Chapter 1

Introduction

1.1 Background

1.1.1 Linear and Nonlinear Waves

Our lives are surrounded by various natural phenomena such as physical processes, sunrise, fog, thunder, weather, erosion, wave propagation, earthquakes, volcanic eruptions, and many more. Most of these phenomena are non-linear and are represented by non-linear partial differential equations (PDEs). So, the comprehension of the non-linearity is very essential to understand our surroundings. Non-linear models arising in the real world often face serious mathematical difficulties related to the occurrence of discontinuities, singularities, the resonance between wave speeds, etc. From last few decades, there have been significant rise in research and development activities focused on non-linear waves. There is huge difference between ”linear” phenomena of sound, light, or electromagnetic signals to the violent disturbances resulting from explosive detonation, supersonic projectile flights or rocket nozzle flows. Its evolution is governed by non-linear differential equations, and as a result, the familiar laws of superposition, reflection, and refraction no longer apply; instead, more novel features emerge, among which the shock fronts occurrence is most evident.

A shock wave can be defined as a region of small thickness propagating in gas across which flow variables change adruptly. The thickness of shock wave region is very small and so we may consider the shock wave as a surface of discontinuity. Shock waves can form through the steepening of ordinary waves. For example, ocean waves break on the shore, and during supersonic explosions caused by heavy explosives like TNT, known as detonations. They also occur during the gravitational collapse of massive stars, leading to the formation of black holes, neutron stars and supernovae. In fluid dynamics, shock wave can form during cavitation, created by a rapidly moving propeller, collapse suddenly due to the surrounding liquid pressure. One of the key characteristic of shock waves is the ability to accelerate particles at the shock front. Depending on their intensity, shock waves are categorized into weak shocks (shocks of small amplitude) and strong shocks (shocks of large amplitude).

These categories are further defined by shock strength $k = \frac{p}{p_0}$ where p and p_0 are the pressures just behind and ahead of the shock, respectively. For weak shocks, k is slightly greater than one, causing them to move at nearly the speed of sound. In contrast, for the strong shocks, k is much greater than one, resulting in supersonic speed. Unlike acoustic waves, shock waves have a finite amplitude and exhibit several unusal features. When analyzed in a coordinate system moving with the shock, the flow is always subsonic behind the shock and supersonic ahead of it. The supersonic velocity of the flow is highly dependent on the pressure. Across the shock, there is an abrupt change in the flow variables such as density, velocity, pressure, temperature along with the formation of steep wavefront. For non-planar shocks, such as spherical blast waves, the flow velocity decreases significantly as the distance from the centre increases. This decrease occurs because some of the shock wave's energy is used to raise the temperature of the medium in which it travels. The real exmaples of the existence of the shock wave on the earth are the Tunguska event in 1908, the 2013 Rusian meteor event, both caused by meteors. In hyperbolic quasi linear PDEs, classical solutions cannot be expected globally. The solution may become multivalued in finite time due to gradient blow-up, which is physically unrealistic. So, a discontinuity is introduced in the solution using Rankine-Hugoniot jump conditions, ensuring the solution remains single valued.

The study and analysis of nonlinear wave motion have long been of significant importance in both mathematics and physics. Over a century ago, pioneering work by mathematicians and physicists such as Stokes, Earnshaw, Riemann, Rankine, Hugoniot, Lord Rayleigh, and later contributors like Hadamard, Von Neumann, Courant, Friedrichs, G. B. Whitham, and others laid the fundamental concepts and wrote research papers and books that initiated this field. In recent years, there has been a renewed focus on nonlinear wave motion, particularly regarding phenomena such as shock waves and expansion waves. The most familiar concept of a wave pertains to the transmission of a disturbance or variation that progressively conveys energy from one point to another within a medium. This disturbance may manifest as elastic deformation or variations in pressure, electric or magnetic intensity, electric potential, or temperature. Such waves are intrinsically linked to movement through the spatial domain $\mathbb{R}^n$ and the time t. Consequently, it is essential to differentiate the time variable from other independent variables. In the context of linear wave motion, such as sound transmission, disturbances propagate at a specific speed relative to the medium. This is a local characteristic of the medium and remains constant for all possible linear wave motions within that medium. Conversely, in nonlinear wave motion, the notion of sound speed is pivotal. Small disturbances, or wavelets, that slightly alter a primary wave motion propagate at a certain speed, also referred to as the sound speed. In nonlinear wave motion, this speed is dependent not only on the position within the medium but also on the medium's state as affected by the primary wave motion. A key aspect of nonlinear waves is the handling of disturbances or discontinuities, which may not necessarily be small.

