|_X)$ is pure in N_2. Hence, $Im(f) \leq^{\oplus} N_2$. □

Proposition 3.1.7. *Let N be a pure $C3$ module. Then the following statements hold:*

1. *If $A \leq^{\oplus} N$, $B \leq^{\oplus} N$ such that $A \cap B = 0$, then $N = A_1 \oplus B = A \oplus B_1$ for submodules $A \leq A_1$ and $B \leq B_1$.*
2. *If $A \leq^{\oplus} N$, $B \leq^{\oplus} N$ and $A \cap B \leq^{\oplus} N$, then $A + B \leq^{\oplus} N$.*

Proof. It is clear. □

By definition of $C3$ modules, we know that the $C3$ module implies pure $C3$, whereas we have given a counterexample that shows that a pure $C3$ module does not imply $C3$ module in general.

Example 3.1.8. *Suppose* $\mathbb{Z}\oplus 2\mathbb{Z}$ *as* $\mathbb{Z}$*-module, [Example 2.5, [2]] is not a* $C3$ *module whereas it is pure* $C3$ *since* $\mathbb{Z}\oplus 2\mathbb{Z}$ *module is finitely generated over the Noetherian ring, so the pure submodules of a module* $\mathbb{Z}\oplus 2\mathbb{Z}$ *are the only direct summands [Corollary 4.91, [32]]. Hence, it is a pure* $C3$ *module.*

Proposition 3.1.9. *If* N *is a pure* $C3$ *module with* $K \leq^{\oplus} N$*, then* K *is also a pure* $C3$ *module.*

Proof. Let N be a pure $C3$ module and $K \leq^{\oplus} N$. Let $N_1,\ N_2 \leq^{\oplus} K$ such that $N_1 \cap N_2 = 0$, then $N_1 \oplus N_2$ is a pure submodule in K. Since $N_1 \oplus N_2 \leq^p K \leq^p N \Rightarrow N_1 \oplus N_2 \leq^p N$. As N is a pure $C3$ module, it follows that $N_1 \oplus N_2 \leq^{\oplus} N \Rightarrow N = (N_1 \oplus N_2) \oplus L$ for some submodule L of N. Now, by modular law $N \cap K = [(N_1 \oplus N_2) \oplus L] \cap K \Rightarrow K = (N_1 \oplus N_2) \oplus [L \cap K]$ $(\underline{\text{as } N_1 \oplus N_2 \leq K})$. Hence, $N_1 \oplus N_2 \leq^{\oplus} K$, so K is a pure $C3$ module.

$\square$

Theorem 3.1.10. *Any pure* $C2$ *module is a pure* $C3$ *module.*

Proof. Let N be a pure $C2$ module and $N_1,\ N_2 \leq^{\oplus} K$ such that $N_1 \cap N_2 = 0$. Let $N = N_1 \oplus N_1'$ for some submodule N_1' of N and $f : N = N_1 \oplus N_1' \to N_1'$ be a projection map. As $f|_{N_2}$ is a monomorphism (since $f|_{N_1'}$ is a monomorphism and $N_2 \leq N_1'$), so we get $f(N_2) \cong N_2 \leq^{\oplus} N$ and N is a pure $C2$, so $f(N_2) \leq^{\oplus} N$. As $f(N_2) \leq^{\oplus} N \Rightarrow N = f(N_2) \oplus L$ for some submodule $L \leq N$. Therefore by modular law, $N \cap N_1' = [f(N_2) \cap L] \cap N_1' \Rightarrow N_1' = f(N_2) \oplus (L \oplus N_1')$ $(\underline{\text{as } f(N_2) \leq N_1'})$ $\Rightarrow f(N_2) \leq^{\oplus} N_1'$ which implies that $N_1' = f(N_2) \oplus K$ for some submodule K of

N. So, $N = N_1 \oplus (f(N_2) \oplus K) \Rightarrow N = (N_1 \oplus (f(N_2)) \oplus K$. Hence, $N_1 \oplus N_2 \cong N_1 \oplus f(N_2) \leq^{\oplus} N \Rightarrow N_1 \oplus N_2 \leq^{\oplus} N$ (because N is pure $C2$). So, N is a pure $C3$ module. □

Remark 3.1.11. *As every $C3$ module is not a $C2$ module [4], intuitively, we observe that every pure $C3$ module need not be a pure $C2$ module, but unfortunately, we have not found any counter-example.*

Proposition 3.1.12. *Let N be a pure simple module. Then N is a pure $C3$ module.*

Proof. If N is a pure-simple module, then it does not contain any proper pure submodule. Hence N is a pure $C3$ module. □

Corollary 3.1.13. *Every indecomposable module is a pure $C3$ module.*

In the next theorem, we show that every finitely generated R-module is dual-Rickart if and only if every finitely generated R-module is a pure $C3$ module.

