CHAPTER-4

Triple compound synchronization among eight chaotic systems with external disturbances via nonlinear approach

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4.1 Introduction

The Chaos theory is a field of study in mathematics and has useful applications in several areas including physics, engineering, economics, biology, and philosophy. The term chaos in highly associated with nonlinear systems and it creates the occurrences of irregular solution when the equation of motion is deterministic. Chaos is an interesting phenomenon of nonlinear systems. The effect of chaos in nonlinear dynamics is studied during last few decades by the researchers from different parts of the world. In general chaotic system is a bounded nonlinear deterministic system which has aperiodic long term behaviour that exhibits highly sensitive to initial conditions and it is more popularly known as the name of 'butterfly effect'. A chaotic system is described by a set of mathematical equations, which may include two types of variables viz., dynamic variables and static variables. Dynamic variables represent the fundamental properties of the system that are changing all the time. Static variables, which are also called parameters, denote a set of the points in parameter space which are not functions of time, and has become an active research for the nonlinear science community for the last few decades. Nowadays the study of chaos in nonlinear science has become a popular and important topic and received a great deals of interests amongst the researcher, scientists, engineers in the last few decades in various fields and considerable research efforts have been devoted to chaos control and chaos synchronization problems in many dynamical systems.

The contents of this chapter have been communicated in Journal of ***Differential Equations and Dynamical Systems***

Synchronization of chaotic systems is a naturally occurring phenomenon where one chaotic dynamical system mimics the dynamical behaviour of another chaotic system. It is a process of making two identical or non-identical chaotic systems structurally stable through the adjustment of a given property. The idea of synchronizing chaotic systems was first introduced by Pecora and Carroll (1990); where the possibility of synchronization of chaotic systems through a simple coupling is shown. The synchronization of chaotic dynamical systems has been intensively studied by many researchers and has attracted a great deal of interest in various field due to its important applications in secure communications (Murali and Lakshmanan (2003)), ecological system, chemical system, physical system, modeling brain activity, system identification, pattern recognition phenomena (Blasius et al. (1999), Han et al. (1995), Lakshmanan and Murali (1996), Cuomo and Oppenheim (1993), Kocarev and Parlitz (1995) etc.). In 2015, Liu et al. (2015) studied that finite-time synchronization for high dimensional chaotic systems in secure communication, which shows that the systems can realize monotonous synchronization and the information signal can be recovered undistorted.

Recently, more works have been done in the study of chaos synchronization. So far, different types of synchronization in chaotic dynamical systems such as complete synchronization, anti-synchronization, phase synchronization, lag synchronization, projective synchronization, function projective synchronization, dual synchronization, combination synchronization, dual combination synchronization, compound synchronization and double compound synchronization (Muhammad (2013), Yadav et al. (2017), Wang et al. (2011), Li (2007), Du et al. (2009), Ning et al. (2007), Runzi et al. (2011), Singh et al. (2017), Sun et al. (2014b), Zhang and Deng (2014)) have been observed in coupled systems. Several methods are used to achieve such types of synchronizations viz., active control method, adaptive control method, OGY method, sliding mode control method, linear and nonlinear feedback method, time-delay feedback approach, backstepping approach (Haeri et al. (2007), Singh et al. (2016b), Chen and Lu (2002), Ott et al. (1990), Mahmoud et al. (2016), Bai and Lonngren (1997)) etc.

In secure communication, the synchronization of chaotic systems has remarkable contributions. During synchronization if more drive and more response systems are taken, then the transmitted signals will be very strong and have own stronger anti-attack and anti-translated ability. Complexity of the drive systems and formation of the driving signals are important features to ensure security of communication. So far, the researchers were mainly focused on synchronization between one drive and one response systems, but in recent years the researchers are more interested to study the synchronization among more than two systems. In 2011, Runzi et al. (2011) proposed the combination synchronization scheme, in which two drive systems synchronized with one response system and Zhou et al. (2013) investigated combination synchronization among three nonlinear complex hyper-chaotic systems in the year 2013. Recently Yadav et al. (2017) studied the combined synchronization among delay chaotic systems in the presence of uncertain parameters. The Chinese scientists Sun et al. (2013b, 2014b) extended the compound synchronization and combination synchronization scheme to achieve synchronization between four chaotic systems. In the year 2016, Sun et al. (2016a) proposed compound-combination synchronization scheme for five chaotic systems, while compound-combination anti-synchronization scheme for five simplest memristor chaotic systems was studied by Sun and Shen (2016b) in the same year. In 2014, Double-compound synchronization of six memristor-based Lorenz systems was studied by Zhang and Deng (2014).

Overall the above facts have motivated the author to study the triple compound synchronization among eight chaotic systems with external disturbances. In secure communication, the receiver plants will definitely suffer from external disturbance, which will definitely influence the accuracy of the communication. Therefore, the synchronization among chaotic systems with external disturbances are not easy jobs for researchers since there are always possibilities of destroying synchronization under the effects of those terms in chaotic systems. The triple compound synchronization is the generalization of double compound synchronization (Zhang and Deng (2014)), and it is the one of the kind of synchronizations, where more than two identical (or different) chaotic systems can synchronize with different initial values.

In this chapter, a sincere attempt has been made to study triple compound synchronization among non-identical chaotic systems in the presence of external disturbances using nonlinear control method.

4.2 Problem formulation of triple compound synchronization

In this section a scheme to triple compound synchronization among five master and three response systems in the presence external disturbances is designed,
Let us consider five chaotic systems with external disturbances as master systems as

$$\frac{dx_{1i}}{dt} = a_{1i}x_{1i} + f_{1i}(x_{1i}) + d_{1i}(t), \qquad i = 1, 2, 3,.....,n \tag{4.1}$$

$$\frac{dx_{2i}}{dt} = a_{2i}x_{2i} + f_{2i}(x_{2i}) + d_{2i}(t), \qquad i = 1, 2, 3,.....,n \tag{4.2}$$

$$\frac{dx_{3i}}{dt} = a_{3i}x_{3i} + f_{3i}(x_{3i}) + d_{3i}(t), \qquad i = 1, 2, 3,.....,n \tag{4.3}$$

$$\frac{dx_{4i}}{dt} = a_{4i}x_{4i} + f_{4i}(x_{4i}) + d_{4i}(t), \qquad i = 1, 2, 3,.....,n \tag{4.4}$$

$$\frac{dx_{5i}}{dt} = a_{5i}x_{5i} + f_{5i}(x_{5i}) + d_{5i}(t), \qquad i = 1, 2, 3,.....,n \tag{4.5}$$

and three chaotic systems with disturbances are considered as slave systems are defined as

$$\frac{dy_{1i}}{dt} = b_{1i}y_{1i} + g_{1i}(y_{1i}) + d_{6i}(t) + u_{1i}(t), \qquad i = 1, 2, 3,.....,n \tag{4.6}$$

$$\frac{dy_{2i}}{dt} = b_{2i}y_{2i} + g_{2i}(y_{2i}) + d_{7i}(t) + u_{2i}(t), \qquad i = 1, 2, 3,.....,n \tag{4.7}$$

$$\frac{dy_{3i}}{dt} = b_{3i}y_{3i} + g_{3i}(y_{3i}) + d_{8i}(t) + u_{3i}(t), \qquad i = 1, 2, 3,.....,n, \tag{4.8}$$

where x_{1i}, x_{2i}, x_{3i}, x_{4i}, x_{5i} and y_{1i}, y_{2i}, y_{3i} are the state vectors of chaotic systems, $a_{1i}, a_{2i}, a_{3i}, a_{4i}, a_{5i}, b_{1i}, b_{2i}, b_{3i}$ are constant parameters of the systems, $f_{1i}(x_{1i})$, $f_{2i}(x_{2i})$, $f_{3i}(x_{3i})$, $f_{4i}(x_{4i})$, $f_{5i}(x_{5i})$ and $g_{1i}(y_{1i})$, $g_{2i}(y_{2i})$, $g_{2i}(y_{2i})$ are the nonlinear functions of the systems and $d_{ji}(t)$, $j = 1,2,......,8$, $i = 1, 2, 3,.....,n$ are the external disturbances of chaotic systems with $|d_{ji}(t)| \le \rho_{ji}$, $j = 1,2,....,8$ and $i = 1, 2, 3,.....,n$ where $\rho_{ji} > 0$, $j = 1,2,....,8$ and $i = 1, 2, 3,.....,n$ and $u_{1i}(t)$, $u_{2i}(t)$, $u_{3i}(t)$, $i = 1, 2, 3,.....,n$ are the control functions of the chaotic system (4.6)-(4.8). Now controllers $u_{1i}(t)$, $u_{2i}(t)$, $u_{3i}(t)$ are to be designed in such a way that considered master and slave systems (4.1)-(4.8) are synchronized through the proper definition of the errors.

Definition: If the triple compound synchronization error is defined as

$$\lim_{t\to\infty}\| e \| = \lim_{t\to\infty}\| AY_1 + BY_2 + CY_3 - (KX_1 + LX_2 + MX_3)(NX_4 + OX_5)\| = 0, \tag{4.9}$$

where $\|.\|$ denotes matrix norm,

$X_1 = diag(x_{11}, x_{12}, ... x_{1n})$, $X_2 = diag(x_{21}, x_{22}, ... x_{2n})$, $X_3 = diag(x_{31}, x_{32}, ... x_{3n})$, $X_4 = diag(x_{41}, x_{42}, ... x_{4n})$, $X_5 = diag(x_{51}, x_{52}, ... x_{5n})$, $Y_1 = diag(y_{11}, y_{12}, ... y_{1n})$, $Y_2 = diag(y_{21}, y_{22}, ... y_{2n})$, $Y_3 = diag(y_{31}, y_{32}, ... y_{3n})$ are *n*-dimensional diagonal matrices and $A, B, C, K, L, M, N, O \in R^n \times R^n$ are constant diagonal matrices, then the considered master systems (4.1)-(4.5) and response systems (4.6)-(4.8) are said to be triple compound synchronization.

