CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Mathematical Modeling and its Applications

Mathematical modeling is the process of transforming a real-world phenomenon into a mathematical problem using a mathematical model. A mathematical model is a mathematical representation, such as an equation, formula, graph, or table, that captures the essential characteristics of a particular scenario, whereas mathematical modeling refers to the procedure of constructing a mathematical model. The process of translation of a real-life problem into a mathematical form can give a better representation and solution of certain problems. Mathematical modeling provides many advantages to engineers, scientists, mathematicians, and other professionals who use it, including:

- Mathematics is a very precise language. This aids in the formulation of concepts and the identification of underlying assumptions.
- Mathematics is a concise language. Also, it provides a direction when trying to solve a problem.
- Calculations and other tasks can be carried out using computers.
- The results provide an in-depth understanding of how a system or an object works.
- It may facilitate timely and accurate decision-making.

A schematic representation of the mathematical modeling process can be shown in Fig. 1.1.1.

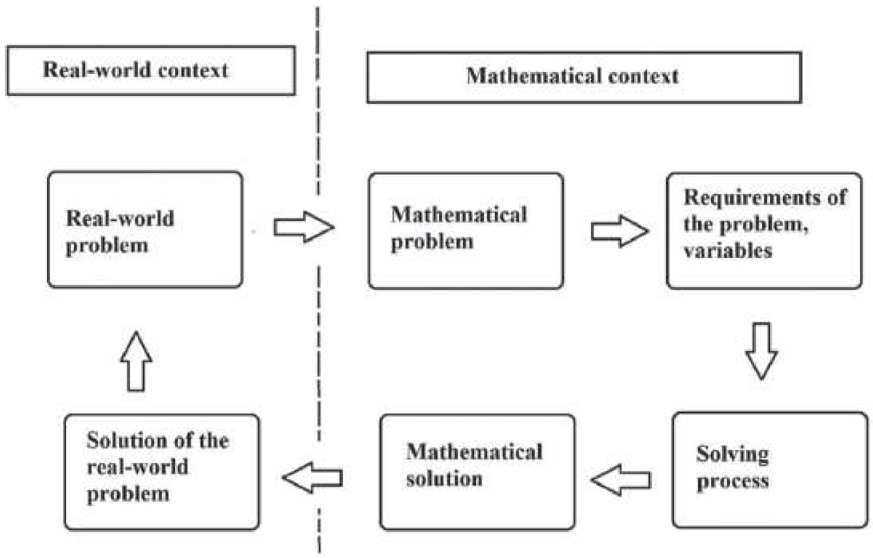

Figure 1.1.1: Flowchart of the process of mathematical modeling

Mathematical modeling holds significant importance in the fields of engineering and science, as it plays a crucial role in understanding and analyzing devices and phenomena. Consequently, engineers and scientists are motivated by practical considerations to engage in mathematical modeling. Mathematical models are frequently used to forecast future outcomes in scientific and engineering design practices. For instance, every new building or airplane reflects a model-based prediction that the structure will stand or the aircraft will fly without experiencing severe, unforeseen repercussions. Thus, prediction in engineering design goes beyond simply confirming a model and assumes that resources such as time, imagination, and money can be invested with confidence because the expected outcome would be excellent. There are several other applications for mathematical modeling, which can be found in a variety of contexts. The accuracy of the model is determined by both the level of understanding regarding a given system and the proficiency with which the modeling process is executed. As a result of utilizing the aforementioned modeling concept, this thesis examines the physical behavior of various thermoelastic systems and determines some fundamental aspects about them.

1.2 Classical Fourier's Law of Heat Conduction and its Drawback

The phenomenon of heat conduction occurs as a result of molecular vibrations or disturbances, which leads to the transmission of energy to neighboring molecules. Heat energy is transported from a higher temperature area to a lower temperature area as surrounding molecules collide. As it is well understood, this process follows Fourier's law of thermal conduction, according to which the rate of heat flow through a material is proportional to the negative temperature gradient and the area of the surface through which the heat flows. Thus, Fourier's law of heat conduction in the isotropic and homogeneous medium can be defined as

$$q_i = -K\theta_{,i}. \tag{1.2.1}$$

As shown above, the heat flux q_i is the instantaneous result of temperature gradient $\theta_{,i}$. This Eq. (1.2.1), when combined with the law of energy conservation

$$-q_{i,i} + \rho H = \rho C_E \dot{\theta} \tag{1.2.2}$$

yields the following parabolic-type heat transfer equation in the absence of heat source:

$$K\theta_{,ii} = \rho C_E \dot{\theta}. \tag{1.2.3}$$

The above law of heat conduction is widely employed by engineers, physicists, and mathematicians to solve conventional heat conduction problems that involve large spatial dimensions and long-time behavior. However, due to the infinite velocity of heat wave propagation, Fourier's law predicts unsatisfactory outcomes in the contexts of the problems incorporating short-time behavior, high heat flux, and extreme thermal gradients, like laser-material interactions. Furthermore, rapid development of nanotechnology has led to the creation of nano-scale devices and it has been discovered that the heat conduction of these small devices exhibits a variety of different phenomena, such as the size effect and wave phenomena, that are not covered by the classical Fourier's law.

Since the 1900s, efforts have been made to eradicate the paradox of infinite heat propagation speed. It is worth recalling that in 1867 when formulating a kinetic theory of gases, Maxwell (1867) hypothesized the appearance of wave-type heat flow and suggested a modification to Fourier's law (see Chandrasekharaiah (1986b)). The phenomenon of thermal signals like wave type is now known as the "second sound" effect. Later, Nernst (1917) suggested that heat waves may occur in good thermal conductors at low temperatures (see Ward and Wilks (1951)). Landau (1941) identified the second sound for superfluid helium as the propagation of phonon density disturbance and deduced that it should have a speed of $v_p/\sqrt{3}$ at 0 K, where v_p denotes the speed of the conventional sound (first sound). Additionally, Tisza (1947) made a prediction regarding the possibility of extremely low rates of heat propagation in liquid helium. Experimentally, Peshkov (1944) discovered the second sound for the first time in liquid helium and determined that its speed was 19 m/s at 1.4 K. Furthermore, the experimental verification of Tisza's and Landau's predictions was conducted by Maurer and Herlin (1949), Pellam and Scott (1949), and Atkins and Osborne (1950). The occurrence of a second sound in fluid helium has been established at temperatures below 2.2 K, as documented by Lifshitz (1958). The effect was subsequently noted in some other crystals (see McNelly et al. (1970), Jackson et al. (1970), Jackson and Walker (1971), Rogers (1971)).

1.3 Modified Fourier's Law of Heat Conduction

In order to address the limitations of Fourier's law, several efforts have therefore been carried out to develop various non-Fourier heat conduction models. These models specifically consider the wave-like phenomenon that occurs in the heat transfer process. We must recall some achievements in this direction as described below.

1.3.1 Cattaneo-Vernotte (CV) Law of Heat Conduction

Cattaneo (1958) and Vernotte (1958; 1961) put forth a generalized version of Fourier's law to explain the occurrence of second sound from a theoretical perspective. This generalized law in the case of an isotropic and homogeneous material is given as

$$q_i + \tau_q \frac{\partial q_i}{\partial t} = -K\theta_{,i}, \tag{1.3.1}$$

where τ_q is called the thermal relaxation time. When a temperature gradient is suddenly imposed on a volume element, τ_q depicts the time lag required to achieve the steady state of heat conduction in that element. Combining Eq. (1.3.1) with the energy equation (1.2.2), the following hyperbolic-type heat conduction equation is obtained:

$$K\theta_{,ii} = \left(1 + \tau_q \frac{\partial}{\partial t}\right)\left(\rho C_E \dot{\theta} - \rho H\right). \tag{1.3.2}$$

Unlike the classical Fourier's law of heat conduction, Eq. (1.3.2) predicts a finite speed of thermal wave which is equal to $(K/\rho C_E \tau_q)^{1/2}$.