Theorem 3.1.14. *Let R be a ring and every R module be a finitely generated module. Then the following statements are equivalent :*

1. *Every R-module is dual-Rickart;*
2. *Every R-module has SSP property;*
3. *Every submodule of an R-module is a direct summand;*
4. *Every R-module is pure $C3$.*

Proof. (1) $\Rightarrow$ (2) Every dual-Rickart module has the SSP property [Proposition 2.11, [33]].

(2) $\Rightarrow$ (3) Let N be finitely generated R-module such that $K \leq N$. By the given hypothesis, $N \oplus K$ has SSP property. By Corollary 3.1.4, $K \leq^{\oplus} N$.

(3) $\Rightarrow$ (4) Let N be a finitely generated R-module with K and L as direct summands of N with zero intersection. Now, $K+L$ is finitely generated as K and L are finitely generated . It follows that $K+L \leq^{\oplus} N$. So N is $C3$ i.e. pure $C3$.

(4) $\Rightarrow$ (1) Let N be a finitely generated R-module and f be an endomophism of N. Then $N + f(N)$ is finitely generated. By hyphothesis, $N + f(N)$ is pure $C3$. So by Corollary 3.1.4, $f(N) \leq^{\oplus} N$. Hence, N is dual-Rickart. □

Remark 3.1.15. *In the above proposition,* (2) $\Rightarrow$ (4) *holds without taking* N *to be a finitely generated* R*-module.*

Corollary 3.1.16. *Let* N *be an* R*-module with* SSP *property. Then* N *is a pure* $C3$ *module.*

Proof. Let N be an R-module having the SSP property. It follows that the sum of any two direct summands is again a summand. Thus, N is pure $C3$. □

3.2 Characterization of rings using pure C3 modules

This section deals with some characterizations of several rings in terms of pure $C3$ modules. In the first three propositions, we obtain new characterizations of the von Neumann regular rings with respect to pure $C3$ modules.

Proposition 3.2.1. *For a ring* R*, the following statements are equivalent :*

1. *R is a von Neumann regular ring;*

2. *Every pure $C3$ R-module is $C3$ module.*

Proof. (1) ⇒ (2) Let N be a pure $C3$ module and $A \leq^{\oplus} N$, $B \leq^{\oplus} N$ such that $A \cap B = 0$. Since, over a von Neumann regular ring R, every submodule of an R-module N is pure. As N is pure $C3$, it follows that $A \oplus B \leq^{\oplus} N$. Hence, N is a $C3$ module.

(2) ⇒ (1) Let N be a pure-injective module. Then $E(N)$ is pure-injective. So $N \oplus E(N)$ is pure-injective. Therefore, the inclusion map $i : N \to E(N)$ splits, so $N = E(N)$ is an injective module. Hence, R is a von Neumann regular ring [Proposition 37.6, [57]]. □

Proposition 3.2.2. *For a ring R, the following conditions are equivalent:*

1. *R is a von Neumann regular ring;*
2. *Every pure $C3$ R-module is a flat.*

Proof. (1) ⇒ (2) It follows from [Theorem 4.21, [32]].