Remark 4.1: If $A = 0, B = 0, K = 0, L = 0$ or $A = 0, B = 0, L = 0, M = 0$ or $A = 0, B = 0, K = 0, M = 0$ or $B = 0, C = 0, K = 0, L = 0$ or $B = 0, C = 0, L = 0, M = 0$ or $B = 0, C = 0, K = 0, M = 0$ or $A = 0, C = 0, K = 0, L = 0$ or $A = 0, C = 0, L = 0, M = 0$ $A = 0, C = 0, K = 0, M = 0$, then the triple compound synchronization reduced into compound synchronization.

Remark 4.2: If $A = 0,\ K = 0,$ or $B = 0,\ K = 0,$ or $C = 0,\ K = 0,$ or $A = 0,\ L = 0,$ or $B = 0,\ L = 0,$ or $C = 0,\ L = 0,$ or $A = 0,\ M = 0,$ or $B = 0,\ M = 0$ or $C = 0,\ M = 0$, then the triple compound synchronization changed into double compound synchronization.

Remark 4.3: If $K = 0,\ L = 0,\ M = 0,\ A = 0,\ B = 0$ or $K = 0,\ L = 0,\ M = 0,\ B = 0,\ C = 0$ or $K = 0,\ L = 0,\ M = 0,\ A = 0,\ C = 0$ or $N = 0,\ O = 0,\ A = 0,\ B = 0$ or $N = 0,\ O = 0,\ B = 0,\ C = 0$ or $N = 0,\ O = 0,\ A = 0,\ C = 0$, then the triple compound synchronization will be turn into a chaos control problem.

Remark 4.4: If $A=0,\ B=0,\ K=0,\ L=0,\ N=0$ or $A=0,\ B=0,\ K=0,\ L=0,\ O=0$ or $A=0,\ B=0,\ L=0,\ M=0,\ N=0$ or $A=0,\ B=0,\ L=0,\ M=0,\ O=0$ or $A=0,\ B=0,\ K=0,\ M=0,\ N=0$ or $A=0,\ B=0,\ K=0,\ M=0,\ O=0$ or $B=0,\ C=0,\ K=0,\ L=0,\ N=0$ or $B=0,\ C=0,\ K=0,\ L=0,\ O=0$ or $B=0,\ C=0,\ L=0,\ M=0,\ N=0$ or $B=0,\ C=0,\ L=0,\ M=0,\ O=0$ or $B=0,\ C=0,\ K=0,\ M=0,\ N=0$ or $B=0,\ C=0,\ K=0,\ M=0,\ O=0$ or $A=0,\ C=0,\ K=0,\ L=0,\ N=0$ or $A=0,\ C=0,\ K=0,\ L=0,\ O=0$ or $A=0,\ C=0,\ L=0,\ M=0,\ N=0$ or $A=0,\ C=0,\ L=0,\ M=0,\ O=0$ or $A=0,\ C=0,\ K=0,\ M=0,\ N=0$ or $A=0,\ C=0,\ K=0,\ M=0,\ O=0$, then the triple compound synchronization problem changed into function projective synchronization problem.

Remark 4.5: The error $e=AY_1+BY_2+CY_3-(KX_1+LX_2+MX_3)(NX_4+OX_5)$ can be rewritten as
$e=[AY_1-KX_1(NX_4+OX_5)]+[BY_2-LX_2(NX_4+OX_5)]+[CY_3-MX_3(NX_4+OX_5)]$, this is the sum of three errors of compound synchronization. Hence we can say this problem is triple compound synchronization.

Now the error function for triple compound synchronization is taken as

$$e=AY_1+BY_2+CY_3-(KX_1+LX_2+MX_3)(NX_4+OX_5),$$

where $e=diag(e_1,\ e_2,\ e_3,....,\ e_n)$ is the diagonal matrix and $A=diag(a_1,\ a_2,\ a_3,.....,a_n)$, $B=diag(b_1,\ b_2,\ b_3,.......,\ b_n)$, $C=diag(c_1,\ c_2,\ c_3,....,c_n)$, $K=diag(k_1,\ k_2,\ k_3,.......,k_n)$, $L=diag(l_1,\ l_2,\ l_3,.....,l_n)$, $M=diag(m_1,\ m_2,\ m_3,........,m_n)$, $N=diag(n_1,\ n_2,\ n_3,.......,n_n)$, $O=diag(o_1,\ o_2,\ o_3,........,o_n)$ are the constant diagonal matrices. After putting the values of error state variables and scaling matrices, we will get the following error functions

$$\left[\begin{array}{ccc|c} e_1 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & e_3 & 0 \\ \hline 0 & 0 & 0 & e_n \end{array}\right]=$$

$$= \left[\begin{array}{ccc|c} a_1 y_{11} + b_1 y_{21} + c_1 y_{31} & 0 & 0 & 0 \\ 0 & a_2 y_{12} + b_2 y_{22} + c_2 y_{32} & 0 & 0 \\ 0 & 0 & a_3 y_{13} + b_3 y_{23} + c_3 y_{33} & 0 \\ \hline 0 & 0 & 0 & a_n y_{1n} + b_n y_{2n} + c_n y_{3n} \end{array}\right]$$

$$- \left[\begin{array}{ccc|c} (k_1 x_{11} + l_1 x_{21} + m_1 x_{31})(n_1 x_{41} + o_1 x_{51}) & 0 & 0 & 0 \\ 0 & (k_2 x_{12} + l_2 x_{22} + m_2 x_{32})(n_2 x_{42} + o_2 x_{52}) & 0 & 0 \\ 0 & 0 & (k_3 x_{13} + l_3 x_{23} + m_3 x_{33})(n_3 x_{43} + o_3 x_{53}) & 0 \\ \hline 0 & 0 & 0 & (k_n x_{1n} + l_n x_{2n} + m_n x_{3n})(n_n x_{4n} + o_n x_{5n}) \end{array}\right]$$

Comparing both sides, we get the error functions as

$$e_i = a_i y_{1i} + b_i y_{2i} + c_i y_{3i} - (k_i x_{1i} + l_i x_{2i} + m_i x_{3i})(n_i x_{4i} + o_i x_{5i}), \tag{4.10}$$

where $i = 1, 2, 3, ..., n$.

Taking derivative of error functions (4.10) as

$$\begin{aligned} \frac{de_i}{dt} &= a_i \frac{dy_{1i}}{dt} + b_i \frac{dy_{2i}}{dt} + c_i \frac{dy_{3i}}{dt} - (k_i x_{1i} + l_i x_{2i} + m_i x_{3i})(n_i \frac{dx_{4i}}{dt} + o_i \frac{dx_{5i}}{dt}) \\ &\quad - (k_i \frac{dx_{1i}}{dt} + l_i \frac{dx_{2i}}{dt} + m_i \frac{dx_{3i}}{dt})(n_i x_{4i} + o_i x_{5i}), \text{ where } i = 1, 2, 3, ...n. \end{aligned} \tag{4.11}$$

Putting the values of $\frac{dx_{1i}}{dt}$, $\frac{dx_{2i}}{dt}$, $\frac{dx_{3i}}{dt}$, $\frac{dx_{4i}}{dt}$, $\frac{dx_{5i}}{dt}$ and $\frac{dy_{1i}}{dt}$, $\frac{dy_{2i}}{dt}$, $\frac{dy_{3i}}{dt}$, $i = 1, 2, 3, ..., n$ in equation (4.11) from equation (4.1)-(4.8), then we get

$$\begin{aligned} \frac{de_i}{dt} &= a_i b_{1i} y_{1i} + a_i g_{1i}(y_{1i}) + a_i d_{6i}(t) + b_i b_{2i} y_{2i} + b_i g_{2i}(y_{2i}) + b_i d_{7i}(t) + c_i b_{3i} y_{3i} \\ &\quad + c_i g_{3i}(y_{3i}) + c_i d_{8i}(t) - (k_i x_{1i} + l_i x_{2i} + m_i x_{3i})[n_i a_{4i} x_{4i} + n_i f_{4i}(x_{4i}) + n_i d_{4i}(t) \\ &\quad + o_i a_{5i} x_{5i} + o_i f_{5i}(x_{5i}) + o_i d_{5i}(t)] - [k_i a_{1i} x_{1i} + k_i f_{1i}(x_{1i}) + k_i d_{1i}(t) + l_i a_{2i} x_{2i} \\ &\quad + l_i f_{2i}(x_{2i}) + l_i d_{2i}(t) + m_i a_{3i} x_{3i} + m_i f_{3i}(x_{3i}) + m_i d_{3i}(t)](n_i x_{4i} + o_i x_{5i}) + u_i^*(t), \end{aligned} \tag{4.12}$$

where $i = 1, 2, 3, ...n$.

Now we design the control functions $u_i^*(t)$, where $u_i^*(t) = a_i u_{1i}(t) + b_i u_{2i}(t) + c_i u_{3i}(t)$, $i = 1, 2, 3, \ldots.., n$ in such a way that the error systems (4.11) will be stabilized.

Theorem 1: If the control functions are chosen as

$$\begin{aligned} u_i^*(t) = & -k_i e_i - a_i b_{1i} y_{1i} - a_i g_{1i}(y_{1i}) - a_i d_{6i}(t) - b_i b_{2i} y_{2i} - b_i g_{2i}(y_{2i}) - b_i d_{7i}(t) - c_i b_{3i} y_{3i} \\ & - c_i g_{3i}(y_{3i}) - c_i d_{8i}(t) + (k_i x_{1i} + l_i x_{2i} + m_i x_{3i})[n_i a_{4i} x_{4i} + n_i f_{4i}(x_{4i}) + n_i d_{4i}(t) \\ & + o_i a_{5i} x_{5i} + o_i f_{5i}(x_{5i}) + o_i d_{5i}(t)] + [k_i a_{1i} x_{1i} + k_i f_{1i}(x_{1i}) + k_i d_{1i}(t) + l_i a_{2i} x_{2i} \\ & + l_i f_{2i}(x_{2i}) + l_i d_{2i}(t) + m_i a_{3i} x_{3i} + m_i f_{3i}(x_{3i}) + m_i d_{3i}(t)](n_i x_{4i} + o_i x_{5i}) \end{aligned} , \quad (4.13)$$

where $i = 1, 2, 3, \ldots, n$, then the triple compound synchronization among chaotic systems (4.1)-(4.8) is achieved for any positive constant k_i, $i = 1, 2, 3, \ldots, n$ and satisfy the condition $\lim_{t \to \infty} \| e_i(t) \| = 0$, $i = 1, 2, 3, \ldots, n$.

