Chapter 2

The Weinstein transform associated with a family of generalized distributions

2.1 Introduction

The Weinstein transform has made a considerable impact on various problems in the pure and applied mathematical, physical, and engineering sciences. It is also heavily used in many other related scientific fields. Exploiting the theory of the Weinstein transform, many applications have been done by many authors. In particular, Salem and Nasr [57], Mejjaoli et al. [32, 33], Mohamed [35] and Mehrez [30] observed various properties related to theWeinstein transform and thereby found several useful observations.

The theory of ultradistributions was introduced by Beurling [5], Björck [8], and Roumieu [53], which is an interesting and potentially useful generalization of the relatively more familiar Schwartz distributions. A unification of the Beurling-Björck theory and the Roumieu theory was made by Komatsu [29]. The Hankel transform of ultradistributions in the Roumieu setting was contributed by Pathak and Pandey [42] and Pathak and Shrestha [46] and presented their several important results. Motivated by the results of Beurling [5], Björck [8], and Roumieu [53], in the present chapter, the author provides a brief description of spaces of types D_ω and D'_ω and finds their various properties by exploiting the theory of the Weinstein transform.

2.2 Spaces of type D_ω, D'_ω and its properties

In this section, spaces of types D_ω and D'_ω are defined and examined their various properties.

Definition 2.2.1. A real valued function ω on $\mathbb{R}^{n+1}_+$ is called subadditive function if it satisfies

$$0 = \omega(0) = \lim_{x\to 0} \omega(x) \leq \omega(\xi+\eta) \leq \omega(\xi) + \omega(\eta), \forall\, \xi,\, \eta \in \mathbb{R}^{n+1}_+. \tag{2.2.1}$$

Definition 2.2.2. $M_0 = M_0(n)$ be the set of all continuous real valued functions ω on $\mathbb{R}^{n+1}_+$ satisfying (2.2.1), and

$$J_n(\omega) = \int_{|\xi|\geq 1} \frac{\omega(\xi)}{\mid \xi \mid^{n+1}}\, d\mu_\beta(\xi) < \infty. \tag{2.2.2}$$

Definition 2.2.3. If $\phi \in L^1(\mathbb{R}^{n+1}_+)$, and ω satisfy (2.2.1), and λ is a real number then

$$\|\phi\|_{\lambda,\omega} = \int_{\mathbb{R}^{n+1}_+} \mid \mathcal{F}_w\phi(\xi) \mid e^{\lambda\omega(\xi)} d\mu_\beta(\xi), \tag{2.2.3}$$

D_ω is the set of all $\phi \in L^1(\mathbb{R}^{n+1}_+)$ such that ϕ has compact support and

$$||\phi||_{\lambda,\omega} < \infty \qquad (\forall\ \lambda > 0).$$

The elements of D_ω is represents test functions.

Definition 2.2.4. Let Ω is an open subset of $\mathbb{R}^{n+1}_+$, then

$$D_\omega(\Omega) = \{\phi : \phi \in D_\omega \text{ and } \text{ supp } \phi \subset \Omega\}.$$

Theorem 2.2.5. *Let $\omega \in M_0(n)$, $\phi \in D_\omega$ and u be an integrable function with compact support then $u\#_\beta\phi \in D_\omega$.*

Proof: From (1.4.1), we have

$$(\mathcal{F}_w u)(\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) u(x) d\mu_\beta(x).$$

Then

$$\begin{aligned} \mid (\mathcal{F}_w u)(\xi) \mid &= \mid \int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) u(x)\, d\mu_\beta(x) \mid \\ &\leq \int_{\mathbb{R}^{n+1}_+} \mid e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) u(x) \mid\ d\mu_\beta(x) \\ &\leq \int_{\mathbb{R}^{n+1}_+} \mid e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) \mid \cdot \mid u(x) \mid d\mu_\beta(x) \\ &\leq \int_{\mathbb{R}^{n+1}_+} \mid u(x) \mid\ d\mu_\beta(x), \quad as \ \mid e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) \mid \leq 1. \end{aligned}$$

Which implies that

$$\mid (\mathcal{F}_w u)(\xi) \mid \leq \int_{\mathbb{R}^{n+1}_+} \mid u(x) \mid d\mu_\beta(x)\,. \tag{2.2.4}$$

Next, we have to show that $\|u\#_\beta\phi\|_{\lambda,\omega} < \infty$.