Waves are present in various scientific and engineering fields, including fluid mechanics, optics, electromagnetism, solid mechanics, structural mechanics, and quantum mechanics. These waves are characterized by solutions to either linear or nonlinear partial differential equations. For homogeneous linear PDEs, the principle of superposition is applicable: a linear combination of solutions yields another solution. This characteristic indicates that the solution space of homogeneous linear PDEs constitutes a vector space. The linear framework of this space can be utilized to develop solutions with particular attributes that meet a range of boundary and initial conditions. However, for nonlinear PDEs, the principle of superposition does not hold. This makes it impossible to combine solutions straightforward as is done with linear PDEs. The focus of the present work is on solutions to problems involving hyperbolic systems of PDEs in gas dynamics. Specifically, it examines nonlinear hyperbolic waves and the discontinuities that arise during their propagation. Under certain conditions, these waves can be represented by a quasilinear system of first order equations, which are linear in the first derivatives of the dependent variables, but their coefficients may depend on the dependent variables themselves. Notably, when viscosity and heat conduction are neglected, these equations simplify to a hyperbolic system known as Euler's equations.

1.1.2 Hyperbolic system of PDEs

Hyperbolic partial differential equations serve as fundamental models in numerous applications, particularly within various domains of fluid dynamics involving conservation laws. The one-dimensional hyperbolic system of first-order PDEs covers a wide range of scientific and technological fields. Notably, it finds extensive applications in gas dynamics, fluid dynamics, aerodynamics, multiphase flows, astrophysics, and plasma physics. A key characteristics of the quasilinear hyperbolic system of PDEs is that smooth solutions break down after a finite time. This breakdown leads to one of the most interesting nonlinear phenomena in nature: the formation of shocks, characterized by abrupt changes in density, pressure, and velocity. Another interesting aspect of quasilinear hyperbolic systems is the interaction of nonlinear waves. For a detailed exploration of mathematical properties and applications of nonlinear wave propagation problems witnin the context of hyperbolic systems of PDEs, one can refer to the works of Courant and Friedrichs [3] , Lax [4], Jeffrey [5], Bressan [6], Dafermos [7], Sharma [8] and Smoller [9] etc.

In order to present the mathematical description, let us consider first order partial differential equation of the form

$$U_{i,t} + \sum_{j=1}^{k} A_{ij}(x,t,U_i)U_{i,x} + B_i(x,t,U_i) = 0 \tag{1.1}$$

for $i = 1, ..., k$. This represents a system of k equations involving k unknowns U_i, which are functions of the spatial variable x and time variable t. The variables U_i are dependent, while x and t serve as independent variables, denoted as $U_i = U_i(x,t)$. The notation $U_{i,t}$ indicates the partial derivative of U_i with respect to t, whereas $U_{i,x}$ represents the partial derivative with respect to x. This system (1.1) can be represented in matrix form as follows:

$$\hat{U}_t + \hat{A}\hat{U}_x + \hat{B} = 0, \tag{1.2}$$

where

$$\hat{U} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_k \end{bmatrix}, \hat{B} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_k \end{bmatrix}, \hat{A} = \begin{bmatrix} a_{11} & . & . & a_{1k} \\ a_{21} & . & . & a_{2k} \\ . & . & . & . \\ . & . & . & . \\ a_{k1} & . & . & a_{kk} \end{bmatrix}.$$

If the elements a_{ij} of the matrix $\hat{A}$ are constant and the components b_j of the vector $\hat{B}$ are also constant, then system described in equation (1.2) is linear with constant coefficients. Conversely, if $a_{ij} = a_{ij}(x,t)$ and $b_i = b_i(x,t)$, the system transitions to being linear with variable coefficients. The system retains its linearity if $\hat{B}$ is linearly dependent on $\hat{U}$ and is classified as quasi–linear if the coefficient matrix $\hat{A}$ is a function of the vector $\hat{U}$, expressed as $\hat{A}{=}\hat{A}(\hat{U})$. It is important to note that quasi–linear systems are typically systems of non–linear equations. In the context of equation (1.2), the condition $\hat{B} = 0$ signifies a homogeneous system.

Definition 1.1.1. (**Hyperbolic system**) The system (1.2) is referred to as a first-order hyperbolic system of partial differential equations if the matrix $\hat{A}(\hat{U})$ possesses k real eigenvalues

$$\lambda_1(\hat{U}) \leq \lambda_2(\hat{U}) \leq \lambda_3(\hat{U}) \leq ....., \leq \lambda_k(\hat{U}),$$

along with set of k linearly independent eigenvectors J_1, J_2, J_3.....,J_k. These eigenvalues are also known as characteristic speeds or wave speeds associated with (1.2).

