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An infeasible interior-point technique to generate the nondominated set for multiobjective optimization problems

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ABSTRACT

In this paper, an infeasible interior-point technique is proposed to generate the nondominated set of nonlinear multi-objective optimization problems with the help of the direction-based cone method. We derive the proposed method for both convex and nonconvex problems. In order to solve the parametric optimization problems of the cone method, the infeasible interior-point method starts with an initial iterate outside the feasible region, and then gradually reduces the primal and dual infeasibility measures and the objective function value across the iterations with the help of a merit function. Estimates of the reduction of primal and dual infeasibility parameters per iteration are given. The convergence analysis of the method and an estimate of the number of iterations to reach an ϵ-precise solution are also provided. We provide the performance of the proposed methods on a variety of convex and nonconvex multi-objective test problems. Performance comparison between the proposed method and popular existing solvers is provided with respect to two performance measures and the corresponding relative efficiency measures. The reduction of a combined infeasibility measure, as the iterations progress, on the test problems is also shown graphically.

1. Introduction

In order to achieve the nondominated set of multi-objective optimization problems, there are several methods such as weighted sum (Kim and De Weck, 2005, 2006; Zadeh, 1963), ϵ-constraint (Ehrgott, 2005; Miettinen, 2012), normal constraint (Messac et al., 2003), normal boundary intersection (Das and Dennis, 1998; Motta et al., 2012), physical programming (Messac, 2006), directed search domain (Erfani and Utyuzhnikov, 2011), etc. The classical methods for solving multi-objective optimization problems (MOPs) are described in Ansari et al. (2018), Ehrgott (2005), Kim and De Weck (2006), Marler and Arora (2004), Miettinen (2012), Pascoletti and Serafini (1984), Rosinger (1981), Wang and Yang (1991) and Zhao et al. (2021) and the references therein.

In the recent past, many researchers have started extending the classical methods of scalar optimization to solve MOPs. These extensions include steepest descent (Drummond and Svaiter, 2005; Fliege and Svaiter, 2000), conditional gradient (Assuncao et al., 2021), Newton (Fleige et al., 2009; Wang et al., 2019), quasi-Newton (Povalej, 2014), conjugate gradient (Goncalves and Prudente, 2020; Lucambio Pérez and Prudente, 2018), projected gradient (Drummond and Iusem, 2004; Fukuda and Drummond, 2013), and proximal methods (Bonnel et al., 2005). However, as the approach of these methods is not related to the approach proposed in this paper, we include very short details of these papers.

Fliege and Svaiter (2000) introduced the steepest descent for both constrained and unconstrained MOPs. In Fliege and Svaiter (2000), the objective function is considered to be continuously differentiable for unconstrained problems, and Lipschitz-continuously differentiable for a constrained problem.

The conditional gradient method or Frank-Wolfe method was proposed in Assuncao et al. (2021) for constrained MOPs. It is assumed in Assuncao et al. (2021) that the objective functions are continuously differentiable, and the constraint set is convex and compact.

In the extension of Newton's method (Fleige et al., 2009) and quasi-Newton's method (Povalej, 2014) for unconstrained MOPs, the objective functions are assumed to be twice continuously differentiable with the local strongly convex property.

Lucambio Pérez and Prudente (2018) generalize various conjugate gradient methods, such as the Fletcher-Reeves, conjugate descent,

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Dai-Yuan, Polak-Ribiére-Polyak, and Hestenes-Stiefel methods for continuously differentiable MOPs.

In Drummond and Iusem (2004), an extension of the projected gradient method has been introduced for MOPs. In Drummond and Iusem (2004), vector-valued functions are used directly instead of scalar-valued objectives.

In the extension of the proximal point method (Bonnel et al., 2005) for scalar-valued convex optimization, the subproblems consist of finding weakly efficient points for suitable regularizations; in this paper, an exact and an inexact version of subproblems are solved approximately.

It is reported in Marler and Arora (2004) that the conventional methods such as those mentioned above either cannot find all nondominated solutions or need some prior information about the location of the nondominated set. Recently, Ghosh and Chakraborty (2014) introduced a nondominated set generating method, known as the cone method (Ghosh and Chakraborty, 2014), for MOPs. The cone method is theoretically (i.e., in the limit) able to retrieve all the nondominated points (which are infinite, in general). However, in practice, as we need to employ a direction-based discretization to solve an MOP by the cone method, only a discrete representative of the whole nondominated set is obtained. The formulation of the cone method is similar to the Pascoletti–Serafini technique (Pascoletti and Serafini, 1984) for vector optimization. The cone method has the ability to generate both convex and nonconvex parts of the nondominated set. Although the formulation of the cone method is detailed in Ghosh and Chakraborty (2014), how to effectively solve the set of direction-based parametric problems of the cone method is not given therein. In this paper, we are concerned with this topic and derive an interior point approach.

Interior-point algorithms have been a large and exciting area of research for continuous optimization techniques because of their polynomial-time complexity. After Khachiyan's ellipsoid method (Khachiyan, 1979), Karmarkar (1984) published a new polynomial-time algorithm (projective method)—the *interior-point method* (IPM). It is practically more efficient than the ellipsoid method. Karmarkar (1984) also claimed a superior performance compared to the simplex method. In the early days, interior-point methods were applicable for problems whose feasible region has a nonempty interior and where the starting point is an interior point of the feasible region. However, computation of an interior point of the feasible region is not easy, and the interior of the feasible region might be an empty set.

Lustig (1990) and Tanabe (1990) introduced an interior-point algorithm in which the initial point is not feasible. Such a method is known as the *infeasible interior-point method* (IIPM). Kojima et al. (1993) proved its global convergence and Zhang (1994) proved its polynomiality. In 1999, Vanderbei (1994) described a software package called LOQO, which implements a primal–dual interior-point method for general quadratic programming. Vanderbei and Shanno (1999) extended the work in Vanderbei (1994) for convex and nonconvex optimization problems.

In this article, we propose an infeasible interior-point method combined with the cone method to find nondominated points of multi-objective optimization problems. The proposed method exploits efficiencies of the cone method (Ghosh and Chakraborty, 2014) and the infeasible interior-point method (Vanderbei and Shanno, 1999). As a consequence, the proposed method captures a discrete approximation of the complete nondominated set.

1.1. Delineation

The work in this paper is presented in the following sequence. In Section 2, we provide the required terminologies and notations and the parametric subproblems of the cone method. In Section 3, we formulate an infeasible interior-point method to solve the set of parametric problems of the cone method; in addition, we find search direction formulas to the subproblems. In Section 4, we formulate the approach which is developed in Section 3 for convex multi-objective optimization. The choice of a merit function, initial point, barrier parameter, and step length in the development of IIPM are detailed in Section 4; Section 4.6 provides the convergence analysis of the proposed algorithm. In Section 5, we explain the algorithm for nonconvex multi-objective optimization. Numerical examples and the performance of the proposed algorithms are presented in Section 6.

2. Preliminaries and terminologies

Consider the following MOP:

$$\left.\begin{array}{ll}\text{minimize} & F(x) = \left(f_1(x), f_2(x), \dots, f_s(x)\right)^\top,\ s \geq 2\\ \text{subject to} & h_i(x) \geq 0,\ i = 1, 2, \dots, m,\end{array}\right\} \tag{1}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ and $h_i : \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable functions for all $j = 1, 2, \dots, s$ and $i = 1, 2, \dots, m$.

We refer to $x = \left(x_1, x_2, \dots, x_n\right)^\top$ as the vector of decision variables and the set $\mathcal{X} = \{x \in \mathbb{R}^n : h_i(x) \geq 0,\ i = 1, 2, \dots, m\}$ as the feasible set in decision space. We denote the image of $\mathcal{X}$ under the vector-valued objective function F by $\mathcal{Y}$, i.e., $\mathcal{Y} = F(\mathcal{X}) = \left\{\left(f_1(x), f_2(x), \dots, f_s(x)\right)^\top : x \in \mathcal{X}\right\}$. The set $\mathcal{Y}$ is referred to as the feasible set in objective space. Note that $\mathcal{X}$ is a subset of $\mathbb{R}^n$ and $\mathcal{Y}$ is a subset of $\mathbb{R}^s$.

Due to the conflicting nature of the objective functions $f_1, f_2, \dots, f_s$, and the nonexistence of a linear ordering in $\mathbb{R}^s$, the optimality concept for an MOP is different than that of a conventional single objective optimization problem. The notion of optimality for an MOP is Pareto optimality. The definition of Pareto optimality is based on a dominance relation in $\mathbb{R}^s$. For the required dominance relation for Pareto optimality, we use the following notations ($s > 1$):

- $\mathbb{R}^s_{\geqq} = \{y \in \mathbb{R}^s : y \geqq 0\}$, the nonnegative orthant of $\mathbb{R}^s$; by $y \geqq 0$ we mean all the components of y are nonnegative.
- $\mathbb{R}^s_{\geq} = \{y \in \mathbb{R}^s : y \geq 0\}$, where $y \geq 0$ denotes $y \geqq 0$ but $y \neq 0$.
- $\mathbb{R}^s_{>} = \{y \in \mathbb{R}^s : y > 0\}$ represents the interior of $\mathbb{R}^s_{\geqq}$, where $y > 0$ means all the components of y are positive.
- The relations $\leqq, \leq$ and $<$ can also be defined in a similar way.
- For two vectors $y^1, y^2 \in \mathbb{R}^s$, we say that the vector y^1 dominates y^2, for a minimization problem, if $y^1 \leq y^2$.

Throughout the paper, we also use the following notations. For an $x = (x_1, x_2, \dots, x_n)^\top \in \mathbb{R}^n$,

- $\|x\|_1 = \sum_{i=1}^n |x_i|$,
- $\|x\|_\infty = \max_i |x_i|$ and
- $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$.

Further, $o(t)$ denotes a function $f : \mathbb{R} \to \mathbb{R}$ for which $\lim_{t\to\infty} \frac{f(t)}{t} = 0$.

Definition 2.1 (*Pareto Optimality (e.g. Ehrgott, 2005)*). A feasible solution $\hat{x} \in \mathcal{X}$ is called efficient or Pareto optimal if there is no $x \in \mathcal{X}$ such that $F(x) \leq F(\hat{x})$. If $\hat{x}$ is efficient, $F(\hat{x})$ is called a nondominated point.

The set of all efficient solutions of the MOP (1) is denoted by $\mathcal{X}_E$. The set of all nondominated points is represented by $\mathcal{Y}_N$. Evidently, $\mathcal{Y}_N = F(\mathcal{X}_E)$.

Definition 2.2 (*Weak Pareto Optimality (e.g. Ehrgott, 2005)*). A feasible solution $\hat{x} \in \mathcal{X}$ is called weakly efficient or weakly Pareto optimal if there is no $x \in \mathcal{X}$ such that $F(x) < F(\hat{x})$. The point $\hat{y} = F(\hat{x})$ is then said to be weakly nondominated.

The set of all weakly efficient solutions of the MOP (1) is denoted by $\mathcal{X}_{wE}$. The collection of all weakly nondominated points is represented by $\mathcal{Y}_{wN}$.

To obtain a (weakly) non-dominated point of $F(\mathcal{X})$, the *cone method* (Ghosh and Chakraborty, 2014) suggests to solve the following minimization problem corresponding to a particular $\hat{\beta} \in \mathbb{S}^{s-1}_{\geqq} = \mathbb{S}^{s-1} \cap \mathbb{R}^s_{\geqq}$ (where $\mathbb{S}^{s-1}$ represents the unit sphere in $\mathbb{R}^s$):

$$\mathrm{CM}(\hat{\beta}) \left\{ \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & t\hat{\beta} \geqq F(x), \\ & h_i(x) \geq 0,\ i = 1, 2, \ldots, m, \\ & t \geq 0. \end{array} \right\} \tag{2}$$

In fact,

$$\mathcal{Y}_{wN} = \bigcup_{\hat{\beta} \in \mathbb{S}^{s-1}_{\geqq}} \left\{ t_{\hat{\beta}} \hat{\beta} : t_{\hat{\beta}} \text{ is 'min } t\text{' of (2)} \right\}.$$

From the complete weakly nondominated set $\mathcal{Y}_{wN}$, the process to filter out the set of all nondominated points is detailed in Ghosh and Chakraborty (2014).

We note that problem (2) is a single-objective parametric problem, with parameter $\hat{\beta}$, corresponding to the MOP (1). To generate the complete nondominated set, one needs to solve the problem (2) for all $\hat{\beta}$'s in $\mathbb{S}^{s-1}_{\geqq}$. For obtaining a uniformly spreaded nondominated points, the work in Ghosh and Chakraborty (2014) suggests to take the following expression of $\hat{\beta}$:

$$\left(\cos\theta_1, \cos\theta_2 \sin\theta_1, \cos\theta_3 \sin\theta_2 \sin\theta_1, \ldots, \cos\theta_{s-1} \prod_{i=1}^{s-2} \sin\theta_i, \prod_{i=1}^{s-1} \sin\theta_i \right), \tag{3}$$

where $0 \leq \theta_i \leq \frac{\pi}{2}, i = 1, 2, \ldots, s-1$. In the proposed algorithm, we follow the suggested way in Ghosh and Chakraborty (2014) on the number of grid points for θ_i's to obtain a discrete subset of nondominated points.

Note that the MOP (1) is convex if f_j and $-h_i$ are convex functions for all $j = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, m$; otherwise, the MOP (1) is nonconvex. For each $\hat{\beta}$, the parametric subproblem (2) of the cone method is convex (respectively, nonconvex) if the problem (1) is convex (respectively, nonconvex). The case of convex problems (Section 4) and noncovex problems (Section 5) are considered separately. Initially, we build an Algorithm 1 to deal with the convex case of the parametric problem (2), and later we extend it to solve the nonconvex case (see Algorithm 3). In fact, Algorithm 3 eventually merges the algorithms for convex and nonconvex case in one single algorithm.

The next section formulates the Newton scheme for the IPM to solve the parametric problem $\mathrm{CM}(\hat{\beta})$.

3. Analysis of CM($\hat{\beta}$) based on the infeasible interior-point method

This section investigates a parametric scalar optimization problem that is equivalent to $\mathrm{CM}(\hat{\beta})$. In the sequel, we formulate a log-barrier problem corresponding to $\mathrm{CM}(\hat{\beta})$. Afterwards, to solve the formulated log-barrier problem, Karush–Kuhn–Tucker (KKT) conditions are derived. In order to find an approximate solution of KKT conditions, IIPM utilizes Newton's method to find the direction along which to proceed. Thus, an explicit expression of the search direction to the Newton system is also derived.

We note that by denoting $\mathrm{x} = \left(x_1, x_2, \ldots, x_n, t\right)^\top$, $c = (0, 0, \ldots, 0, 1)^\top$, $\hat{\beta} = (\beta_1, \beta_2, \ldots, \beta_s)^\top$, $f_j(\mathrm{x}) = f_j(x)$, $j = 1, 2, \ldots, s$ and $F(\mathrm{x}) = F(x)$, the parametric problem (2) can be written in the following form:

$$\left. \begin{array}{ll} \text{minimize} & c^\top \mathrm{x} \\ \text{subject to} & \hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v = 0, \\ & h(\mathrm{x}) - w = 0, \\ & v \geqq 0, \quad w \geqq 0, \end{array} \right\} \tag{4}$$

where $h(\mathrm{x}) = \left(h_1(\mathrm{x}), h_2(\mathrm{x}), \ldots, h_m(\mathrm{x}), h_{m+1}(\mathrm{x})\right)^\top$ with $h_{m+1}(\mathrm{x}) = c^\top \mathrm{x}$, $v = \left(v_1, v_2, \ldots, v_s\right)^\top$; $w = (w_1, w_2, \ldots, w_m, w_{m+1})^\top$ being vectors of slack variables. We eliminate the inequality constraints in (4) by placing them inside a barrier term as follows:

$$\left. \begin{array}{ll} \text{minimize} & b_\mu(\mathrm{x}, v, w) \\ \text{subject to} & \hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v = 0, \\ & h(\mathrm{x}) - w = 0, \end{array} \right\} \tag{5}$$

where

$$b_\mu(\mathrm{x}, v, w) = c^\top \mathrm{x} - \mu \left(\sum_{j=1}^{s} \log(v_j) + \sum_{i=1}^{m+1} \log(w_i) \right)$$

and $\mu > 0$ is the barrier parameter.

In the rest of the paper, we use the notations $p = (\mathrm{x}, v, w)$ *and* $\Omega = (p, y, z) = (\mathrm{x}, v, w, y, z)$.

The Lagrangian for the problem (5) is given by

$$L_{\hat{\beta}}(\Omega, \mu) = b_\mu(\mathrm{x}, v, w) - y^\top \left(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v \right) - z^\top \left(h(\mathrm{x}) - w \right), \quad y \in \mathbb{R}^s \text{ and } z \in \mathbb{R}^{m+1}.$$

The first-order KKT conditions for a minimum of (5) are

$$\left. \begin{array}{l} \nabla_{\mathrm{x}} L_{\hat{\beta}}(\Omega, \mu) \equiv c - \left(\nabla_{\mathrm{x}} \left(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) \right) \right)^\top y - (\nabla_{\mathrm{x}} h(\mathrm{x}))^\top z = 0, \\ \nabla_v L_{\hat{\beta}}(\Omega, \mu) \equiv -\mu V^{-1} e + y = 0, \quad y \geqq 0, \\ \nabla_w L_{\hat{\beta}}(\Omega, \mu) \equiv -\mu W^{-1} e + z = 0, \quad z \geqq 0, \\ \nabla_y L_{\hat{\beta}}(\Omega, \mu) \equiv F(\mathrm{x}) - \hat{\beta} c^\top \mathrm{x} + v = 0, \\ \nabla_z L_{\hat{\beta}}(\Omega, \mu) \equiv -h(\mathrm{x}) + w = 0, \end{array} \right\} \tag{6}$$

where $V = \mathrm{diag}(v_1, v_2, \ldots, v_s)$, $W = \mathrm{diag}(w_1, w_2, \ldots, w_{m+1})$, e is a vector of all 1's of dimension s or $(m+1)$ according to the context, and $\nabla_{\mathrm{x}} \left(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) \right)$ and $\nabla_{\mathrm{x}} h(\mathrm{x})$ are the Jacobian matrices of the functions $\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x})$ and $h(\mathrm{x})$, respectively.