Therefore, from (2.2.3) we have

$$\|u\#_\beta\phi\|_{\lambda,\omega} = \int_{\mathbb{R}^{n+1}_+} \mid \mathcal{F}_w(u\#_\beta\phi)(\xi) \mid e^{\lambda\omega(\xi)}\, d\mu_\beta(\xi). \tag{2.2.5}$$

Also by using (1.4.13), we get

$$\begin{aligned}
\|u\#_\beta\phi\|_{\lambda,\omega} &= \int_{\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w u)(\xi)(\mathcal{F}_w\phi)(\xi) \mid e^{\lambda\omega(\xi)}\, d\mu_\beta(\xi)\\
&\leq \int_{\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w u)(\xi) \mid\mid (\mathcal{F}_w\phi)(\xi) \mid e^{\lambda\omega(\xi)}\, d\mu_\beta(\xi)\\
&= \int_{\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w\phi)(\xi) \mid e^{\lambda\omega(\xi)} \mid (\mathcal{F}_w u)(\xi) \mid \, d\mu_\beta(\xi)\\
&\leq \sup_{\xi\in\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w u)(\xi) \mid \int_{\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w\phi)(\xi) \mid e^{\lambda\omega(\xi)}\, d\mu_\beta(\xi).
\end{aligned}$$

Now, in veiw of (2.2.3), we obtain

$$\|u\#_\beta\phi\|_{\lambda,\omega} \leq \sup_{\xi\in\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w u)(\xi) \mid \|\phi\|_{\lambda,\omega}. \tag{2.2.6}$$

Moreover by using (2.2.4), we get

$$\begin{aligned}
\|u\#_\beta\phi\|_{\lambda,\omega} &\leq \|\phi\|_{\lambda,\omega} \int_{\mathbb{R}^{n+1}_+} \mid u(x) \mid \, d\mu_\beta(x)\\
&\leq \|\phi\|_{\lambda,\omega}\|u\|_{L^1_\beta(\mathbb{R}^{n+1}_+)}\\
&< \infty.
\end{aligned}$$

which shows that

$$\|u\#_\beta\phi\|_{\lambda,\omega} < \infty.$$

Hence we have

$$u\#_\beta\phi \in D_\omega.$$

Theorem 2.2.6. *Let $\omega_1, \omega_2 \in M_0(n)$. If for some real number A and postive constant C we have*

$$\omega_2(\xi) \leq A + C\omega_1(\xi) \quad \forall \xi \in \mathbb{R}_+^{n+1}. \tag{2.2.7}$$

Then, D_{ω_1} is dense subset of D_{ω_2}.

Proof: Let $\phi \in D_{\omega_1}$. Then we have

$$\|\phi\|_{\lambda,\omega_1} = \int_{\mathbb{R}_+^{n+1}} \mid \mathcal{F}_w\phi(\xi) \mid e^{\lambda\omega_1(\xi)}\, d\mu_\beta(\xi) < \infty, for\ \lambda \in \mathbb{R}. \tag{2.2.8}$$

For $\phi \in D_{\omega_2}$, and from (2.2.3) we find

$$\|\phi\|_{\lambda,\omega_2} = \int_{\mathbb{R}_+^{n+1}} \mid (\mathcal{F}_w\phi)(\xi) \mid e^{\lambda\omega_2(\xi)}\, d\mu_\beta(\xi). \tag{2.2.9}$$

With the help of (2.2.7), the above yields

$$\begin{aligned} \|\phi\|_{\lambda,\omega_2} &\leq \int_{\mathbb{R}_+^{n+1}} \mid (\mathcal{F}_w\phi)(\xi) \mid e^{\lambda(A+C\omega_1(\xi))}\, d\mu_\beta(\xi) \\ &= e^{A\lambda} \int_{\mathbb{R}_+^{n+1}} \mid (\mathcal{F}_w\phi)(\xi) \mid e^{C\lambda\omega_1(\xi)}\, d\mu_\beta(\xi) \\ &= e^{A\lambda} \int_{\mathbb{R}_+^{n+1}} \mid (\mathcal{F}_w\phi)(\xi) \mid e^{\Lambda\omega_1(\xi)}\, d\mu_\beta(\xi) \qquad (\Lambda = C\lambda \in \mathbb{R}). \end{aligned}$$