This law yields favorable outcomes in situations when there are localized moving heat sources exhibiting high intensity, shock wave propagation, fast propagating fracture tips, laser material processing, laser surgery, and similar applications. Numerous researchers reported the physical values of the τ_q for various materials while conducting their experiments (Chandrasekharaiah (1986b; 1998)). Chester (1963) described the physical interpretation of Eq. (1.3.1) and provided an estimation for the value of τ_q in a subsequent manner:

$$\tau_q = \frac{3K}{\rho C_E v_p^2},$$

where v_p is the speed of ordinary sound. The range of τ_q for metals and gases was discovered to be between $10^{-14}s$ to $10^{-10}s$ by various researchers (see Nettleton (1960), Chester (1963; 1966), Maurer (1969), Mengi and Turhan (1978) and references therein). Several researchers such as Baumeister and Hamill (1969; 1971), Chan et al. (1971), Maurer and Thompson (1973), and Sadd and Cha (1982) provided experimental proofs supporting this modified heat conduction theory in the context of very high heat-flux and very short time intervals. According to their reports, the hyperbolic type heat conduction equation (Eq. (1.3.2)) associated with the modified Fourier's law (Eq. (1.3.1)) yields more physically meaningful outcomes in these situations when compared to the parabolic type diffusion equation based on Fourier's law of heat conduction.

1.3.2 Dual-Phase-Lag (DPL) Law of Heat Conduction

High-rate heating on thin film structures has gained a lot of attention from researchers due to the development of short-pulse laser technologies and their applicability to modern micro-fabrication technologies (Tzou (1995a)). A shorter response time leads to a non-equilibrium thermodynamic transition and considerable implications on energy exchange during heat transmission; consequently, the model formulation becomes microscopic in nature. Based on several experimental findings, it has been observed that the CV heat conduction law is not applicable in certain situations, particularly when considering the heating of thin films. Hence, Tzou (1995a; 1995b) has proposed the following heat conduction law by introducing two phase-lag parameters:

$$q_i(\boldsymbol{x}, t + \tau_q) = -K\theta_{,i}(\boldsymbol{x}, t + \tau_\theta). \tag{1.3.3}$$

The delay time τ_θ is referred to as the phase-lag of the temperature gradient and it is viewed as the time that is induced by the microstructural interactions. The other delay time τ_q, also known as the phase-lag of the heat flux, is interpreted as the relaxation time owing to the fast-transient effects of thermal inertia. Therefore, this heat conduction law is called as the dual-phase-lag (DPL) heat conduction law. Based on the Eq. (1.3.3), if a temperature gradient is imposed at a point inside the medium at time $t + \tau_\theta$, the heat-flux will be experienced at that point at time $t + \tau_q$, provided that $\tau_q > \tau_\theta$. However, when $\tau_\theta > \tau_q$, the outcomes are the opposite. It is worth mentioning that the conventional Fourier law performs at a macroscopic scale in both space and time, whereas the relation (1.3.3) performs at a microscopic scale in both space and time.

1.4 Bio-Heat Transfer Models

The application of mathematical modeling has proven to be effective in numerous practical biomedical contexts, including the characterization of tumors, the optimization of thermal dosage for cancer treatments, and the computation of thermal damage resulting from medical therapy. As a result of the rapid advancement of laser technology, focused ultrasound, microwave, and radio-frequency, a variety of modern thermal therapies are now extensively used in medical treatment. In order to develop effective treatment plans and innovative therapeutic heating systems, precise predictions of the heat transmission mechanism and associated soft tissue deformation are crucial. Due to the fact that the thermal behavior of biological tissues is influenced by a number of complex phenomena, including blood flow, metabolic heat production, and sweating, various governing equations have been established by researchers.

In an attempt to provide an explanation for this process, Pennes (1948) first developed a bioheat transfer equation to predict the temperature distribution in the human forearm. Pennes bioheat transfer equation is stated in the following form:

$$\rho C_E \dot{\theta} = -q_{i,i} + w_b c_b (T_b - T) + q_{\text{met}} + q_{\text{ext}} \,. \tag{1.4.1}$$

The classical Fourier's law, which predicts an infinite speed propagation of thermal signals, provided the basis for Pennes bioheat transfer equation. Nevertheless, experimental results suggest that the basic framework of blood is non-homogeneous, resulting in a finite rate of heat exchange between tissues and blood (see Peshkov (1960), Bertman and Sandiford (1970), Mitra et al. (1995)). Therefore, Liu et al. (1995) developed a thermal-wave model of bio-heat transfer based on the CV law in living tissues. This modified Pennes' bioheat transfer equation can be written as

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right)\left(\rho C_E \dot{\theta} - w_b \rho_b c_b (T_b - T) - q_{\text{met}} - q_{\text{ext}}\right) = K\theta_{,ii}. \tag{1.4.2}$$

The dual-phase-lag model has also been utilized more frequently in bioheat transfer models (see Ho et al. (2003), Antaki (2005), Liu et al. (2012), Majchrzak and Mochnacki (2018)).

1.5 Thermoelasticity: Definition and Applications

Thermoelasticity is a field of mechanics that studies the behavior of materials under the combined effects of mechanical deformation and temperature variations. This is an advancement in the field of elasticity theory that incorporates the consideration of heat effects, including thermal stress, strain, and deformation. When the temperature of a material changes, materials has the potential to experience either expansion or contraction, causing it to deform mechanically. In the same way, adding mechanical loads to a material can cause its temperature to change because the mechanical energy is turned into heat. Consequently, thermoelasticity theory is utilized to forecast the thermomechanical interactions within an elastic body. The theory of thermoelasticity differs from the conventional theory of elasticity by considering the influence of internal forces on the temperature field, as well as the impact of temperature changes on deformation. Therefore, thermoelasticity theory relies on two separate but equally important theories: elasticity theory and heat conduction theory.

Due to its broad applications across numerous sectors, thermoelasticity theory has attracted a great deal of interest from engineers and researchers in a variety of scientific and technology disciplines. Thermoelasticity holds significant importance in diverse engineering disciplines, including structural engineering, thermal analysis, and materials science. The significance of this factor is of utmost importance in the design and analysis of components that experience fluctuations in temperature and mechanical loads, such as pipelines, bridges, engines, and aerospace components. Thermoelastic concepts are useful in the study of geological processes and seismic events because they explain how rocks and Earth materials respond to variations in temperature and stress. The use of thermoelasticity in the development of medical devices and implants helps to ensure that they are able to survive the temperature variations and mechanical stress that occur within the body. Thermoelasticity theory also has applications in other engineering disciplines and technologies, including nuclear, mining, chemical, and acoustical engineering. Moreover, other scientific disciplines such as electro-thermoelasticity, magneto-thermoelasticity, viscothermoelasticity, thermo-piezoelectric theory, poro-thermoelasticity, aero-thermoelasticity, etc. are built on the foundation of thermoelasticity.