We modify (6) by multiplying the second and third equations by V and W, respectively. Accordingly, we get the following standard primal–dual system:

$$\left. \begin{array}{r} c - \left(\nabla_{\mathrm{x}} \left(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) \right) \right)^\top y - (\nabla_{\mathrm{x}} h(\mathrm{x}))^\top z = 0, \\ -\mu e + V Y e = 0, \quad y \geqq 0, \\ -\mu e + W Z e = 0, \quad z \geqq 0, \\ -F(\mathrm{x}) + \hat{\beta} c^\top \mathrm{x} - v = 0, \\ h(\mathrm{x}) - w = 0, \end{array} \right\} \tag{7}$$

where $Y = \mathrm{diag}(y_1, y_2, \ldots, y_s)$ and $Z = \mathrm{diag}(z_1, z_2, \ldots, z_{m+1})$. To find a solution to the primal–dual system (7), we apply Newton's method. For a simplified appearance of the expressions below, we introduce the following notations:

$$A_{\hat{\beta}}(\mathrm{x}) = \nabla_{\mathrm{x}} \left(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) \right), \quad B(\mathrm{x}) = \nabla_{\mathrm{x}} h(\mathrm{x})$$

and

$$H(\mathrm{x}, y, z) = \sum_{j=1}^{s} y_j \nabla^2 f_j(\mathrm{x}) - \sum_{i=1}^{m+1} z_i \nabla^2 h_i(\mathrm{x}), \quad y \geqq 0, \quad z \geqq 0. \tag{8}$$

For a given barrier parameter $\mu > 0$, the Newton direction $(\Delta\mathrm{x}, \Delta v, \Delta w, \Delta y, \Delta z)$ at a point (x, v, w, y, z) is obtained by solving the following Newton system for (7):

$$\begin{bmatrix} H(\mathrm{x}, y, z) & 0 & 0 & -(A_{\hat{\beta}}(\mathrm{x}))^\top & -(B(\mathrm{x}))^\top \\ 0 & Y & 0 & V & 0 \\ 0 & 0 & Z & 0 & W \\ A_{\hat{\beta}}(\mathrm{x}) & -I & 0 & 0 & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} c - (A_{\hat{\beta}}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z \\ -\mu e + V Y e \\ -\mu e + W Z e \\ \hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v \\ h(\mathrm{x}) - w \end{bmatrix}. \tag{9}$$

The matrix on the left of (9) is not symmetric. However, it can be easily symmetrized by multiplying the first equation by -1, the second equation by $-V^{-1}$ and the third equation by $-W^{-1}$. Accordingly, we get

$$\begin{bmatrix} -H(\mathrm{x},y,z) & 0 & 0 & (A_{\hat\beta}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ 0 & -V^{-1}Y & 0 & -I & 0 \\ 0 & 0 & -W^{-1}Z & 0 & -I \\ A_{\hat\beta}(\mathrm{x}) & -I & 0 & 0 & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sigma_{\hat\beta} \\ -\gamma_1 \\ -\gamma_2 \\ \varrho_{\hat\beta} \\ \rho \end{bmatrix}, \tag{10}$$

where

$$\left.\begin{aligned} \sigma_{\hat\beta}(\mathrm{x},y,z) &= c - (A_{\hat\beta}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z, \\ \gamma_1(v,y) &= \mu V^{-1}e - y, \\ \gamma_2(w,z) &= \mu W^{-1}e - z, \\ \varrho_{\hat\beta}(\mathrm{x},v) &= F(\mathrm{x}) + v - \hat\beta c^\top \mathrm{x}, \\ \text{and}\quad \rho(\mathrm{x},w) &= w - h(\mathrm{x}). \end{aligned}\right\} \tag{11}$$

The reason behind making the system symmetric is that we can use the Cholesky factorization (Vanderbei, 1995) to solve the system. Note that the notations $\sigma_{\hat\beta}$, $\varrho_{\hat\beta}$, ρ, γ_1 and γ_2 depend on x, y, z, v and w. If $\rho_{\hat\beta}$ and ρ do not vanish at a point, then the point is *primal infeasible*. Therefore, $\varrho_{\hat\beta}$ and ρ together denote *primal infeasibility*. In contrast, if $\rho_{\hat\beta}$ and ρ vanish at a point, then the point is primal feasible. Similarly, If $\sigma_{\hat\beta}$ does not vanish at a point, then the point is dual infeasible. Therefore, $\sigma_{\hat\beta}$ denotes the *dual infeasibility*.

We note that second and third equations of (10) can be used to eliminate Δv and Δw without producing any off-diagonal fill-in in the remaining system with the help of the following equations:

$$\left.\begin{aligned} \Delta v &= VY^{-1}(\gamma_1 - \Delta y) \\ \Delta w &= WZ^{-1}(\gamma_2 - \Delta z). \end{aligned}\right\} \tag{12}$$

Accordingly, from (10), the resulting *reduced KKT system* is given by

$$\begin{bmatrix} -H(\mathrm{x},y,z) & (A_{\hat\beta}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ A_{\hat\beta}(\mathrm{x}) & VY^{-1} & 0 \\ B(\mathrm{x}) & 0 & WZ^{-1} \end{bmatrix} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sigma_{\hat\beta} \\ \varrho_{\hat\beta} + VY^{-1}\gamma_1 \\ \rho + WZ^{-1}\gamma_2 \end{bmatrix}. \tag{13}$$

To find a solution to (13), the algorithm that we propose below (Algorithm 1) starts from a given initial point $\left(\mathrm{x}^{(0)}, v^{(0)}, w^{(0)}, y^{(0)}, z^{(0)}\right)$; then, at the kth iteration, it determines a search direction $\left(\Delta\mathrm{x}^{(k)}, \Delta v^{(k)}, \Delta w^{(k)}, \Delta y^{(k)}, \Delta z^{(k)}\right)$ by solving (13) at $\left(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, y^{(k)}, z^{(k)}\right)$; lastly, it chooses a step length $\alpha^{(k)}$ and then finds the next iterate by

$$\left.\begin{aligned} \mathrm{x}^{(k+1)} &= \mathrm{x}^{(k)} + \alpha^{(k)}\Delta\mathrm{x}^{(k)} \\ v^{(k+1)} &= v^{(k)} + \alpha^{(k)}\Delta v^{(k)} \\ w^{(k+1)} &= w^{(k)} + \alpha^{(k)}\Delta w^{(k)} \\ y^{(k+1)} &= y^{(k)} + \alpha^{(k)}\Delta y^{(k)} \\ z^{(k+1)} &= z^{(k)} + \alpha^{(k)}\Delta z^{(k)}, \end{aligned}\right\} \tag{14}$$

where $\alpha^{(k)}$ is the step length that is detailed in Section 4.3.

In order to compute a search direction, we need to solve a system (13). The following theorem gives the explicit form of the search direction.

Theorem 1. *Let the point $\Omega = (\mathrm{x}, y, z, v, w)$ be such that $y > 0,\ z > 0,\ v > 0$ and $w > 0$. We denote*

$$N_{\hat\beta}(\Omega) = H(\mathrm{x},y,z) + (A_{\hat\beta}(\mathrm{x}))^\top V^{-1}YA_{\hat\beta}(\mathrm{x}) + (B(\mathrm{x}))^\top W^{-1}ZB(\mathrm{x}).$$

If at a point Ω, $N_{\hat\beta}$ is nonsingular, then the system (10) has a unique solution. In particular,

$$\left.\begin{aligned} \Delta\mathrm{x} &= -N_{\hat\beta}^{-1}c + N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}Y\varrho_{\hat\beta} + N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top \mu V^{-1}e \\ &\quad + \mu N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}e \\ &\quad + N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\rho, \\ \Delta v &= -A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}c - \left(I - A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}Y\right)\varrho_{\hat\beta} \\ &\quad + \mu A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}e, \\ &\quad + A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\rho + \mu A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}e \ \text{ and} \\ \Delta w &= -B(\mathrm{x})N_{\hat\beta}^{-1}c - \left(I - B(\mathrm{x})N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\right)\rho \\ &\quad + B(\mathrm{x})N^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}Y\varrho_{\hat\beta} \\ &\quad + \mu B(\mathrm{x})N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}e + \mu B(\mathrm{x})N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}e. \end{aligned}\right\} \tag{15}$$

Proof. By solving the second and the third equations of (13) for Δy and Δz, we get

$$\begin{aligned} \Delta y &= V^{-1}Y\varrho_{\hat\beta} + \gamma_1 - V^{-1}YA_{\hat\beta}(\mathrm{x})\Delta\mathrm{x} \\ \text{and}\ \Delta z &= W^{-1}Z\rho + \gamma_2 - W^{-1}ZB(\mathrm{x})\Delta\mathrm{x}. \end{aligned}$$

Eliminating Δy and Δz from the first block of the system (13), we get

$$\begin{aligned} \Delta\mathrm{x} = -N_{\hat\beta}^{-1}c + \left(\mu N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}Y\varrho_{\hat\beta}\right) \\ + \left(\mu N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}e + N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\rho\right). \end{aligned}$$

We notice that the square matrix of order $(n+m+s+1)\times(n+m+s+1)$ on the left side of the system (13) is quasi-definite and therefore nonsingular in nature (see Vanderbei, 1995). Hence, using $\Delta\mathrm{x}$, we can compute Δy and Δz, and finally Δv and Δw uniquely as follows:

$$\begin{aligned} \Delta v &= -A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}c - \left(I - A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}Y\right)\varrho_{\hat\beta} \\ &\quad + \mu A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}e \\ &\quad + A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\rho + \mu A_{\hat\beta}(\mathrm{x})N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}e \\ \text{and}\quad \Delta w &= -B(\mathrm{x})N_{\hat\beta}^{-1}c - \left(I - B(\mathrm{x})N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\right)\rho \\ &\quad + B(\mathrm{x})N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}Y\varrho_{\hat\beta} \\ &\quad + \mu B(\mathrm{x})N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}e + \mu B(\mathrm{x})N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}e. \quad \square \end{aligned}$$

In Section 4, we describe an algorithm to solve convex multi-objective optimization problems with the help of the reduced KKT system (13), the sequence of iterations (14) and Theorem 1.

4. Algorithm for convex multi-objective optimization

In this section, we consider the MOP (1) where each f_j, $j = 1, 2, \dots, s$ is convex and each h_i, $i = 1, 2, \dots, m$ is concave. Hence, the MOP under consideration in this section is a convex problem. Evidently, the single objective parametric problem (2) is convex and the Hessian matrix $H(\mathrm{x}, y, z)$ is positive semi-definite. Then, the matrix

$$N_{\hat\beta}(\mathrm{x},y,z,v,w) = H(\mathrm{x},y,z) + (A_{\hat\beta}(\mathrm{x}))^\top V^{-1}YA_{\hat\beta}(\mathrm{x}) + (B(\mathrm{x}))^\top W^{-1}ZB(\mathrm{x})$$

is positive semi-definite. Since the algorithm that we propose (Algorithm 1) solves the system (13) with the help of Cholesky factorization, it is important that the algorithm preserves the positive semi-definiteness of the Hessian matrix $H(\mathrm{x}, y, z)$ across the iterations. If the Hessian matrix is not positive definite, the algorithm suitably perturbs the Hessian (see Section 5). Therefore, Section 4 assumes that the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite, and hence the matrix $N_{\hat\beta}(\mathrm{x}, y, z, v, w)$ is positive definite. Consequently, the search directions can be determined by Theorem 1.

Once an iterative point becomes feasible, in order to retain the feasibility of the next iterates, we need to maintain nonnegativity of the slack variables v and w and of the dual variables y and z. Hence, the step length for movement at each iteration needs to be chosen carefully. Usually, merit functions (Akrotirianakis and Rustem, 2005; Argáez and Tapia, 2002; El-Bakry et al., 1996) are used to determine suitable step lengths along the search directions.

4.1. Merit functions

Feasible interior-point algorithms (Karmarkar, 1984; Monteiro and Adler, 1989) are known to maintain feasibility of every iterate. Thus, in feasible IPMs, one focuses only on finding suitable descent directions. In infeasible IPMs, however, the iterations start from outside the feasible region. Therefore, if we attempt to apply an infeasible IPM, in addition to finding a descent direction, we need to reduce infeasibility at every iteration. Progress of both of the objective function and infeasibility can be done by monitoring a suitable merit function reduction at every iteration. These progresses are obtained by suitably truncating the step length along the search directions (13) so that there is substantial reduction in the merit function.

Some variants (El-Bakry et al., 1996; Fiacco and McCormick, 1990) of merit functions for solving the system (7) are as follows:

$$\phi_1(\mathrm{x}, v, w, \eta, \mu) = b_\mu(\mathrm{x}, v, w) + \eta\left(\|\varrho_{\hat\beta}(\mathrm{x}, v)\|_1 + \|\rho(\mathrm{x}, w)\|_1\right),$$

$$\phi_2(\mathrm{x}, v, w, y, z) = \|\sigma_{\hat\beta}\|_2^2 + \|YVe\|_2^2 + \|WZe\|_2^2 + \|\varrho_{\hat\beta}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2,$$

$$\psi_{\eta,\mu}(\mathrm{x}, v, w) = b_\mu(\mathrm{x}, v, w) + \frac{\eta}{2}\left(\|\varrho_{\hat\beta}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2\right), \quad \text{etc.},$$

where $\mu > 0$, $\eta > 0$.

The merit function $\phi_1(\mathrm{x}, v, w, \eta, \mu)$ is exact but nondifferentiable due to the l_1 norm. The word 'exact' refers to the existence of a positive scalar η_0 such that for every $\eta \geq \eta_0$, a local minimum point of the problem (5) is also a local minimum point of $\phi_1(\mathrm{x}, v, w, \eta, \mu)$.

The l_2 merit function $\phi_2(\mathrm{x}, v, w, y, z)$ was introduced by El-Bakry et al. (1996). They showed that the algorithm is convergent whenever the Jacobian of the system (10) is nonsingular. The proposed algorithm in El-Bakry et al. (1996) was tested on the Hock and Schittkowski test problems by Shanno and Simantiraki (1997). They found that the algorithm fails on some problems.

In this paper, we use the differentiable l_2 merit function $\psi_{\eta,\mu}$ (see Fiacco and McCormick, 1990). A mild disadvantage of the merit function $\psi_{\eta,\mu}$ is that η needs to be large enough to guarantee the convergence to a feasible point. The following theorem verifies that if the problem (1) is such that the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite, the search direction given by Theorem 1 is a descent direction for $\psi_{\eta,\mu}(\mathrm{x}, v, w)$ provided η is large enough.

Theorem 2. *Consider the barrier problem (5) and the point $\Omega = (\mathrm{x}, v, w, y, z)$ is such that $v > 0$, $w > 0$, $y > 0$ and $z > 0$. Suppose that $N_{\hat\beta}$ is positive definite at Ω. Then, for any $\mu > 0$, the following results hold:*

(i) If the point Ω is primal feasible, i.e., $\varrho_{\hat\beta} = \rho = 0$, then either $(\Delta\mathrm{x}, \Delta v, \Delta w)$ is a descent direction for the barrier function $b_\mu(\mathrm{x}, v, w)$ or the point Ω satisfies the KKT conditions (7).

(ii) If the point Ω is not primal feasible, then there exists an $\eta_{\min} \geq 0$ such that for each $\eta > \eta_{\min}$, $(\Delta\mathrm{x}, \Delta v, \Delta w)$ is a descent direction for the merit function $\psi_{\eta,\mu}(\mathrm{x}, v, w)$.

Proof.

(i) From the expression of $b_\mu(\mathrm{x}, v, w)$, we obtain $\nabla_\mathrm{x} b_\mu = c$, $\nabla_v b_\mu = -\mu V^{-1}e$ and $\nabla_w b_\mu = -\mu W^{-1}e$.
Denote $y = \mu V^{-1}e$, $z = \mu W^{-1}e$ and $\sigma_{\hat\beta} = c - (A_{\hat\beta}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z$.
By the expression of $\Delta\mathrm{x}$, Δv and Δw from Theorem 1, we obtain

$$\begin{bmatrix}\nabla_\mathrm{x} b_\mu\\ \nabla_v b_\mu\\ \nabla_w b_\mu\end{bmatrix}^\top \begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\end{bmatrix} = -\sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}\sigma_{\hat\beta} + \sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}(A_{\hat\beta}(\mathrm{x}))^\top V^{-1}Y\varrho_{\hat\beta} + y^\top\varrho_{\hat\beta} + z^\top\rho + \sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\rho.$$

Since the given point (x, v, w, y, z) is primal feasible, $\varrho_{\hat\beta} = \rho = 0$.

Hence, we get

$$\begin{bmatrix}\nabla_\mathrm{x} b_\mu\\ \nabla_v b_\mu\\ \nabla_w b_\mu\end{bmatrix}^\top \begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\end{bmatrix} = -\sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}\sigma_{\hat\beta}. \tag{16}$$

As $N_{\hat\beta}$ is positive definite, $N_{\hat\beta}^{-1}$ is positive definite. Thus, $\sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}\sigma_{\hat\beta} \geq 0$.

- *Case* 1. If $\sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}\sigma_{\hat\beta} > 0$, then (16) shows that $(\Delta\mathrm{x}, \Delta v, \Delta w)$ is a descent direction for the barrier function.
- *Case* 2. If $\sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}\sigma_{\hat\beta} = 0$, then $\sigma_{\hat\beta} = 0$. In this case, $\varrho_{\hat\beta} = \rho = 0$ and $\sigma_{\hat\beta} = 0$ together satisfy the KKT system (7).