(2) ⇒ (1) Let N be an R-module and $PE(N)$ be the pure-injective hull of N. So $0 \to N \to PE(N) \to PE(N)/N \to 0$ is a pure exact sequence. From the given hypothesis, $PE(N)$ is a flat module so by [Corollary 4.86(1), [32]], $PE(N)/N$ is flat module. Therefore, by [Corollary 4.86 (2), [32]], N is a flat module which implies R is a von Neumann regular ring [Theorem 4.21, [32]]. □

Proposition 3.2.3. *If every pure $C3$ R-module is quasi-injective, then R is von Neumann regular ring.*

Proof. Let N be a pure-injective R-module. Then $E(N)$ is pure-injective. So, $N \oplus E(N)$ is pure-injective. Hence, $N \oplus E(N)$ is quasi-injective. Now, the inclusion map $i : N \to E(N)$, splits. Thus, $N = E(N)$ it implies N is injective. By [Proposition 3.7, [45]], R is a von Neumann regular ring. □

In the following proposition, we obtain a characterization of the PDS ring in terms of a pure $C3$ module.

Proposition 3.2.4. *Over a PDS ring R, all R-modules are pure $C3$.*

Proof. Let N be an R-module and $A, B \leq^{\oplus} N$ such that $A \cap B = 0$ with $A \oplus B$ pure in N. By the given hypothesis, R is a PDS ring in which all pure submodules are direct summands, it follows that $A \oplus B \leq^{\oplus} N$. Hence N be pure $C3$.

□

Theorem 3.2.5. *For a ring R, the following conditions are equivalent:*

1. *R is a semisimple ring;*
2. *Every pure $C3$ R-module is projective;*
3. *Every pure $C2$ R-module is projective;*
4. *Every quasi-pure-injective R-module is projective;*
5. *Every pure-injective R-module is projective.*

Proof. (1) ⇒ (2) If R is a semisimple ring then every R-module N is projective [Proposition 20.7,[57]], so (2) holds.

(2) ⇒ (3) Every pure $C2$ module is pure $C3$, so (3) holds.

(3) ⇒ (4) Every quasi-pure-injective R-module is pure $C2$, so (4) holds.

(4) ⇒ (5) Every pure-injective R-module is quasi-pure-injective, so (5) holds.

(5) ⇒ (1) Every pure-injective right R-module is projective, which implies every injective is projective. Therefore, (1) holds by [Proposition 20.7, [57]]. □

For a semisimple ring R, an R-module is projective if and only if it is injective. So, we give an immediate consequence of Theorem 3.2.5 in terms of injective modules in the following proposition.

Theorem 3.2.6. *For a ring R, the following conditions are equivalent:*

1. *R is a semisimple ring;*
2. *Every pure $C3$ R-module is injective;*
3. *Every pure $C2$ R-module is injective;*
4. *Every quasi-pure-injective R-module is injective;*
5. *Every pure-injective R-module is injective.*

Proof. Straightforward. □

In the next proposition, we give a new characterization of pure-semisimple rings in respect of pure $C3$ modules.

Proposition 3.2.7. *For a ring R, the following conditions are equivalent:*

1. *R is a pure-semisimple ring;*
2. *Every R-module is pure-injective;*
3. *Every R-module is pure $C3$.*

Proof. (1) $\Leftrightarrow$ (2) from [Theorem 8.4, [27]].

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (2) Let K be an R-module, $N = K \oplus PE(K)$ and $\phi : K \to PE(K)$ be the inclusion map. By Proposition 3.1.6, $Im(\phi) = K \leq^{\oplus} PE(K)$ and $PE(K)$ is pure essential extension of K, $K = PE(K)$ i.e. K is pure-injective. So, (2) holds. □

Proposition 3.2.8. *For a uniserial ring, every R-module N is a pure $C3$ module.*

Proof. Since every uniserial ring is pure-semisimple [Theorem 10.4, [17]], N is a pure $C3$ module. □

Proposition 3.2.9. *Let R to be a Noetherian ring, then every finitely generated R-module is a pure $C3$ module.*

Proof. Pure submodules of a finitely generated module over a Noetherian ring are just direct summands [Corollary 4.91, [32]] it implies a pure $C3$ module. □

In the following proposition, we give a new characterization of pure-hereditary rings in terms of pure $C3$ modules.