The system is termed as strictly hyperbolic if its eigenvalues are distinct:

$$\lambda_1(\hat{U}) < \lambda_2(\hat{U}) < \lambda_3(\hat{U}) < ....., < \lambda_k(\hat{U}).$$

We present the concepts of linearity and nonlinearity for each j-wave family.

Definition 1.1.2. For every pair of indices i and j where i, j ranges from 1 to k, we define the j-characteristic field associated with the j-characteristic of (1.2) is **genuinely nonlinear** if

$$\nabla \lambda_j(\hat{U}).L_j(\hat{U}) \neq 0,$$

and **linearly degenerate** if

$$\nabla \lambda_j(\hat{U}).L_j(\hat{U}) = 0,$$

where $\nabla = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, ...\frac{\partial}{\partial u_k}\right)$.

1.1.3 Non-ideal Gases

The ideal gas law is represented by the equation

$$PV = nRT,$$

where n denotes the number of gas molecules, R signifies the gas constant, T represents the absolute temperature, P indicates the pressure, and V stands for the volume of the gas. This equation effectively characterizes the behavior of most gases under low-density conditions, where the molecules are typically spaced apart.

For real gases, the equation of state is given by $\lim_{P \to 0} PV/RT = 1$ with compressibility factor Z defined as

$$Z(P,T) = PV/RT.$$

The deviation of Z from unity indicates the degree of departure from ideal gas behavior. To theoretically derive the ideal gas law, two key assumptions are made: the gas molecules are negligibly small (having no volume) and there are no interactions between the molecules.

When temperatures are extremely high and densities are significantly low, the initial assumptions become invalid, rendering the ideal gas law ineffective. An established alternative is the van der Waals model, formulated by the Dutch physicist van der Waals, which is represented by the van der Waals equation of state:

$$\left(P + \frac{na}{V^2}\right)(V - nb) = nRT,$$

where a and b are constants determined through experimentation, and n denotes the number of gas molecules. The van der Waals model provides a close approximation of the behavior of real gases across a broad spectrum of temperatures and pressures.

1.1.4 Dusty Gas

The study of a two-phase flows involving gas and dust particles becomes important to understand various engineering problems. These flows, where gas carries a significant amount of solid particles, exhibit unique behavior due to which rapid changes in gas velocity and temperature effect the overall flow properties. When the mass concentration of the solid particles and gas are comparable then the flow properties diverge significantly from pure gas flow. In this scenario, we consider a combination of a perfect gas and numerous small, uniformly spherical dust particles.

The dusty gas represents a blend of gaseous components and minute solid dust particles, with the latter making up less than five percent of the overall volume [10]. At high speed of fluid, these small solid particles behave as a pseudo fluid [11]. We conceptualize the mixture as consisting of two distinct fluids: one being the gas and the other the pseudo fluid generated by the solid particles. The solid particles are uniform spheres characterized by identical mass m_{sp}, radius r_{sp}, and specific heat c_{sp}. We define an element of the gas-solid mixture with a total mass $M = M_g + M_{sp}$ and a total volume $V = V_g + V_{sp}$, where the subscript g refers to gas properties and the subscript sp pertains to the solid particle properties. The volume occupied by the solid particles within the mixture is expressed as follows:

$$V_{sp} = n_{sp}.V.\tau_{sp},$$

where τ_{sp} is the volume of solid particle and n_{sp} is the number of solid dust particles per unit volume of dusty gas. The mass of solid particles in the volume V of the mixture is expressed as:

$$M_{sp} = n_{sp}.V.m_{sp}.$$

The species density of the solid particles is defined as:

$$\rho_{sp} = \frac{M_{sp}}{V_{sp}} = \frac{m_{sp}}{\tau_{sp}}.$$

Additionally, the partial density of the pseudo-fluid of solid particles is defined as:

$$\overline{\rho}_{sp} = \frac{M_{sp}}{V_{sp}} = n_{sp}.m_{sp} = Z\rho_{sp} = n_{sp}.\rho_{sp}.\tau_{sp},$$

where Z represents the fraction of volume occupied by the solid particles in the mixture. Furthermore, the volume fraction of solid particles is given as:

$$Z = \frac{V_{sp}}{V} = n_{sp}.\tau_{sp}.$$

The species density of the fluid is defined as:

$$\rho_g = \frac{M_g}{V_g}.$$

Similarly, the partial density of a gas is defined as:

$$\overline{\rho}_g = \frac{M_g}{V} = (1 - Z)\rho_g.$$

Considering the thermodynamic equilibrium condition:

$$T_{sp} = T_g = T.$$

The density of the mixture is then obtained as:

$$\rho = Z\rho_{sp} + (1 - Z)\rho_g = \overline{\rho}_{sp} + \overline{\rho}_g.$$