(ii) We note that

$$\nabla_\mathrm{x}\left(\|\varrho_{\hat\beta}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2\right) = -2\left(\varrho_{\hat\beta}(\mathrm{x}, v)^\top A_{\hat\beta}(\mathrm{x}) + \rho(\mathrm{x}, w)^\top B(\mathrm{x})\right)$$

Therefore,

$$\begin{bmatrix}\nabla_\mathrm{x}\left(\|\varrho_{\hat\beta}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2\right)\\ \nabla_v\left(\|\varrho_{\hat\beta}(\mathrm{x}, v)\|_2^2\right)\\ \nabla_w\left(\|\rho(\mathrm{x}, w)\|_2^2\right)\end{bmatrix}^\top \begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\end{bmatrix} = -2\varrho_{\hat\beta}^\top(A_{\hat\beta}(\mathrm{x})\Delta\mathrm{x} - \Delta v) - 2\rho^\top(B(\mathrm{x})\Delta\mathrm{x} - \Delta w) = -2\left(\|\varrho_{\hat\beta}\|_2^2 + \|\rho\|_2^2\right).$$

Further,

$$\begin{bmatrix}\nabla_\mathrm{x}\psi_{\eta,\mu}\\ \nabla_v\psi_{\eta,\mu}\\ \nabla_w\psi_{\eta,\mu}\end{bmatrix}^\top \begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\end{bmatrix} = \begin{bmatrix}\nabla_\mathrm{x} b_\mu\\ \nabla_v b_\mu\\ \nabla_w b_\mu\end{bmatrix}^\top \begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\end{bmatrix} + \begin{bmatrix}-\eta\varrho_{\hat\beta}^\top A_{\hat\beta}(\mathrm{x}) - \eta\rho^\top B(\mathrm{x})\\ \eta\varrho_{\hat\beta}\\ \eta\rho\end{bmatrix}^\top \begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\end{bmatrix}$$
$$= -\sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}\sigma_{\hat\beta} + \sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}(A(\mathrm{x}))^\top V^{-1}Y\varrho_{\hat\beta} + y^\top\varrho_{\hat\beta} + z^\top\rho + \sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\rho - \eta\left(\|\varrho_{\hat\beta}\|_2^2 + \|\rho\|_2^2\right).$$

We denote

$$\Gamma_{\hat\beta}(\Omega) = \sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}(A(\mathrm{x}))^\top V^{-1}Y\varrho_{\hat\beta} + y^\top\varrho_{\hat\beta} + z^\top\rho + \sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}(B(\mathrm{x}))^\top W^{-1}Z\rho - \sigma_{\hat\beta}^\top N_{\hat\beta}^{-1}\sigma_{\hat\beta}. \tag{17}$$

Then,

$$\begin{bmatrix}\nabla_\mathrm{x}\psi_{\eta,\mu}\\ \nabla_v\psi_{\eta,\mu}\\ \nabla_w\psi_{\eta,\mu}\end{bmatrix}^\top \begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\end{bmatrix} = \Gamma_{\hat\beta}(\Omega) - \eta\left(\|\varrho_{\hat\beta}\|_2^2 + \|\rho\|_2^2\right).$$

Since the given point (x, v, w, y, z) is not a feasible point, i.e., either or both of $\|\varrho_{\hat\beta}\|_2 \neq 0$ and $\|\rho\|_2 \neq 0$, the following two cases arise:

- *Case* 1. In this case, we consider $\Gamma_{\hat\beta}(\Omega) \leq 0$. This implies

$$\begin{bmatrix}\nabla_\mathrm{x}\psi_{\eta,\mu}\\ \nabla_v\psi_{\eta,\mu}\\ \nabla_w\psi_{\eta,\mu}\end{bmatrix}^\top \begin{bmatrix}\Delta\mathrm{x}\\ \Delta v\\ \Delta w\end{bmatrix} < 0.$$

Hence, $(\Delta\mathrm{x}, \Delta v, \Delta w)$ is a descent direction of $\psi_{\eta,\mu}$ for any $\eta > 0$.
In this case, the result follows with $\eta_{\min} = 0$.

- *Case* 2. Let $\Gamma_{\hat{\beta}}(\Omega) > 0$. In this case, by choosing an η so that

$$\eta > \frac{\Gamma_{\hat{\beta}}(\Omega)}{\left(\|\varrho_{\hat{\beta}}\|_2^2 + \|\rho\|_2^2\right)} = \eta_{\min},$$

we see that $(\Delta x, \Delta v, \Delta w)$ is a descent direction for $\psi_{\eta,\mu}$. Hence, the result follows. □

Note 1. Theorem 2 suggests to keep η as zero for the merit function $\psi_{\eta,\mu}$ as long as either the current iterate is feasible or the direction $(\Delta x, \Delta v, \Delta w)$ is descent. In case the current iterate is neither feasible nor $(\Delta x, \Delta v, \Delta w)$ is descent, choose η not smaller than

$$\eta_{\min} = \frac{\Gamma_{\hat{\beta}}(\Omega)}{\left(\|\varrho_{\hat{\beta}}\|_2^2 + \|\rho\|_2^2\right)}, \tag{18}$$

where $\Gamma_{\hat{\beta}}(\Omega)$ is given by (17); in the proposed algorithm, we typically choose $\eta = 10\eta_{\min}$.

So far, we have not discussed the choice of the barrier parameter μ, step length, and strategy for choosing the initial point. Next, we detail these concerns in Sections 4.2, 4.3 and 4.4.

4.2. Choice of the barrier parameter μ

The selection of the barrier parameter has an important role in IPMs, as the optimality is achieved whenever the barrier parameter decreases to zero (see Den Hertog, 2012). Feasible interior point methods (Den Hertog, 2012) are generally based on the strategy of central path. Therefore, whenever the primal–dual feasible point (x, v, w, y, z) lies on the central path, it has to satisfy the second and third equations of (7), i.e.,

$$v^\top y = s\mu \text{ and } w^\top z = (m+1)\mu.$$

Hence, the exact value of μ at any iteration is given by $\mu = \frac{v^\top y + w^\top z}{s+m+1}$. In the proposed algorithm, as the iterations start from outside the feasible region, an estimate of the barrier parameter μ is chosen as (see Vanderbei, 2020):

$$\mu = r\frac{v^\top y + w^\top z}{s+m+1} \text{ for some } r \in (0,1). \tag{19}$$

This choice of μ shows the convergence (Theorem 4) of the iterative sequence generated by the proposed Algorithm 1. Computational evidence, as demonstrated in Section 6, also shows that this choice of barrier parameter (19) works well for the merit function $\psi_{\eta,\mu}$.

4.3. Choice of the step length

The proposed algorithm updates the iteration point at the end of each iteration by (14). When choosing the step length at every iteration, attention must be given so that the slack variables v_j and w_i, and the dual variables y_j and z_i, stay positive for all $j = 1,2,\dots,s$ and $i = 1,2,\dots,m+1$ across the iterations, otherwise the barrier function (see (5)) will not be well defined. For this positivity, we choose the step length α at every iteration by the following standard ratio formula (see Vanderbei, 2020):

$$\alpha = \min\left\{\delta\left(\max_{i,j}\left\{-\frac{\Delta v_j}{v_j}, -\frac{\Delta y_j}{y_j}, -\frac{\Delta w_i}{w_i}, -\frac{\Delta z_i}{z_i}\right\}\right)^{-1}, 1\right\}, \tag{20}$$

where $0 < \delta \le 1$. The results in Vanderbei and Shanno (1999) demonstrate that $\delta = 0.95$ works well in practice. Therefore, we also select this value of δ in the proposed algorithm.

The choice of step length α by (20) is generally used to maintain positivity of the nonnegative variables for linear and quadratic programming (Vanderbei, 2020). However, for general nonlinear programming, it might be the case that a slight decrease in the objective function contributes to a large rise of infeasibility. Therefore, at each iteration, the interior-point method chooses $\bar{\alpha} \in (0, \alpha]$ such that the merit function follows the following Armijo rule:

$$\psi_{\eta,\mu}(p + \bar{\alpha}\Delta p) - \psi_{\eta,\mu}(p) \le \kappa\bar{\alpha}\left(\nabla_p \psi_{\eta,\mu}(p)\right)^\top \Delta p, \tag{21}$$

where $\kappa \in (0,1)$, $p = (x, v, w)$ and $\Delta p = (\Delta x, \Delta v, \Delta w)$. If the point p is such that the Hessian matrix $H(x, y, z)$ is positive definite, then the matrix $N_{\hat{\beta}}(\Omega)$ is positive definite, and hence the direction Δp calculated by Theorem 1 is descent for the merit function $\psi_{\eta,\mu}(p)$ (see Theorem 2). Consequently, the right side of (21) is negative, which confirms that the merit function $\psi_{\eta,\mu}(p)$ reduces at the point $p + \bar{\alpha}\Delta p$.

Note 2. Note that as Δp is a descent direction for the merit function $\psi_{\eta,\mu}$, there exists an $\alpha^* > 0$ that satisfies the condition (21) (see Section 2.5 in Fletcher, 2013). In finding an estimate of the largest α^* (which we denote as $\bar{\alpha}$) that satisfies the condition (21), a common way is to choose the largest value in $\{\zeta^2\alpha, \zeta^3\alpha, \dots\}$ such that the condition (21) holds, where $\zeta \in (0,1)$. Notice that there exist $l \in \{2,3,\dots\}$ such that $\bar{\alpha}_l = \zeta^l\alpha \le \alpha^*$ that will satisfy the condition (21).

4.4. Choice of the initial point

To solve the convex multi-objective optimization problem (1), the method that we propose first converts the MOP into a set of parametric scalar optimization problems CM($\hat{\beta}$)'s (see (2)). Subsequently, the method chooses the parameter $\hat{\beta}$ by (3) and then initializes the algorithm by selecting an appropriate starting point. Interior-point methods not only require a suitable initialization of x but also of the slack variables $v_j, j = 1,2,\dots,s$, and $w_i, i = 1,2,\dots,m+1$. For a given $x^{(0)}$, an initialization of x, the slack variable values are given by (see (7))

$$v^{(0)} = \hat{\beta}c^\top x^{(0)} - F(x^{(0)}) \text{ and } w^{(0)} = h(x^{(0)}). \tag{22}$$

With the computed values of $v^{(0)}$ and $w^{(0)}$, two difficulties may occur:

(i) if $x^{(0)}$ is not feasible, then at least one component of each of the vectors $v^{(0)}$ and $w^{(0)}$ becomes negative, which makes the barrier function in (5) undefined, and
(ii) if $x^{(0)}$ is feasible, then it may be possible that one or more components of $v^{(0)}$ and $w^{(0)}$ are very close to zero. This impedes progress towards the solution.

To resolve both the issues, one way of initializing $v^{(0)}$ and $w^{(0)}$ (see Vanderbei and Shanno, 1999) is

$$\left.\begin{aligned} v_j^{(0)} &= \begin{cases} \xi_1 & \text{if } \beta_j c^\top x^{(0)} - f_j(x^{(0)}) = 0, \quad j = 1,2,\dots,s \\ |\beta_j c^\top x^{(0)} - f_j(x^{(0)})| & \text{otherwise} \end{cases} \\ w_i^{(0)} &= \begin{cases} \xi_2 & \text{if } h_i(x^{(0)}) = 0, \quad i = 1,2,\dots,m+1, \\ |h_i(x^{(0)})| & \text{otherwise,} \end{cases} \end{aligned}\right\} \tag{23}$$

where $\xi_1 > 0$ and $\xi_2 > 0$. This method of initializing v_j's and w_i's holds the variables v and w at least as large as ξ_1 and ξ_2, and it eliminates both of the above-mentioned difficulties. In practice, we use $\xi_1 = 1$ and $\xi_2 = 1$.

Recall that the solution of the multi-objective problem (1) will be achieved by solving the parametric single objective optimization problem CM(β). The interior-point method reformulates the problem CM($\hat{\beta}$) as a parametric barrier problem (5). Algorithm 1 solves the parametric barrier problem (5), via solving (10), and saves the objective vectors of the Pareto optimal solutions in 'Nondominated Set' for each $\hat{\beta} \in \mathbb{S}_{\geqq}^{s-1}$. The set 'Nondominated Set' in the Algorithm 1 gives a discrete approximation of the complete nondominated set of the problem (1). By increasing the number of $\hat{\beta}$, the entire $\mathbb{S}_{\geqq}^{s-1}$ can be covered, and hence Algorithm 1 is capable of capturing the complete nondominated set of the problem (1). This is due to the fact that for each nondominated point, there is a $\hat{\beta} \in \mathbb{S}_{\geqq}^{s-1}$ (see Theorem 2 in Ghosh and Chakraborty, 2014).

As we aim to find a solution of (5), the solution point is required to satisfy primal feasibility, dual feasibility and complementary slackness. We, thus, use the following merit function

$$\nu(\Omega) = \max\{\|\sigma_{\hat{\beta}}\|_1, \|\varrho_{\hat{\beta}}\|_1, \|\rho\|_1, \|VYe\|_1, \|WZe\|_1\} \quad (24)$$

to measure overall progress of the iterative points. For a given $\epsilon > 0$, we define the approximated KKT point as $\Omega = (\mathrm{x}, v, w, y, z)$ such that $v > 0$, $w > 0$, $y > 0$, $z > 0$ and $\nu(\Omega) \leq \epsilon$. Note that $\nu(\Omega) = 0$ if and only if Ω is a KKT point. Thus, Algorithm 1 attempts to reduce merit function (24) until it is less than a precision parameter ϵ. In addition, Algorithm 1 also reduces the merit function

$$\psi_{\eta,\mu}(p) = c^{\top}\mathrm{x} - \mu\left(\sum_{i=1}^{k}\log(v_i) + \sum_{j=1}^{m+1}\log(w_j)\right) + \frac{\eta}{2}\left(\|\varrho_{\hat{\beta}}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2\right)$$

with adaptive μ and η to find a minimum of the problem (5).

Algorithm 1 is applicable to MOPs for which the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite at every iteration. For example, if all the objective functions f_i's are twice continuously differentiable and convex, and the constraint functions h_i's are twice continuously differentiable and concave, then $H(\mathrm{x}, y, z)$ is positive definite. In case we are not sure about the positive definiteness of $H(\mathrm{x}, y, z)$, Algorithm 1 is not applicable for such a problem, and for such a case, we need to apply Algorithm 3. In fact, Algorithm 3 is applicable irrespective of the positive definiteness of $H(\mathrm{x}, y, z)$. If, however, we somehow know apriori that $H(\mathrm{x}, y, z)$ is positive definite, we must prefer to apply Algorithm 1 over Algorithm 3 since in Algorithm 3 there are some extra computations due to the line numbers 8 to 14.

4.5. Well-definedness of Algorithm 1

The well-definedness of Algorithm 1 depends on line numbers 8 and 9. For Algorithm 1, the Hessian matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ is positive definite at any kth iteration. Hence, the system (13) is consistent and Cholesky factorization (see Vanderbei, 1995) is applicable to obtain $(\Delta\mathrm{x}^{(k)}, \Delta y^{(k)}, \Delta z^{(k)})$. Thus, line 8 is well-defined. For the line 9, we initially take the values of $v^{(0)}$, $w^{(0)}$, $y^{(0)}$ and $w^{(0)}$ positive and thus nonzero. Then the rule (20) is applicable to obtain the step length α at the initial iteration. Rule (20) also keeps the values of $v^{(k)}$, $w^{(k)}$, $y^{(k)}$ and $w^{(k)}$ positive at any kth iteration (see Section 4.3). This implies that line number 9 is also well-defined.

4.6. Convergence analysis

In this subsection, we examine the convergence of the proposed algorithm. Algorithm 1 generates the sequence (14) by computing the direction and step length at every iteration and minimizes the merit function $\psi_{\eta,\mu}(p)$. Apart from the reduction of $\psi_{\eta,\mu}(p)$, it is essential to calculate the reduction of $\nu(\Omega)$ because we use $\nu(\Omega) < \epsilon$ as the stopping criterion of Algorithm 1. The following questions arise concerning the merit function $\nu(\Omega)$:

(i) Does the Algorithm 1 reduce the values of primal infeasibilities $(\rho, \varrho_{\hat{\beta}})$, dual infeasibility $(\sigma_{\hat{\beta}})$ and complementarity (γ_1, γ_2) at each iteration?
(ii) Does the sequence (14) generated by the Algorithm 1 converge to an optimal point of the problem(5)?
(iii) How many iterations does the algorithm need to reach an ϵ-precise solution for a given $\epsilon > 0$?

Answers to these three questions are addressed in Section 4.7. Answer to (i) is provided by Theorem 3. For the answer to (ii) and (iii), see Theorem 4.

Algorithm 1 IIPM for the MOPs when $H(\mathrm{x}, y, z)$ is positive definite.

Aim: To generate a discrete approximation of the nondominated set $\mathcal{D}$ of the MOP (1)
1: **Inputs**
Provide $f_1, f_2, \ldots, f_s$ which are twice continuously differentiable objective functions of (1)
Provide $h_1, h_2, \ldots, h_m$ which are twice continuously differentiable constraints functions of (1)
Provide N, the number of different $\hat{\beta}$'s corresponding to which the subproblem (5) is to be solved
2: **Initialization**
Set nondominated set $\mathcal{D} \leftarrow \emptyset$
To solve (5), for a given $\hat{\beta}$, provide an initial point $\mathrm{x}^{(0)}$
Choose any positive ξ_1 and ξ_2 for (23)
Initialize $v^{(0)}$, $w^{(0)}$ by (23) and $y^{(0)}, z^{(0)}$ by any positive values
Choose any positive r, δ and $\kappa \in (0, 1)$ for (19), (20) and (21), respectively
Give a value of the precision parameter $\epsilon > 0$ for the optimum solutions to (5)
Set $k \leftarrow 0$
Set $\mu^{(k)} \leftarrow r\frac{(v^{(k)})^{\top}y^{(k)} + (w^{(k)})^{\top}z^{(k)}}{m+s+1}$
3: **Main Steps to solve (5) for N different $\hat{\beta}$'s**
4: **for** $i = 1 : 1 : N$ **do**
(Lines 5 to 36 solves (5) for a given $\hat{\beta}$ and find $\mathrm{x}^{(k)}$ as a solution to (5). Line 38 keeps stacking up the nondominated points of the MOP (1) in the set $\mathcal{D}$)
5: Choose a direction $\hat{\beta}$ by using (3)
6: Set $\nu(\Omega^{(k)}) \leftarrow \max\{\|\sigma_{\hat{\beta}}^{(k)}\|_1, \|\varrho_{\hat{\beta}}^{(k)}\|_1, \|\rho^{(k)}\|_1, \|V^{(k)}Y^{(k)}e\|_1, \|W^{(k)}Z^{(k)}e\|_1\}$ (according to (24))
7: **while** $\nu(\Omega^{(k)}) \geq \epsilon$ **do**
8: Solve the system (13) for $(\Delta\mathrm{x}^{(k)}, \Delta v^{(k)}, \Delta w^{(k)})$ with the help of Cholesky factorization
9: Choose step length α by the formula (20)
10: Set $p^{(k+1)} \leftarrow p^{(k)} + \alpha\Delta p^{(k)}$, $y^{(k+1)} \leftarrow y^{(k)} + \alpha\Delta y^{(k)}$, $z^{(k+1)} \leftarrow z^{(k)} + \alpha\Delta z^{(k)}$
11: Calculate $\Gamma_{\hat{\beta}}(\Omega^{(k)})$ by the expression (17)
12: **if** $\varrho_{\hat{\beta}}(\mathrm{x}^{(k)}, v^{(k)}) = 0$, $\rho(\mathrm{x}^{(k)}, w^{(k)}) = 0$ and $\psi_{\eta,\mu}(\Omega^{(k+1)}) \leq \psi_{\eta,\mu}(\Omega^{(k)})$ **then**
13: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^{\top}y^{(k+1)} + (w^{(k+1)})^{\top}z^{(k+1)}}{m+s+1}$
14: Update $\nu(\Omega^{(k+1)})$ according to (24)
15: Set $k \leftarrow k + 1$
16: **else**
17: **if** $\Gamma_{\hat{\beta}}(\Omega^{(k)}) < 0$ **then**
18: Set $\eta \leftarrow 0$
19: Backtrack $\alpha^{(k)} \in [0, \alpha]$ until the condition (21) holds
20: Update $\Omega^{(k+1)} \leftarrow \Omega^{(k)} + \alpha^{(k)}\Delta\Omega^{(k)}$
21: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^{\top}y^{(k+1)} + (w^{(k+1)})^{\top}z^{(k+1)}}{m+s+1}$
22: Update $\nu(\Omega^{(k+1)})$ according to (24)
23: Set $k \leftarrow k + 1$
24: **else**
25: Calculate $\eta_{\min}$ by equation (18)
26: Set $\eta = 10\eta_{\min}$
27: Backtrack $\alpha^{(k)} \in [0, \alpha]$ until the condition (21) holds
28: Update $\Omega^{(k+1)} \leftarrow \Omega^{(k)} + \alpha^{(k)}\Delta\Omega^{(k)}$
29: **if** $\psi_{\hat{\beta}}(\Omega^{(k+1)}) \leq \psi_{\hat{\beta}}(\Omega^{(k)})$ **then**
30: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^{\top}y^{(k+1)} + (w^{(k+1)})^{\top}z^{(k+1)}}{m+s+1}$
31: Update $\nu(\Omega^{(k+1)})$ according to (24)
32: Set $k \leftarrow k + 1$
33: **end if**
34: **end if**
35: **end if**
36: **end while**
37: Calculate $F(\mathrm{x}^{(k)}) = \hat{\beta}c^{T}\mathrm{x}^{(k)} - v^{(k)}$
38: Update the nondominated set $\mathcal{D} \leftarrow \mathcal{D} \bigcup \{F(\mathrm{x}^{(k)})\}$
39: **end for**
40: **return** the set $\mathcal{D}$ (a discrete approximation of the whole nondominated set)

4.7. Measures of progress

The progress of the interior-point method towards a solution of the KKT system (7) can be observed by estimating the following three criteria at consecutive iterates:

(i) primal infeasibility ($\|\varrho_{\hat{\beta}}\|_1, \|\rho\|_1$),
(ii) dual infeasibility ($\|\sigma_{\hat{\beta}}\|_1$) and
(iii) complementarity ($\|VYe\|_1, \|WZe\|_1$).