1.5.1 Classical Coupled Theory of Thermoelasticity and its Limitations

According to the uncoupled thermoelasticity theory, heat conduction in a material is solely influenced by a temperature gradient. This theory disregards the impact of other mechanical factors, such as the effect of the elastic property of the material on heat transmission. On the contrary, the coupled thermoelasticity theory addresses the effects of deformation on temperature distribution as well as the influence of temperature on stress and strain distributions. The stress and temperature distributions are simultaneously evaluated in coupled theory, whereas in uncoupled theory the physical fields are analyzed one at a time. Therefore, the primary objective of the coupled thermoelasticity theory is to overcome the limitation seen in the uncoupled thermoelasticity theory, wherein changes in elasticity do not impact the temperature and vice versa.

It is worth to recall that the coupling between thermal and mechanical fields was first introduced by Duhamel (1837), who constructed equations that incorporated a temperature gradient term for the strain fields. In his work, the governing equations of thermoelasticity were derived focusing solely on the influence of the temperature field on deformation. Similar stress-strain and temperature relations were obtained by Neumann (1841). Later, Thomson (1853) became the pioneer in utilizing the principles of thermodynamics to investigate the distributions of stress and strain within an elastic body when subjected to a thermal gradient. Later on Biot (1956b) employed thermodynamics laws and established a satisfactory mathematical formulation of coupled thermoelasticity theory. This theory is therefore referred to as the classical coupled dynamical thermoelasticity theory. The author also provided a methodology for deriving a general solution to the thermoelastic problem in a homogeneous and isotropic medium. According to Biot (1956b), the basic governing equations and constitutive relations for an anisotropic medium under the theory of linear classical coupled thermoelasticity can be summarized as follows:

Equation of motion:

$$\sigma_{ij} + F_i = \rho \ddot{u}_i. \tag{1.5.1}$$

Energy equation:

$$\rho T_0 \dot{S} = -q_{i,i} + \rho H. \tag{1.5.2}$$

Constitutive relations:

$$\sigma_{ij} = C_{ijkl}e_{kl} - \gamma_{ij}\theta, \tag{1.5.3}$$

$$\rho T_0 S = \rho C_E \theta + \gamma_{ij} T_0 e_{ij}, \tag{1.5.4}$$

$$q_i = -K_{ij}\theta_{,j}. \tag{1.5.5}$$

Now, based on the Eqs. (1.5.1)-(1.5.5), the coupled field equations can be expressed in terms of u_i and θ as

$$C_{ijkl}e_{kl,j} - \gamma_{ij}\theta_{,j} + F_i = \rho\ddot{u}_i, \tag{1.5.6}$$

$$K_{ij}\theta_{,ij} = \rho C_E \dot{\theta} + T_0\gamma_{ij}\dot{e}_{ij} - \rho H. \tag{1.5.7}$$

Thus, Eqs. (1.5.6) and (1.5.7) represent the displacement equation of motion and heat conduction equation, respectively in the context of the classical coupled thermoelasticity for homogeneous and anisotropic medium.

Further, for homogeneous and isotropic medium, the above Eqs. (1.5.6) and (1.5.7) take the following forms:

$$\mu u_{i,jj} + (\lambda + \mu)\, u_{j,ji} - \gamma\theta_{,i} + F_i = \rho\ddot{u}_i, \tag{1.5.8}$$

$$K\theta_{,ii} = \rho C_E \dot{\theta} + T_0\gamma\dot{e}_{kk} - \rho H. \tag{1.5.9}$$

The above-mentioned Biot's thermoelasticity theory is regarded as an elegant framework for investigating a variety of issues involving the coupling effects of mechanical and thermal fields. Pioneering works reported by Chadwick (1960), Boley and Weiner (1960), Nowacki (1962; 1975a), Parkus (1976), Nowinski (1978), Dhaliwal and Singh (1980), Chandrasekharaiah (1986b), etc. provide a comprehensive and detailed discussion with interesting applications and theorems based on the Biot's theory. In this manner, although the deficiency of uncoupled thermoelasticity was eliminated, the parabolic type partial differential equation of heat conduction was still there, leading to the paradox of infinite speed of thermal wave propagation. This behavior implies that the observed effects will be instantaneously noticed at an infinite distance far from the source, which is inconsistent with the actual physical phenomena. As a result, this theory does not provide an adequate or sufficient description of a solid's response when subjected to rapid transient loading, such as short laser pulses, and when the temperature is low. Due to the defects of this theory in various circumstances, several modifications in the classical thermoelasticity theory have been proposed over the past six decades. The term "generalized thermoelasticity theories" is frequently used to describe these modified thermoelasticity theories. The next subsections provide a concise overview of the generalized thermoelasticity theories that are pertinent to the present thesis.

1.5.2 Generalized Thermoelasticity

Generalized thermoelasticity theories have been constructed as modified or alternative versions of the classical coupled thermoelasticity theory. These theories aim to address the apparent paradox of thermal wave propagation at an infinite speed. These theories can be classified into two types. The first approach is founded upon the modification of the heat conduction law. This means that Fourier's law is replaced with a more appropriate modification by incorporating the concept of phase-lags/thermal relaxation parameters or new constitutive field variables. The second type of thermoelasticity theory uses thermodynamic principles to improve the conventional theory in order to derive modified constitutive equations. In the second type, Fourier's law remains unaltered. The following provides a brief overview of some modifications to classical thermoelasticity theory:

1.5.2.1 Lord-Shulman (LS) thermoelasticity theory

The theory of generalized thermoelasticity presented by Lord and Shulman (1967) has been extensively investigated and is considered one of the prominent modified theories in the field of thermoelasticity. This is the most basic form of generalized thermoelasticity, achieved by substituting the classical Fourier law of heat conduction with a CV heat conduction law in classical thermoelasticity (CTE). The present model incorporates a hyperbolic heat conduction equation and therefore resolves the paradox of infinite speed of heat propagation inherent in Biot's theory. It is also known as extended thermoelasticity or thermoelasticity with one thermal relaxation parameter.

1.5.2.2 Green-Lindsay (GL) thermoelasticity theory

The Green-Lindsay (GL) thermoelasticity theory is the second most prominent thermoelasticity theory after the LS theory. This model was established by Green and Lindsay (1972) and takes into account the "second sound" effects. Prior to this theory, Muller (1967) discovered the entropy production inequality with some constraints on constitutive equations. Subsequently, Green and Laws (1972) presented a re-formulation of this inequality in a more generalized manner. Employing the concept of Green and Laws, Green and Lindsay (1972) developed a new modification to coupled thermoelasticity theory. This theory admits the finite speed of thermal waves without violating the Fourier law on the assumption that the medium contains a center of symmetry. In Green and Lindsay's (GL) model, two constants which are known as thermal relaxation parameters have been imposed in the constitutive equations. This GL theory is also recognized as a temperature-rate dependent thermoelasticity theory due to the involvement of temperature-rate terms. The following are the constitutive relations in the context of GL theory:

$$\sigma = C_{ijkl}e_{kl} - \gamma_{ij}(\theta + \tau_1\dot{\theta}), \tag{1.5.10}$$

$$T_0\rho S = \rho C_E(\theta + \tau_2\dot{\theta}) + \gamma_{ij}T_0 e_{ij}, \tag{1.5.11}$$

$$q_i = -K_{ij}\theta_{,j}, \tag{1.5.12}$$

where τ_1 and τ_2 are the two thermal relaxation time parameters satisfying the property that $\tau_1 \geq \tau_2 > 0$.