Proof: Let us define the Lyapunov function as

$$V = \frac{1}{2} \sum_{i=1}^{n} e_i^2 .$$

Taking the derivative of V w. r. to t, we get

$$\frac{dV}{dt} = \frac{1}{2} \sum_{i=1}^{n} \frac{de_i^2}{dt} = \sum_{i=1}^{n} e_i \frac{de_i}{dt} . \quad (4.14)$$

Putting the values of $\frac{de_i}{dt}$ and $u_i^*(t)$ in equation (4.14) from equations (4.12) and (4.13), we obtain

$$\frac{dV}{dt} = \sum_{i=1}^{n} - k_i e_i^2 < 0 . \quad (4.15)$$

Thus, it may be concluded that since the control parameter $k_i > 0$, $V \in R$ is positive definite function and $\frac{dV}{dt} \in R$ is negative definite function, then according to Lyapunov stability theorem, the error system is asymptotically stable. Consequently, the state trajectories of master and slave systems will be triple compound synchronized.

4.3 Systems' descriptions

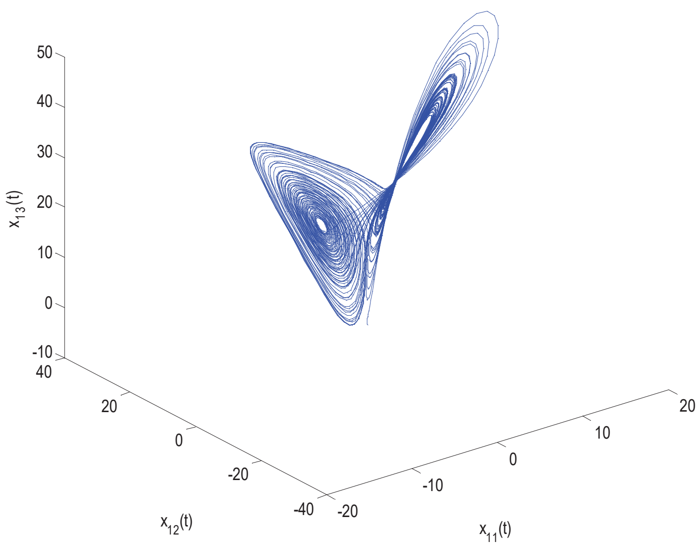
4.3.1 Lorenz chaotic system

The example of a nonlinear dynamic system is Lorenz chaotic system related to the long term behaviour of the Lorenz oscillation, which is a three dimensional dynamical system exhibits that lemniscate type shaped chaotic flow and it describes the state of dynamical system how evolves over time in a complex and non-repeating pattern. The model describes the stability of fluid flows in the atmosphere and also has the important implications for climate and weather predictions. This is also applicable for simplified models for lasers (Lorenz (1963)) and dynamos (Knobloch (1981)).

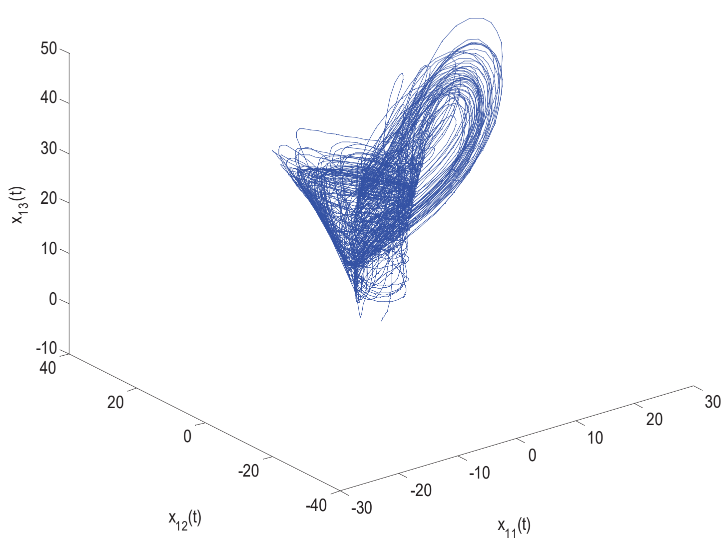
The Lorenz system (Wu and Shen (2009), Grigorenko and Grigorenko (2003)) is given by

$$\begin{aligned}\frac{dx_{11}}{dt} &= a_{11}(x_{12}-x_{11}),\\ \frac{dx_{12}}{dt} &= x_{11}(a_{13}-x_{13})-x_{12},\\ \frac{dx_{13}}{dt} &= x_{11}x_{12}-a_{12}x_{13},\end{aligned} \tag{4.16}$$

where a_{11} is the Prandtl number, a_{13} is the Rayleigh number and a_{12} is the size of the region approximated by the system. The phase portraits of Lorenz system is shown through Fig. 4.1(a) for the parameters' values $a_{11}=10,$ $a_{12}=8/3$, $a_{13}=28$ and initial condition $(0.1, 0.1, 0.1)$. The chaotic attractors in the $x_{11}-x_{12}-x_{13}$ space are shown in Fig. 4.1(a).



(a)



(b)

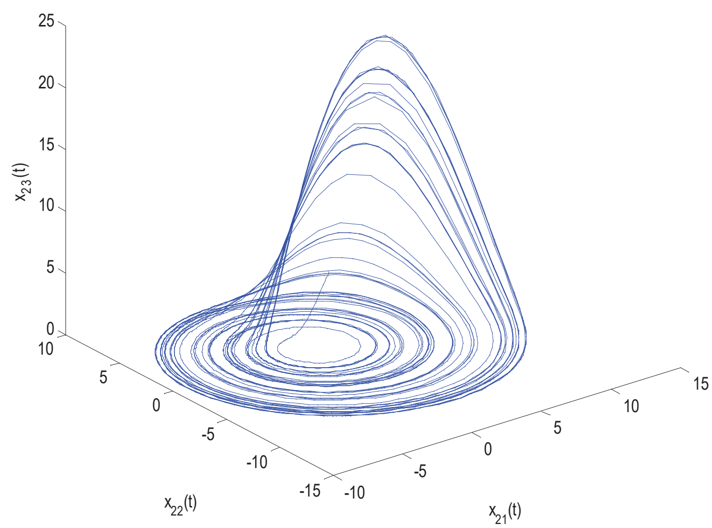
Fig. 4.1. Phase portraits of the Lorenz system: (a) In $x_{11} - x_{12} - x_{13}$ space, (b) In $x_{11} - x_{12} - x_{13}$ space with external disturbances.

4.3.2 Rossler chaotic systems

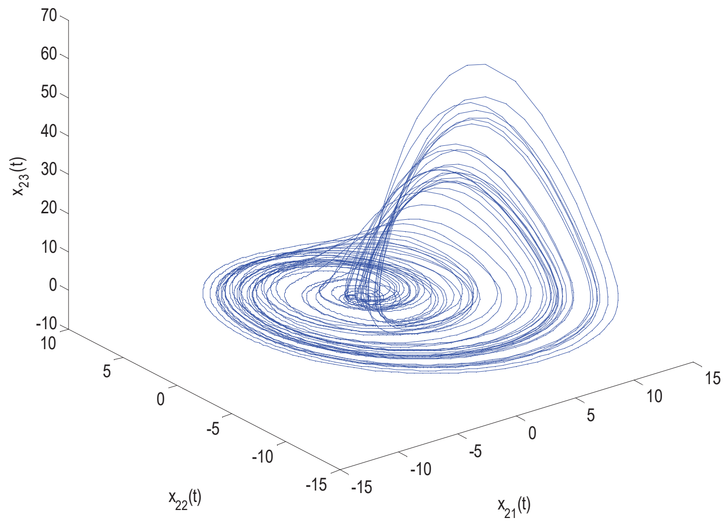
The Rossler system (Yan and Li (2007)) is given by

$$
\begin{aligned}
\frac{dx_{21}}{dt} &= -x_{22} - x_{23}, \\
\frac{dx_{22}}{dt} &= x_{21} + a_{21}x_{22}, \qquad (4.17) \\
\frac{dx_{23}}{dt} &= a_{22} + x_{21}x_{23} - a_{23}x_{23},
\end{aligned}
$$

where x_{21}, x_{22}, x_{23} are the state variables, for the parametric values $a_{21} = 0.2,\ a_{22} = 0.2,\ a_{23} = 5.7$, the system (4.17) is chaotic. The phase portrait of Rossler system is shown through Fig. 4.2(a).



(a)



(b)

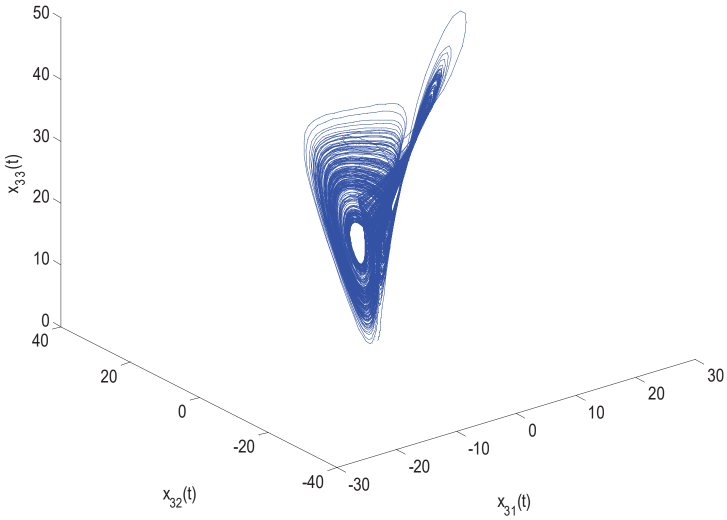
Fig. 4.2. Phase portraits of the Rossler system: (a) In $x_{21}-x_{22}-x_{23}$ space, (b) In $x_{21}-x_{22}-x_{23}$ space with external disturbances.