In the derivation below, we use the combined complementarity by $\gamma = v^\top y + w^\top z$. The following theorem calculates the reduction of criteria (i), (ii) and (iii) in one iteration.

Theorem 3. *Consider the barrier problem (5). At the kth iterative point* $\Omega^{(k)} = \left(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, y^{(k)}, z^{(k)}\right)$, *let*

(i) the Hessian matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ *be positive definite,*
(ii) the direction $\Delta\Omega^{(k)} = (\Delta\mathrm{x}^{(k)}, \Delta v^{(k)}, \Delta w^{(k)}, \Delta y^{(k)}, \Delta z^{(k)})$ *in Algorithm 1 be given by solving the system (13),*
(iii) $\alpha^{(k)}$ *be given by (20),*
(iv) for some $\zeta \in (0,1)$, *the largest value in* $\left\{ \zeta^2\alpha^{(k)}, \zeta^3\alpha^{(k)}, \zeta^4\alpha^{(k)}, \dots \right\}$ *that satisfies the condition (21) be* $\bar{\alpha}^{(k)}$, *and*
(v) the next iterative point $\Omega^{(k+1)} = \left(\mathrm{x}^{(k+1)}, v^{(k+1)}, w^{(k+1)}, y^{(k+1)}, z^{(k+1)}\right)$ *of* $\Omega^{(k)}$ *of the proposed Algorithm 1 be given by*

$$\Omega^{(k+1)} = \Omega^{(k)} + \bar{\alpha}^{(k)}\Delta\Omega^{(k)}.$$

Suppose the primal infeasibilities, dual infeasibility and complementarity at $\Omega^{(k)}$ *and* $\Omega^{(k+1)}$ *are* ($\varrho_{\hat{\beta}}^{(k)}$, $\rho^{(k)}$, $\sigma_{\hat{\beta}}^{(k)}$, $\gamma^{(k)}$) *and* ($\varrho_{\hat{\beta}}^{(k+1)}$, $\rho^{(k+1)}$, $\sigma_{\hat{\beta}}^{(k+1)}$, $\gamma^{(k+1)}$), *respectively. If there exists* $M > 0$ *such that* $\Delta\Omega^{(k)}$ *is bounded by* M, *i.e.,* $\|\Delta\Omega^{(k)}\|_\infty \le M$, *then*

$$\|\rho^{(k+1)}\|_1 \le (1-\bar{\alpha}^{(k)})\|\rho^{(k)}\|_1$$
$$\|\sigma_{\hat{\beta}}^{(k+1)}\|_1 \le (1-\bar{\alpha}^{(k)})\|\sigma_{\hat{\beta}}^{(k)}\|_1 \text{ and}$$
$$\gamma^{(k+1)} \le \gamma^{(k)}(1-\bar{\alpha}^{(k)}(1-r)) + M(\|\varrho_{\hat{\beta}}^{(k)}\|_1 + \|\rho^{(k)}\|_1 + \|\sigma_{\hat{\beta}}^{(k)}\|_1).$$

Proof. We note that

$$\begin{aligned}
\varrho_{\hat{\beta}}^{(k+1)} &= F(\mathrm{x}^{(k+1)}) + v^{(k+1)} - \hat{\beta}c^\top \mathrm{x}^{(k+1)} \\
&= F(\mathrm{x}^{(k)} + \bar{\alpha}^{(k)}\Delta\mathrm{x}^{(k)}) + v^{(k)} + \bar{\alpha}^{(k)}\Delta v^{(k)} - \hat{\beta}c^\top(\mathrm{x}^{(k)} + \bar{\alpha}_l^{(k)}\Delta\mathrm{x}^{(k)}) \\
&\ge F(\mathrm{x}^{(k)}) + \bar{\alpha}^{(k)}\nabla F(\mathrm{x}^{(k)})\Delta\mathrm{x}^{(k)} + v^{(k)} + \bar{\alpha}^{(k)}\Delta v^{(k)} - \hat{\beta}c^\top\mathrm{x}^{(k)} - \bar{\alpha}^{(k)}\hat{\beta}c^\top\Delta\mathrm{x}^{(k)} \\
&\quad (\text{since } F \text{ is a convex function }) \\
&= \left(F(\mathrm{x}^{(k)}) + v^{(k)} - \hat{\beta}c^\top\mathrm{x}^{(k)}\right) - \bar{\alpha}^{(k)}\left(\nabla\left(\hat{\beta}c^\top\mathrm{x}^{(k)} - F(\mathrm{x}^{(k)})\right)\Delta\mathrm{x}^{(k)} - \Delta v^{(k)}\right).
\end{aligned} \tag{25}$$

As the quantities $F(\mathrm{x}^{(k)}) + v^{(k)} - \hat{\beta}c^\top\mathrm{x}^{(k)}$ and $\nabla_{\mathrm{x}}\left(\hat{\beta}c^\top\mathrm{x}^{(k)} - F(\mathrm{x}^{(k)})\right)$ are the primal infeasibility $\varrho_{\hat{\beta}}^{(k)}$ (see the second last expression of (11)) and $A_{\hat{\beta}}(\mathrm{x}^{(k)})$, respectively, we get from (25) that

$$\varrho_{\hat{\beta}}^{(k+1)} \ge \varrho_{\hat{\beta}}^{(k)} - \bar{\alpha}^{(k)}\left(A_{\hat{\beta}}(\mathrm{x}^{(k)})\Delta\mathrm{x}^{(k)} - \Delta v^{(k)}\right).$$

The quantity $A_{\hat{\beta}}(\mathrm{x}^{(k)})\Delta\mathrm{x}^{(k)} - \Delta v^{(k)}$ is equal to $\varrho_{\hat{\beta}}^{(k)}$ (see the second last equation of system (10)). Hence,

$$\varrho_{\hat{\beta}}^{(k+1)} \ge \left(1-\bar{\alpha}^{(k)}\right)\varrho_{\hat{\beta}}^{(k)}. \tag{26}$$

As $\hat{\beta}c^\top\mathrm{x}^{(k)} \ge F(\mathrm{x}^{(k)})$ (see (4)), we have $\varrho_{\hat{\beta}}^{(k)} \le 0$ for all $k = 0, 1, 2 \dots$. Hence, the inequality (26) yields

$$\|\varrho_{\hat{\beta}}^{(k+1)}\|_1 \le (1-\bar{\alpha}^{(k)})\|\varrho_{\hat{\beta}}^{(k)}\|_1. \tag{27}$$

Next, notice that

$$\begin{aligned}
\rho^{(k+1)} &= w^{(k+1)} - h(\mathrm{x}^{(k+1)}) = w^{(k)} + \bar{\alpha}^{(k)}\Delta w^{(k)} - h(\mathrm{x}^{(k)} + \bar{\alpha}^{(k)}\Delta\mathrm{x}^{(k)}) \\
&\ge (1-\bar{\alpha}^{(k)})\rho^{(k)} \quad (\text{as } -h \text{ is a convex function}).
\end{aligned} \tag{28}$$

As $h(\mathrm{x}^{(l)}) - w^{(l)} \ge 0$ (see (4)), we have $\rho^{(l)} \le 0$ for all $l = 0, 1, 2 \dots$. Hence, the inequality (28) becomes

$$\|\rho^{(k+1)}\|_1 \le (1-\bar{\alpha}^{(k)})\|\rho^{(k)}\|_1. \tag{29}$$

Similarly, a reduction in dual infeasibility is determined by

$$\begin{aligned}
\sigma_{\hat{\beta}}^{(k+1)} &= c - (A_{\hat{\beta}}(\mathrm{x}^{(k+1)}))^\top y^{(k+1)} - (B(\mathrm{x}^{(k+1)}))^\top z^{(k+1)} \\
&= c - A_{\hat{\beta}}(\mathrm{x}^{(k)} + \bar{\alpha}^{(k)}\Delta\mathrm{x}^{(k)})^\top(y^{(k)} + \bar{\alpha}^{(k)}\Delta y^{(k)}) \\
&\quad - B(\mathrm{x}^{(k)} + \bar{\alpha}^{(k)}\Delta\mathrm{x}^{(k)})^\top(z^{(k)} + \bar{\alpha}^{(k)}\Delta z^{(k)}) \\
&\le c - (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top y^{(k)} \\
&\quad + \bar{\alpha}^{(k)}\left(\sum_{i=1}^{s} y_i^{(k)}\nabla^2 f_i(\mathrm{x}^{(k)})\right)\Delta\mathrm{x}^{(k)} - \bar{\alpha}^{(k)}(A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top\Delta y^{(k)} \\
&\quad - (B(\mathrm{x}^{(k)}))^\top z^{(k)} - \bar{\alpha}^{(k)}\left(\sum_{j=1}^{m} z_j^{(k)}\nabla^2 h_j(\mathrm{x}^{(k)})\right)\Delta\mathrm{x}^{(k)} \\
&\quad - \bar{\alpha}^{(k)}(B(\mathrm{x}^{(k)}))^\top\Delta z^{(k)} \quad (\because F \text{ and } -h \text{ are convex}) \\
&= \bar{\alpha}^{(k)}\left(H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)} - (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top\Delta y^{(k)} - (B(\mathrm{x}^{(k)}))^\top\Delta z^{(k)}\right) \\
&\quad + \left(c - (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top y^{(k)} - (B(\mathrm{x}^{(k)}))^\top z^{(k)}\right).
\end{aligned} \tag{30}$$

Since $\sigma_{\hat{\beta}}^{(k)} = -H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)} + (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top\Delta y^{(k)} + (B(\mathrm{x}^{(k)}))^\top\Delta z^{(k)}$ (from the first component of (10)) and $\sigma_{\hat{\beta}}^{(k)} = c - (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top y^{(k)} - (B(\mathrm{x}^{(k)}))^\top z^{(k)}$, therefore (30) yields

$$\sigma_{\hat{\beta}}^{(k+1)} \le (1-\bar{\alpha}^{(k)})\sigma_{\hat{\beta}}^{(k)},$$

and hence

$$\|\sigma_{\hat{\beta}}^{(k+1)}\|_1 \le (1-\bar{\alpha}^{(k)})\|\sigma_{\hat{\beta}}^{(k)}\|_1. \tag{31}$$

For the complementarity values,

$$\begin{aligned}
\gamma^{(k+1)} &= (v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)} \\
&= (v^{(k)})^\top y^{(k)} + (w^{(k)})^\top z^{(k)} \\
&\quad + \bar{\alpha}^{(k)}\left((v^{(k)})^\top\Delta y^{(k)} + (\Delta v^{(k)})^\top y^{(k)} + (w^{(k)})^\top\Delta z^{(k)} + (\Delta w^{(k)})^\top z^{(k)}\right) \\
&\quad + (\bar{\alpha}^{(k)})^2\left((\Delta v^{(k)})^\top\Delta y^{(k)} + (\Delta w^{(k)})^\top\Delta z^{(k)}\right).
\end{aligned}$$

We note that

$$\begin{aligned}
(v^{(k)})^\top\Delta y^{(k)} + (\Delta v^{(k)})^\top y^{(k)} &= e^\top(V^{(k)}\Delta y^{(k)} + Y^{(k)}\Delta v^{(k)}) \\
&= e^\top(\mu e - V^{(k)}Y^{(k)}e) = \mu s - (v^{(k)})^\top y^{(k)}.
\end{aligned}$$

Similarly,

$$(w^{(k)})^\top\Delta z^{(k)} + (\Delta w^{(k)})^\top z^{(k)} = \mu(m+1) - (w^{(k)})^\top z^{(k)}$$

and

$$\begin{aligned}
(\Delta v^{(k)})^\top\Delta y^{(k)} + (\Delta w^{(k)})^\top\Delta z^{(k)} = \; &(\Delta\mathrm{x}^{(k)})^\top\sigma_{\hat{\beta}}^{(k)} + (\Delta\mathrm{x}^{(k)})^\top H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)} \\
&- (\varrho_{\hat{\beta}}^{(k)})^\top\Delta y^{(k)} - (\rho^{(k)})^\top\Delta z^{(k)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma^{(k+1)} &= \gamma^{(k)}(1-\alpha^{(k)}(1-r)) + (\bar{\alpha}^{(k)})^2\left((\Delta\mathrm{x}^{(k)})^\top\sigma_{\hat{\beta}}^{(k)}\right. \\
&\quad \left. + (\Delta\mathrm{x}^{(k)})^\top H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)} - (\varrho_{\hat{\beta}}^{(k)})^\top\Delta y^{(k)} - (\rho^{(k)})^\top\Delta z^{(k)}\right).
\end{aligned}$$

By Hölder inequality, we have

$$|(\Delta\mathrm{x}^{(k)})^\top\sigma_{\hat{\beta}}^{(k)}| \le \|\sigma_{\hat{\beta}}^{(k)}\|_1\|\Delta\mathrm{x}^{(k)}\|_\infty, \qquad |(\Delta\mathrm{x}^{(k)})^\top H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)}| \le \lambda_{\max}\|\Delta\mathrm{x}^{(k)}\|_\infty^2,$$

$$|(\Delta y^{(k)})^\top\varrho_{\hat{\beta}}^{(k)}| \le \|\varrho_{\hat{\beta}}^{(k)}\|_1\|\Delta y^{(k)}\|_\infty \quad \text{and} \quad |(\Delta z^{(k)})^\top\rho^{(k)}| \le \|\rho^{(k)}\|_1\|\Delta z^{(k)}\|_\infty,$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$. Hence,

$$\gamma^{(k+1)} \le \gamma^{(k)}(1-\bar{\alpha}^{(k)}(1-r)) + \left(\|\varrho_{\hat{\beta}}^{(k)}\|_1\|(\bar{\alpha}^{(k)})^2\Delta y^{(k)}\|_\infty\right.$$

$$+ \|\rho^{(k)}\|_1 \|(\bar{\alpha}^{(k)})^2 \Delta z^{(k)}\|_\infty + \|\sigma^{(k)}_{\hat{\beta}}\|_1 \|(\bar{\alpha}^{(k)})^2 \Delta \mathrm{x}^{(k)}\|_\infty + \lambda_{\max} \|(\bar{\alpha}^{(k)})^2 \Delta \mathrm{x}^{(k)}\|_\infty \Big) \tag{32}$$

because $0 < \bar{\alpha}^{(k)} \le 1$ (see (20)). As we have considered that $\Delta\Omega^{(k)}$ is bounded by M, i.e., $\|\Delta\Omega^{(k)}\|_\infty \le M$, thus

$$\|\Delta \mathrm{x}^{(k)}\|_\infty \le M, \quad \|\Delta v^{(k)}\|_\infty \le M, \quad \|\Delta w^{(k)}\|_\infty \le M,$$
$$\|\Delta y^{(k)}\|_\infty \le M, \quad \text{and} \quad \|\Delta z^{(k)}\|_\infty \le M.$$

Hence, the inequality (32) can be written as follows:

$$\gamma^{(k+1)} \le \gamma^{(k)}(1 - \bar{\alpha}^{(k)}(1-r)) + M \left(\|\varrho^{(k)}_{\hat{\beta}}\|_1 + \|\rho^{(k)}\|_1 + \|\sigma^{(k)}_{\hat{\beta}}\|_1 + \lambda_{\max} \right). \quad \square \tag{33}$$

Note 3. The dual problem corresponding to the primal problem (4) is

$$\left.\begin{array}{ll} \text{maximize} & c^\top \mathrm{x} - y^\top(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x})) - z^\top h(\mathrm{x}) + ((A_{\hat{\beta}}(\mathrm{x}))^\top y + (B(\mathrm{x}))^\top z - c)^\top \mathrm{x} \\ \text{subject to} & (A_{\hat{\beta}}(\mathrm{x}))^\top y + (B(\mathrm{x}))^\top z = c, \\ & y, z \ge 0. \end{array}\right\} \tag{34}$$

The difference between the values of the dual objective function and the primal objective function is

$$G = y^\top(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x})) + z^\top h(\mathrm{x}) - ((A_{\hat{\beta}}(\mathrm{x}))^\top y + (B(\mathrm{x}))^\top z - c)^\top \mathrm{x}$$
$$= \gamma - y^\top \varrho_{\hat{\beta}} - z^\top \rho + \sigma^\top_{\hat{\beta}} \mathrm{x}.$$

At any point (x, v, w, y, z), the value of G can be estimated by

$$\begin{aligned} |G| &\le \gamma + |\varrho^\top_{\hat{\beta}} y| + |\rho^\top z| + |\sigma^\top_{\hat{\beta}} \mathrm{x}| \\ &\le \gamma + \|\varrho_{\hat{\beta}}\|_1 \|y\|_\infty + \|\rho\|_1 \|z\|_\infty + \|\sigma_{\hat{\beta}}\|_1 \|\mathrm{x}\|_\infty. \end{aligned} \tag{35}$$

The last inequality holds by the Hölder inequality. For a precision parameter $\epsilon > 0$, if $\|\varrho_{\hat{\beta}}\|_1 < \epsilon$, $\|\rho\|_1 < \epsilon$, $\|\sigma_{\hat{\beta}}\|_1 < \epsilon$ and $\gamma < \epsilon$, then the gap G becomes very close to zero, and hence the current solution is approximately optimal for (5).