1.5.2.3 Green-Naghdi (GN) thermoelasticity theory

During the 1990s, Green and Naghdi (1991; 1992; 1993) made significant contributions to the field of coupled thermoelasticity by introducing distinct ideas that expanded upon existing concepts. They introduced three alternative versions of thermoelasticity theory which are classified as GN-I, GN-II, and GN-III. One notable aspect of Green and Naghdi's theories is that it does not rely on an entropy production inequality to derive the constitutive equations, however, it is based on the entropy balance law. In these theories, thermal displacement (ν), gradient of thermal displacement ($\nabla\nu$), and temperature (θ) are considered as the constitutive variables, satisfying the property that $\dot{\nu} = \theta$. For the homogeneous and isotropic medium, the laws of heat conduction used in these thermoelasticity theories are given as follows:

- **GN-I theory:** $q_i = -K\theta_{,i}$
- **GN-II theory:** $q_i = -K^*\nu_{,i}$
- **GN-III theory:** $q_i = -\left[K\theta_{,i} + K^*\nu_{,i}\right].$

The GN-I model is equivalent to the CTE model by taking the linearized form of this model. The temperature and the gradient of temperature are considered as the independent thermal variables in this model. For GN-II and GN-III models, thermal displacement has a great role. In the GN-II model, the temperature and the gradient of thermal displacement are considered as the independent thermal variables. This model predicts the finite propagation of thermal signals. The authors also deduced that there is no energy dissipation in the relationship of model GN II. Therefore, this theory is referred to as the thermoelasticity theory without energy dissipation. Moreover, the temperature, the gradient of the temperature, and the gradient of thermal displacement are taken as the independent thermal variables in the GN-III model. This model includes both GN-I and GN-II as limiting cases. The exponential decay of solutions has been obtained in this case, however, this theory also predicts the instantaneous propagation of thermal waves (see Puri and Jordan (2004)) and has the same drawback as the conventional theory.

1.5.2.4 Dual-Phase-Lag (DPL) thermoelasticity theory

In order to expand the concept of generalized thermoelasticity theory, Chandrasekharaiah (1998a) suggested a dual phase-lag thermoelasticity theory that admits a finite speed of thermal wave propagation. This theory is constructed by using the dual phase-lag heat conduction law in place of classical Fourier's law. The author provides a further explanation that the DPL thermoelasticity theory, which is founded on Eq. (1.3.3), will admit thermal disturbances to propagate at a finite speed when $\tau_q > \tau_\theta$. If this condition is not fulfilled, the dilemma of infinite speed will continue to exist under this theory.

1.5.2.5 Three-Phase-Lag (TPL) thermoelasticity theory

Roychoudhury (2007a) developed this three-phase-lag thermoelasticity theory by incorporating three distinct phase-lags into the heat conduction law given by Green and Naghdi (1993) (GN-III model). In addition to τ_q and τ_θ, the author included an additional phase-lag parameter τ_ν, for the gradient of thermal displacement. Therefore, the modified heat conduction law under the TPL thermoelasticity theory is expressed as follows:

$$q_i(\boldsymbol{x}, t+\tau_q) = -\left[K\theta_{,i}(\boldsymbol{x}, t+\tau_\theta) + K^*\nu_{,i}(\boldsymbol{x}, t+\tau_\nu)\right]. \tag{1.5.13}$$

According to the author, the heat-flux is obtained at time $t+\tau_q$, whereas the temperature gradient and thermal displacement gradient are noticed at time $t+\tau_\theta$ and $t+\tau_\nu$, respectively.

1.5.2.6 Moore-Gibson-Thompson (MGT) thermoelasticity theory

The increasing number of applications of generalized thermoelasticity theory has prompted more investigation into alternate theories that demonstrate the finite speed of thermal signals. To overcome the apparent shortcoming inherent in the GN-III model, the concept of a thermal relaxation parameter is introduced in the heat conduction law of the GN-III model. In 2019, Quintanilla (2019) employed this concept and introduced a totally innovative theory of thermoelasticity which is known as the Moore-Gibson-Thompson thermoelasticity theory. The MGT heat conduction law in this theory can be written as follows:

$$q_i + \tau_q \frac{\partial q_i}{\partial t} = -K\theta_{,i} - K^*\nu_{,i}. \tag{1.5.14}$$

Clearly, Eq. (1.5.14) is the generalization of heat conduction laws defined in the LS theory and GN-III theory. The author also discussed the stability of MGT thermoelasticity theory and mentioned that the solution under this theory is exponentially stable if the material parameters τ_q, K and K^* satisfy the relation $K > K^*\tau_q$ (see Quintanilla (2019)).

1.6 Two-Temperature Thermoelasticity

Gurtin and Williams (1966) have drawn serious attention to the modification of the Clausis-Duhem inequality which represents the second law of thermodynamics as a suitable form. This proposed inequality involves the distinction between two heat transfer mechanisms implying that the external heat supply and the internal heat flux over the body are independent. Due to this distinction, in the second law of thermodynamics, the separation of entropy flow can be considered and it can also be assumed that the two different temperatures occur from the same proportionality factor. Then the conductive temperature is assumed to be the volume-relevant temperature and the thermodynamic temperature is assumed to be the surface-relevant temperature. Subsequently, Chen and Gurtin (1968) used this modified inequality and derived the heat conduction theory including thermodynamic and conductive temperatures. The proportionality of the difference between thermodynamic and conductive temperatures to the supply of heat under steady-state situations is also proved by Chen and Gurtin (1968). Two temperatures coincide in steady-state situations with the absence of external heat supply. However, these two temperatures do not coincide in the conditions of time-dependent, even if the heat supply is absent.

The thermoelasticity theories based on the two temperatures have dragged considerable attention in the last few decades. Chen et al. (1969) used the fundamental laws of thermodynamics and obtained the governing equations for the two-temperature thermoelasticity theory. Based on Biot's theory of thermoelasticity for isotropic medium, the heat conduction law and the two-temperature relation are as follows:

$$q_i = -K\phi_{,i}, \tag{1.6.1}$$

$$\phi - \theta = a\phi_{,ii}, \tag{1.6.2}$$

where ϕ and θ represent the conductive temperature and thermodynamic temperature, respectively. a denotes the two-temperature parameter. Further, Youssef (2006) constructed a generalization of the LS theory as well as GL theory by taking into account the involvement of two temperatures and derived the corresponding constitutive equations. Recently, Shivay and Mukhopadhyay (2019) have formulated the temperature-rate dependent two-temperature (TRDTT) thermoelasticity theory with the help of thermodynamical laws and derived all the basic governing equations and constitutive relations for this theory. In this theory, a generalized two-temperature relation associated with temperature-rate terms has been obtained and it has been shown that the two-temperature relation used by Youssef (2006) is identical to this relation in a specific situation. The uniqueness of solutions under this modified TRDTT thermoelasticity theory has further been discussed by Shivay and Mukhopadhyay (2019).

1.7 Poro-Thermoelasticity

Porous materials are defined as solid materials that include a large number of interconnected pores, often on the order of micrometers, with sufficient interconnectivity to allow fluid movement through the material. Porous materials are most commonly associated with clays and rocks, but foams, biological tissue, and paper products also fit into this group. Researchers have taken an interest in porous materials due to their wide range of applications in disciplines connected to their study, including osseous tissue, polyurethane foam, water-saturated soil, sound-absorbing materials, etc. To explain the mechanical behavior of water-saturated soil, Biot (1956a; 1962) first proposed the idea of a poroelastic material. This material is made up of fluid and deformable solid, with the solid forming a porous skeleton with numerous microscopic spaces that are connected to one another and filled with the fluid. Consequently, poroelasticity is focused on characterizing the relationship between elastic solid deformation and viscous fluid flow within a porous medium. Subsequently, the interaction between thermoelastic solid and fluid, also known as porothermoelasticity, is receiving considerable attention due to various applications in the geophysics area and significant topics. Numerous applications exist for the topic of poro-thermoelasticity, particularly in examining the effect of employing waste materials on the disintegration of asphalt concrete mixture (ACM). Also, coupled thermal and poromechanical processes play a significant role in a number of geomechanics-related issues, such as hydraulic fracturing in geothermal reservoirs or highly heated petroleum-bearing formations and borehole stability. Furthermore, it has applications in a variety of domains, including hydrology, petroleum engineering, soil dynamics, earthquake engineering, and biomechanics.