Proposition 3.2.10. *For a ring R, the following conditions are equivalent:*

1. *R is a pure-hereditary ring;*
2. *Every quotient module of pure-injective R-module is pure $C3$ module.*

Proof. (1) $\Leftrightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (1) Let N be an injective R-module and $K \leq N$. Let $PE(N/K) \oplus N$ be an R-module. Then $PE(N/K) \oplus N/K$ is the factor module of $PE(N/K) \oplus N$. By the given hypothesis, $PE(N/K) \oplus N/K$ is pure $C3$ module and a inclusion map $\phi : N/K \to PE(N/K)$. (By Proposition 3.1.6), $Im(\phi) = N/K \leq^{\oplus} PE(N/K)$ and $PE(N/K)$ is pure-essential extension of N/K, $N/K = PE(N/K)$ i.e. N/K is pure-injective as required. Hence, R is a pure-hereditary ring. □

Proposition 3.2.11. *If every finitely cogenerated R-module is pure $C3$ then R is a pure-V-ring.*

Proof. Let N be a simple R-module and $N \oplus PE(N)$ is finitely cogenerated. By hypothesis, $N \oplus PE(N)$ is pure $C3$ module. By Corollary 3.1.5, N is pure-injective. So, every simple module is pure-injective, i.e., R is pure-V-ring. ☐

Remark 3.2.12. *The converse of the Proposition 3.2.11 may not hold in general. However, in particular, if we take ring R to be a von Neumann ring and pure-V-ring, then every cofinitely generated R-module is pure $C3$.*

Proposition 3.2.13. *Every quasi-pure-injective module is a pure $C3$ module.*

Proof. Clear from [Corollary 3.9, [36]]. ☐

Proposition 3.2.14. *The following statements are equivalent:*

1. *Every pure $C3$ module is pure-injective;*
2. *The direct sum of any two pure $C3$ modules is a pure $C3$ module.*

Proof. (1) ⇒ (2) Let N_1 and N_2 be two pure $C3$ modules. By the given hypothesis, N_1 and N_2 are pure-injective then $N_1 \oplus N_2$ is a pure-injective module. Hence, $N_1 \oplus N_2$ is pure $C3$ module.

(2) ⇒ (1) Let N be a pure $C3$ module, by the given hypothesis $N \oplus PE(N)$ is pure $C3$. By Corollary 3.1.5, N is pure-injective. ☐

Recall a ring R is said to be a Xu ring if every cotorsion R-module is a pure-injec. module [22]

Proposition 3.2.15. *For a ring R, the following statements are equivalent:*

1. *R is a Xu ring;*

2. *Every cotorsion R-module is pure $C3$ module.*

Proof. (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (1) Let N be a cotorsion R-module. Then there exists a pure exact sequence $0 \to N \to PE(N) \to PE(N)/(N) \to 0$ where $PE(N)$ is the pure-injective hull of N. Since $N \oplus PE(N)$ is cotorsion module, by hypothesis, $N \oplus PE(N)$ is pure $C3$. Hence, R is the Xu ring. □

Motivated by [2], and [4], the notion of a Pure C3 Envelope and Pure C3 Cover of a module are introduced here.

3.3 Pure C3 Envelopes and Pure C3 Covers of a Module

In this section, we study pure $C3$ envelopes and pure $C3$ covers of a module.

Definition 3.3.1. *A homomorphism $f : N \longrightarrow E$ is called a pure $C3$ envelope of a module N, if E is a pure $C3$ module and any diagram below*

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "g"'] & E \arrow[dl, dashed, "h"] \\ F & \end{tikzcd}$$

with F as a pure $C3$ module and a homomorphism $g : N \longrightarrow F$ can be completed by a homomorphism $h : E \longrightarrow F$, and the diagram

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, "g"'] & E \arrow[dl, dashed, "k"] \\
E &
\end{tikzcd}$$

can be completed by an automorphism $k : E \longrightarrow E$.

Proposition 3.3.2. *The following statements are equivalent:*

1. *Every pure $C3$ module is pure-injective;*
2. *Every module has a pure $C3$ envelope.*

Proof. (1) $\Rightarrow$ (2) Since every module has a pure-injective envelope; by hypothesis, every pure $C3$ module is pure-injective. Hence, Every module has a pure $C3$ envelope.