4.3.3 Lu chaotic system

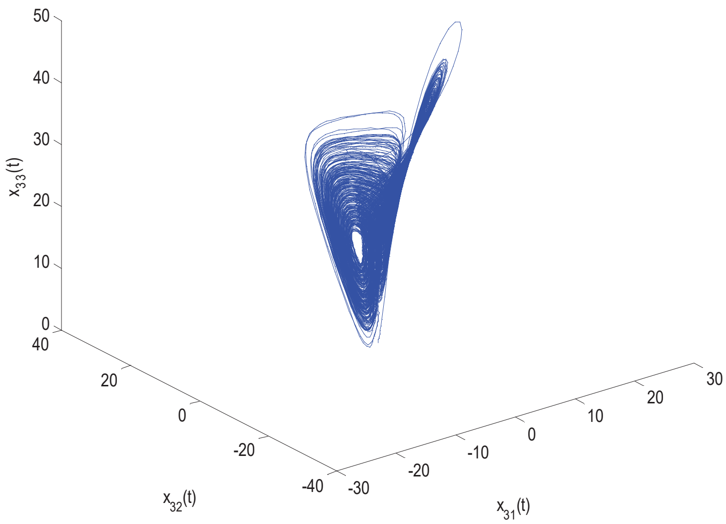
The Lu chaotic system (Petras (2011)) is defined as

$$\begin{aligned}\frac{dx_{31}}{dt} &= a_{31}(x_{32}-x_{31}) \\ \frac{dx_{32}}{dt} &= -x_{31}x_{33}+a_{33}x_{32} \\ \frac{dx_{33}}{dt} &= x_{31}x_{32}-a_{32}x_{33},\end{aligned} \tag{4.18}$$

where a_{31}, a_{32}, a_{33} are system's parameters. The phase portraits of the system (4.18) in $x_{31}-x_{32}-x_{33}$ space is shown through Fig. 4.3(a) for the parameters' values $a_{31}=36$, $a_{32}=3$ and $a_{33}=20$, and initial condition $(x_{31}(0), x_{32}(0), x_{33}(0)) = (0.2,\ 0.5,\ 0.3)$.



(a)



(b)

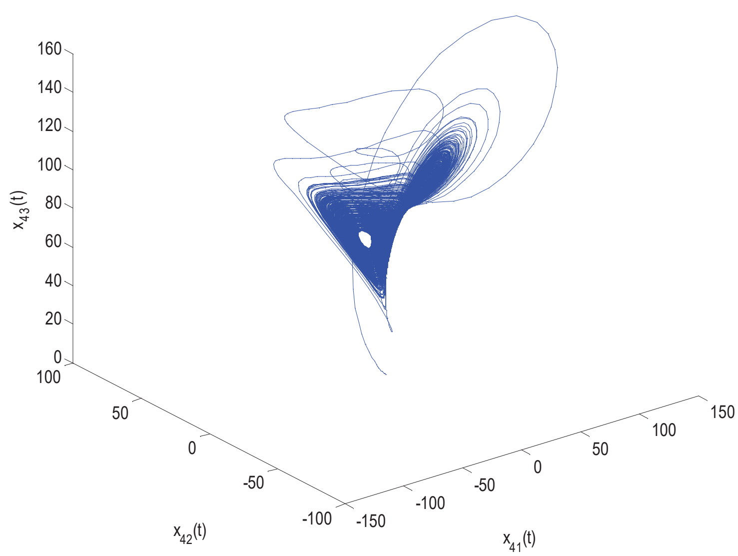
Fig. 4.3. Phase portraits of the Lu system: (a) In $x_{31} - x_{32} - x_{33}$ space, (b) In $x_{31} - x_{32} - x_{33}$ space with external disturbances.

4.3.4 Qi chaotic system

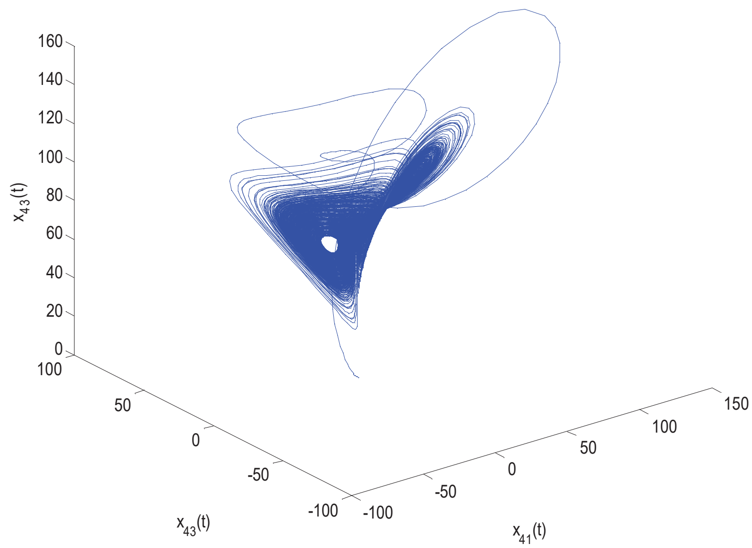
The Qi system (Qi et al. (2005)) is described as

$$\begin{aligned}
\frac{dx_{41}}{dt} &= a_{41}(x_{42} - x_{41}) + x_{42}x_{43} \\
\frac{dx_{42}}{dt} &= a_{43}x_{41} - x_{42} - x_{41}x_{43} \\
\frac{dx_{43}}{dt} &= -a_{42}x_{43} + x_{41}x_{42}\ .
\end{aligned} \tag{4.19}$$

The phase portrait of the system (4.19) is depicted through Fig. 4.4(a) for the values of the parameters' $a_{41} = 35,\ a_{42} = 8/3,\ a_{43} = 80$ and initial condition $(x_{41}(0),\ x_{42}(0),\ x_{43}(0)) = (3,\ 2,\ 1)$.



(a)



(b)

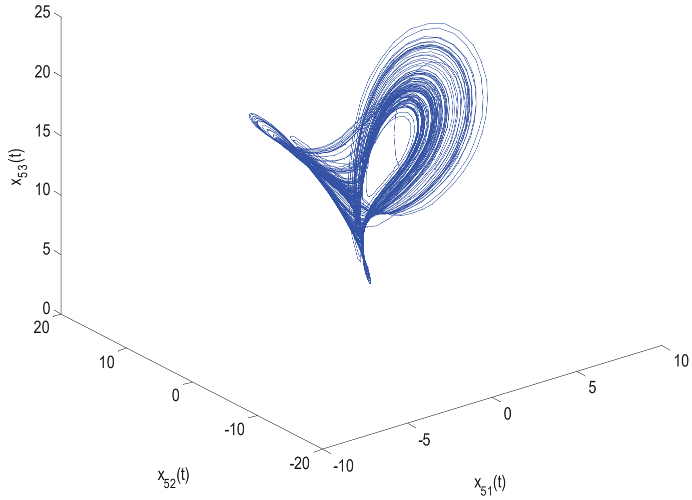
Fig. 4.4. Phase portraits of the Qi system: (a) In $x_{41}-x_{42}-x_{43}$ space, (b) In $x_{41}-x_{42}-x_{43}$ space with external disturbances.

4.3.5 T-system

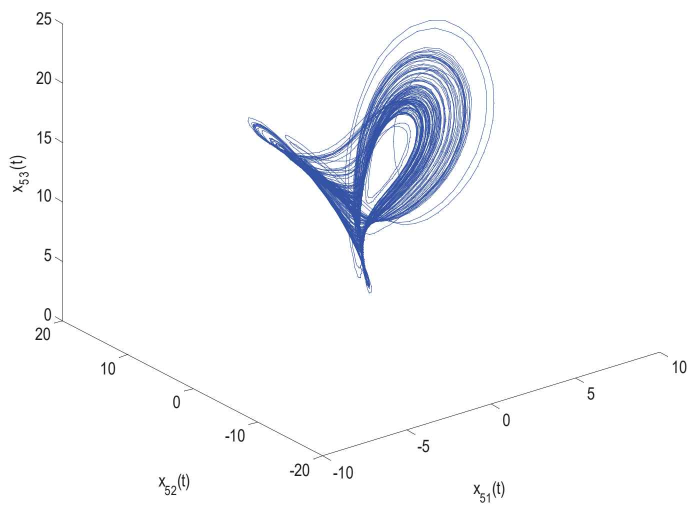
The chaotic dynamical T-system is introduced by G. Tigan and D. Opris (2008), G. Tigan and C. Dana (2009), which is described by

$$\begin{aligned}\frac{dx_{51}}{dt} &= a_{51}(x_{52}-x_{51})\\ \frac{dx_{52}}{dt} &= (a_{53}-a_{51})x_{51}-a_{51}x_{51}x_{53}\\ \frac{dx_{53}}{dt} &= -a_{52}x_{53}+x_{51}x_{52},\end{aligned} \tag{4.20}$$

where a_{51}, a_{52}, a_{53} are the parameters and x_{51}, x_{52}, x_{53} are state variables of the system. When the values of the parameters' are taken as $(a_{51}, a_{52}, a_{53}) = (2.1, 0.6, 30)$ and the maximal Lyapunov exponent of system is 0.37, the T-system exhibits chaos. Fig. 4.5(a) depicts that the T-system shows the regular chaotic behaviour.



(a)



(b)

Fig. 4.5. Phase portraits of the T-system: (a) In $x_{51}-x_{52}-x_{53}$ space, (b) In $x_{51}-x_{52}-x_{53}$ space with external disturbances.