Next, we calculate the reduction in primal infeasibilities, dual infeasibility and complementarity. In addition, the convergence of Algorithm 1 is addressed in Theorem 4.

Theorem 4. *Consider the barrier problem (5). Assume that the sequence* $(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, y^{(k)}, z^{(k)})$ *generated by Algorithm 1 is bounded and Hessian matrix* $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ *is positive definite at every kth iteration. Let the quantities* $\varrho^{(k)}_{\hat{\beta}}$, $\rho^{(k)}$, $\sigma^{(k)}_{\hat{\beta}}$ *and* $\gamma^{(k)}$ *denote infeasibility and complementarity values at the kth iterate. Then, the following three results hold.*

(i) Suppose $\tau > 0$ *and* $M > 0$ *are such that for all* k,

$$\alpha^{(k)} \ge \tau, \quad \|\mathrm{x}^{(k)}\|_\infty \le M, \quad \|v^{(k)}\|_\infty \le M, \quad \|w^{(k)}\|_\infty \le M,$$
$$\|y^{(k)}\|_\infty \le M \quad \text{and} \quad \|z^{(k)}\|_\infty \le M.$$

Then,

$$\|\varrho^{(k)}_{\hat{\beta}}\|_1 \le (1-\tau)^k \|\varrho^{(0)}_{\hat{\beta}}\|_1, \quad \|\rho^{(k)}\|_1 \le (1-\tau)^k \|\rho^{(0)}\|_1, \quad \|\sigma^{(k)}_{\hat{\beta}}\|_1 \le (1-\tau)^k \|\sigma^{(0)}_{\hat{\beta}}\|_1 \tag{36}$$

and there exists an $\tilde{M} > 0$ *such that* $\gamma^{(k)} \le (1-\bar{\tau})^k \tilde{M}$, *where* $\bar{\tau} = \tau(1-r)$,

(ii) For $k > \bar{K}$, *infeasibilities and complementarity become less than a precision value* $\epsilon > 0$, *where*

$$\bar{K} = \max \left\{ \frac{\log\left(\frac{\epsilon}{\|\varrho^{(0)}_{\hat{\beta}}\|_1}\right)}{\log(1-\tau)}, \frac{\log\left(\frac{\epsilon}{\|\rho^{(0)}\|_1}\right)}{\log(1-\tau)}, \frac{\log\left(\frac{\epsilon}{\|\sigma^{(0)}_{\hat{\beta}}\|_1}\right)}{\log(1-\tau)}, \frac{\log\left(\frac{\epsilon}{\tilde{M}}\right)}{\log(1-\bar{\tau})} \right\}.$$

(iii) If $\sum_{k=1}^{\infty} \alpha^{(k)} = \infty$, *then*

$$\lim_{k\to\infty} \|\varrho^{(k)}_{\hat{\beta}}\|_1 = 0, \quad \lim_{k\to\infty} \|\rho^{(k)}\|_1 = 0, \quad \lim_{k\to\infty} \|\sigma^{(k)}_{\hat{\beta}}\|_1 = 0 \quad \text{and} \quad \lim_{k\to\infty} \gamma^{(k)} = 0.$$

Proof.

(i) For the kth iteration, inequality (27) and $\alpha^{(k)} \ge \tau$ lead to

$$\|\varrho^{(k)}_{\hat{\beta}}\|_1 \le (1-\tau)\|\varrho^{(k-1)}_{\hat{\beta}}\|_1 \le (1-\tau)^2 \|\varrho^{(k-2)}_{\hat{\beta}}\|_1 \le \cdots \le (1-\tau)^k \|\varrho^{(0)}_{\hat{\beta}}\|_1.$$

Similarly, from (29) and (31), we obtain

$$\|\rho^{(k)}\|_1 \le (1-\tau)^k \|\rho^{(0)}\|_1 \quad \text{and} \quad |\sigma^{(k)}_{\hat{\beta}}\|_1 \le (1-\tau)^k \; \| \; \sigma^{(0)}_{\hat{\beta}}\|_1.$$

An estimate of the complementarity decrease at the kth iteration is calculated by (33):

$$\begin{aligned} \gamma^{(k)} &\le \gamma^{(k-1)}(1 - \tau(1-r)) + M(1-\tau)^{k-1}(\|\varrho^{(0)}_{\hat{\beta}}\|_1 + \|\rho^{(0)}\|_1 + \|\sigma^{(0)}_{\hat{\beta}}\|_1 + \lambda_{\max}) \\ &= (1-\bar{\tau})\gamma^{(k-1)} + \bar{M}(1-\tau)^{k-1}, \end{aligned}$$

where $\bar{\tau} = \tau(1-r)$ and $\bar{M} = M(\|\varrho^{(0)}_{\hat{\beta}}\|_1 + \|\rho^{(0)}\|_1 + \|\sigma^{(0)}_{\hat{\beta}}\|_1 + \lambda_{\max})$. Similarly,

$$\begin{aligned} \gamma^{(k)} &\le (1-\bar{\tau}) \left[(1-\bar{\tau})\gamma^{(k-2)} + \bar{M}(1-\tau)^{k-2}\right] + \bar{M}(1-\tau)^{k-1} \\ &= (1-\bar{\tau})^2 \gamma^{(k-2)} + \bar{M}(1-\tau)^{k-1} \left[\frac{1-\bar{\tau}}{1-\tau} + 1\right]. \end{aligned}$$

Proceeding in a similar way, we see that

$$\begin{aligned} \gamma^{(k)} &\le (1-\bar{\tau})^2 \left[(1-\bar{\tau})\gamma^{(k-3)} + \bar{M}(1-\tau)^{k-3}\right] + \bar{M}(1-\tau)^{k-1} \left[\frac{1-\bar{\tau}}{1-\tau} + 1\right] \\ &= (1-\bar{\tau})^3 \gamma^{(k-3)} + \bar{M}(1-\tau)^{k-1} \left[\left(\frac{1-\bar{\tau}}{1-\tau}\right)^2 + \frac{1-\bar{\tau}}{1-\tau} + 1\right] \\ &\ldots \\ &\le (1-\bar{\tau})^k \gamma^{(0)} + \bar{M}(1-\tau)^{k-1} \left[\left(\frac{1-\bar{\tau}}{1-\tau}\right)^{k-1} + \cdots + \frac{1-\bar{\tau}}{1-\tau} + 1\right] \\ &= (1-\bar{\tau})^k \gamma^{(0)} + \bar{M} \frac{(1-\bar{\tau})^k - (1-\tau)^k}{\tau - \bar{\tau}}. \end{aligned}$$

Since $\bar{\tau} = \tau(1-r)$, we can write

$$\frac{(1-\bar{\tau})^k - (1-\tau)^k}{\tau - \bar{\tau}} \le \frac{(1-\bar{\tau})^k}{r\tau}.$$

Hence,

$$\gamma^{(k)} \le (1-\bar{\tau})^k \left(\gamma^{(0)} + \frac{\bar{M}}{r\tau}\right).$$

Denoting $\tilde{M} = \left(\gamma^{(0)} + \frac{\bar{M}}{r\tau}\right)$, we get

$$\gamma^{(k)} \le (1-\bar{\tau})^k \tilde{M}.$$

(ii) We note from (36) that

$$\|\varrho^{(k)}_{\hat{\beta}}\|_1 \le (1-\tau)^k \|\varrho^{(0)}_{\hat{\beta}}\|_1 < \epsilon$$

holds if

$$(1-\tau)^k < \frac{\epsilon}{\|\varrho^{(0)}_{\hat{\beta}}\|_1}$$

i.e., if $k \log(1-\tau) < \log\left(\frac{\epsilon}{\|\varrho^{(0)}_{\hat{\beta}}\|_1}\right)$

i.e., if $k > \dfrac{\log\left(\frac{\epsilon}{\|\varrho_{\hat{\beta}}^{(0)}\|_1}\right)}{\log(1-\tau)}$ since $0 < \tau < 1$.

Similarly, from other inequalities in (36), $\|\rho^{(k)}\| < \epsilon$, $\|\sigma_{\hat{\beta}}^{(k)}\|_1 < \epsilon$ and $\gamma^{(k)} < \epsilon$ hold if

$$k > \frac{\log\left(\frac{\epsilon}{\|\rho^{(0)}\|_1}\right)}{\log(1-\tau)}, \quad k > \frac{\log\left(\frac{\epsilon}{\|\sigma_{\hat{\beta}}^{(0)}\|_1}\right)}{\log(1-\tau)} \quad \text{and} \quad k > \frac{\log\left(\frac{\epsilon}{\bar{M}}\right)}{\log(1-\bar{\tau})}, \text{ respectively.}$$

Hence, the result follows.

(iii) We first prove that $\prod_{k=1}^{\infty}(1-\alpha^{(k)}) = 0$ if $\sum_{k=1}^{\infty}\alpha^{(k)} = \infty$.
For $k \in \mathbb{N}$, we denote

$$p_k = (1-\alpha^{(1)})(1-\alpha^{(2)})\cdots(1-\alpha^{(k)}) \text{ and } s_k = \alpha^{(1)} + \alpha^{(2)} + \cdots + \alpha^{(k)}.$$

Using the fact that $1 - x \le e^{-x}$ for every $x \in \mathbb{R}$, we have

$$p_k \le e^{-\alpha^{(1)}} e^{-\alpha^{(2)}} \cdots e^{-\alpha^{(k)}} = e^{-s_k}.$$

Hence,

$$\prod_{k=1}^{\infty}(1-\alpha^{(k)}) \le e^{-\sum_{k=1}^{\infty}\alpha^{(k)}} = 0$$

Since $0 \le \alpha^{(k)} \le 1$ for all k (see (20)),

$$\prod_{k=1}^{\infty}(1-\alpha^{(k)}) = 0. \tag{37}$$

Now, we proceed with the main part of the theorem. By , we can write that

$$\|\varrho_{\hat{\beta}}^{(k)}\|_1 \le (1-\alpha^{(1)})(1-\alpha^{(2)})\cdots(1-\alpha^{(k)})\|\varrho_{\hat{\beta}}^{(0)}\|_1.$$

As k approaches to infinity, we obtain

$$0 \le \lim_{k\to\infty}\|\varrho_{\hat{\beta}}^{(k)}\|_1 \le \prod_{k=1}^{\infty}(1-\alpha^{(k)})\|\varrho_{\hat{\beta}}^{(0)}\|_1 = 0.$$

Thus, $\lim_{k\to\infty}\|\varrho_{\hat{\beta}}^{(k)}\|_1 = 0$. Similarly, the remaining limits can be shown. □

Section 5 extends Algorithm 1 for nonconvex multi-objective optimization problems.

5. Algorithm for nonconvex multi-objective optimization problems

So far, we discussed that the nondominated points of the multi-objective optimization problem (1) are obtained by solving the single objective parametric problem CM($\hat{\beta}$) (see (2)). Algorithm 1 has the ability to handle those single objective parametric problems in which the Hessian matrix $H(\mathrm{x}, y, z)$ (calculated by (8)) is positive definite at every iteration. However, in case of nonconvex problems, $H(\mathrm{x}, y, z)$ and hence $N_{\hat{\beta}}(\mathrm{x}, y, z, v, w)$ may fail to be positive definite. In such a case, we lose the descent property given in Theorem 2. To maintain the descent property, the following substitution is taken in place of the Hessian matrix $H(\mathrm{x}, y, z)$

$$\tilde{H}(\mathrm{x}, y, z) = H(\mathrm{x}, y, z) + \lambda I, \tag{38}$$

where I is the identity matrix of order $(n+1)\times(n+1)$ and the choice of λ (see Section 5.1) is such that $\tilde{H}(\mathrm{x}, y, z)$ is positive definite.

The primal and dual directions are determined by solving the following system

$$\begin{bmatrix} -\tilde{H}(\mathrm{x}, y, z) & (A_{\hat{\beta}}(\mathrm{x}))^\top & B((\mathrm{x}))^\top \\ A_{\hat{\beta}}(\mathrm{x}) & VY^{-1} & 0 \\ B(\mathrm{x}) & 0 & WZ^{-1} \end{bmatrix}\begin{bmatrix}\Delta\mathrm{x} \\ \Delta y \\ \Delta z\end{bmatrix} = \begin{bmatrix}\sigma_{\hat{\beta}} \\ \varrho_{\hat{\beta}} + VY^{-1}\gamma_1 \\ \rho + WZ^{-1}\gamma_2\end{bmatrix}. \tag{39}$$

The solution of the system (39) provides the explicit formulas (40) for the primal–dual directions

$$\left.\begin{aligned} \Delta\mathrm{x} &= \tilde{N}_{\hat{\beta}}^{-1}\left(-\sigma_{\hat{\beta}} + \left(A_{\hat{\beta}}(\mathrm{x})\right)^\top(\gamma_1 + V^{-1}Y\varrho_{\hat{\beta}}) + (B(\mathrm{x}))^\top(\gamma_2 + W^{-1}Z\rho)\right), \\ \Delta v &= -\varrho_{\hat{\beta}} + A_{\hat{\beta}}(\mathrm{x})\Delta\mathrm{x}, \\ \Delta w &= -\rho + B(\mathrm{x})\Delta\mathrm{x}, \\ \Delta y &= \gamma_1 + V^{-1}Y\left(\varrho_{\hat{\beta}} - A_{\hat{\beta}}(\mathrm{x})\Delta\mathrm{x}\right), \\ \Delta z &= \gamma_2 + W^{-1}Z\left(\rho - B(\mathrm{x})\Delta\mathrm{x}\right), \end{aligned}\right\} \tag{40}$$

where $\tilde{N}_{\hat{\beta}} = \tilde{N}_{\hat{\beta}}(\Omega) = \tilde{H}(\mathrm{x}, y, z) + (A_{\hat{\beta}}(\mathrm{x}))^\top V^{-1}YA_{\hat{\beta}}(\mathrm{x}) + (B(\mathrm{x}))^\top W^{-1}ZB(\mathrm{x})$. Note that primal–dual directions (40) are determined in a similar way to Theorem 2.

Undoubtedly, the following question arises naturally: what is the effect of perturbation (38) on primal infeasibility, dual infeasibility and complementarity? Theorem 5 addresses the reduction in primal and dual infeasibility due to the perturbation. In this result, we assume that the search directions are computed with $\tilde{H}(\mathrm{x}, y, z)$ instead of $H(\mathrm{x}, y, z)$.

Theorem 5. *Consider the following perturbations in* x, v, w, y *and* z:

$$\bar{\mathrm{x}} = \mathrm{x} + \varepsilon\Delta\mathrm{x}, \quad \bar{v} = v + \varepsilon\Delta v, \quad \bar{w} = w + \varepsilon\Delta w, \quad \bar{y} = y + \varepsilon\Delta y, \quad \bar{z} = z + \varepsilon\Delta z,$$

where ε *is calculated according to Section 4.3. If* $\bar{\varrho}_{\hat{\beta}} = \varrho_{\hat{\beta}}(\bar{\mathrm{x}}, \bar{v})$, $\bar{\rho} = \rho(\bar{\mathrm{x}}, \bar{w})$ *and* $\bar{\sigma}_{\hat{\beta}} = \sigma_{\hat{\beta}}(\bar{y}, \bar{z})$, *then*

$$\begin{aligned} \bar{\varrho}_{\hat{\beta}} &= (1-\varepsilon)\varrho_{\hat{\beta}} + o(\varepsilon^2), \\ \bar{\rho} &= (1-\varepsilon)\rho + o(\varepsilon^2), \\ \bar{\sigma}_{\hat{\beta}} &= (1-\varepsilon)\sigma_{\hat{\beta}} - \varepsilon\lambda\Delta\mathrm{x} + o(\varepsilon^2), \\ \bar{v}^\top\bar{y} &= (1-\varepsilon(1-\delta_1))v^\top y + o(\varepsilon^2), \text{ and} \\ \bar{w}^\top\bar{z} &= (1-\varepsilon(1-\delta_2))w^\top y + o(\varepsilon^2), \end{aligned}$$

where $\delta_1 = \frac{s\mu}{v^\top y}$ *and* $\delta_2 = \frac{(m+1)\mu}{w^\top z}$.

Proof. We have

$$\begin{aligned} \bar{\varrho}_{\hat{\beta}} &= F(\bar{\mathrm{x}}) + \bar{v} - \hat{\beta}c^\top\bar{\mathrm{x}} \\ &= F(\mathrm{x} + \varepsilon\Delta\mathrm{x}) + v + \varepsilon\Delta v - \hat{\beta}c^\top(\mathrm{x} + \varepsilon\Delta\mathrm{x}) \\ &= F(\mathrm{x}) + \varepsilon\nabla F(\mathrm{x})\Delta\mathrm{x} + o(\varepsilon^2) + v + \varepsilon\Delta v - \hat{\beta}c^\top\mathrm{x} - \varepsilon\hat{\beta}c^\top\Delta\mathrm{x} \\ &= \left(F(\mathrm{x}) + v - \hat{\beta}c^\top\mathrm{x}\right) - \varepsilon\left(\left(\hat{\beta}c^\top - \nabla F(\mathrm{x})\right)\Delta\mathrm{x} - \Delta v\right) + o(\varepsilon^2) \\ &= \varrho_{\hat{\beta}} - \varepsilon\left(A_{\hat{\beta}}(\mathrm{x})\Delta\mathrm{x} - \Delta v\right) + o(\varepsilon^2) \\ &= (1-\varepsilon)\varrho_{\hat{\beta}} + o(\varepsilon^2), \end{aligned} \tag{41}$$

where the last equality holds from the third block of Eq. (9). Similarly, $\bar{\rho}$ can be determined. Next, we have

$$\begin{aligned} \bar{\sigma}_{\hat{\beta}} &= c - (A_{\hat{\beta}}(\bar{\mathrm{x}}))^\top\bar{y} - (B(\bar{\mathrm{x}}))^\top\bar{z} \\ &= c - (A_{\hat{\beta}}(\mathrm{x} + \varepsilon\Delta\mathrm{x}))^\top(y + \varepsilon\Delta y) - (B(\mathrm{x} + \varepsilon\Delta\mathrm{x}))^\top(z + \varepsilon\Delta z) \\ &= c - (A_{\hat{\beta}}(\mathrm{x}))^\top y + \varepsilon\left(\sum_{i=1}^{k} y_i\nabla^2 f_i(\mathrm{x})\right)\Delta\mathrm{x} - \varepsilon(A_{\hat{\beta}}(\mathrm{x}))^\top\Delta y - (B(\mathrm{x}))^\top z \\ &\quad - \varepsilon\left(\sum_{j=1}^{m} z_j\nabla^2 h_j(\mathrm{x})\right)\Delta\mathrm{x} - \varepsilon(B(\mathrm{x}))^\top\Delta z + o(\varepsilon^2) \\ &= c + \varepsilon\left(\tilde{H}(\mathrm{x}, y, z)\Delta\mathrm{x} - (A_{\hat{\beta}}(\mathrm{x}))^\top\Delta y - (B(\mathrm{x}))^\top\Delta z\right) \\ &\quad - (A_{\hat{\beta}}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z + o(\varepsilon^2) \\ &= (1-\varepsilon)\sigma_{\hat{\beta}} - \varepsilon\lambda\Delta\mathrm{x} + o(\varepsilon^2), \end{aligned} \tag{42}$$

where the last equality holds from the first block of Eq. (9) in which $H(\mathrm{x}, y, z)$ is replaced by $\tilde{H}(\mathrm{x}, y, z)$. Again, we see that

$$\begin{aligned} \bar{v}^\top\bar{y} &= (v + \varepsilon\Delta v)^\top(y + \varepsilon\Delta y) \\ &= v^\top y + \varepsilon(v^\top\Delta y + \Delta v^\top y) + o(\varepsilon^2) \end{aligned}$$

$$\begin{aligned}
&= v^{\top}y + \varepsilon e^{\top}(V\Delta y + Y\Delta v) + o(\varepsilon^2) \\
&= v^{\top}y + \varepsilon e^{\top}(\mu e - VYe) + o(\varepsilon^2) \\
&= v^{\top}y + \varepsilon(s\mu - v^{\top}y) + o(\varepsilon^2) \\
&= v^{\top}y - \varepsilon(1-\delta_1)v^{\top}y + o(\varepsilon^2) \\
&= (1-\varepsilon(1-\delta_1))v^{\top}y + o(\varepsilon^2).
\end{aligned} \tag{43}$$

Similarly, we can compute the reduction in $\bar{w}^{\top}\bar{z}$ by

$$\bar{w}^{\top}\bar{z} = (1-\varepsilon(1-\delta_2))w^{\top}z + o(\varepsilon^2). \quad \square \tag{44}$$

Note that Theorem 5 shows dual infeasibility $\sigma_{\hat{\beta}}$ may fail to reduce whenever $\lambda > 0$, even if an $\varepsilon \in (0,1)$ is taken along the search direction. This behavior can affect the overall performance of the algorithm. However, when $o(\lambda) = o(\varepsilon)$, then $o(\varepsilon\lambda\Delta x) = o(\varepsilon^2)$ and hence the convergence of $\bar{\sigma}_{\hat{\beta}}$ to 0 is guaranteed (by (42) and $\varepsilon \in (0,1)$).