The mass concentration of the pseudo fluid of the solid particles is given by:

$$k_p = \frac{\overline{\rho}_{sp}}{\rho} = \frac{Z\rho_{sp}}{\rho}.$$

The pressure of the mixture is expressed as:

$$p = p_{sp} + p_g.$$

The total pressure of the mixture p is obtained from the ideal gas law as:

$$p = R\rho_g T_g.$$

Using above analysis, the pressure of the mixture as a whole is:

$$p_m = p = R\rho_g T_g = R\left(\frac{\rho_m - Z\rho_{sp}}{1-Z}\right)T_g = R\rho_m\left(\frac{1-k_{sp}}{1-Z}\right)T.$$

Therefore, $p_m = \frac{\rho_m R_m T}{1-Z}$ where, $R_m = (1-k_p)R$. Here, R may be considered as an effective gas constant of the mixture and the subscript m refers to the value of the gas constant for the mixture as a whole.

1.1.5 Magnetogasdynamics

When a gas that conducts electricity moves through a magnetic field, it produces an electric field that induces currents within the gas. The magnetic field applies a force on these currents, which can alter the flow characteristics. In turn, these currents can influence the magnetic field itself. Consequently, magnetogasdynamics examines the relationship between the magnetic field and the motion of electrically conducting gases. In numerous cases, the energy present in the electric field is significantly lower than that in the magnetic field, allowing for the electric field's energy to be frequently disregarded in analyses. Thus, the equations governing gas flow are formulated by integrating the field equations with the equations of gas dynamics. The study of shock waves in the presence of magnetic field becomes more complex due to additional interaction between magnetic field and gas. It's presence changes the conditions for shock formation and its structure. The study of shock wave in magnetogasdynamics is crucial for understanding various astrophysical and geophysical phenomena such as supernova explosions, solar flares, geomagnetic storms etc. The interest in issues related to the behavior of high-temperature plasma in the presence of a magnetic field has increased due to the advent of research on thermonuclear fusion reactions([12] and [13]).

The interaction between gas dynamics phenomena and magnetic fields is examined by integrating the field equations with those of gas dynamics. In most electromagnetic problems concerning conductors, Maxwell's displacement currents are often disregarded([14],[15]). The magnetic permeability of the media considered in magnetogasdynamics deviates only marginally from unity, hence unity is assumed in this context.

The field equations are as follows:

$$\nabla \times E = -\frac{1}{c}B, \tag{1.3}$$

$$\nabla \times B = -\frac{4\pi}{c}J = \frac{4\pi}{c}\sigma(E + \frac{u \times B}{c}), \tag{1.4}$$

$$\nabla \cdot B = 0, \tag{1.5}$$

where c denotes the speed of light, u indicates the fluid velocity, B represents the magnetic field, σ signifies the electrical conductivity, E refers to the electric field intensity, and J stands for the current density. Assuming a uniform σ across the medium, substituting equation (1.3) into equation (1.4) results in:

$$B_t - \nabla \times (u \times B) = \frac{c^2\nabla^2 B}{4\pi\sigma}. \tag{1.6}$$

Due to the complexity of analyzing non-linear differential equations, initial research in this domain concentrated on the propagation of gas dynamic shocks and electromagnetic waves. In the study of hydrodynamic shocks, it is presumed that the electrical conductivity of the medium is infinite. This presumption indicates that self-induction will prevent changes to the magnetic field in a stationary medium ([16], [17], [18]). Furthermore, the governing equations converts into a non-convex hyperbolic system, where the characteristic surface may display unexpected singularities, thereby complicating the wave structure more than that of aerodynamic shocks ([19], [20]). Ideal magneto-hydrodynamics have great potential for applications, yet it also presents numerous unresolved questions and uncertainties([21]).

1.1.6 The concept of Self Similarity

The non-linear systems involving discontinuties, such as shocks, do not always permit complete exact solutions, therefore, we must depend on approximate, analytical or numerical methods for our analysis. These approaches can yield valuable insights into the complex physical processes involved.