4.3.6 Newton-Leipnik chaotic system

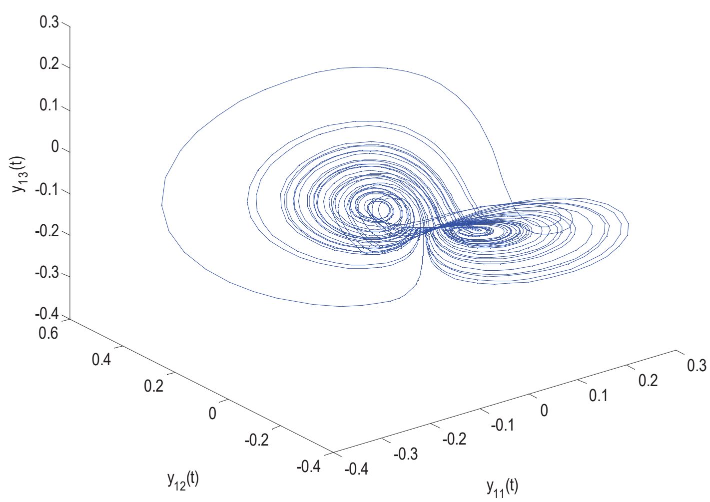
The Newton-Leipnik chaotic system (Leipnik and newton (1981), Sheu et al. (2008)) is given by

$$\frac{dy_{11}}{dt} = -b_{11}y_{11} + y_{12} + 10y_{12}y_{13}$$

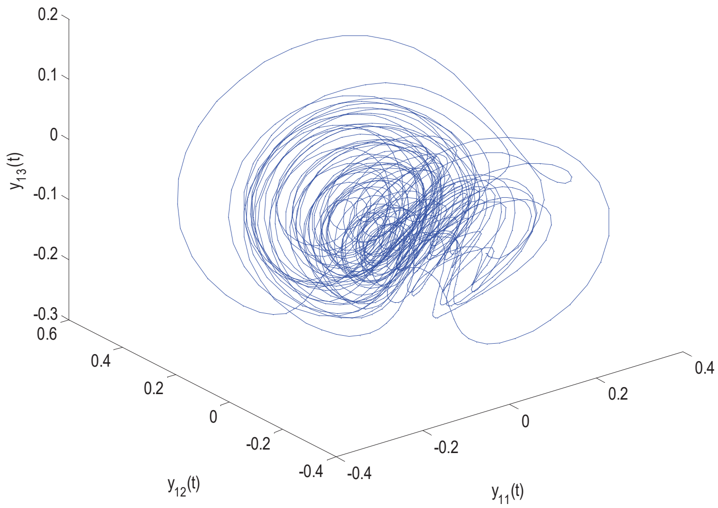
$$\frac{dy_{12}}{dt} = -y_{11} - 0.4y_{12} + 5y_{11}y_{13} \tag{4.21}$$

$$\frac{dy_{13}}{dt} = b_{12}y_{13} - 5y_{11}y_{12},$$

where b_{11} and b_{12} are the variable parameters' and $b_{12} \in (0, 0.8)$. The system is ill-behaved when b_{12} takes the values outside of this interval. If b_{12} becomes close to zero, the system shows uninteresting dynamic and if $b_{12} \geq 0.8$, the given system becomes explosive i.e., the solution of this system diverges to infinity for any initial condition other than the critical points. For the parameters' values $b_{11} = 0.4$, $b_{12} = 0.175$ and the initial condition $(0.19, 0, -0.18)$, the Newton-Leipnik system shows chaotic behaviour which is depicted through Fig. 4.6(a).



(a)



(b)

Fig. 4.6. Phase portraits of the Newton-Leipnik chaotic system: (a) In $y_{11}-y_{12}-y_{13}$ space, (b) In $y_{11}-y_{12}-y_{13}$ space with external disturbances.

4.3.7 Liu chaotic system

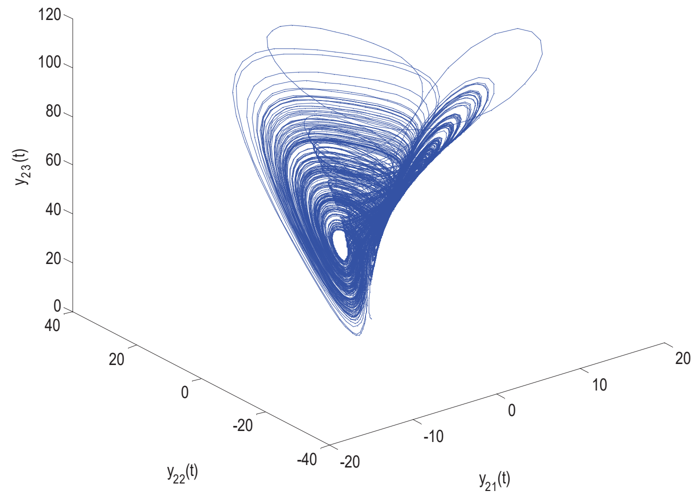
The Liu chaotic system (Liu et al. (2009)) is given as

$$\frac{dy_{21}}{dt}=b_{21}(y_{22}-y_{21})$$

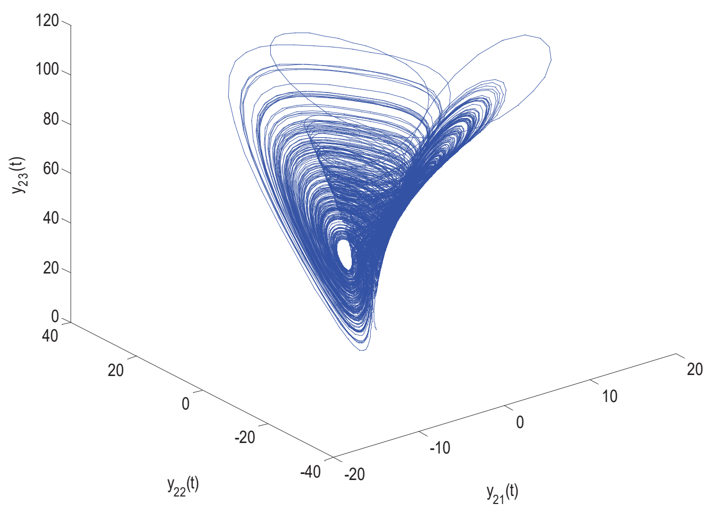
$$\frac{dy_{22}}{dt}=b_{22}y_{21}-y_{21}y_{23} \tag{4.22}$$

$$\frac{dy_{23}}{dt}=-b_{23}y_{23}+b_{24}y_{21}^2 .$$

The chaotic attractor of the system (4.22) is described through Fig. 4.7(a) for the values of the parameters $b_{21}=10,\ b_{22}=40,\ b_{23}=2.5,\ b_{24}=4,$ and initial condition $(1,\ 1.4,\ 1)$.



(a)



(b)

Fig. 4.7. Phase portraits of the Liu chaotic system: (a) In $y_{21} - y_{22} - y_{23}$ space, (b) In $y_{21} - y_{22} - y_{23}$ space with external disturbances.

4.3.8 Chen chaotic system

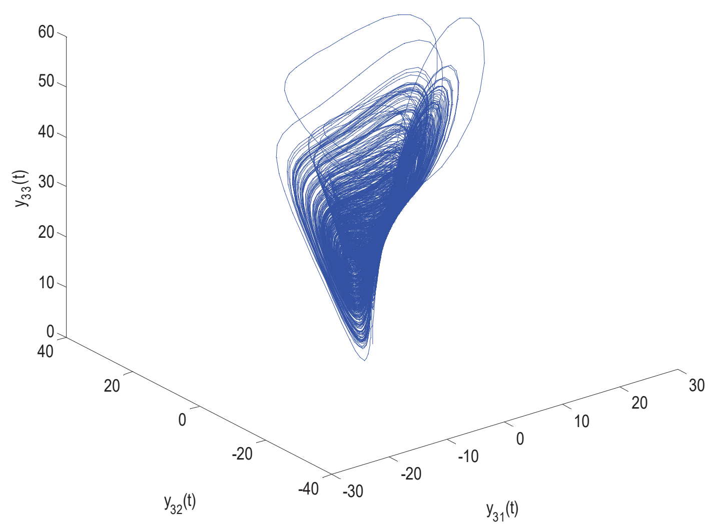
The Chen system (Yassen (2003)) is considered as

$$\frac{dy_{31}}{dt} = b_{31}(y_{32} - y_{31})$$

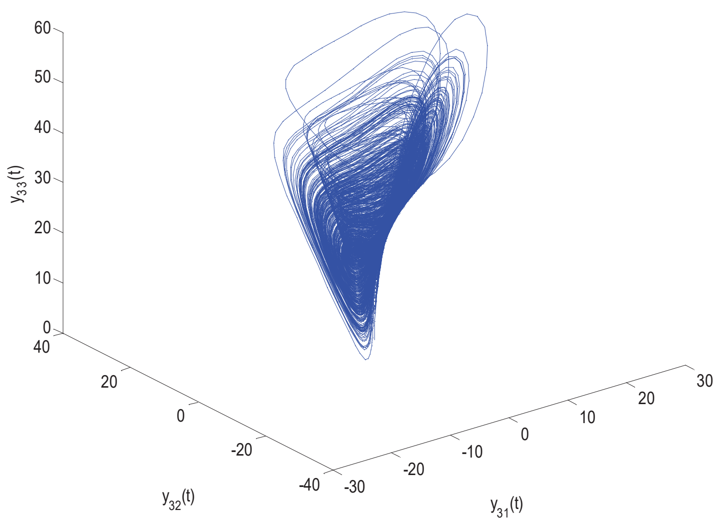
$$\frac{dy_{32}}{dt} = (b_{33} - b_{31})y_{31} - y_{31}y_{33} + b_{33}y_{32} \tag{4.23}$$

$$\frac{dy_{33}}{dt} = y_{31}y_{32} - b_{32}y_{33}\,.$$

Fig. 4.8(a) shows the chaotic attractors of the system (4.23) for the parameters' values $b_{31} = 35,\ b_{32} = 3,\ b_{33} = 28$ and the initial condition $(1,\ 1.4,\ 1)$.



(a)



(b)

Fig. 4.8. Phase portraits of the Chen chaotic system: (a) In $y_{31} - y_{32} - y_{33}$ space, (b) In $y_{31} - y_{32} - y_{33}$ space with external disturbances.

4.4 Triple compound synchronization among chaotic systems with external disturbances via nonlinear approach

In this section the Triple compound synchronization among five master and three response systems in the presence of external disturbances using nonlinear control method is studied.