In the following lemma, we introduce a choice of λ such that $o(\lambda) = o(\varepsilon)$ and hence show that the dual infeasibility $\sigma_{\hat{\beta}}$ reduces at every iteration.

Lemma 1. *Let at any iteration the Hessian matrix $H(x,y,z)$ is not positive definite. Also, assume that $\lambda = \zeta^l|\lambda_{\min}|$, where $\zeta \in (0,1)$, $l \in \{2,3,4\ldots\}$ and $\lambda_{\min}$ is the minimum eigenvalue of the Hessian matrix. Then there exists a natural number $\hat{K}$ such that for all $l \geq \hat{K}$ the following inequality holds*

$$\|\bar{\sigma}_{\hat{\beta}}\|_1 \leq (1-\bar{\alpha}_l)\|\sigma_{\hat{\beta}}\|_1.$$

Proof. From Note 2, we have

$$\varepsilon = \bar{\alpha}_l = \zeta^l\alpha \text{ for some } l \in \{2,3,4,\ldots\} \text{ and } \zeta \in (0,1). \tag{45}$$

Similarly, from line 6 of Algorithm 2, we have

$$\lambda = \zeta^l|\lambda_{\min}| \text{ for some } l \in \{2,3,4,\ldots\} \text{ and } \zeta \in (0,1). \tag{46}$$

From (45) and (46), it is clear that $\varepsilon = o(\zeta)$ and $\lambda = o(\zeta)$. Now Eq. (42) can be written as follows:

$$\bar{\sigma}_{\hat{\beta}} = (1-\bar{\alpha}_l)\sigma_{\hat{\beta}} + o(\zeta^2).$$

As $\lim_{l\to\infty}\bar{\alpha}_l = 0$, so there exists a natural number $\hat{K}$ such that for every $l \geq \hat{K}$, the following inequality holds:

$$\|\bar{\sigma}_{\hat{\beta}}\|_1 \leq (1-\bar{\alpha}_l)\|\sigma_{\hat{\beta}}\|_1. \tag{47}$$

The inequality (47) confirms that the value of λ calculated by Algorithm 2 reduces the dual infeasibility ($\sigma_{\hat{\beta}}$) at each iteration. $\square$

Now, based on Lemma 1 a method is presented (see Algorithm 2) to determine λ so that $N_{\hat{\beta}}(x,y,z,v,w)$ remains positive definite as well as $o(\lambda) = o(\varepsilon)$ (line number 6 of Algorithm 2). Also, note that the formulas of the primal–dual directions (40) involve the perturbation form of Hessian (38), where λ is taken as described in Section 5.1.

5.1. Determination of λ

We initially start with an $LDL^{\top}$ factorization of the following symmetric matrix of the reduced KKT system (13):

$$S = \begin{bmatrix} -H(x,y,z) & (A_{\hat{\beta}}(x))^{\top} & (B(x))^{\top} \\ A_{\hat{\beta}}(x) & VY^{-1} & 0 \\ B(x) & 0 & WZ^{-1} \end{bmatrix}.$$

The matrix S is quasidefinite whenever $H(x,y,z)$ is positive definite (Vanderbei, 1995), and therefore there is a guarantee of LDL^T factorization. When $H(x,y,z)$ is not positive definite then the LDL^T factorization will not be applicable. Therefore, we choose the value of λ so that the matrix $\tilde{H}(x,y,z) = H(x,y,z) + \lambda I$ becomes positive definite. Thereafter, we replace the matrix $H(x,y,z)$ by $\tilde{H}(x,y,z)$ in the matrix S and solve the system (39) with the help of a variant of LDL^T factorization, i.e., Cholesky factorization. The value of λ is computed by the following algorithm:

Algorithm 2 A method to determine a value of λ so that $\tilde{H}(x,y,z)$ becomes positive definite.

Aim: To determine a value of λ to be used in (38) to make $\tilde{H}(x,y,z)$ positive definite matrix

1: **Inputs**
Provide $n+1$, the dimension of the vector x
Provide the matrix $H(x,y,z)$ (see (8)) of dimension $(n+1)\times(n+1)$
(In the below, I is the identity matrix of order $(n+1)\times(n+1)$)

2: **Initialization**
Choose any $\zeta \in (0,1)$

3: **Main Steps** (Steps 4 to 7) to find a λ such that $\tilde{H}(x,y,z)$ is positive definite

4: Calculate the minimum eigenvalue ($\lambda_{\min}$) of the symmetric matrix $H(x,y,z)$

5: Set $H(x,y,z) \leftarrow H(x,y,z) + |\lambda_{\min}|I$

6: Choose λ as the smallest value in $\{\zeta^2|\lambda_{\min}|, \zeta^3|\lambda_{\min}|, \zeta^4|\lambda_{\min}|, \ldots\}$ such that

$$H(x,y,z) + \lambda I \quad \text{is positive definite}$$

7: **return** the value λ (for which $\tilde{H}(x,y,z) = H(x,y,z) + \lambda I$ is positive definite)

In the case of convex MOP, if the Hessian matrix $H(x,y,z)$ is positive definite then the Newton's direction (15) is a descent direction for the merit function $\psi_{\eta,\mu}(\Omega)$ (see Theorem 2). However, if the problem is nonconvex and the Hessian matrix $H(x,y,z)$ fails to be positive definite at any point of iteration, then we perturb the Hessian matrix and calculate the reduction in primal and dual infeasibilities (Theorem 5). Theorem 5 does not guarantee a reduction in dual infeasibility. The following Algorithm 3 has the ability to handle the Hessian matrix $H(x,y,z)$ whether it is positive definite or not.

5.2. Well-definedness of Algorithm 3

Algorithm 3 not only solves convex MOPs but also nonconvex MOPs. Algorithm 1 works only when the Hessian matrix $H(x,y,z)$ is positive definite at every iteration. Algorithm 3 handles both the cases, whether the Hessian matrix $H(x,y,z)$ is positive definite or not. The well-definedness of Algorithm 3 depends on lines number 10, 14 and 17. The line 10 is well-defined as the Hessian matrix $H(x^{(k)},y^{(k)},z^{(k)})$ is positive definite at any kth iteration, and hence the system (13) is solvable by Cholesky factorization. At the kth iteration, when $H(x^{(k)},y^{(k)},z^{(k)})$ is not positive definite, Algorithm 3 computes the value of $\lambda_{\min}$ by Algorithm 2 so that the matrix $\tilde{H}(x^{(k)},y^{(k)},z^{(k)})$ becomes positive definite. Now Cholesky factorization is applicable to solve the system (39). Hence, the line number 14 is well-defined. The reason of well-definedness of line number 17 is identical to line number 9 in Algorithm 1.

As an illustration, we consider the following problem

$$\begin{aligned}
\text{minimize} \quad & (f_1(x), \quad f_2(x))^{\top} \\
\text{subject to} \quad & (x_1+1)^2 + x_2^2 \leq 4 \\
& (x_1+2)^2 + (x_2+2)^2 \leq 4 \\
& x_1 \in [-5,2], \quad x_2 \in [-5,3],
\end{aligned} \tag{48}$$

where $f_1(x) = (x_1+3)^2 + (x_2-2)^2$ and $f_2(x) = x_1^2 + (x_2+3)^2$. The location of the nondominated set of this problem is not known a priori. The cone formulation of the problem (48) is as follows:

$$\begin{aligned}
\text{minimize} \quad & t \\
\text{subject to} \quad & t\cos\theta \geq f_1(x) \\
& t\sin\theta \geq f_2(x) \\
& (x_1+1)^2 + x_2^2 \leq 4 \\
& (x_1+2)^2 + (x_2+2)^2 \leq 4
\end{aligned} \tag{49}$$

Algorithm 3 IIPM for MOPs when $H(\mathrm{x}, y, z)$ is not necessarily positive definite.

Aim: To generate a discrete approximation of the nondominated set $\mathcal{D}$ of the MOP (1)

1: **Inputs**
Provide $f_1, f_2, \dots, f_s$ which are twice continuously differentiable objective functions of (1)
Provide $h_1, h_2, \dots, h_m$ which are twice continuously differentiable constraints functions of (1)
Provide N, the number of different $\hat{\beta}$'s corresponding to which the subproblem (5) is to be solved

2: **Initialization**
Set nondominated set $\mathcal{D} \leftarrow \emptyset$
To solve (5) for a given $\hat{\beta}$, provide an initial point $\mathrm{x}^{(0)}$
Choose any positive ξ_1 and ξ_2 for (23)
Initialize $v^{(0)}$, $w^{(0)}$ by (23) and $y^{(0)}, z^{(0)}$ by any positive values
Give the values of parameters r, δ and $0 < \kappa < 1$ according to (19), (20) and (21), respectively
Give a value of the precision parameter $\epsilon > 0$
Set $k \leftarrow 0$
Set $\mu^{(k)} \leftarrow r\frac{(v^{(k)})^\top y^{(k)} + (w^{(k)})^\top z^{(k)}}{m+s+1}$

3: **Main Steps to solve (5) for N different $\hat{\beta}$'s**

4: **for** $i = 1 : 1 : N$ **do**
(Lines 5 to 42 solves (5) for a $\hat{\beta}$ and find $\mathrm{x}^{(k)}$ as a solution to (5). Line 44 keeps stacking up the nondominated points of the MOP (1) in the set $\mathcal{D}$)

5: Choose a direction $\hat{\beta}$ by using (3)
6: Set $\nu(\Omega^{(k)}) \leftarrow \max\{\|\sigma_{\hat{\beta}}^{(k)}\|_1, \|\varrho_{\hat{\beta}}^{(k)}\|_1, \|\rho^{(k)}\|_1, \|V^{(k)}Y^{(k)}e\|_1, \|W^{(k)}Z^{(k)}e\|_1\}$ (by (24))
7: **while** $\nu(\Omega^{(k)}) \geq \epsilon$ **do**
8: Calculate the Hessian matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ by using (8)
9: **if** $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ is positive definite **then**
10: Solve the system (13) by Cholesky factorization
11: **else**
12: Compute the value of perturbation parameter $\lambda_{\min}$ for $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ by Algorithm 2
13: Set $\tilde{H}(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)}) \leftarrow H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)}) + \lambda_{\min} I$
14: Solve the system (39) by Cholesky factorization
15: **end if**
16: Find primal–dual directions $\Delta\Omega^{(k)}$
17: Choose step length α by formula (20)
18: Set $p^{(k+1)} \leftarrow p^{(k)} + \alpha\Delta p^{(k)}$, $y^{(k+1)} \leftarrow y^{(k)} + \alpha\Delta y^{(k)}$, $z^{(k+1)} \leftarrow z^{(k)} + \alpha\Delta z^{(k)}$
19: Calculate $\Gamma_{\hat{\beta}}(\Omega^{(k)})$ by the expression (17)
20: **if** $\varrho_{\hat{\beta}}(\mathrm{x}^{(k)}, v^{(k)}) = 0$, $\rho(\mathrm{x}^{(k)}, w^{(k)}) = 0$ and $\psi_{\eta,\mu}(\Omega^{(k+1)}) \leq \psi_{\eta,\mu}(\Omega^{(k)})$ **then**
21: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{m+s+1}$
22: Update $\nu(\Omega^{(k+1)})$ according to (24)
23: Set $k \leftarrow k+1$
24: **else**
25: **if** $\Gamma_{\hat{\beta}}(\Omega^{(k)}) < 0$ **then**
26: Set $\eta = 0$ and backtrack $\alpha^{(k)} \in [0, \alpha]$ until the condition (21) holds
27: Update $\Omega^{(k+1)} \leftarrow \Omega^{(k)} + \alpha^{(k)}\Delta\Omega^{(k)}$
28: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{m+s+1}$
29: Update $\nu(\Omega^{(k+1)})$ according to (24)
30: Set $k \leftarrow k+1$
31: **else**
32: Calculate $\eta_{\min}$ by equation (18)
33: Set $\eta = 10\eta_{\min}$ and backtrack $\alpha^{(k)} \in [0, \alpha]$ until the condition (21) holds
34: Update $\Omega^{(k+1)} \leftarrow \Omega^{(k)} + \alpha^{(k)}\Delta\Omega^{(k)}$
35: **if** $\psi_{\hat{\beta}}(\Omega^{(k+1)}) \leq \psi_{\hat{\beta}}(\Omega^{(k)})$ **then**
36: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{m+s+1}$
37: Update $\nu(\Omega^{(k+1)})$ according to (24)
38: Set $k \leftarrow k+1$
39: **end if**
40: **end if**
41: **end if**
42: **end while**
43: Calculate $F(\mathrm{x}^{(k)}) = \hat{\beta}c^T\mathrm{x}^{(k)} - v^{(k)}$
44: Update nondominated set $\mathcal{D} \leftarrow \mathcal{D} \bigcup \{F(\mathrm{x}^{(k)})\}$
45: **end for**
46: **return** the set $\mathcal{D}$ (a discrete approximation of the nondominated set)

$$x_1 \in [-5, 2], \ x_2 \in [-5, 3],$$

where $\theta \in [0, \pi/2]$. The convergence of Algorithm 3 towards the nondominated points of the problem (48) is shown in Fig. 1(a). To obtain the nondominated points of the problem (48), Algorithm 3 solves the formulated problem (49) for different values of $\theta \in [0, \pi/2]$. It starts with an initial point $(x_1^{(0)}, x_2^{(0)}, t^{(0)}) = (1.5, 1, 15)$ and a value $\theta \in [0, \pi/2]$, and then gradually moves towards the efficient point. After finding one efficient point, Algorithm 3 changes the value of $\theta \in [0, \pi/2]$ and then converges to another efficient point. We have shown all the iterations in the objective space (see Fig. 1(a)). The blue point $(21.25, 18.25)$ in Fig. 1(a) is the starting point in the objective space and green points are the generated nondominated points corresponding to different values of $\theta \in [0, \pi/2]$. Note that the initial point remains unchanged throughout the entire process (see Fig. 1(a)).

One can also check the performance of the proposed algorithm by taking different strategies of starting points. Here we discuss three strategies of choosing starting points and compare them with respect to the number of iterations and time. For every strategy, we take the problem (48) as a test example and solve its cone formulation (49) for 30 values of $\theta \in [0, \pi/2]$ by Algorithm 3 with accuracy value $\epsilon = 10^{-4}$.

Strategy 1. *In the first strategy, we solve the problem (48) by taking different initial points and check the performance of Algorithm 3 based on the total number of iterations and time. We take four different initial points and calculate the total number of iterations and time corresponding to every initial point (see Table 1). For every initial point, convergence of the iterations is depicted in Figs. 1 and 2. The data reported in Table 1 shows that Algorithm 3 performs better in terms of iterations and time if the image of initial points are taken closer to the nondominated frontier (see Fig. 2(b)), i.e., this strategy will speed up the generation of nondominated points if the chosen initial point is closer to the true frontier.*

Strategy 2. *In this strategy, we chose the initial point as the last generated nondominated point. Under this strategy, we measure the performance of Algorithm 3 to solve the problem (49) by taking* $(0, 0, 15)$ *as an initial point for first* $\hat{\beta}$. *Thereafter, for the next* $\hat{\beta}$, *the initial point is taken as the generated nondominated point by taking* $(0, 0, 15)$ *as the initial point (see Fig. 3(a)).*

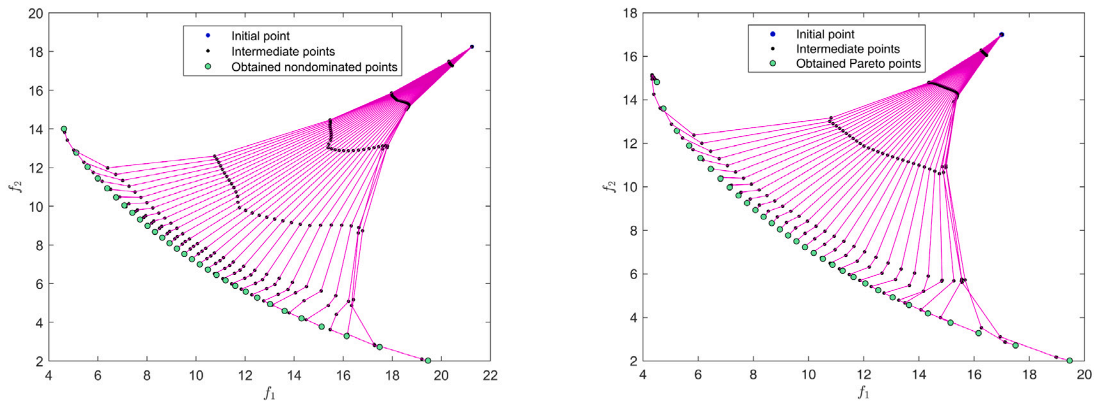
Strategy 3. *In this strategy, we employ random initial points. Under this strategy, we solve the problem (49) by taking 30 random initial points from the set* $[-5, 5] \times [-5, 5] \times [10, 20]$ *for every* $\hat{\beta}$ *(see Fig. 3(b)).*

From these three strategies, we experienced that Strategy 2 is expected to take less iterations and time than the other two strategies to generate the nondominated set.