Using (2.2.3), then, we obtain last expression

$$\|\phi\|_{\lambda,\omega_2} \leq e^{A\lambda}\|\phi\|_{\Lambda,\omega_1}. \tag{2.2.10}$$

Above implies that

$$\|\phi\|_{\lambda,\omega_2} < \infty.$$

Hence

$$\phi \in D_{\omega_2}.$$

Now we have to show that D_{ω_1} is dense in D_{ω_2}. Then, let $u \in D_{\omega_2}$ and apply Theorem (2.2.5), we obtain $u\#_\beta \phi_\epsilon \in D_{\omega_1}$, where

$$\phi_\epsilon(x) = \epsilon^{-(n+1)}\phi(\frac{x}{\epsilon}), \qquad (\epsilon > 0,\, x \in \mathbb{R}_+^{n+1}).$$

From (1.4.1), we can write

$$\begin{aligned}(\mathcal{F}_w\phi_\epsilon)(\xi) &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\phi_\epsilon(x)d\mu_\beta(x)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\epsilon^{-(n+1)}\phi\left(\frac{x}{\epsilon}\right) d\mu_\beta(x).\end{aligned}$$

Taking

$$\frac{x}{\epsilon} = u \text{ and } d\mu_\beta(x) = \epsilon^{n+1}\, d\mu_\beta(u),$$

We have

$$\begin{aligned}(\mathcal{F}_w\phi_\epsilon)(\xi) &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle \epsilon u',\xi'\rangle} J_\beta(\epsilon u_{n+1}\xi_{n+1})\phi(u)\, d\mu_\beta(u)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle u',\epsilon\xi'\rangle} J_\beta(u_{n+1}\epsilon\xi_{n+1})\phi(u)\, d\mu_\beta(u)\\ &=(\mathcal{F}_w\phi)(\epsilon\xi).\end{aligned}$$

This implies that

$$(\mathcal{F}_w\phi_\epsilon)(\xi) = (\mathcal{F}_w\phi)(\epsilon\xi). \tag{2.2.11}$$

We now consider

$$\|u - (u\#_\beta\phi_\epsilon)\|_{\lambda,\omega_2} = \int_{\mathbb{R}_+^{n+1}} e^{\lambda\omega_2(\xi)} \mid \mathcal{F}_w(u - (u\#_\beta\phi_\epsilon))(\xi) \mid\, d\mu_\beta(\xi).$$

By linear property of the Weinstein transform, we get

$$\|u-(u\#_\beta\phi_\epsilon)\|_{\lambda,\omega_2}=\int_{\mathbb{R}^{n+1}_+} e^{\lambda\omega_2(\xi)} \mid (\mathcal{F}_w u)(\xi)-\mathcal{F}_w(u\#_\beta\phi_\epsilon)(\xi)\mid \, d\mu_\beta(\xi).$$

From (1.4.13), we get

$$\begin{aligned}
&\|u-(u\#_\beta\phi_\epsilon)\|_{\lambda,\omega_2}\\
&=\int_{\mathbb{R}^{n+1}_+} e^{\lambda\omega_2(\xi)} \mid (\mathcal{F}_w u)(\xi)-(\mathcal{F}_w u)(\xi)(\mathcal{F}_w\phi_\epsilon)(\xi)\mid \, d\mu_\beta(\xi)\\
&=\int_{\mathbb{R}^{n+1}_+} e^{\lambda\omega_2(\xi)} \mid (\mathcal{F}_w u)(\xi)(1-(\mathcal{F}_w\phi_\epsilon)(\xi))\mid \, d\mu_\beta(\xi)\\
&\leq \int_{\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w u)(\xi)\mid e^{\lambda\omega_2(\xi)}. \mid (1-(\mathcal{F}_w\phi_\epsilon)(\xi))\mid \, d\mu_\beta(\xi).
\end{aligned}$$

Thus by using (2.2.11), we get

$$\|u-(u\#_\beta\phi_\epsilon)\|_{\lambda,\omega_2}\leq \int_{\mathbb{R}^{n+1}_+} \mid (\mathcal{F}_w u)(\xi)\mid e^{\lambda\omega_2(\xi)}. \mid (1-(\mathcal{F}_w\phi)(\epsilon\xi))\mid \, d\mu_\beta(\xi), \quad (2.2.12)$$

which tends to zero as $\epsilon\to 0$, by the Dominated Convergence Theorem .