Fluid motion is classified as one-dimensional when its characteristics are solely dependent on a single geometric coordinate and time. This type of motion is referred to as self-similar when the distributions of flow variables retain their similarity over time, changing only in scale. The principle of self-similar motion holds significant relevance in the field of gas dynamics. In this context, the flow variables are not solely dependent on the coordinates and time independently, but rather on specific combinations of these factors. Dimensional analysis techniques can be employed to derive precise solutions for particular issues related to one-dimensional unsteady motion of a compressible fluid. In the Eulerian framework, the flow variables consist of velocity v, density ρ, and pressure p. The defining parameters includes the linear coordinate r, time t, and constants integral to the equations, along with the boundary and initial conditions relevant to the problems. As the dimensions of the quantities ρ and p involve mass, hence, for at least one constant 'A', which also has mass dimensions, to be included as a characteristic parameter. Thus, we can make this assumption without any loss of generality.

$$dim A = ML^mT^n$$

We can express the velocity, density and pressure as:

$v = \frac{r}{t}V,$ $\rho = \frac{A}{r^{m+3}t^n}D$ $p = \frac{A}{r^{m+1}t^{n+2}}U,$

where V, D, and U represent arbitrary quantities that are dependent only on one-dimensional combinations of r, t, and other relevant parameters of the problem. Usually, these quantities are functions of two non-dimensional variables. However, if an additional characteristic parameter $'C'$ is introduced, which has dimensions that are independent of those of $'A'$, the number of independent variables formed by the combinations of r, t, $'A'$, and $'C'$ is reduced to one. Given that the dimensions of the constant $'A'$ includes mass, we select the constant $'C'$ such that its dimensions exclude mass, i.e.

$$dim C = L^sT^k.$$

In this context, the single non-dimensional independent variable is represented by r^st^k/C, which can be replaced (s$\neq$ 0) by the variable,

$$\eta = \frac{r}{c^{1/s}t^{\alpha}}, \quad \text{where} \quad \alpha = \frac{-k}{s}.$$

If s=0, then V, D and U depend only on the time, and the velocity v is proportional to r. The solution that relies on the independent variables may include multiple arbitrary constants.

1.2 Motivation and Overview

Any mathematical representation of a continuum is articulated through a set of partial differential equations (PDEs). These equations are important for modeling several fundamental natural processes, such as convection, dispersion, diffusion, and dissipation. In physics, for example, wave propagation and thermal conditions are both regulated by PDEs. Furthermore, PDEs are integral to the majority of population models utilized in ecology. They also govern the majority of physical phenomena studied in fields such as quantum mechanics, electricity, non-linear fiber optics, and the dynamics of shallow-water waves, as well as in quantum field theory and numerous other models. The complexity of these non-linear equations has attracted extensive research from mathematicians, physicists, and other scientists over many years, leading to the development of various methods for addressing these challenges. In recent decades, notable advancements have been made in this domain. Throughout the centuries, significant progress have been achieved in the theoretical and numerical analysis of non-linear PDEs; however, a universal methodology capable of addressing all types of non-linear PDEs has not been discovered yet. The principles of conservation for mass, momentum, and energy establish a unified framework, with each medium defined by its specific constitutive laws. These conservation principles, along with the constitutive equations for the field variables and the underlying assumptions, lead to partial differential equations that are typically non-linear and non-homogeneous. There are three main reasons to concentrate on hyperbolic conservation laws: First, the equations governing compressible fluid flow reduce to the Euler equations when neglecting viscosity and heat conduction effects, representing a significant category of hyperbolic conservation laws. Second, the numerical methods utilized for hyperbolic partial differential equations are heavily reliant on discretization techniques. Finally, the theory surrounding hyperbolic systems is considerably more advanced than that of complete mathematical models, such as the Navier-Stokes equations. The set of non-linear PDEs is categorized into hyperbolic, parabolic, and elliptic types. Among these, the hyperbolic system of conservation laws represents one of the most important classes of non-linear PDEs. A prominent example of hyperbolic PDEs is represented by Euler's equations in gas dynamics. Obtaining analytical solutions for initial value problems or boundary value problems in hyperbolic systems of non-linear partial differential equations (PDEs) is a highly challenging task. Consequently, each non-linear PDE problem requires a specific technique for resolution. Both numerical and analytical methods for addressing non-linear PDEs possess their own advantages and disadvantages. While numerical methods for analyzing non-linear equations were once effective, advancements in numerical techniques have also led to improvements in analytical methods in recent years. A considerable number of researchers in scientific and mathematical disciplines employ a combination of analytical and numerical approaches, which have the potential to yield valuable results. From a mathematical perspective, one of the most fascinating aspects of such systems is their susceptibility to shock wave formation, which are discontinuities in the solution that can arise even from the initial conditions.