Next, we provide a list of λ-values taken up by Algorithm 3 (in line 12) for a nonconvex problem. We consider a typical tri-objective optimization problem (comet problem Khorram et al., 2014) which is nonconvex in nature and which is difficult to be tackled by state-of-the-art solvers. The mathematical formulation of the comet problem is as follows:

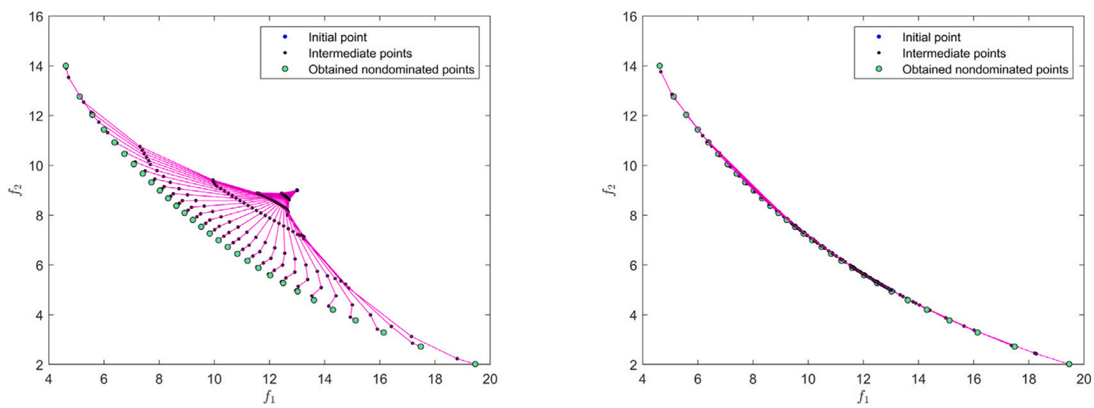
$$\begin{aligned} &\text{minimize} \quad \left(f_1(x), \quad f_2(x), \quad f_3(x)\right)^\top \\ &\text{subject to} \quad 1 \leq x_1 \leq 3.5 \\ &\qquad\qquad -2 \leq x_2 \leq 2 \\ &\qquad\qquad 0 \leq x_3 \leq 1, \end{aligned} \tag{50}$$

where $f_1(x) = (1 + x_3)(x_1^3 x_2^2 - 10x_1 - 4x_2)$, $f_2(x) = (1 + x_3)(x_1^3 x_2^2 - 10x_1 + 4x_2)$ and $f_3(x) = 3(1 + x_3)x_2^2$. The nondominated surface of the problem (50) looks like a comet (see Fig. 4(a)). One end of the nondominated set of the comet problem is a wide-spreaded region which continuously reduces to a thinner region on the other end. Capturing the nondominated solutions simultaneously from both of the



(a) (b)

Fig. 1. Obtained nondominated points by taking initial points $(1.5, 1, 15)$ and $(1, 1, 15)$, respectively.



(a) (b)

Fig. 2. Obtained nondominated points by taking initial points $(0, 0, 15)$ and $(-1, -1, 15)$, respectively.

Table 1
Performance of Algorithm 3 on problem (48) by choosing different strategies of initial points.

	Initial point ($x^{(0)}$)	$(f_1(x^{(0)}), f_2(x^{(0)}))$	Total iterations	Total time taken (in seconds)
Strategy 1	$(1.5, 1, 15)$	$(21.25, 18.25)$	333	488.83
	$(1, 1, 15)$	$(17, 17)$	327	441.55
	$(0, 0, 15)$	$(13, 9)$	300	401.62
	$(-1, -1, 15)$	$(13, 5)$	254	356.65
Strategy 2	$(0, 0, 15)$	$(13, 9)$	251	352.36
Strategy 3	random		307	529.90

wide-spreaded and thin areas is a challenging task (see Fig. 4(b)). The generated nondominated set of the comet problem by Algorithm 3 is depicted in Fig. 4(a). As the comet problem is a nonconvex problem, the Hessian matrix is not positive definite across the iterations of Algorithm 3. However, Algorithm 3 takes a positive definite approximation of the Hessian by adding λI with it (see the line 12). In Table 2, we provide the value of λ that is computed by Algorithm 2 (by taking $\zeta = 0.95$) with the precision parameter $\epsilon = 10^{-5}$ for an exemplary direction $\hat{\beta} = (0.7071, 0.5721, 0.4156)$, the value of ε and the value of $\nu(\Omega)$ (defined in (24)). From Table 2 we see that as the number of iterations progresses, the value of $\nu(\Omega)$ reduces and becomes smaller than 10^{-5}, i.e., a point on the front has been found.

Table 2
Description of the λ values of the comet problem for $\hat{\beta} = (0.7071, 0.5721, 0.4156)$.

Number of iterations (k)	Value of λ (by Algorithm 2)	Value of ε	$\nu(\Omega^{(k+1)})$
0	0.7578	$2.341E-4$	5997.96
1	0.9389	$1.543E-8$	5997.33
2	0.1497	$5.410E-1$	62.04
3	0.0641	$1.321E-1$	0.9666
4	0.2725	$8.321E-8$	0.9324
5	0.3585	$7.003E-10$	0.8797
6	0.3434	$6.543E-8$	0.0022
7	0.0224	$3.785E-9$	0.000006

6. Computational tests

In this section, a MATLAB implementation of Algorithm 3 is tested on various test problems (listed in Table 4) from the literature. We solve some widely used multiobjective constrained test problems (see Table 3) and unconstrained test problems (See Table 4) to test the performance of Algorithm 3. The test problem Kita in Table 3 is a

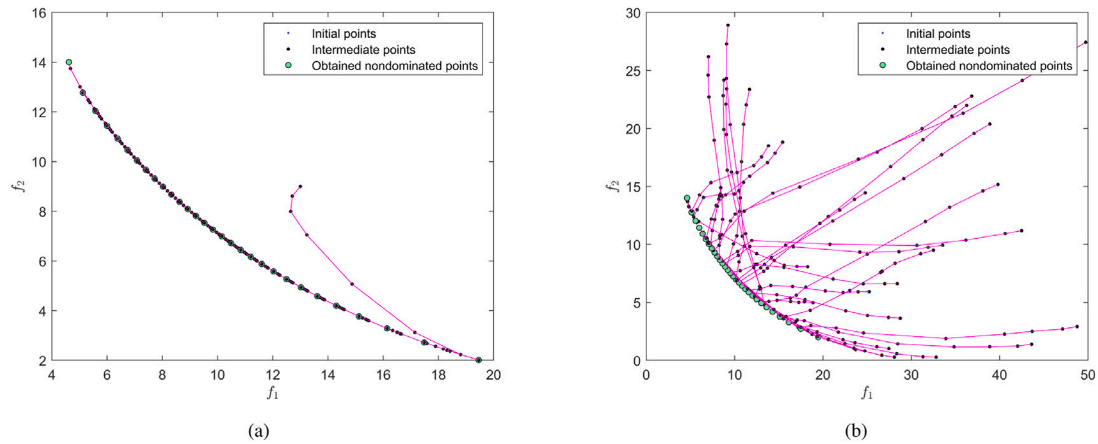


Fig. 3. Obtained nondominated points by taking initial points according to Strategy 2 and 3, respectively.

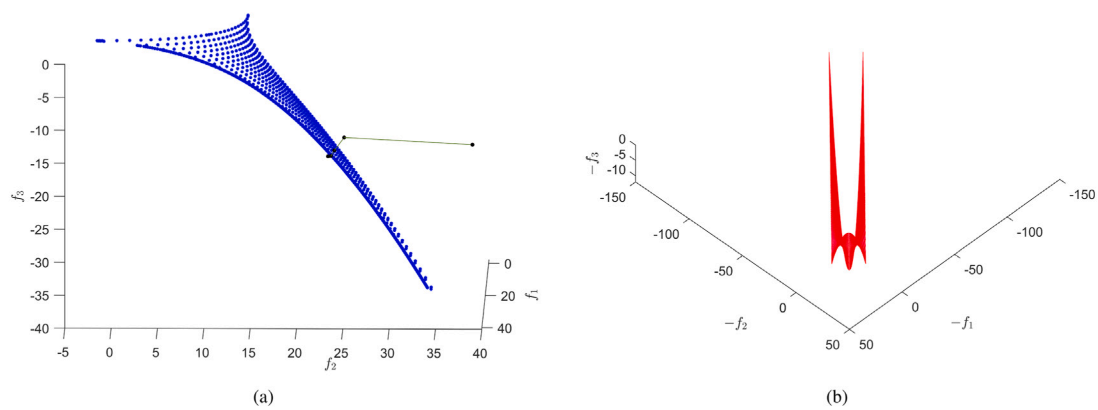


Fig. 4. Obtained nondominated points (left) of the comet problem by Algorithm 3 and the feasible region in the objective space (right).

Table 3
Constrained test problems used in this study.

Problem	n	s	Nondominated set type	Number of $\hat{\beta}$'s	Average iteration	Nondominated set
BNH (Binh and Korn, 1997)	2	2	convex	200	8.6	Not known a priori
TNK (Tanaka et al., 1995)	2	2	nonconvex and disconnected	100	9.8	Not known a priori
SRN (Srinivas and Deb, 1994)	2	2	convex	75	8.1	Not known a priori
Kita (Deb, 2011)	2	3	nonconvex	150	10.2	Not known a priori

maximization problem and remaining are the minimization problems. The nondominated sets of the constrained test problems in Table 3 are priorly unknown, i.e., the closed form of nondominated sets are unknown and the nondominated sets of unconstrained test problems are priorly known to us. The generated nondominated points of these problems together with feasible objective space are shown in Figs. 5 and 6. For unconstrained test problems, the generated nondominated set along with reduction in infeasibility for a fixed $\hat{\beta}$, are shown in Figs. 7–14.

For each test problem, we use $\xi_1 = 1$ and $\xi_2 = 1$ to initiate the values of slack variables v and w (see (23)). Also, we use $\epsilon = 10^{-6}$ as the precision value. The results in this section are obtained by taking $r = 0.1$, $\kappa = 0.5$ and $\delta = 0.95$.

6.1. Performance metrics and relative efficiency

To check the performance of Algorithm 3, two performance metrics, namely Inverted Generational Distance (IGD) and Hyper-Volume (HV) (Sierra and Coello, 2005) are used to assess the quality of the generated solution set in terms of optimality and diversity. For the IGD value, the smaller, the better. For the HV value, the larger, the better.

Table 4 reports the considered test problems and average number of iterations (per one $\hat{\beta}$) taken by Algorithm 3 to generate a nondominated point.

In Table 4, the parameter n is the number of decision variables and s is the number of objectives. In order to calculate the performance metrices for Algorithm 3, we use the same initial point for different $\hat{\beta}$'s to solve all the parametrized problems.

For ZDT and DTLZ test suits (listed in Table 4), the median of the IGD values and HV values of the generated solution sets by Algorithm 3 and other existing efficient solvers are reported in Tables 5 and 6, respectively. From Table 5, we see that the IIPM (CM) has least values for all the test problems. From Table 6, we observe that the IIPM (CM) has the largest values for all the test problems. Hence, the proposed method outperforms the existing efficient methods.

Based on the result reported in Table 5, we evaluate the relative efficiency (Yuan and Lu, 2009) of the proposed method. It is computed as follows: for the ith test problem, we compute the median of IGD values by the jth solver and denote it by MIGD(i, j). Then, we calculate the ratio

$$r(i, j) = \frac{\text{MIGD}(i, j)}{\text{MIGD}(i, \text{IIPM(CM)})}.$$

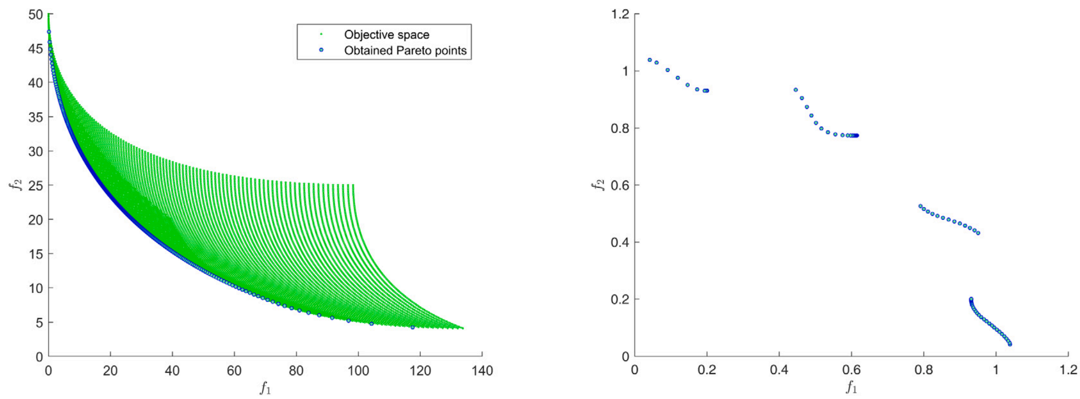


Fig. 5. Obtained nondominated points of BNH and TNK problems by Algorithm 3.

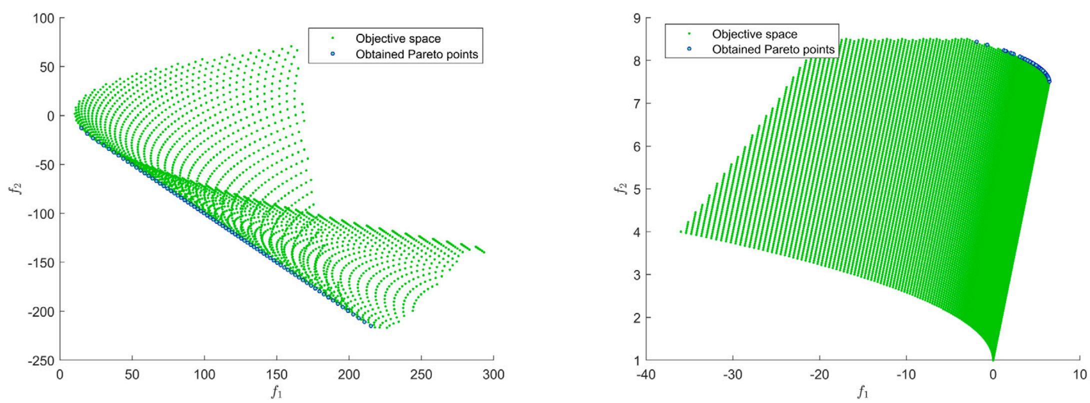


Fig. 6. Obtained nondominated points of SRN and Kita problems by Algorithm 3.

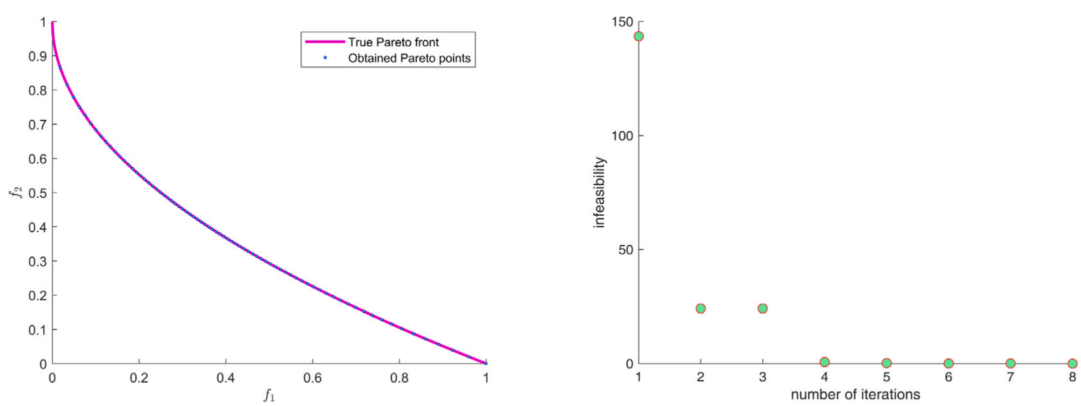


Fig. 7. Obtained nondominated points of ZDT1 by Algorithm 3, and the reduction of infeasibility for $\hat{\beta} = (0.978, 0.207)^{\top}$.

Table 4
Data for the test problems that are used to check performance of Algorithm 3.

Problem	n	s	Average iteration	No. of $\hat{\beta}$	Nondominated set type	Nondominated set	Source
ZDT1	10	2	8.1	75	Convex	A priori known	Zitzler et al. (2000)
ZDT2	10	2	13.4	75	Nonconvex	A priori known	Zitzler et al. (2000)
ZDT3	10	2	16.4	75	Nonconvex and disconnected	A priori known	Zitzler et al. (2000)
ZDT4	10	2	14.4	75	Convex	A priori known	Zitzler et al. (2000)
DTLZ1	5	3	33.2	359	Convex	A priori known	Deb et al. (2005)
DTLZ2	5	3	6.9	285	Nonconvex	A priori known	Deb et al. (2005)
DTLZ3	5	3	9.9	285	Nonconvex	A priori known	Deb et al. (2005)
DTLZ5	5	3	12.7	100	Nonconvex	A priori known	Deb et al. (2005)

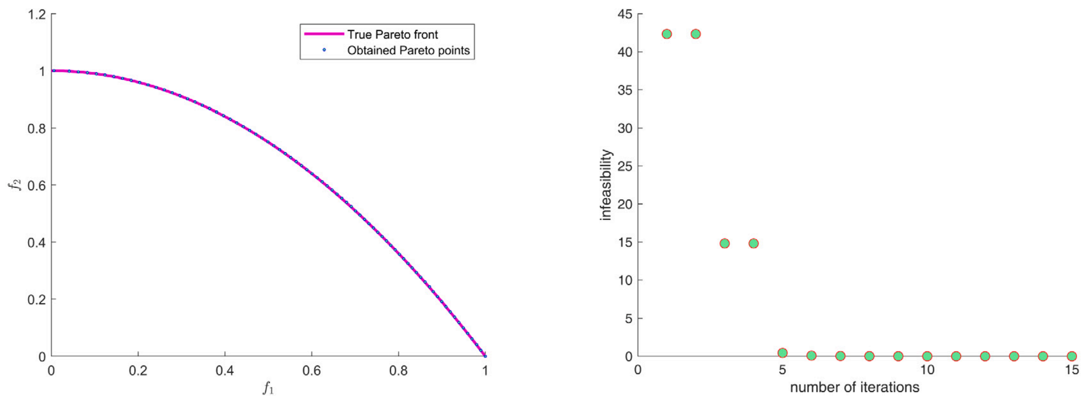


Fig. 8. Obtained nondominated points of ZDT2 by Algorithm 3, and the reduction of infeasibility for $\hat{\beta} = (0.923, 0.382)^{\top}$.