Hence, D_{ω_1} is dense in D_{ω_2} .

Definition 2.2.7. The set of all continous real-valued functions ω satisfying (2.2.1), (2.2.2) and (2.2.13), is denoted by M

$$\omega(\xi)\geq a+b\log(1+\mid\xi\mid),\ \forall\,\xi\in\mathbb{R}^{n+1}_+,\ a\in\mathbb{R},\ b>0. \qquad (2.2.13)$$

Definition 2.2.8. Let $\omega\in M,\ \phi\in L^1(\mathbb{R}^{n+1}_+)$ and λ is a real number then we define

$$|||\phi|||_\lambda=|||\phi|||_{\lambda,\omega}=\sup_{\xi\in\mathbb{R}^{n+1}}\mid(\mathcal{F}_w\phi)(\xi)\mid e^{\lambda\omega(\xi)}. \qquad (2.2.14)$$

Theorem 2.2.9. *Let $\omega \in M$. Then there exists a positive constant Λ such that*

$$C_\Lambda = \int_{\mathbb{R}_+^{n+1}} e^{-\Lambda\omega(\xi)} d\mu_\beta(\xi) < \infty. \tag{2.2.15}$$

Then

$$||\phi||_\lambda \leq C_\Lambda |||\phi|||_{\lambda+\Lambda},\ \forall\, \lambda \in \mathbb{R}, \qquad (\forall \phi \in L^1(\mathbb{R}_+^{n+1})).$$

Proof: From (2.2.14), we have

$$C_\Lambda |||\phi|||_{\Lambda+\lambda} = C_\Lambda \sup_{\xi \in \mathbb{R}_+^{n+1}} |\mathcal{F}_w\phi(\xi)| e^{(\Lambda+\lambda)\omega(\xi)} = \sup_{\xi \in \mathbb{R}_+^{n+1}} |\mathcal{F}_w\phi(\xi)| e^{(\Lambda+\lambda)\omega(\xi)} C_\Lambda.$$

Also by taking (2.2.15), we get

$$\begin{aligned} C_\Lambda |||\phi|||_{\Lambda+\lambda} &= \sup_{\xi \in \mathbb{R}_+^{n+1}} |\mathcal{F}_w\phi(\xi)| e^{(\Lambda+\lambda)\omega(\xi)} \Big(\int_{\mathbb{R}^{n+1}} e^{-\Lambda\omega(\xi)} d\mu_\beta(\xi) \Big) \\ &\geq \int_{\mathbb{R}_+^{n+1}} e^{-\Lambda\omega(\xi)} |\mathcal{F}_w\phi(\xi)| e^{(\Lambda+\lambda)\omega(\xi)}\, d\mu_\beta(\xi) \\ &= \int_{\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)} |\mathcal{F}_w\phi(\xi)|\, d\mu_\beta(\xi) \\ &= ||\phi||_\lambda. \end{aligned}$$

Above shows that

$$||\phi||_\lambda \leq C_\Lambda |||\phi|||_{\Lambda+\lambda}.$$

Theorem 2.2.10. *Let $\omega \in M$. Let $\phi \in D_\omega$ then for any multi-index α, we have*

$$(\Delta_{W,\beta}^n)^\alpha \phi \in D_\omega.$$

Proof: Let $\phi \in D_\omega$. Then from (2.2.3), we have

$$||\phi||_{\lambda,\omega} = \int_{\mathbb{R}^{n+1}_+} |\mathcal{F}_w\phi(\xi)|e^{\lambda\omega(\xi)}\, d\mu_\beta(\xi) < \infty,\ \lambda \in \mathbb{R}. \tag{2.2.16}$$