Let us consider the Lorenz chaotic system, Rossler chaotic systems, Lu chaotic system, Qi chaotic system and T chaotic system with external disturbances as master systems and defined by the equations (4.24)-(4.28) as

$$\frac{dx_{11}}{dt} = a_{11}(x_{12} - x_{11}) + 10\sin(10\,t)$$

$$\frac{dx_{12}}{dt} = x_{11}(a_{13} - x_{13}) - x_{12} + 10\cos(10\,t)$$

(4.24)

$$\frac{dx_{13}}{dt} = x_{11}x_{12} - a_{12}x_{13} + 10\sin(10\,t)\,,$$

where $d_1(t) = \begin{bmatrix} d_{11}(t) \\ d_{12}(t) \\ d_{13}(t) \end{bmatrix} = \begin{bmatrix} 10\sin(10\,t) \\ 10\cos(10\,t) \\ 10\sin(10\,t) \end{bmatrix}$ is the external disturbances, Fig. 4.1(b) show the phase portrait of the system (4.24).

$$\frac{dx_{21}}{dt} = -x_{22} - x_{23} - \sin(t),$$

$$\frac{dx_{22}}{dt} = x_{21} + a_{21}x_{22} + \cos(t), \tag{4.25}$$

$$\frac{dx_{23}}{dt} = a_{22} + x_{21}x_{23} - a_{23}x_{23} - \sin(t)\,,$$

where $d_2(t) = \begin{bmatrix} d_{21}(t) \\ d_{22}(t) \\ d_{23}(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}$ is the external disturbances, the phase portrait of system (4.25) is depicted through Fig. 4.2(b).

$$\frac{dx_{31}}{dt} = a_{31}(x_{32} - x_{31}) + 5\sin(20\,t)$$

$$\frac{dx_{32}}{dt} = -x_{31}x_{33} + a_{33}x_{32} + 5\cos(20\,t) \tag{4.26}$$

$$\frac{dx_{33}}{dt} = x_{31}x_{32} - a_{32}x_{33} + 5\sin(20\,t)\,,$$

where $d_3(t) = \begin{bmatrix} d_{31}(t) \\ d_{32}(t) \\ d_{33}(t) \end{bmatrix} = \begin{bmatrix} 5\sin(20\,t) \\ 5\cos(20\,t) \\ 5\sin(20\,t) \end{bmatrix}$ is the external disturbances and phase portrait of systems (4.26) with disturbances is shown through Fig. 4.3(b).

$$\frac{dx_{41}}{dt} = a_{41}(x_{42} - x_{41}) + x_{42}x_{43} + 0.3\sin(30\,t)$$

$$\frac{dx_{42}}{dt} = a_{43}x_{41} - x_{42} - x_{41}x_{43} + 0.3\cos(30\,t) \qquad (4.27)$$

$$\frac{dx_{43}}{dt} = -a_{42}x_{43} + x_{41}x_{42} + 0.3\sin(30\,t).$$

The phase portrait of system (4.27) with external disturbances $d_4(t) = \begin{bmatrix} d_{41}(t) \\ d_{42}(t) \\ d_{43}(t) \end{bmatrix} = \begin{bmatrix} 0.3\sin(30\,t) \\ 0.3\cos(30\,t) \\ 0.3\sin(30\,t) \end{bmatrix}$ is depicted through Fig. 4.4(b).

$$\frac{dx_{51}}{dt} = a_{51}(x_{52} - x_{51}) + 0.5\sin(20\,t)$$

$$\frac{dx_{52}}{dt} = (a_{53} - a_{51})x_{51} - a_{51}x_{51}x_{53} + 0.5\cos(20\,t) \qquad (4.28)$$

$$\frac{dx_{53}}{dt} = -a_{52}x_{53} + x_{51}x_{52} + 0.5\sin(20\,t).$$

The phase portrait of system (4.28) with $d_5(t) = \begin{bmatrix} d_{51}(t) \\ d_{52}(t) \\ d_{53}(t) \end{bmatrix} = \begin{bmatrix} 0.5\sin(20\,t) \\ 0.5\cos(20\,t) \\ 0.5\sin(20\,t) \end{bmatrix}$ is shown in Fig. 4.5(b).

The Newton-Leipnik chaotic system, Liu chaotic system and Chen chaotic system with external disturbances are considered as response systems and defined as

$$\frac{dy_{11}}{dt} = -b_{11}y_{11} + y_{12} + 10\,y_{12}y_{13} + 0.1\sin(2t) + u_{11}(t)$$

$$\frac{dy_{12}}{dt} = -y_{11} - 0.4\,y_{12} + 5\,y_{11}y_{13} + 0.1\cos(2t) + u_{12}(t) \qquad (4.29)$$

$$\frac{dy_{13}}{dt} = b_{12}y_{13} - 5\,y_{11}y_{12} + 0.1\sin(2t) + u_{13}(t),$$

where $d_6(t) = \begin{bmatrix} d_{61}(t) \\ d_{62}(t) \\ d_{63}(t) \end{bmatrix} = \begin{bmatrix} 0.1\sin(2t) \\ 0.1\cos(2t) \\ 0.1\sin(2t) \end{bmatrix}$ is the external disturbances, $u_{1j}(t)$ for $(j = 1,2,3)$ are control functions and phase portraits of system (4.29) without control functions are depicted through Fig. 4.6(b).

$$\frac{dy_{21}}{dt} = b_{21}(y_{22} - y_{21}) + 0.3\sin(40t) + u_{21}(t)$$

$$\frac{dy_{22}}{dt} = b_{22}y_{21} - y_{21}y_{23} + 0.3\cos(40t) + u_{22}(t) \tag{4.30}$$

$$\frac{dy_{23}}{dt} = -b_{23}y_{23} + b_{24}y_{21}^2 + 0.3\sin(40t) + u_{23}(t).$$

with external disturbances $d_7(t) = \begin{bmatrix} d_{71}(t) \\ d_{72}(t) \\ d_{73}(t) \end{bmatrix} = \begin{bmatrix} 0.3\sin(40\,t) \\ 0.3\cos(40\,t) \\ 0.3\sin(40\,t) \end{bmatrix}$, the phase portraits of equation (4.30) without control functions $u_{2,j}(t)$ $(j = 1,2,3)$ are shown through Fig. 4.7(b).

$$\frac{dy_{31}}{dt} = b_{31}(y_{32} - y_{31}) + 0.5\sin(20t) + u_{31}(t)$$

$$\frac{dy_{32}}{dt} = (b_{33} - b_{31})y_{31} - y_{31}y_{33} + b_{33}y_{32} + 0.5\cos(20t) + u_{32}(t) \tag{4.31}$$

$$\frac{dy_{33}}{dt} = y_{31}y_{32} - b_{32}y_{33} + 0.5\sin(20t) + u_{33}(t).$$

The phase portraits of system (4.31) without control functions $u_{3,j}(t)$ $(j = 1,2,3)$ with $d_8(t) = \begin{bmatrix} d_{81}(t) \\ d_{82}(t) \\ d_{83}(t) \end{bmatrix} = \begin{bmatrix} 0.5\sin(20\,t) \\ 0.5\cos(20\,t) \\ 0.5\sin(20\,t) \end{bmatrix}$ are depicted through Fig. 4.8(b).

Now defining the error function as

$e = AY_1 + BY_2 + CY_3 - (KX_1 + LX_2 + MX_3)(NX_4 + OX_5)$,

where $e = diag(e_1, e_2, e_3)$ is the diagonal matrix and $A = diag(a_1, a_2, a_3)$, $B = diag(b_1, b_2, b_3)$, $C = diag(c_1, c_2, c_3)$, $K = diag(k_1, k_2, k_3)$, $L = diag(l_1, l_2, l_3)$, $M = diag(m_1, m_2, m_3)$, $N = diag(n_1, n_2, n_3)$, $O = diag(o_1, o_2, o_3)$ are the constant diagonal matrices and these are also called the scaling matrices. After putting the values of error state variables and scaling matrices, we will get the following error functions

$$\begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} = \begin{bmatrix} a_1 y_{11} + b_1 y_{21} + c_1 y_{31} & 0 & 0 \\ 0 & a_2 y_{12} + b_2 y_{22} + c_2 y_{32} & 0 \\ 0 & 0 & a_3 y_{13} + b_3 y_{23} + c_3 y_{33} \end{bmatrix} -$$

$$\begin{bmatrix} (k_1x_{11}+l_1x_{21}+m_1x_{31})(n_1x_{41}+o_1x_{51}) & 0 & 0 \\ 0 & (k_2x_{12}+l_2x_{22}+m_2x_{32})(n_2x_{42}+o_2x_{52}) & 0 \\ 0 & 0 & (k_3x_{13}+l_3x_{23}+m_3x_{33})(n_3x_{43}+o_3x_{53}) \end{bmatrix}.$$

Comparing both sides, then we get the error functions as

$$\begin{aligned} e_1 &= a_1 y_{11} + b_1 y_{21} + c_1 y_{31} - (k_1 x_{11} + l_1 x_{21} + m_1 x_{31})(n_1 x_{41} + o_1 x_{51}) \\ e_2 &= a_2 y_{12} + b_2 y_{22} + c_2 y_{32} - (k_2 x_{12} + l_2 x_{22} + m_2 x_{32})(n_2 x_{42} + o_2 x_{52}) \\ e_3 &= a_3 y_{13} + b_3 y_{23} + c_3 y_{33} - (k_3 x_{13} + l_3 x_{23} + m_3 x_{33})(n_3 x_{43} + o_3 x_{53}). \end{aligned} \tag{4.32}$$