(2) $\Rightarrow$ (1) Let N be a pure $C3$ module and $f : N \oplus PE(N) \to E$ is a pure $C3$ envelope of $N \oplus PE(N)$. Now, $\pi_1 : N \oplus PE(N) \to N$ and $\pi_2 : N \oplus PE(N) \to PE(N)$ be canonical projections. Since N and $PE(N)$ are pure $C3$ modules, so there are homomorphisms $\theta_1 : E \to N$ and $\theta_2 : E \to PE(N)$ such that the following diagram commutes.

$$\begin{tikzcd}
N \oplus PE(N) \arrow[r, "\mu"] \arrow[d, "\pi_1"'] & E \arrow[dl, dashed, "\theta_1"] \\
N &
\end{tikzcd}$$

$$\begin{tikzcd}
N \oplus PE(N) \arrow[r, "\mu"] \arrow[d, "\pi_2"'] & E \arrow[dl, dashed, "\theta_2"] \\
PE(N) &
\end{tikzcd}$$

Now, from diagrams $\theta_1\mu = \pi_1$ and $\theta_2\mu = \pi_2$. Also define $\theta : E \to N \oplus PE(N)$ by $\theta(x) = \theta_1(x) + \theta_2(x)$ such that $\theta\mu = id_E$, it implies μ is a monomorphism, and $N \oplus PE(N) \cong Im(\mu) \leq^{\oplus} E$. By Proposition 3.1.9, $N \oplus PE(N)$ is pure $C3$ module. Hence, by Corollary 3.1.5, N is a pure-injective module.

□

Definition 3.3.3. *A homomorphism $f : C \to N$ is called a pure $C3$ cover of a module N, if C is a pure $C3$ module such that the following diagram commutes*

$$\begin{tikzcd}
& C \arrow[d, "f"] \\
C' \arrow[ur, dashed, "g"] \arrow[r, "h"] & N
\end{tikzcd}$$

with C' is a pure $C3$ module can be completed

$$\begin{tikzcd}
& C \arrow[d, "f"] \\
C \arrow[ur, dashed, "g"] \arrow[r, "h"] & N
\end{tikzcd}$$

and above diagram commutes where $g : C \longrightarrow C$ automorphism.

Proposition 3.3.4. *Let R be a pure-semisimple ring. Then every R-module has a pure $C3$ cover.*

Proof. Every R-module over a pure-semisimple ring is pure $C3$. Hence, every R-module has a pure $C3$ cover. □

Proposition 3.3.5. *Let R be a Noetherian ring. Then every finitely generated R-module has a pure $C3$ cover.*

Proof. As we know every finitely generated module over the Noetherian ring is pure-injective. So, every finitely generated module over the Noetherian ring is a pure $C3$ module. Hence, every finitely generated R-module has pure $C3$ cover. ☐

3.4 Pure Quasi-Continuous Modules

In this section, we define pure quasi-continuous modules which is the generalization of the quasi-continuous modules by the use of pure $C3$ modules.

Definition 3.4.1. *A pure quasi-continuous module is a module N in which every pure submodule is essential in a direct summand of N, and if K, L are direct summands of N with zero intersection and $K \oplus L$ a pure submodule of N, then $K \oplus L$ is a direct summand of N.*

In other words, a module that is pure $C1$ and pure $C3$ is known as a pure quasi-continuous module.

Example 3.4.2. *Any Quasi-continuous module or Continuous module is a pure quasi-continuous module.*

It is easily seen from the definition of pure quasi-continuous modules that quasi-continuous modules imply pure quasi-continuous modules, whereas the converse is not true. We show this in the following example.

Example 3.4.3. *Let $N = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ as $\mathbb{Z}$-module, here p is any prime. In particular take $p = 2$, $P = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$-module. P is pure $C1$ whereas P is not $C1$ [Example 2.1.7]. Moreover, by Example 3.1.2 (5), P is a pure $C3$, so P is a pure quasi-continuous but not a quasi-continuous module.*

This example shows that a pure quasi-continuous module is a proper generalization of the quasi-continuous module.

Proposition 3.4.4. *Every pure continuous module is a pure quasi-continuous module.*

Proof. As we know every pure $C2$ module is a pure $C3$ module (Theorem 3.1.10, so a pure continuous module is a quasi-pure-continuous module. ☐

Proposition 3.4.5. *Every direct summand of a pure quasi-continuous module is a pure quasi-continuous module.*

Proof. The proof easily follows from Proposition 2.1.4 and Proposition 3.1.9. ☐