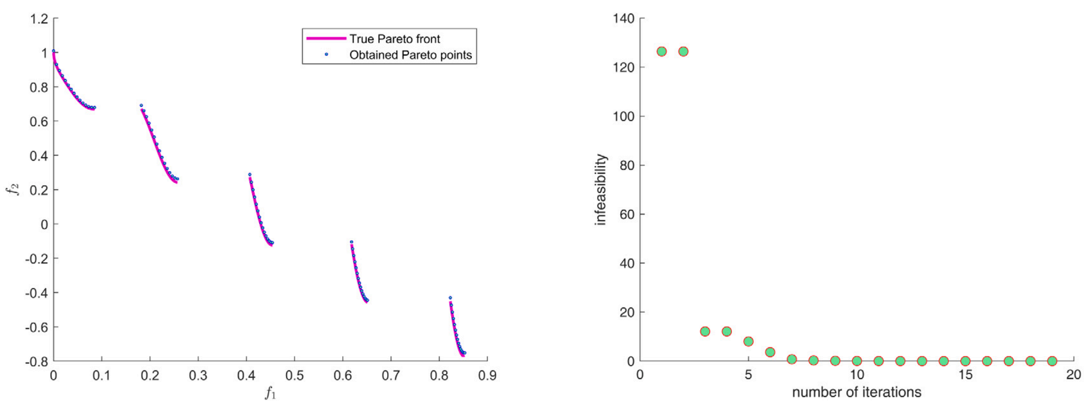


Fig. 9. Obtained nondominated points of ZDT3 by Algorithm 3, and the reduction of infeasibility for $\hat{\beta} = (0.707, 0.707)^{\top}$.

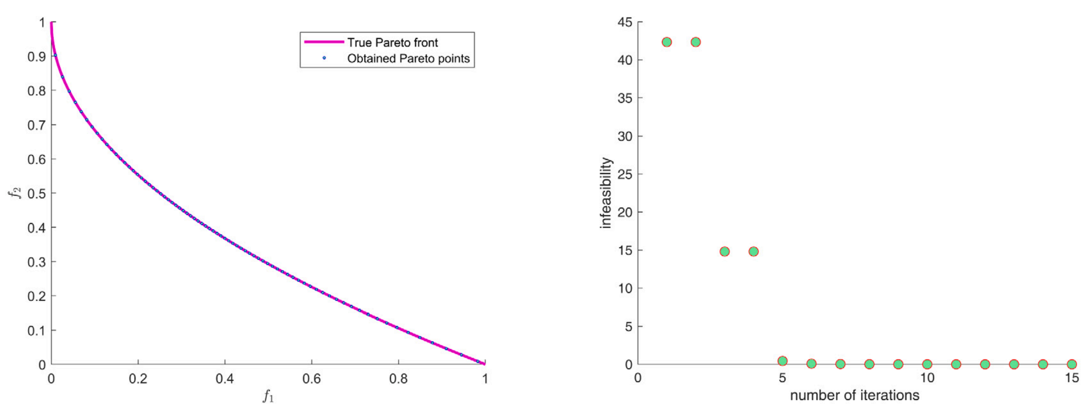


Fig. 10. Obtained nondominated points of ZDT4 by Algorithm 3, and the reduction of infeasibility for $\hat{\beta} = (0.341, 0.940)^{\top}$.

Table 5
Median of IGD values for the ZDT and DTLZ benchmark suite obtained by different algorithms.

Problem	MOEA/D (WS) (Siwei et al., 2011)	MOEA/D (TE) (Siwei et al., 2011)	MOEA/D (PBI) (Siwei et al., 2011)	NSGA-II (Siwei et al., 2011)	$pa\lambda$-MOEA/D (Siwei et al., 2011)	IIPM (CM)
ZDT1	$5.42E-4$	$6.84E-4$	$1.14E-3$	$7.94E-4$	$5.80E-4$	$1.14E-4$
ZDT2	$1.30E-2$	$5.84E-4$	$7.03E-4$	$8.16E-4$	$6.02E-4$	$2.95E-6$
ZDT3	$4.93E-3$	$2.01E-3$	$2.06E-3$	$1.20E-3$	$1.97E-3$	$2.04E-4$
ZDT4	$7.02E-3$	$6.54E-4$	$7.86E-4$	$8.14E-4$	$5.94E-4$	$1.98E-7$
DTLZ1	$4.03E-3$	$6.93E-4$	$4.21E-4$	$7.91E-4$	$4.52E-4$	$4.74E-6$
DTLZ2	$5.35E-3$	$7.42E-4$	$6.15E-4$	$7.58E-4$	$5.78E-4$	$1.43E-4$
DTLZ3	$1.42E-2$	$1.24E-3$	$1.90E-3$	$3.59E-3$	$1.18E-3$	$1.27E-4$
DTLZ5	$1.38E-3$	$5.22E-5$	$1.11E-4$	$1.89E-5$	$2.79E-5$	$3.78E-5$

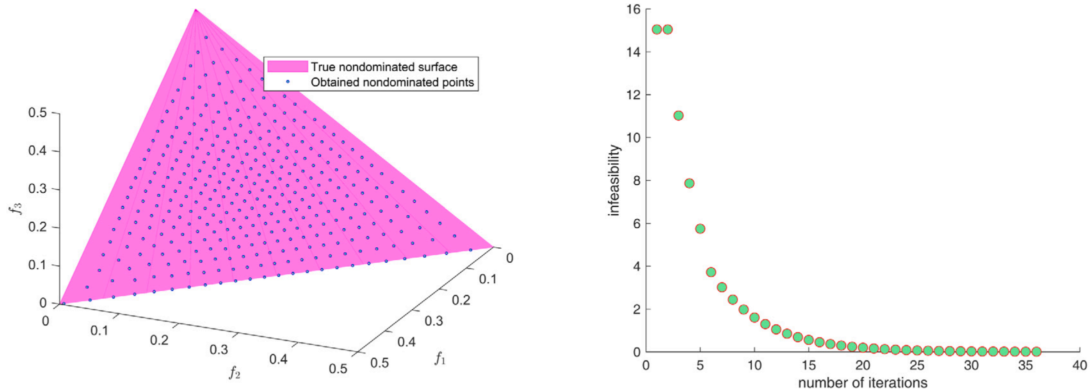


Fig. 11. Obtained nondominated set of DTLZ1 by Algorithm 3, and the reduction of infeasibility for $\hat{\beta} = (0.877, 0.421, 0.229)^{\top}$.

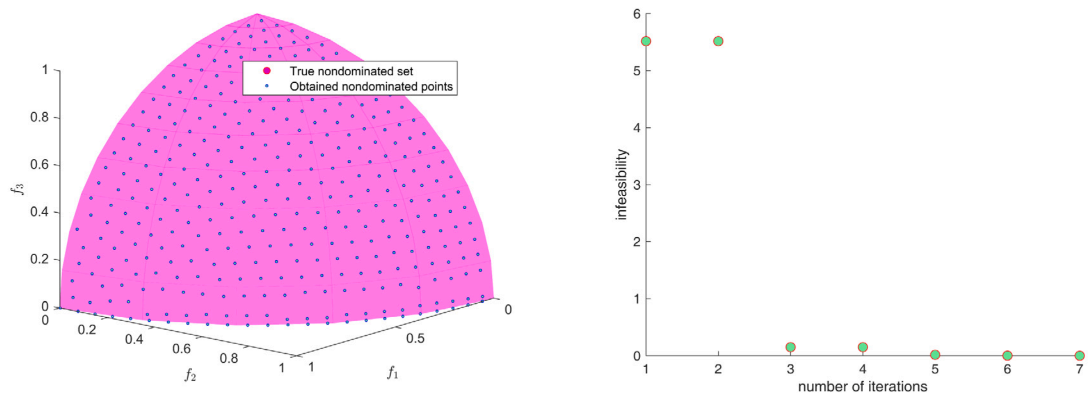


Fig. 12. Obtained nondominated set of DTLZ2 by Algorithm 3, and the reduction of infeasibility for $\hat{\beta} = (1, 0)^{\top}$.

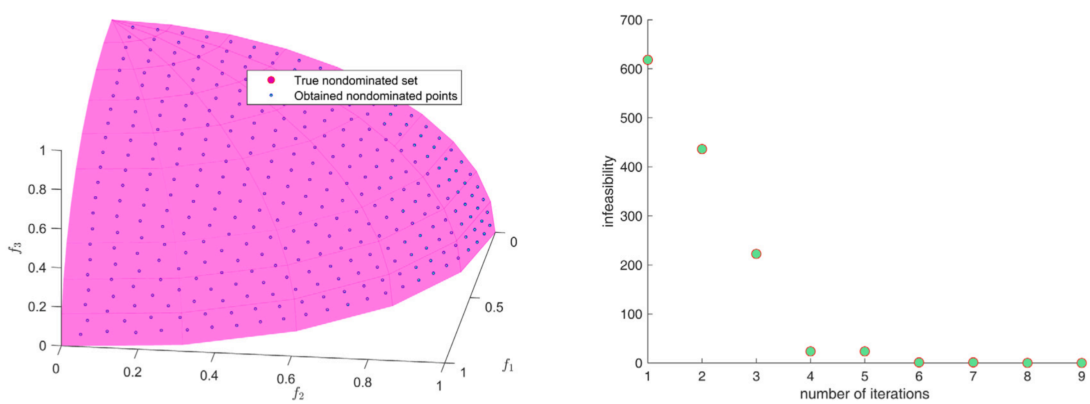


Fig. 13. Obtained nondominated set of DTLZ3 by Algorithm 3, and the reduction of infeasibility for $\hat{\beta} = (1, 0, 0)^{\top}$.

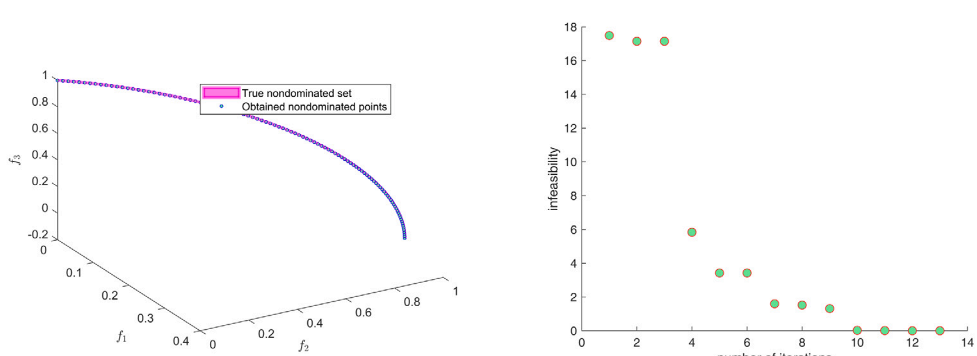


Fig. 14. Obtained nondominated set of DTLZ5 by Algorithm 3, and the reduction of infeasibility for $\hat{\beta} = (0.809, 0.558, 0.181)^{\top}$.

Table 6
Median of HV values for the ZDT and DTLZ benchmark suite obtained by different algorithms.

Problem	MOEA/D (WS) (Siwei et al., 2011)	MOEA/D (TE) (Siwei et al., 2011)	MOEA/D (PBI) (Siwei et al., 2011)	NSGA-II (Siwei et al., 2011)	$pa\lambda$-MOEA/D (Siwei et al., 2011)	IIPM (CM)
ZDT1	0.6521	0.6392	0.6057	0.6381	0.6412	0.6585
ZDT2	0.0000	0.3097	0.2957	0.3060	0.3107	0.3218
ZDT3	0.4863	0.4807	0.4642	0.5066	0.4873	0.5101
ZDT4	0.3534	0.6360	0.6244	0.6359	0.6391	0.6593
DTLZ1	0.2259	0.7434	0.7835	0.7262	0.7814	0.9119
DTLZ2	0.0000	0.3777	0.3812	0.3766	0.4074	0.4562
DTLZ3	0.0000	0.3605	0.2633	0.1901	0.3817	0.4002
DTLZ5	0.0000	0.0894	0.0779	0.0930	0.0916	0.2010

Table 7
Relative efficiency of MOEA/D(WS), MOEA/D(TE), MOEA/D(PBI), NSGA-II, $pa\lambda$-MOEA/D and IIPM (CM) methods with respect to median of IGD values.

MOEA/D(WS)	MOEA/D(TE)	MOEA/D(PBI)	NSGA-II	$pa\lambda$-MOEA/D	IIPM (CM)
148.228	28.156	33.542	29.555	23.041	1

Table 8
Relative efficiency of MOEA/D(WS), MOEA/D(TE), MOEA/D(PBI), NSGA-II, $pa\lambda$-MOEA/D and IIPM (CM) methods with respect to median of HV values.

MOEA/D(WS),	MOEA/D(TE)	MOEA/D(PBI)	NSGA-II	$pa\lambda$-MOEA/D	IIPM (CM)
0	0.831	0.776	0.772	0.855	1

Geometric mean of these ratios for the jth solver over all the test problems is defined by

$$r(j) = \left(\prod_{i \in P} r(i,j) \right)^{\frac{1}{|P|}}, \tag{51}$$

which is referred to as *relative efficiency*, where P denotes the set of the test problems and $|P|$ is the cardinality of P.

The values of r(MOEA/D(WS)), r(MOEA/D(TE)), r(MOEA/D(PBI)), r(NSGA-II) and r($pa\lambda$-MOEA/D) are mentioned in Table 7. Similarly, based on the median HV values reported in Table 6, we calculate the relative efficiency with respect to median of HV values, which is mentioned in Table 8.

From Tables 7 and 8 for relative efficiencies, we notice that IIPM (CM) has the least relative efficiency with respect to IGD values and has the greatest relative efficiency with respect to HV values. Hence, the proposed IIPM (CM) outperforms the existing efficient techniques.

7. Conclusion

This paper has introduced an infeasible interior-point approach to find a discrete subset of nondominated points of an MOP with the help of the cone method. Towards the derivation, a log-barrier problem corresponding to each parametrized problem of the cone method has been formulated and solved by the Newton method. To find a solution to the Newton system, two merit functions are used in the proposed algorithms: one is to find an appropriate step length ($\psi_{\eta,\mu}$) and another for the stopping criterion ($\nu(\Omega)$).

Theorem 2 has shown that the search directions (15) computed by Theorem 1 are descent for the merit function $\psi_{\eta,\mu}$. Reductions in primal infeasibilities ($\varrho_{\hat{\beta}}$, ρ), dual infeasibility ($\sigma_{\hat{\beta}}$) and complementarity (γ) after one iteration are shown in Theorem 3. Theorem 4 extends the conclusion of Theorem 3 for the kth iteration and provides the total number of iterations needed to obtain an ϵ-precise solution.

It is important to notice that the proposed infeasible interior-point method is capable of handling convex (Algorithm 1) and nonconvex multi-objective optimization (Algorithm 3). Whenever the problem is not convex, a diagonal perturbation to the Hessian matrix has been applied to make the search directions descent. Consequently, the reductions in primal infeasibilities ($\varrho_{\hat{\beta}}$, ρ), dual infeasibility ($\sigma_{\hat{\beta}}$) and complementarity (γ) have been shown in Theorem 5.

7.1. Pros and cons of the proposed method

In this section, we discuss some merits and demerits of the proposed algorithm. These are the following advantages of the proposed algorithm:

(i) The construction of the proposed method (Algorithm 3) is such that it can generate the convex as well as nonconvex parts of the nondominated set.
(ii) The method does not require prior knowledge about the location of the nondominated set.
(iii) Algorithm 3 is capable to solve the convex and nonconvex MOPs.
(iv) By an appropriate strategy to choose the initial point (as discussed in Strategy 1, 2 and 3), one can save the computational time by not calculating many dominated points before converging to an efficient point corresponding to a given $\hat{\beta}$ (see Figs. 2(b) and 3(a)).
(v) According to the derived theoretical results and our computational experience with the test problems, Algorithm 3 ends up generating a nondominated point corresponding to each $\hat{\beta}$ "starting with any initial point (feasible or *infeasible*)".

The proposed algorithm has the following limitations:

(i) The method requires that the minimum of each objective function must exist. If it is not the case then the method will not be applicable.
(ii) The choice of uniformly distributed $\hat{\beta}$ (see (3)) does not guarantee that the obtained nondominated points by Algorithm 3 will be uniformly distributed over the true nondominated surface.
(iii) The method is based on the formulation of Newton method and therefore the objective functions and constraints have been considered twice continuously differentiable. When the objective functions and constraints do not fulfill this requirement, the proposed algorithm will not be applicable.
(iv) Although theoretical results (Theorem 4) convey that any choice of ζ in $(0,1)$ will provide global convergence of the proposed method and does not influence the number of iterations to reach at an ϵ-precise solution, the authors have experienced during the numerical computations that a moderate value (around 0.7) of the reduction factor ζ (see Note 2) can influence the number of required iterations for convergence and thereby influencing the speed of the proposed method to reach the optimality.

7.2. Future work

- Numerical results of the test problems (mentioned in Mittelmann, 2021) demonstrate that the merit function $\psi_{\eta,\mu}$ fails to reduce significantly when the step length is small along the search direction (see Vanderbei, 1994). However, this behavior is not observed with the test problems discussed in Section 6. Nevertheless, for the algorithm's betterment, a new merit function with better convergence results can be implemented in future.
- Since interior-point methods are known to work well in higher dimensions and with structured or sparse problems, our future work will aim to rigorously investigate if the proposed IIPM (CM) will work well for high-dimension and sparse multi-objective optimization problems.
- To extend the efficiency of the proposed algorithms, in future, we will consider nonsmooth MOPs with a nonsmooth merit function and apply inexact Newton methods.

CRediT authorship contribution statement

Jauny: Study was designed, Writing of the original manuscript, Implementing the method, Carrying out the numerical experiments. **Debdas Ghosh:** Study was designed, Writing of the original manuscript, Providing ideas for the mathematical proofs, Implementing the method, Carrying out the numerical experiments. **Qamrul Hasan Ansari:** Providing ideas for the mathematical proofs, Proofreading the manuscript, Discussion of the design of the manuscript. **Matthias Ehrgott:** Providing ideas for the mathematical proofs, Proofreading the manuscript, Discussion of the design of the manuscript. **Ashutosh Upadhayay:** Carrying out the numerical experiments, Proofreading the manuscript.

Data availability

Data will be made available on request.

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