The thermo-mechanical coupling in the poroelastic material is more complicated than it is in the classical case since it involves mechanical and thermal interaction both inside each phase as well as coupling between the phases. The complete system of equations for porothermoelasticity and a study of the heat effects on wave propagation in liquid-filled porous medium were both accomplished by Pecker and Deresiewicz (1973). Further, McTigue (1986), Kurashige (1989), and Wang and Papamichos (1994) investigated the thermoelasticity of a fluid-saturated porous medium and discussed the fluid and heat flow in a poroelastic medium. In the framework of LS thermoelasticity theory, Youssef (2007) formulated the generalized theory of porothermoelasticity. The governing equations in this theory were derived by following the LS theory of thermoelasticity, and therefore, this theory also predicts the finite speed of thermal waves.

1.8 Other Extended Thermoelasticity Theories

In addition to the aforementioned discussions on thermoelasticity and poro-thermoelasticity theories, various modified theories have been formulated by incorporating the coupling of thermoelasticity with other fields, such as piezoelectric fields, magnetic fields, viscosity, and so forth. The concept of fractional calculus, non-locality, and micropolar materials are also used to extend the theory of thermoelasticity. Several references that discuss fractional order thermoelasticity include the works of Sherief et al. (2010), Youssef (2010a; 2016), El-Karamany and Ezzat (2011a; 2011b), Abbas (2014b; 2015b), and Povstenko (2015; 2019). Several researchers have contributed to the advancement of generalized thermoelasticity within the context of the micropolar theory which can be found in the articles reported by Eringen (1970), Boschi and Iesan (1973), Chandrasekharaiah (1986a), Ciarletta (1999), Sherief et al. (2005), Othman and Singh (2007), etc.

1.9 Literature Review

Due to the wide range of applications of thermoelasticity in engineering, science, and technology, various studies on thermomechanical interactions have been conducted in both mathematical and physical terms in the last few decades. Some distinct features of generalized thermoelasticity theories, which are developed to overcome the limitations of classical thermoelasticity theory and get suitable results in extreme heat and mechanical circumstances, have been reported in literature. Numerous researchers have devoted considerable attention to the investigation of thermoelasticity theories in various engineering contexts, and their findings have been documented in existing literature. In this respect, several books and review articles by Nowacki (1969; 1975b), Chandrasekharaiah (1986b; 1998a), Joseph and Prezios (1989; 1990), Hetnarski and Ignaczak (1999), Ignaczak and Ostoja-Starzewski (2010), Straughan (2011), Parkus (2012), and Iesan and Scalia (2013) are worth to be mentioned. In addition, PhD theses by Roushan Kumar (2010), Rajesh Prasad (2012), Shweta Kothari (2013), Rakhi Tiwari (2017), Bharti Kumari (2017), Shashi Kant (2018), Anil Kumar (2018), Manushi Gupta (2021), and Om Namha Shivay (2021) provide a detailed investigation on some generalized thermoelasticity theories and their applicability to various engineering problems. Some interesting work that are relevant to the present thesis is recalled in the following:

In order to investigate a semi-infinite elastic medium under Biot's theory, Sternberg and McDowell (1957) used Green's function method and showed how stress and displacement fields behave in a steady state. For the conventional coupled theory, Lessen (1957; 1959) investigated the propagation of thermoelastic wave propagation in the thermoelastic medium and also studied the problem considering thermal shock. Herrmann (1958) obtained the complementary energy principle in the classical thermoelasticity theory. In the study conducted by Paria (1958), the aforementioned theory was employed to get a general solution for the distribution of stress and temperature within an isotropic thermoelastic solid half space. Further, Chadwick (1962) performed an investigation into the thermoelastic interaction that occurs within thin plates and rods. Parkus (1963) conducted a survey examining different approaches to address thermoelastic problems, categorized according to linear and nonlinear, stationary and non-stationary, isotropic and anisotropic, deterministic and random methods. Biot's variational concept was applied by Lardner (1963) to a variety of problems for one-dimensional heat conduction. These examples demonstrate how the variational approach can be used to solve the problems with temperature-dependent material properties and heat flow boundary conditions. Soler and Brull (1965) used perturbation techniques to obtain the solution to the classical coupled thermoelastic problem. The solution of the equations of classical coupled thermoelasticity was reported by Ignaczak and Nowacki (1966) using surface integrals. The continuous spatial solutions to the two-dimensional heat conduction problem were derived by Wilson and Nickell (1966) using the variational technique of the finite element method. Purushotham (1968) used integral transform techniques to address the problem of classical thermoelasticity. Tanaka et al. (1970) utilized a direct technique of the boundary element method for the quasi-static problems in coupled thermoelasticity and performed analysis on various three-dimensional problems. The findings and applications of this classical thermoelasticity theory were well documented in books by Chadwick (1960), Boley and Wiener (1960), Carlson (1973), Nowacki (1975a), Parkus (2012), Nowinski (1978), and Dhaliwal and Singh (1980). However, researchers gradually shifted their focus from studying classical thermoelasticity to modifying classical thermoelasticity in order to address the problem of infinite heat propagation speed.

Lord and Shulman (1967) presented the extended thermoelasticity theory, which generalizes the classical thermoelasticity theory by applying an appropriate modification to Fourier's law of heat conduction. Following the LS theory, Achenbach (1968) investigated the propagation of stress and temperature discontinuities. Chen and Gurtin (1970) investigated the second sound effect in thermoelastic materials with memory using the LS theory. Lord and Lopez (1970) conducted an investigation into the one-dimensional scenario involving an elastic half space that experiences both thermal and mechanical disturbances. They derived alternative expressions of the solutions, which were subsequently employed to address the specific issue of a step-strain input. Norwood (1972) used the LS theory to investigate the problem of a half space that was exposed to step-time inputs of temperature. Nayfeh and Nemat-Nasser (1972) studied Lamb's problem in two dimensions and pointed out the impact that relaxation times have on the speed and amplitude of waves. Tokuoka (1973) discussed thermo-acoustical wave propagation within the linear thermoelastic material in the framework of LS theory. Under this generalized thermoelasticity theory, Ignaczak (1979b) showed the uniqueness results. The theory of LS thermoelasticity for the general anisotropic medium was further extended by Dhaliwal and Sherief (1980). Sherief (1987) established a uniqueness theorem and the stability of the null solution in the Lyapunov sense for this generalized thermoelasticity. He also discussed the results of the LS theory as a special case. In the framework of Lord-Shulman thermoelasticity theory, Chandrasekharaiah (1986b) provided a comprehensive elaboration of the uniqueness of the solution, domain of influence results, variational principle, and reciprocity theorem. In the domain of Laplace-transforms, Anwar and Sherief (1988) derived a boundary integral equation formulation for generalized thermoelasticity. Furukawa et al. (1990) investigated a one-dimensional problem within an infinite body containing a circular cylindrical hollow. They derived a short-time approximated solution and examined the influence of relaxation time on the physical fields. Also, Mukhopadhyay et al. (1991) conducted a study on thermoelastic waves in an infinite solid containing a spherical cavity. Authors examined the scenario when the inner boundary of the cavity experiences a sudden increase in temperature and dynamic pressure on its surface.