We now take $\omega \in M$. Then from (1.3.3), we find that

$$\begin{aligned}
&\omega(\xi) \geq a + b\ log(1 + ||\xi||), \qquad (a \in \mathbb{R},\ b > 0)\\
&\omega(\xi) - a \geq b\ log(1 + ||\xi||)\\
&\frac{1}{b}\big(\omega(\xi) - a\big) \geq log(1 + ||\xi||)\\
&e^{\frac{1}{b}(\omega(\xi)-a)} \geq (1 + ||\xi||) > ||\xi||\\
&e^{\frac{2\alpha}{b}(\omega(\xi)-a)} \geq ||\xi||^{2\alpha}.
\end{aligned}$$

We thus obtain

$$||\xi||^{2\alpha} < e^{\frac{-2a\alpha}{b}} e^{\frac{2\alpha\omega(\xi)}{b}}. \tag{2.2.17}$$

Next, by the Weinstein transform of the derivative (1.4.5) of ϕ, we have

$$|\mathcal{F}_w((\Delta^n_{W,\beta})^\alpha\phi)(\xi)| = ||\xi||^{2\alpha}|(\mathcal{F}_w\phi)(\xi)|. \tag{2.2.18}$$

From (2.2.3) we have

$$||((\Delta^n_{W,\beta})^\alpha\phi)(\xi)||_{\lambda,\omega} = \int_{\mathbb{R}^{n+1}_+} e^{\lambda\omega(\xi)}|\mathcal{F}_w((\Delta^n_{W,\beta})^\alpha\phi)(\xi)|\, d\mu_\beta(\xi).$$

Using (2.2.18), we get

$$\begin{aligned}
||((\Delta^n_{W,\beta})^\alpha\phi)(\xi)||_{\lambda,\omega} &= \int_{\mathbb{R}^{n+1}_+} e^{\lambda\omega(\xi)}||\xi||^{2\alpha}|(\mathcal{F}_w\phi)(\xi)|\, d\mu_\beta(\xi)\\
&= \int_{\mathbb{R}^{n+1}_+} e^{\lambda\omega(\xi)}|(\mathcal{F}_w\phi)(\xi)|||\xi||^{2\alpha}\, d\mu_\beta(\xi).
\end{aligned}$$

Thus, in veiw of (2.2.17), we obtain

$$\begin{aligned}||(\Delta_{W,\beta}^n)^\alpha\phi)(\xi)||_{\lambda,\omega} &\leq \int_{\mathbb{R}_+^{n+1}} e^{\lambda\omega(\xi)}|(\mathcal{F}_w\phi)(\xi)|e^{\frac{-2a\alpha}{b}}e^{\frac{2\alpha}{b}\omega(\xi)}\,d\mu_\beta(\xi)\\ &= e^{\frac{-2a\alpha}{b}}\int_{\mathbb{R}_+^{n+1}} e^{(\lambda+\frac{2\alpha}{b})\omega(\xi)}|(\mathcal{F}_w\phi)(\xi)|\,d\mu_\beta(\xi)\\ &= e^{\frac{-2a\alpha}{b}}||\phi||_{\lambda+\frac{2\alpha}{b},\omega}\\ &< \infty.\end{aligned}$$

Therefore,

$$(\Delta_{W,\beta}^n)^\alpha\phi \in D_\omega\,.$$

Theorem 2.2.11. *Let $\omega \in M$ and $\phi \in D_\omega$ be given. Then the mapping from $\mathbb{R}_+^{n+1}$ into D_ω defind by*

$$y \to \tau_y(\phi)$$

is continuous.

Proof: Let $\omega \in M,\ \phi \in D_\omega$ and for $x \in \mathbb{R}_+^{n+1}$ then from (2.2.14), we have

$$||\tau_y(\phi) - \tau_x(\phi)||_{\lambda,\omega} = \int_{\mathbb{R}_+^{n+1}} |\mathcal{F}_w\big(\tau_y(\phi)(\xi) - \tau_x(\phi)(\xi)\big)|e^{\lambda\omega(\xi)}d\mu_\beta(\xi)\,.$$

Also, by the linear property of the Weinstien transform (1.4.1), we get

$$||\tau_y(\phi) - \tau_x(\phi)||_{\lambda,\omega} = \int_{\mathbb{R}_+^{n+1}} |\mathcal{F}_w\tau_y(\phi)(\xi) - \mathcal{F}_w\tau_x(\phi)(\xi)|e^{\lambda\omega(\xi)}d\mu_\beta(\xi)\,. \tag{2.2.19}$$