From a mathematical perspective, a shock can be defined as a discontinuous solution to the hyperbolic system of conservation laws, which complies with the 'Rankine-Hugoniot condition' and the 'entropy condition'. Physically, the emergence of a shock wave in fluid dynamics can be illustrated by abrupt variations in essential flow parameters such as pressure, velocity, density, and temperature. Various types of discontinuities are observed in nature. Non-propagating discontinuities occur at the boundaries between free surfaces and materials, referred to as 'contact discontinuities'. The dynamics of converging shock waves have gathered considerable interest in recent decades. Owing to their remarkable capacity to rapidly increase temperature and pressure, these waves are employed in numerous innovative industrial applications and material synthesis. In the study of ideal gas dynamics, the non-linear equations pertain to three categories of non-linear waves: shock fronts, rarefactions, and contact discontinuities. Shock waves are predominantly observed in gases due to phenomena such as gaseous electrical discharges, the motion of an object traveling faster than the speed of sound, and explosions. The movement of a high-velocity object or the rapid expansion of explosion products generates elastic compression waves. Under specific conditions, these compression waves merge to create a singular wave characterized by an exceptionally sharp front, which propagates at supersonic speeds. These waves are marked by discontinuities in pressure, density, and temperature, and are referred to as 'Shock Waves'.

During the rapid phenomena of explosion and implosion, both temperature and pressure increase significantly. In these scenarios, the front of the condensed wave becomes steeper due to the resulting overpressure, leading to the formation of a shock wave. From a mathematical perspective, implosion is characterized as a convergent process where pressure escalates rapidly over time, resulting in unstable fluid dynamics directed towards the center of the implosion. In contrast, explosion is a divergent process where pressure diminishes as the distance from the source of the explosion increases. Shock waves, which differ from acoustic waves due to their limited amplitude, exhibit several unique characteristics:

- When analyzed within a coordinate system that moves with the shock, the flow remains subsonic behind the shock and transitions to supersonic ahead of it.
- The supersonic flow velocity is highly influenced by pressure.

- There is a sudden change in flow variables such as density, velocity, pressure, and temperature across the shock, leading to the formation of a steep wave front.
- In the case of non-planar shocks, such as spherical blast waves, a significant reduction in flow velocity occurs as one moves away from the center, as some energy from the shock wave is utilized to raise the temperature of the medium.
- Entropy consistently increases across a shock.

Additionally, shock waves have the effect of accelerating particles adjacent to the shock front. Shock waves can be categorized into weak shocks (with modest amplitude) and strong shocks (with large amplitude) based on their intensity. The strength of a shock is defined by the ratio $k = \frac{p}{p_0}$, where p represents the pressure immediately before the shock and p_0 is the pressure just ahead of it. For small shocks, where k is slightly greater than one, shock waves propagate at the speed of sound, while for strong shocks, where $k \geqq 1$, they travel at supersonic speeds.

Non-linear hyperbolic conservation laws present a considerable challenge in modern mathematical research. The determination of exact solutions for hyperbolic systems of partial differential equations (PDEs) is notably complex. This complexity arises primarily from the emergence of discontinuities, such as shocks and slip surfaces, in the solutions, even when the initial conditions are smooth. There are two principal methodologies for addressing hyperbolic systems of PDEs: analytical and numerical approaches. Over the last seventy to eighty years, plenty of analytical and numerical techniques have been developed to investigate the non-linear waves characterized by these hyperbolic systems. Various analytical methods, including the method of characteristics, progressive wave approach, wave front analysis, perturbation method, self-similar method, and differential constraint method, have been established to elucidate the physical properties of waves produced by quasilinear hyperbolic systems of PDEs. Furthermore, numerous numerical methods, such as the finite difference method, finite element method, finite volume method, and other numerical techniques, have been created to enhance the exploration of hyperbolic PDEs. Comprehensive examinations of the mathematical properties, analytical and numerical methods, and applications related to non-linear wave propagation issues within the framework of hyperbolic systems of PDEs are documented in the works of Courant and Friedrichs [3], Jeffrey [5], Whitham [22], Bressan [6], Dafermos [7], Sharma [8], Smoller [23], Holden and Risebro [24], LeVeque [25], among others.