In 1972, Green and Lindsay (1972) developed another modification to classical thermoelasticity theory without violating the classical Fourier's law. The authors demonstrated the finite speed of thermal waves as well as the uniqueness of this theory. Boschi and Iesan (1973) developed a generalization of the Green and Lindsay (GL) theory to micropolar thermoelasticity. For Green and Lindsay's linear thermoelasticity, Ignaczak (1978a) established a Boggio-type decomposition theorem. The surface waves in LS and GL theories were investigated by Agarwal (1978) using a homogeneous and isotropic thermoelastic half space. Further, Agarwal (1979) studied the propagation and stability of time-dependent harmonic thermoelastic plane waves under these theories. Igznaczak (1980) provided a study that contains a comprehensive literature analysis of the thermoelasticity theories predicting the finite speed of thermal waves. This review includes the GL theory. A general finite element model was presented by Prevost and Tao (1983) with the purpose of analyzing transient phenomena in a thermoelastic medium while taking GL theory into consideration. The effect of rotation on the properties of wave propagation was studied by Chandrasekharaiah and Srikantiah (1984) in the context of a problem involving a homogeneous and isotropic unbounded thermoelastic body that rotates with uniform angular velocity. Tao and Prevost (1984) used the perturbation technique to investigate the impact of relaxation time parameters on wave propagation under the GL theory. Using the GL theory, Royhoudhuri (1985) investigated the harmonically time-dependent thermoelastic plane wave propagation in infinite rotating media. Both the LS and GL theories were considered by Noda et al. (1989) to examine the problems for an infinite solid containing a hole. Dhaliwal and Rokne (1989) employed this theory to address a boundary value problem concerning an isotropic elastic half space. The plane boundary of the half space was either rigidly fixed or stress-free and it experienced a sudden temperature rise. Sherief (1993) utilized the state space approach to solve a thermal shock problem and a problem involving a layered media, both of which lacked heat sources. In a study conducted by Ezzat (1995), stress and temperature distributions were determined for a situation involving a continuous line heat source within a cylindrical region. Following the GL theory, Sinha and Elsibai (1996; 1998) examined the impact of two distinct relaxation times on the reflection behavior of two different incident waves and on the transmission of stonely waves at the interface between two semi-infinite media, respectively. Suh and Burger (1998) conducted a spectral analysis in order to obtain a deeper understanding of the impacts of thermomechanical coupling and relaxation parameters on thermoelastic responses. Ezzat and El-Karamany (2002) presented a generalized theory of thermoviscoelasticity by extending the theories of generalized thermoelasticity by Lord-Shulman and Green-Lindsay. El-Maghraby (2005) investigated the thermoelastic behavior of two-dimensional thick plate under the Lord-Shulman and Green-Lindsay theories, with a traction-free upper surface that was subjected to a known temperature distribution and a rigid lower surface that was thermally insulated. Within the framework of GL theory, Othman (2010a; 2010b) studied some problems for magneto-thermoelasticity and electro-magneto-thermoelasticity, respectively. The transient thermoelastic response within a functionally graded thick hollow cylinder was explored by Darabseh et al. (2012) using the Galerkin finite element method under the GL theory. The analysis conducted by Zenkour (2015) focused on investigating the impact of thermal shock within a three-dimensional thermoelastic medium. The study also assessed the outcomes within the context of several other thermoelasticity theories.

Over the years, researchers have shown a great deal of interest in attempting to grasp the new generalized thermoelasticity theories (GN-I, GN-II, and GN-III theories), which were developed by Green and Naghdi (1991; 1992; 1993). In the context of the GN-II theory, Chandrasekharaiah (1996c) presented an initial boundary value problem that was described in terms of stress and entropy-flux. The author also demonstrated the uniqueness of the solution for this problem. Chandrasekharaiah (1996a) utilized the GN-II theory to investigate the behavior of one-dimensional waves in a half-space for homogeneous and isotropic materials. The waves under consideration were assumed to be a result of abrupt changes in temperature and stress applied to the boundary. The GN-II thermoelasticity theory was also used by Chandrasekharaiah and Srinath (1997a) to investigate the thermoelastic interactions inside an axisymmetric unbounded medium that contains a cylindrical cavity. The investigation of wave propagation in a rotating thermoelastic body was conducted by Chandrasekharaiah and Srinath (1997b). Ieşan (1998) established the fundamental solutions and continuous dependence result in the theory of thermoelasticity without energy dissipation. Quintanilla (1999) and Quintanilla and Straughan (2000) established various theoretical findings based on the GN thermoelasticity theory, including spatial behavior, growth, and uniqueness. The study conducted by Misra et al. (2000) examined thermoelastic interactions within a homogeneous isotropic elastic half space using the GN model. Mukhopadhyay (2002; 2004) investigated the thermoelastic interactions in an unbounded elastic medium with a spherical cavity under the GN-II model. The investigations focused on the effects of thermal shock and harmonically fluctuating temperature, respectively. In order to examine thermal and mechanical waves in a layer, Taheri et al. (2004) employed the GN thermoelasticity of types II and III for isotropic and homogeneous material. Kumar and Sarthi (2006) examined the phenomena of reflection and refraction of thermoelastic waves across various interface combinations within the context of thermoelasticity without energy dissipation. They specifically focused on five different interface combinations and determined the amplitude ratio associated with an imperfect interface. The study conducted by Mallik and Kanoria (2007) focused on examining the thermoelastic interactions inside a functionally graded isotropic unbounded medium caused by the periodically varying heat source utilizing the GN-II theory. Abbas and Othman (2009) addressed the problem of investigating thermoelastic waves in a rotating homogeneous isotropic hollow cylinder using the finite element method in accordance with the Green and Naghdi theory of types II and III. In the study carried out by Abbas (2014a), an eigenvalue approach was employed to derive the solution within the context of the three-dimensional thermoelasticity of GN-II, taking into account temperature-dependent material properties. Marin and Öchsner (2017) established the Hölder stability of solutions and continuous dependency outcomes in the framework of GN-III thermoelasticity for dipolar bodies. Zampoli (2021) studied the singular surfaces in long and thin radiating rod under GN-II theory of thermoelasticity. Prasad and Kumar (2022) examined thermoelastic interactions in a thick granular plate with an axisymmetric temperature distribution in the context of GN-II thermoelasticity theory. Within the framework of GN-II model, Bayat and Nazari (2023a; 2023b) investigated dynamic crack propagation in anisotropic solids.