First of all, we obtain

$$(\tau_x\phi)(y) = \int_{\mathbb{R}_+^{n+1}} \phi(z)D(x,y,z)\,d\mu_\beta(z)\,.$$

Thus, in veiw of (1.4.8), we find that

$$(\tau_x\phi)(y) = \int_{\mathbb{R}^{n+1}_+} \phi(z)\bigg(\int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) e^{i\langle y',\xi\rangle} J_\beta(y_{n+1}\xi_{n+1}) \times e^{-i\langle z',\xi'\rangle} J_\beta(z_{n+1}\xi_{n+1})\, d\mu_\beta(\xi)\bigg)\, d\mu_\beta(z)\,.$$

By using Fubini's theorem, we get

$$(\tau_x\phi)(y) = \int_{\mathbb{R}^{n+1}_+} \bigg(\int_{\mathbb{R}^{n+1}_+} \phi(z) e^{-i\langle z',\xi'\rangle} J_\beta(z_{n+1}\xi_{n+1})\, d\mu_\beta(z)\bigg) e^{i\langle y',\xi'\rangle} \times J_\beta(y_{n+1}\xi_{n+1}) e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\, d\mu_\beta(\xi)\,.$$

Also, by the definition of the Weinstein transform (1.4.1), we find above expression

$$(\tau_x\phi)(y) = \int_{\mathbb{R}^{n+1}_+} \bigg((\mathcal{F}_w\phi)(\xi) e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\bigg) e^{i\langle y',\xi'\rangle} J_\beta(y_{n+1}\xi_{n+1})\, d\mu_\beta(\xi)\,.$$

Now, in veiw of (1.4.1) and (1.4.6), we get

$$(\tau_x\phi)(y) = \mathcal{F}_w^{-1}\bigg(\mathcal{F}_w\phi(y) e^{-i\langle x',y'\rangle} J_\beta(x_{n+1}y_{n+1})\bigg)(y)\,.$$

Therefore, we have

$$\mathcal{F}_w(\tau_x\phi)(y) = e^{-i\langle x',y'\rangle} J_\beta(x_{n+1}y_{n+1})(\mathcal{F}_w\phi)(y)\,. \tag{2.2.20}$$

With the help of (2.2.19) and (2.2.20), we obtain

$$
\begin{aligned}
&||\tau_y(\phi) - \tau_x(\phi)||_{\lambda,\omega} \\
&= \int_{\mathbb{R}^{n+1}_+} |e^{-i\langle y',\xi'\rangle} J_\beta(y_{n+1\xi_{n+1}})(\mathcal{F}_w\phi)(\xi) - e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) \\
&\times (\mathcal{F}_w\phi)(\xi)|e^{\lambda\omega(\xi)} d\mu_\beta(\xi) \\
&\le \int_{\mathbb{R}^{n+1}_+} |e^{-i\langle y',\xi'\rangle} J_\beta(y_{n+1}\xi_{n+1}) - e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})| \\
&\times |(\mathcal{F}_w\phi)(\xi)|e^{\lambda\omega(\xi)} d\mu_\beta(\xi)\,.
\end{aligned}
$$

Hence, the above expression tends to zero as $y \to x$. This implies $y \to \tau_y(\phi)$ is continuous.

Definition 2.2.12. Let $L = (L_k)_{k=0}^{\infty}$ be an increasing sequence of positive numbers and let Ω be an open subset of $\mathbb{R}^{n+1}_+$ then $C^L(\Omega)$ is the set of all $u \in C^\infty(\Omega)$ such that for each compact subset K of Ω there exists a constant C such that

$$
\sup_K |(\Delta^n_{W,\beta})^\alpha\ u| \le C^{k+1} L_k^k\,, \tag{2.2.21}
$$

where α is multi-index with

$$
|\alpha| = k \qquad (k = 0, 1, 2, ...)\ .
$$

Theorem 2.2.13. *Let $u \in D_\omega(\Omega)$, and*

$$
|(\mathcal{F}_w u)(\xi)| \le \frac{C}{q_L(a\xi)(1 + ||\xi||)^{n+1}}, \tag{2.2.22}
$$

where C and $a > 0$ are positive constants with

$$
q_L(\xi) = \sum_{k=0}^{\infty} \left(\frac{||\xi||^2}{L_k}\right)^k \qquad (\xi \in \mathbb{R}^{n+1}_+). \tag{2.2.23}
$$