After derivative, the error functions (4.32) are reduced in following form of the error functions as

$$\begin{aligned} \frac{de_1}{dt} &= a_1 \frac{dy_{11}}{dt} + b_1 \frac{dy_{21}}{dt} + c_1 \frac{dy_{31}}{dt} - (k_1 x_{11} + l_1 x_{21} + m_1 x_{31})(n_1 \frac{dx_{41}}{dt} + o_1 \frac{dx_{51}}{dt}) \\ &\quad - (k_1 \frac{dx_{11}}{dt} + l_1 \frac{dx_{21}}{dt} + m_1 \frac{dx_{31}}{dt})(n_1 x_{41} + o_1 x_{51}) \\ \frac{de_2}{dt} &= a_2 \frac{dy_{12}}{dt} + b_2 \frac{dy_{22}}{dt} + c_2 \frac{dy_{32}}{dt} - (k_2 x_{12} + l_2 x_{22} + m_2 x_{32})(n_2 \frac{dx_{42}}{dt} + o_2 \frac{dx_{52}}{dt}) \\ &\quad - (k_2 \frac{dx_{12}}{dt} + l_2 \frac{dx_{22}}{dt} + m_2 \frac{dx_{32}}{dt})(n_2 x_{42} + o_2 x_{52}) \end{aligned} \tag{4.33}$$

$$\frac{de_3}{dt} = a_3\frac{dy_{13}}{dt} + b_3\frac{dy_{23}}{dt} + c_3\frac{dy_{33}}{dt} - (k_3x_{13} + l_3x_{23} + m_3x_{33})(n_3\frac{dx_{43}}{dt} + o_3\frac{dx_{53}}{dt}) \\ -(k_3\frac{dx_{13}}{dt} + l_3\frac{dx_{23}}{dt} + m_3\frac{dx_{33}}{dt})(n_3x_{43} + o_3x_{53}).$$

Putting the values of $\frac{dx_{ij}}{dt}$ $(i = 1,2,3,4,5; j = 1,2,3)$ from equations (4.24) - (4.28) and $\frac{dy_{ij}}{dt}$ $(i = 1,2,3; j = 1,2,3)$ from equations (4.29) - (4.31) in equations (4.33), then we obtain the error systems as

$$\begin{aligned}
\frac{de_1}{dt} &= -a_1b_{11}y_{11} + a_1y_{12} + 10a_1y_{12}y_{13} + 0.1a_1\sin(2t) + b_1b_{21}(y_{22} - y_{21}) \\
&\quad + 0.3b_1\sin(40t) + b_{31}c_1(y_{32} - y_{31}) + 0.5c_1\sin(20t) - (k_1x_{11} + l_1x_{21} + m_1x_{31}) \\
&\quad \times[n_1a_{41}(x_{42} - x_{41}) + n_1x_{42}x_{43} + 0.3n_1\sin(30t) + o_1a_{51}(x_{52} - x_{51}) + 0.5o_1\sin(20t)] \\
&\quad -[k_1a_{11}(x_{12} - x_{11}) + 10k_1\sin(10t) - l_1x_{22} - l_1x_{23} - l_1\sin(t) + m_1a_{31}(x_{32} - x_{31}) \\
&\quad + 5m_1\sin(20t)](n_1x_{41} + o_1x_{51}) + u_1^*(t) \\
\frac{de_2}{dt} &= -a_2y_{11} - 0.4a_2y_{12} + 5a_2y_{11}y_{13} + 0.1a_2\cos(2t) + b_2b_{22}y_{21} - b_2y_{21}y_{23} \\
&\quad + 0.3b_2\cos(40t) + c_2(b_{33} - b_{31})y_{31} - c_2y_{31}y_{33} + c_2b_{33}y_{32} + 0.5c_2\cos(20t) \\
&\quad -(k_2x_{12} + l_2x_{22} + m_2x_{32})[n_2a_{43}x_{41} - n_2x_{42} - n_2x_{41}x_{43} + 0.3n_2\cos(30t) \\
&\quad + o_2(a_{53} - a_{51})x_{51} - o_2a_{51}x_{51}x_{53} + 0.5o_2\cos(20t)] - [k_2x_{11}(a_{13} - x_{13}) \\
&\quad - k_2x_{12} + 10k_2\cos(10t) + l_2x_{21} + a_{21}l_2x_{22} + l_2\cos(t) - m_2x_{31}x_{33} + a_{33}m_2x_{32} \\
&\quad + 5m_2\cos(20t)](n_2x_{42} + o_2x_{52}) + u_2^*(t) \\
\frac{de_3}{dt} &= a_3b_{12}y_{13} - 5a_3y_{11}y_{12} + 0.1a_3\sin(2t) - b_3b_{23}y_{23} + b_3b_{24}y_{21}^2 + 0.3b_3\sin(40t) \\
&\quad + c_3y_{31}y_{32} - b_{32}c_3y_{33} + 0.5c_3\sin(20t) - (k_3x_{13} + l_3x_{23} + m_3x_{33})[-n_3a_{42}x_{43} \\
&\quad + n_3x_{41}x_{42} + 0.3n_3\sin(30t) - o_3a_{52}x_{53} + o_3x_{51}x_{52} + 0.5o_3\sin(20t)] - [k_3x_{11}x_{12} \\
&\quad - k_3a_{12}x_{13} + 10k_3\sin(10t) + l_3a_{22} + l_3x_{21}x_{23} - l_3a_{23}x_{23} - l_3\sin(t) + m_3x_{31}x_{32} \\
&\quad - m_3a_{32}x_{33} + 5m_3\sin(20t)](n_3x_{43} + o_3x_{53}) + u_3^*(t),
\end{aligned} \tag{4.34}$$

where $u_1^* = a_1u_{11}(t) + b_1u_{21}(t) + c_1u_{31}(t)$, $u_2^* = a_1u_{12}(t) + b_1u_{21}(t) + c_1u_{31}(t)$, $u_1^* = a_1u_{11}(t) + b_1u_{21}(t) + c_1u_{31}(t)$ are the control functions and these are designed using the nonlinear control method given in theorem 4.2.

Theorem 4.2: If the nonlinear control functions are designed as

$$
\begin{aligned}
u_1^*(t) = & -k_1e_1 + a_1b_{11}y_{11} - a_1y_{12} - 10a_1y_{12}y_{13} - 0.1a_1\sin(2t) - b_1b_{21}(y_{22} - y_{21}) \\
& - 0.3b_1\sin(40t) - b_{31}c_1(y_{32} - y_{31}) - 0.5c_1\sin(20t) + (k_1x_{11} + l_1x_{21} + m_1x_{31}) \\
& \times[n_1a_{41}(x_{42} - x_{41}) + n_1x_{42}x_{43} + 0.3n_1\sin(30t) + o_1a_{51}(x_{52} - x_{51}) + 0.5o_1\sin(20t)] \\
& + [k_1a_{11}(x_{12} - x_{11}) + 10k_1\sin(10t) - l_1x_{22} - l_1x_{23} - l_1\sin(t) + m_1a_{31}(x_{32} - x_{31}) \\
& + 5m_1\sin(20t)](n_1x_{41} + o_1x_{51})
\end{aligned}
$$

$$
\begin{aligned}
u_2^*(t) = & -k_2e_2 + a_2y_{11} + 0.4a_2y_{12} - 5a_2y_{11}y_{13} - 0.1a_2\cos(2t) - b_2b_{22}y_{21} + b_2y_{21}y_{23} \\
& - 0.3b_2\cos(40t) - c_2(b_{33} - b_{31})y_{31} + c_2y_{31}y_{33} - c_2b_{33}y_{32} - 0.5c_2\cos(20t) \\
& + (k_2x_{12} + l_2x_{22} + m_2x_{32})[n_2a_{43}x_{41} - n_2x_{42} - n_2x_{41}x_{43} + 0.3n_2\cos(30t) \\
& + o_2(a_{53} - a_{51})x_{51} - o_2a_{51}x_{51}x_{53} + 0.5o_2\cos(20t)] + [k_2x_{11}(a_{13} - x_{13}) - k_2x_{12} \\
& + 10k_2\cos(10t) + l_2x_{21} + a_{21}l_2x_{22} + l_2\cos(t) - m_2x_{31}x_{33} + a_{33}m_2x_{32} \\
& + 5m_2\cos(20t)](n_2x_{42} + o_2x_{52})
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
u_3^*(t) = & -k_3e_3 - a_3b_{12}y_{13} + 5a_3y_{11}y_{12} - 0.1a_3\sin(2t) + b_3b_{23}y_{23} - b_3b_{24}y_{21}^2 - 0.3b_3\sin(40t) \\
& - c_3y_{31}y_{32} + b_{32}c_3y_{33} - 0.5c_3\sin(20t) + (k_3x_{13} + l_3x_{23} + m_3x_{33})[-n_3a_{42}x_{43} \\
& + n_3x_{41}x_{42} + 0.3n_3\sin(30t) - o_3a_{52}x_{53} + o_3x_{51}x_{52} + 0.5o_3\sin(20t)] \\
& + [k_3x_{11}x_{12} - k_3a_{12}x_{13} + 10k_3\sin(10t) + l_3a_{22} + l_3x_{21}x_{23} - l_3a_{23}x_{23} - l_3\sin(t) \\
& + m_3x_{31}x_{32} - m_3a_{32}x_{33} + 5m_3\sin(20t)](n_3x_{43} + o_3x_{53})
\end{aligned}
$$

then the triple compound synchronization among considered chaotic systems is achieved and satisfy the condition $\lim_{t\to\infty}\| e_i(t) \| = 0$, $i = 1, 2, 3$ for any positive constant k_i , $i = 1, 2, 3$.