1.3 Literature Review

The occurrence of shock waves is primarily linked to aerospace engineering, particularly in the context of supersonic flight. The exploration of this specific area of physics commenced in 1746 when mathematician Robins calculated the velocity of a bullet using a ballistic pendulum and observed an increase in aerodynamic drag as velocity approached the speed of sound. However, during the 19th century, the nature of shock waves remained enigmatic to many scholars. In 1759, Euler corresponded with Lagrange, asserting that the speed of sound is influenced by the amplitude of sound, although this does not hold true when the amplitude is infinitesimally small. Nevertheless, his hypothesis that speed would decrease with rising amplitude was flawed. Although the time was ripe for scientists to engage in experimental investigations and observations of shock wave behavior, the absence of appropriate equipment and limited literature in the field hindered their ability to study shock wave phenomena; it was only in 1897 that Hugoniot [26] undertook this challenge. In 1808, Poisson [27] became the first researcher to derive exact solutions to the Euler equation for one-dimensional unsteady fluid flow. Furthermore, in 1823, Poisson made a significant contribution to non-linear wave theory by formulating the isentropic gas law for sound waves with infinitesimal amplitude. In 1848, Stokes [28] investigated the issue of wave steepening and its behavior in finite amplitude acoustic waves, deriving the jump condition for mass and momentum. However, he was unable to provide a clear theory regarding the conservation of dissipated energy. In 1860, Riemann [29] made a notable advancement in shock wave theory by acknowledging the finite amplitude of waves. He explained that wave steepening leads to the formation of shock waves. Utilizing Monge's method of characteristics, he noted that the original waves divide into two distinct types: 'shock waves' and 'rarefaction waves.' The rarefaction wave becomes thicker, while the shock wave becomes thinner. It was also noted that gas passing through the rarefaction wave cools and expands, whereas it heats and compresses when traversing the shock wave. Riemann incorrectly assumed that entropy remains constant across the shock wave, leading him to conclude that the process is reversible. But according to the second law of thermodynamics, shock waves are, in fact, irreversible. Subsequently, Rankine [30] and Hugoniot [26] acknowledged the irreversible nature of shock wave phenomena, and their theory remains the foundational model for shock wave propagation today. In 1899, Chapman [31] applied Riemann's theory [29] to detonation, publishing his findings that same year. Subsequently, Vielle [32] developed a shock tube to measure shock wave velocities, which were found to exceed the speed of sound. He also recognized the parallels between shock waves and detonations, comparing wave speeds in accordance with Hugoniot's theory [26].

When a body moves relative to a fluid, the resulting disturbance is transmitted through the fluid at the speed of sound. In these scenarios, the speed of sound refers to the rate at which small amplitude rarefaction and compression waves travel.

However, when the compressions in the flow reach a finite amplitude, a pressure discontinuity typically arises, resulting in a shock wave. Finite amplitude gas compressions propagate at speeds exceeding that of sound, as observed in powerful explosions. The phenomenon of shock wave formation has been extensively examined by numerous physicists and mathematicians. Initial studies of supersonic free air jets through high-speed photography have found a 'lyre' type configuration of reflected shock waves, which were subsequently referred to as 'shock diamonds' [33]. In 1910, Taylor [34] derived the essential conditions for the discontinuous motion in gases. During the late 1920s, research driven by the practical requirements of aeroballistics (to reduce wave drag), aeronautics (to enhance high-speed propellers), and steam turbine advancements (to achieve optimal Laval nozzle geometry) ultimately led to the formation of gas dynamics, a novel field within fluid dynamics. A dimensionless parameter that characterizes the flow velocity in relation to the sound velocity of the surrounding medium was introduced, which later became known as the 'Mach number' [35]. In 1949, Broderick [36] examined the general equation governing isentropic, irrotational, axially symmetrical gas flow, disregarding viscosity and conductivity, around a thin body of revolution situated in a uniform, supersonic stream, aligned with the direction of the undisturbed flow. Courant and Friedrichs [3] elucidated the characteristics of supersonic flows and non-linear waves in their work titled 'Supersonic flow and shock waves'. A notable aspect of shock waves is the challenge of assessing the differential impacts of shock fronts on the downstream flow field. Taniuti and Wei [37] introduced a category of nonlinear partial differential equations that can be simplified to nonlinear equations, such as the Burgers and Kortweg-deVries equations, and explored their applications in hydrodynamics and plasma physics. This reduction technique relies on a singular perturbation expansion. Varley and Cumberbatch [38] proposed the method of relatively undistorted waves, which utilizes a series of successive approximations to address systems of hyperbolic equations, focusing on high-frequency waves described by nonlinear equations. Ambika et al. [39] expanded the progressive method theory to examine the evolution of waves with both finite and small amplitudes. Nath et al. [40] conducted a further analysis of the progressive wave solutions in a dusty medium. The foundational concepts of progressive wave theory can be found in [38, 41, 42, 43]. Ram [44] investigated the propagation of acceleration waves along characteristic paths by employing the characteristics of the governing quasilinear system as a reference coordinate system. Additionally, Shankar [45] explored the propagation of acceleration waves within the context of radiation-magneto gasdynamics. For a comprehensive understanding of the behavior and characteristics of acceleration waves, readers are encouraged to consult the following research articles [46, 47, 48, 49, 50, 51]. The investigation of fluid dynamics involving solid particles has gathered significant interest in both engineering and scientific research, encompassing areas such as the centrifugal separation of particulate matter from fluids, various chemical processes, the motion of solid particles in rocket exhaust, and dust flow in geophysical and astrophysical contexts [52, 53, 11, 54]. Carrier [55] has examined the properties of shock waves in dusty gases, focusing on the analysis of a plane steady decelerated flow of a dusty gas mixture in a suitable manner. Singh et al. [56] employed the wavefront analysis technique to explore the formation of shock waves in a two-dimensional steady supersonic flow of a radiating gas around plane and axisymmetric structures, including a beak and a sharp-edged ring. For further information regarding the wavefront method, one may refer to the work of Jeffery [5].