Then

$$u \in C^L(\Omega).$$

Proof: Let $u \in D_\omega(\Omega)$ and $\omega \in M$. Then, from the inversion formula of the Weinstein transform (1.4.2), we have

$$u(x) = \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w u)(\xi) e^{i\langle \xi', x' \rangle} J_\beta(\xi_{n+1}x_{n+1})\, d\mu_\beta(\xi).$$

So that

$$\begin{aligned}
(\Delta^n_{W,\beta})^\alpha_x u(x) &= (\Delta^n_{W,\beta})^\alpha_x \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w u)(\xi) e^{i\langle \xi', x' \rangle} J_\beta(\xi_{n+1}x_{n+1})\, d\mu_\beta(\xi) \\
&= \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w u)(\xi) (\Delta^n_{W,\beta})^\alpha_x \big(e^{i\langle \xi', x' \rangle} J_\beta(\xi_{n+1}x_{n+1}) \big)\, d\mu_\beta(\xi) \\
&= \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w u)(\xi) \big((-1)^\alpha ||\xi||^{2\alpha} \big) e^{i\langle \xi', x' \rangle} J_\beta(\xi_{n+1}x_{n+1})\, d\mu_\beta(\xi) \\
&= (-1)^\alpha \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_w u)(\xi) ||\xi||^{2\alpha} e^{i\langle \xi', x' \rangle} J_\beta(\xi_{n+1}x_{n+1}) d\mu_\beta(\xi)\,.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\max_{|\alpha|=k} \sup_{x\in\Omega} |(\Delta^n_{W,\beta})^\alpha u(x)| \\
&\quad = \max_{|\alpha|=k} \sup_{x\in\Omega} \Big|(-1)^\alpha \int_{\mathbb{R}_+^{n+1}} e^{i\langle \xi', x' \rangle} J_\beta(\xi_{n+1}x_{n+1}) (\mathcal{F}_w u)(\xi) ||\xi||^{2\alpha}\, d\mu_\beta(\xi)\Big| \\
&\quad \leq \max_{|\alpha|=k} \sup_{x\in\Omega} |e^{i\langle \xi', x' \rangle} J_\beta(\xi_{n+1}x_{n+1})| \int_{\mathbb{R}_+^{n+1}} |(\mathcal{F}_w u)(\xi)| ||\xi||^{2\alpha}\, d\mu_\beta(\xi) \\
&\quad \leq \max_{|\alpha|=k} \int_{\mathbb{R}_+^{n+1}} ||\xi||^{2\alpha} |(\mathcal{F}_w u)(\xi)|\, d\mu_\beta(\xi).
\end{aligned}$$

Now, by using (2.2.22) and (2.2.23), we get

$$\begin{aligned}\max_{|\alpha|=k}\sup_{x\in\Omega}|(\Delta^n_{W,\beta})^\alpha u(x)| \leq& \max_{|\alpha|=k}\int_{\mathbb{R}^{n+1}_+}\frac{C||\xi||^{2\alpha}}{q_L(a\xi)(1+|\xi|)^{n+1}}\,d\mu_\beta(\xi)\\ \leq& C\max_{|\alpha|=k}\int_{\mathbb{R}^{n+1}}\frac{||\xi||^{2\alpha}L_k^k}{a^k|\xi|^{2k}((1+|\xi|)^{n+1})}\,d\mu_\beta(\xi)\\ \leq& Ca^{-k}L_k^k\int_{\mathbb{R}^{n+1}_+}\frac{d\mu_\beta(\xi)}{(1+|\xi|)^{n+1}}\\ \leq& C_1a^{-k}L_k^k, \qquad (k=0,1,2...)\quad .\end{aligned}$$

Thus, we find that

$$\max_{|\alpha|=k}\sup_{x\in\Omega}|(\Delta^n_{W,\beta})^\alpha u(x)| \leq C_1a^{-k}L_k^k, \qquad (k=0,1,2...)\quad .$$

Hence we have

$$u\in C^L(\Omega)\,.$$

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