Based on Tzou's dual-phase-lag heat conduction model (1995a; 1995b), Chandrasekharaiah (1998a) developed a new theory of thermoelasticity. Subsequently, thermoelasticity with phase-lag effects have drawn attention of researchers. Hetnarski and Ignaczak (1999) presented an analytical approach to study generalized thermoelasticity theories. They conducted a comparison between the results obtained from several existing thermoelasticity theories and those obtained from the DPL thermoelasticity theory. Qualitative analysis of this theory have been reported by several authors. It is worth to mention that Quintanilla (2003) investigated the stability criteria of solution in terms of phase-lag parameters for a one-dimensional DPL thermoelasticity theory. Quintanilla (2004c) considered the LS and DPL thermoelasticity theories and examined the convergence as well as the structural stability of these theories. Roychoudhary (2007b) examined the thermoelastic wave propagation within an elastic half space under the DPL model. The study specifically examined the behavior of the wave at the plane boundary, which was subjected to various conditions. Prasad et al. (2010) investigated the characteristics of harmonic plane waves within the framework of dual-phase-lag thermoelasticity theory. The authors provided asymptotic expressions for wave properties and subsequently reported numerical results for both high and low frequency values. Abouelregal (2011) looked at thermoelastic responses in an isotropic solid sphere with a constrained boundary that was exposed to a constant heat flux. In order to investigate the reflection of thermoelastic waves in accordance with dual-phase-lag thermoelasticity theory, Zenkour et al. (2013) took into account the solid half space with temperature-dependent material properties. Using normal mode analysis and an eigenvalue technique, Sarkar (2017) studied a three-dimensional half space problem with temperature-dependent material properties. Magana and Quintanilla (2018) and then Liu and Quintanilla (2018) discussed some theoritical results like the existence, uniquness, and time decay for this DPL theory of thermoelasticity. The works of Biswas (2019), Gupta and Mukhopadhyay (2019), Sharma et al. (2020), Maes and Bockstal (2021), Bazarra et al. (2021a), Gupta et al. (2022), Sharma et al. (2022), Sheokand et al. (2023), and Prasad et al. (2023) are also worth to be mentioned here.

Three-phase-lag (TPL) theory, as proposed by Roychoudhuri (2007a), represents a broader and more comprehensive version of the thermoelasticity theory. Quintanilla (2009a; 2009b) performed a thorough qualitative investigation on this model such as the spatial behavior of solutions and well-posedness. Kumar and Mukhopadhyay (2009) addressed a thermoelastic problem involving an infinitely long cylindrical cavity subjected to a step input temperature on its boundary. They emphasized the importance of phase-lag parameters within the context of TPL thermoelasticity theory. In this theory, Mukhopadhyay and Kumar (2010) looked into a two-dimensional thick plate and investigated wave propagation as a result of an axisymmetric temperature distribution. Further, Mukhopadhyay et al. (2010) formulated the expression for a Galerkin-type solution within the framework of the TPL thermoelasticity theory. Subsequently, Kothari et al. (2010) obtained the fundamental solutions based on the Galerkin-type representation of the solution of a problem in the same theory. Kumar and Chawla (2011) examined the propagation of plane waves within an anisotropic medium, focusing on the theoretical frameworks of the TPL and DPL models. El-Karamany and Ezzat (2013) proposed the extension of the TPL theory to encompass inhomogeneous and anisotropic medium within the framework of micropolar thermoelasticity theory. The study conducted by Abbas (2014c) investigates the phenomenon of thermoelastic interaction within an unbounded material that is both fiber-reinforced and anisotropic under the TPL model. The propagation of Rayleigh waves under the TPL thermoelasticity theory was examined by Biswas et al. (2017). Biswas and Mukhopadhyay (2019) conducted a study on the three-dimensional vibration analysis of a transversely isotropic cylinder using the matrix Frobenius method. This analysis was performed within the framework of the three-phase lag model of generalized thermoelasticity. Biswas (2020) used state space approach to solve two-dimensional problem of TPL theory of thermoelasticity. In this context, the articles by Sharma et al. (2021), Singh et al. (2021), Deswal et al. (2022), Abo-Dahab et al. (2022), and Mandal and Pal (2023) can also be referred.

Following the introduction of the Moore-Gibson-Thompson (MGT) thermoelasticity theory proposed by Quintanilla (2019), several researchers have reported some interesting works. Conti et al. (2020) used the history-dependent form of the MGT heat conduction equation to present the theory of MGT thermoelasticity. Pellicer and Quintanilla (2020) discussed the instability and uniqueness of some thermomechanical problems under the MGT thermoelasticity theory. Under MGT thermoelasticity, Abouelregal et al. (2020) considered a continuous line heat source in an unbounded isotropic solid and discussed the effect of heat source on the behavior of field variables. From a numerical perspective, Bazarra et al. (2021b) analyzed the MGT thermoelasticity theory and acquired the property of discrete stability. The results were subsequently compared with those of other generalized thermoelastic models. Chen and Ikehata (2021) investigated the Cauchy problem in the dissipative situation for the linear and semilinear MGT heat conduction equation. Singh and Mukhopadhyay (2021a) obtained the Galerkin-type representation of the solution under the MGT thermoelasticity theory. The spatial behavior of the MGT model was investigated by Ostoja-Starzewski and Quintanilla (2021). In addition, they demonstrated that the solutions decay exponentially in particular directions. Under this model, Singh and Mukhopadhyay (2021b) considered the problem of a line heat source and discussed wave propagation in an infinite solid. Moore-Gibson-Thompson thermoelastic model is used by Abouelregal (2022) to investigate the thermomechanical interactions that occur inside a functionally graded nonhomogeneous unbounded medium including a spherical hole. Bazarra et al. (2022) conducted a numerical investigation on a thermoelastic problem that emerges within the framework of the Moore-Gibson-Thompson theory, taking into account the influence of dielectric effects. Further, Atta et al. (2023) investigated the thermodiffusion behavior within a homogeneous spherical shell. They utilized the Moore-Gibson-Thompson theory to analyze this phenomenon incorporating two-time delays. Srivastava and Mukhopadhyay (2023) used long and thin radiating rod to examine thermoelastic interactions under the MGT thermoelastic model. Recently, Singh and Mukhopadhyay (2023a) derived the fundamental solutions for the MGT thermoelasticity theory.

The theory of two-temperature thermoelasticity represents an extension of the existing thermoelasticity theory to encompass non-simple materials. Iesan (1970) examined a linear theory of two-temperature thermoelasticity proposed by Chen et al. (1969) for a homogeneous and isotropic medium. In doing so, Iesan presented the uniqueness, variational, and reciprocity theorems for this theory. The study conducted by Warren (1972) focused on the examination of the two-temperature theory as it applies to cylindrical and spherical cavities composed of isotropic materials. The two-temperature theory was used to study wave propagation by Warren and Chen (1973). Quintanilla (2004b; 2004a) performed a study on the theoretical characteristics of thermoelasticity with two temperatures, focusing on topics such as existence, structural stability, convergence, spatial behavior, exponential stability, and uniqueness. Youssef (2006) developed an extension of the LS theory that incorporates the concept of two temperatures. Subsequently. Youssef and Al-Lehaibi (2007) employed a state-space technique to address the one-dimensional half space problem in this theory. Youssef (2008; 2010b) solved the problems of two-dimensional half space and moving heat source, respectively, when subjected to ramp-type heating within the framework of the generalized two-temperature thermoelasticity. In view of this two-temperature generalized thermoelasticity, Kumar and Mukhopadhyay (2010) looked at the propagation of harmonic plane waves in an isotropic and homogeneous unbounded medium. El-Karamany and Ezzat (2011c) conducted a study on the two-temperature generalization of the GN model and derived several theoretical findings. Additionally, it was demonstrated that the theory of two-temperature thermoelasticity allows for energy dissipation, while the elasticity theory without energy dissipation is only applicable when the two temperatures are the same. Abbas and Youssef (2013) employed the finite element method to analyze thermomechanical behavior in an isotropic and homogeneous thermoelastic material under ramp-type heating. In the study conducted by Abbas (2015a), an investigation was carried out on the phenomenon of a two-temperature generalized thermoelastic thin slim strip. The study focused on analyzing this topic within the framework of GL theory, while also taking into account the presence of a moving heat source. Youssef and Al-Bary (2018) have introduced a new two-temperature relation that predicts the finite speed of thermal waves. Quintanilla (2020) introduced the Moore-Gibson-Thompson thermoelasticity model, which incorporated two temperatures. The author successfully proved the exponential stability and well-posedness of the solutions. Singh and Kumari (2022) conducted an analysis of the propagation of Rayleigh waves in a homogeneous isotropic half space, considering the presence of mass diffusion in TPL thermoelasticity with two-temperature. The study conducted by Bajpai et al. (2023) investigates the impact of two different temperatures on the simultaneous phenomenon of thermomechanical loading and wave propagation inside an isotropic homogeneous plate. The analysis considers various theories of thermoelasticity, including uncoupled, coupled, LS, and GL theories.