In the field of gas dynamics, the non-isentropic equations dictate the continuous flow that occurs behind a shock wave. At the shock front, these equations must be resolved using the Rankine-Hugoniot conditions, which establish the relationship between the flow variable states on either side of the shock wave. Furthermore, the shock theory imposes boundary conditions along the trajectory of the shock. There are two categories of similarity solutions. The first kind of self-similar solutions is well-known, with their existence solely linked to conservation laws and dimensional analysis. An example of this is the self-similar solution observed during the initial phase of a nuclear explosion. A comprehensive collection of such problems and a general methodology for addressing them can be found in the works of Sedov [57] and Korobeinikov [58]. Conversely, there exists a broader class of self-similar problems, referred to as self-similar problems of the second kind, where the nature of similarity is determined not by conservation laws or dimensional analysis, but rather through the resolution of the equations as a non-linear eigenvalue problem. A prominent example of self-similar solutions of the second kind is the collapse of an imploding shock wave. In 1950, Taylor [59] utilized the self-similar method to analyze a blast wave problem, deriving the similarity solution of the original problem by transforming the non-linear partial differential equations into ordinary differential equations. His research employed the conservation of total energy to ascertain the shock trajectory immediately behind the shock.

In 1942, Guderley [60] was the pioneering researcher to explore converging shock waves in an ideal gas, conducting his investigation through analytical methods. The importance of Guderley's contributions in gas dynamics lies in their ability to reveal a wider range of self-similar solutions within this discipline. In relation to the investigation of self-similar solutions of the second kind, it is important to reference the contributions of Barenblatt and Zel'dovich [61], as well as Barenblatt [62]. Utilizing a self-similar solution approach, Zeldovich and Raizer [63] addressed implosion issues concerning the collapse of a spherical bubbles in liquids and spherical shock waves in gases. Numerous scholars have conducted theoretical analyses on the collapse of converging shock waves in ideal gases. Among the significant body of research that has emerged on shock waves, we would like to highlight the influential works of R. B. Lazarus [64], Jeffrey [65], Pandey et al. [66], Whitham [1], Hirschler and Gretler [67], Sariet al. [68], and Ramsey et al. [69], who have made substantial contributions to this area.

1.4 Aims and Thesis Objectives

The purpose of this thesis is to study various non-linear wave propagation issues that are described by a quasilinear hyperbolic system of partial differential equations. The primary goal of this work is to derive the analytical solution for the Euler system of PDEs and to analyze the behavior of waves as they propagate through specific media. In this study, we have employed several analytical methods, including the progressive wave method, the method of characteristics, and wavefront analysis, to solve both one-dimensional and two-dimensional systems of partial differential equations. The aim of this thesis is to extend the understanding of wave propagation as characterized by hyperbolic system partial differential equations (PDEs) through an analytical framework, and to identify the self-similar solutions for imploding shocks in self-gravitating non-ideal gases. The numerical analysis was conducted utilizing the symbolic software package "Mathematica." The findings are presented in the form of graphs and tables to facilitate clear interpretation. To comprehensively address the entire work of the thesis, we have been motivated to work on the following objectives:

1. Study of the effect of non-ideal relaxing gas on the growth and decay behavior of non-linear waves for generalized geometry of flows.

2. Investigation of the propagation of weak discontinuity waves in non-ideal interstellar environments.
3. To examine the behavior of shock wave in two dimensional planar and axisymmetric in dusty non-ideal gas flow with magnetic field.
4. To determine the self-similar solutions for imploding shocks in non-ideal gas with gravitational effects.

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