There is a growing focus among researchers on investigating the thermal and mechanical interactions within porothermoelastic solids. The poroelastic theory was initially introduced by Biot (1956a) in order to examine the mechanical behavior of soil that is saturated with water. Biot (1962) subsequently expanded his formulation of the poroelasticity theory to encompass a broader solid-phase response. Deresiewicz and Skalak (1963) demonstrated the uniqueness of solutions within the context of Biot's theory. Further, the temperature impact on a poroelastic media was first investigated by Zolotarev (1965). Pecker and Deresiewicz (1973) formulated the whole set of equations within the framework of poro-thermoelasticity. They conducted an investigation on the propagation of waves in a porous media filled with liquid, taking into account the influence of the thermal field. Regarding fluid-saturated porous media, Mctigue (1986) developed the linear thermoelasticity theory, which permits both solid and fluid constituents to experience thermal expansion and compressibility. Kurashige (1989) improved the existing Rice-Cleary theory to develop a thermoelasticity theory specifically applicable to fluid-saturated porous materials. Several authors have further studied various problems based on the porothermoelasticity theory (see Wang and Papamichos (1994), Li et al. (1998), Wang (2000)). Youssef (2007) developed the generalized theory of porothermoelasticity within the framework of the LS theory to account the finite speed of thermal waves. In addition to developing the governing equations for isotropic media, he also proved the uniqueness theorem for this theory. In a study conducted by Sharma (2008), the author examined the phenomenon of wave propagation in a thermoelastic saturated porous media. Under this generalized porothermoelasticity theory, Singh (2011) investigated the propagation of plane wave. Sherief and Hussein (2012) derived a mathematical model for porothermoelasticity theory in the context of short-time filtration instances. They then examined the uniqueness and reciprocity outcomes related to the suggested model. Ezzat and Ezzat (2016) developed a theoretical framework for poro-thermoelasticity on the basis of the fractional-order heat conduction equation. Zampoli and Amendola (2019) demonstrated the spatial behavior of a cylindrical inhomogeneous and anisotropic porothermoelastic solid under the framework of both the DPL and TPL theories. Carcione et al. (2019) provided a comprehensive analysis of the physics and wave modeling in the context of the linear poro-thermoelastic medium. Based on the porothermoelasticity theory, Wei and Fu (2020) explored the fundamental solution. Hobiny (2020) and Alzahrani and Abbas (2020) used the finite element method to solve a one-dimensional problem based on the LS and GN porothermoelasticity theories, respectively. Using the fundamental laws of thermodynamics, Shivay and Mukhopadhyay (2021a; 2021b) developed the governing equations for the theory of temperature-rate-dependent generalized porothermoelasticity and further derived the variational and reciprocity relations for this theory.

The comprehension of the interactions between thermal and mechanical factors is of significant importance in various medical treatments. As a result, thermoelasticity theories have been extensively utilized in the examination of bioheat transfer models. Aksan and McGrath (2003) analyzed the thermal and mechanical behavior of soft-tissue thermotherapy. The study conducted by Xu et al. (2008a; 2008b; 2008c) examined the coupled thermal and mechanical characteristics of skin tissue. Their findings lead them to conclude that the experience of thermal pain is contingent upon the thermal stress generated by an external heat source. The mechanical and thermal behavior of bi-layered skin tissue was investigated by Li et al. (2018) using the GN-II model with varying material properties. To examine the thermoelastic behavior of skin tissue induced by laser irradiation, Li et al. (2019) solved the GN-II and DPL GN-II bioheat transfer theories with the help of the finite element method. For hyperthermia therapy, Ezzat (2020) solved a one-dimensional problem to examine the thermoelastic response in tumorous tissue. Li et al. (2020) studied the transient thermoelastic responses during hyperthermia treatment in the context of the DPL thermo-viscoelastic model. In the context of DPL bioheat transfer model, Zhang et al. (2021a) discussed the thermal and elastic behavior of skin tissue with variable thermal properties. In order to investigate the thermo-physical characteristics of multi-layered biological tissue, Alosaimi et al. (2021) adopted the finite difference method. In the context of TPL model, Zhang et al. (2021b) examined the thermoelastic responses of biological tissue when exposed to a sudden temperature load. Wang et al. (2021) investigated the thermo-mechanical behavior of suddenly heated skin tissue using the dual-phase-lag thermoelastic model of bio heat transfer. The study conducted by Hu et al. (2022) investigated the thermoelastic interactions in biological tissue caused by moving laser heating. The authors employed the fractional dual-phase-lag bioheat conduction model to analyze these interactions. Ghanbari and Rezazadeh (2022) discussed the two-dimensional thermal and mechanical behavior of skin tissue utilizing the Galerkin method for the CV model.

1.10 Objective of the Thesis

The main objective of this thesis is to conduct a comprehensive theoretical and numerical analysis of some generalized thermoelastic models and to explore their applications in several disciplines. The thesis also discusses the implementation of an alternate numerical technique for solving coupled thermoelastic and poro-thermoelastic problems. The work in this thesis is divided into three major parts, each of which investigates various aspects of generalized thermoelasticity theories by solving various unsolved problems. Three distinct theories within the field of thermoelasticity are explored in the thesis. Quintanilla's MGT thermoelastic model (2019) is examined in the first part, while the second part provides a detailed elaboration of the MTRDTT thermoelasticity theory proposed by Shivay and Mukhopadhyay (2019). Further, the porothermoelastic model (Youssef (2007)) is analyzed in a variety of contexts, which is the last part of the thesis.

The domain of influence theorem for a thermoelastic process verifies the hyperbolicity of the concerned model and defines the finite propagation of thermoelastic disturbances. Keeping this fact in mind, the domain of influence theorems are derived in the context of MGT model. To complete the theoretical investigations of the mentioned thermoelastic and porothermoelastic models, the present thesis attempts to establish the variational principle, reciprocity theorem, and continuous dependence results. Further, in order to highlight the specific characteristics of thermal and elastic-mode waves produced in the thermoelastic medium in the context of MGT model, the propagation of plane harmonic waves is thoroughly studied to highlight some important findings regarding wave characteristics. Furthermore, Thermoelastic interactions in a linear, isotropic, and homogeneous unbounded solid in presence of a continuous line heat source are investigated and the effect of the two-temperature parameter is analyzed. A numerical scheme based on the Legendre wavelet is proposed to solve any coupled thermoelastic and poro-thermoleastic problems. The proposed Legendre wavelet method is also implemented to examine the thermo-mechanical responses on biological tissue during laser irradiation. All of the essential aspects of the generalized thermoelasticity and porothermoelasticity theories, which fulfill the objective of the thesis, are clearly illustrated by the current analysis of various thermomechanical problems.