Proof: Let us define the Lyapunov function to stabilize the error systems as

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2).$$

Now taking the derivative of V , we get

$$\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3. \tag{4.36}$$

From equation (4.34) and (4.35), putting the values of error systems and controllers in equation (4.36), we obtain

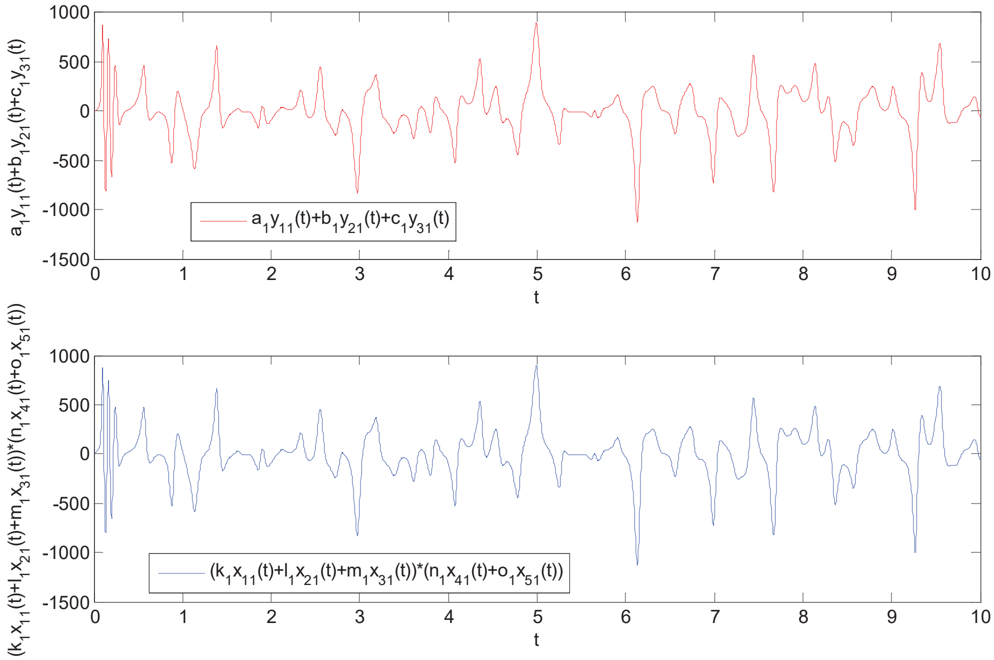
$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 < 0. \tag{4.37}$$

It is clear from the equations (4.37) the derivative of Lyapunov function is negative definite and thus the error system is asymptotically stable according to Lyapunov stability theory for the positive controller parameters k_i , $i = 1, 2, 3$. Consequently the triple compound synchronization among chaotic systems will be achieved. It is seen form the Fig. 4.9 the considered systems are synchronized when error $e_i(t)$, $i = 1, 2, 3$ tends to zero as time becomes large.

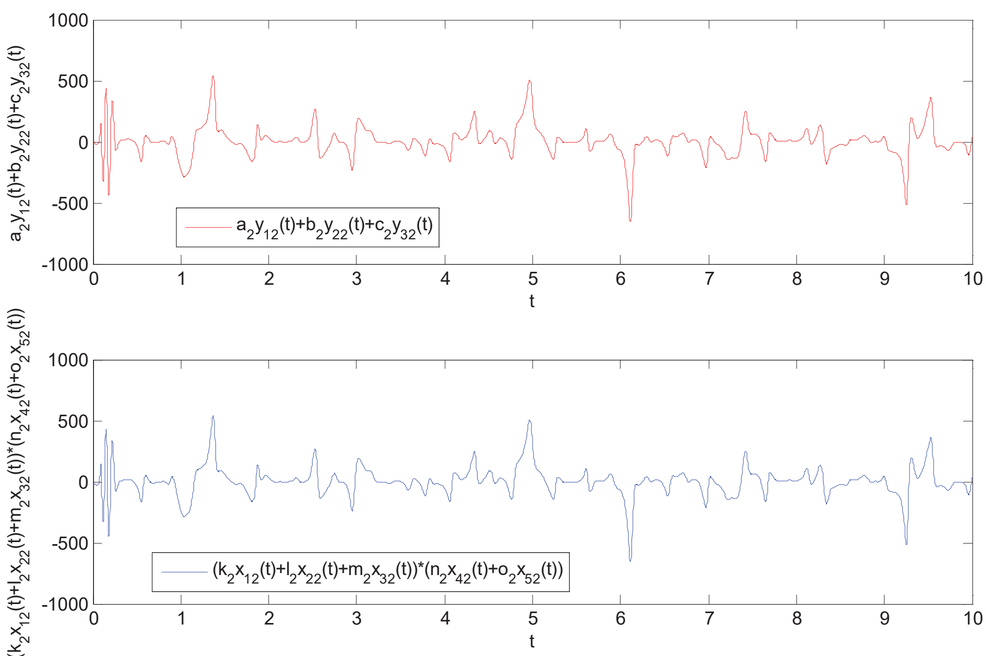
4.5 Numerical simulation and results

In numerical simulation, the earlier values of parameters of Lorenz, Rossler, Lu, Qi, T, Newton-Leipnik, Liu and Chen chaotic systems are taken. The initial condition of five master systems and three slave systems are taken as $(0.1, 0.1, 0.1)$, $(2, -4, 3)$, $(0.2, 0.5, 0.3)$, $(3, 2, 1)$, $(0.1, -0.3, 0.2)$ and $(0.19, 0, -0.18)$, $(0.2, 0, 0.5)$, $(1, 1.4, 1)$ respectively. Thus the initial error according to definition of error functions will be $(-5.74, 7.18, -2.76)$.

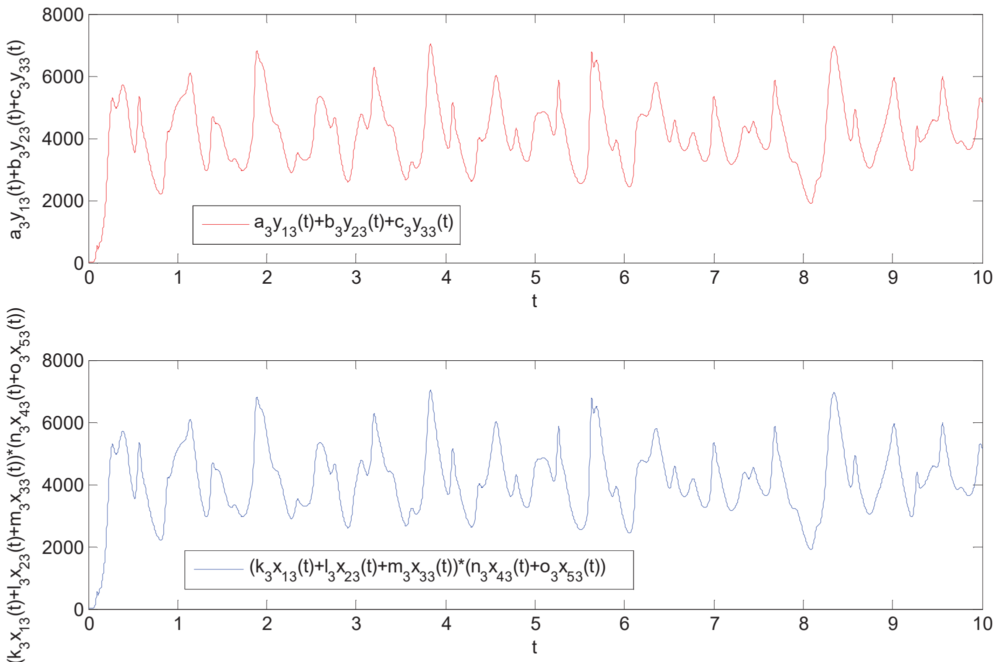
During the triple compound synchronization the parameters are taken as $a_1 = a_2 = a_3 = 1$, $b_1 = b_2 = b_3 = 1$, $c_1 = c_2 = c_3 = 1$, $k_1 = k_2 = k_3 = 1$, $l_1 = l_2 = l_3 = 1$, $m_1 = m_2 = m_3 = 1$, $n_1 = n_2 = n_3 = 1$, $o_1 = o_2 = o_3 = 1$. It is seen from Fig. 4.9(d) that the error functions asymptotically converge to zero after a small time of duration, which shows that the considered master systems (4.24)-(4.28) are triple compound synchronized with the slave systems (4.29)-(4.31).


a1y11(t)+b1y21(t)+c1y31(t)
(k1x11(t)+l1x21(t)+m1x31(t))*(n1x41(t)+o1x51(t))
1000
500
0
-500
-1000
-1500
0 1 2 3 4 5 6 7 8 9 10
t

(a)

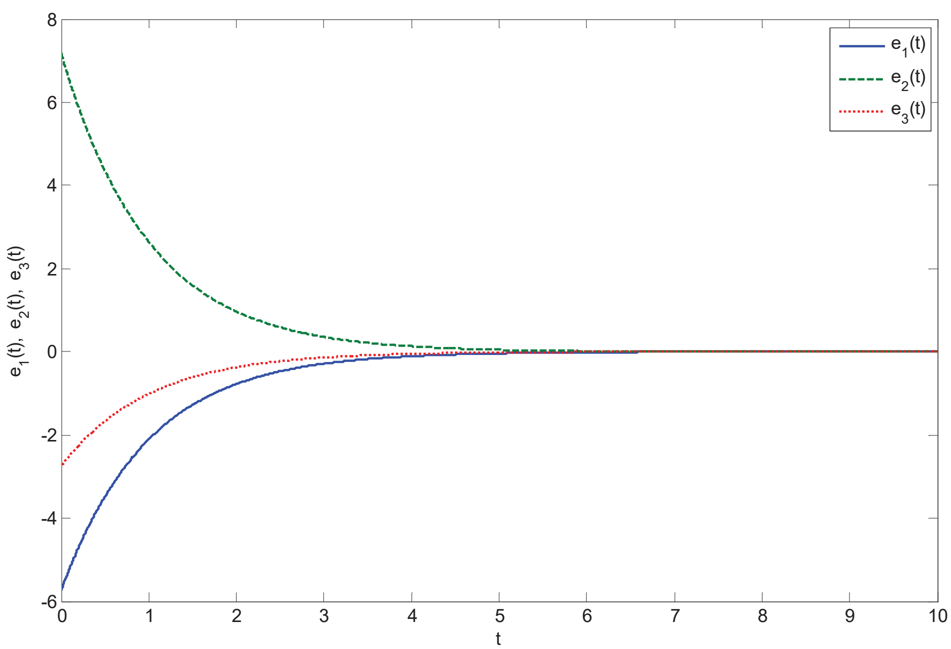

a2y12(t)+b2y22(t)+c2y32(t)
(k2x12(t)+l2x22(t)+m2x32(t))*(n2x42(t)+o2x52(t))
1000
500
0
-500
-1000
0 1 2 3 4 5 6 7 8 9 10
t

(b)



8000
6000
4000
2000
0
0 1 2 3 4 5 6 7 8 9 10
t
$a_3y_{13}(t)+b_3y_{23}(t)+c_3y_{33}(t)$
$a_3y_{13}(t)+b_3y_{23}(t)+c_3y_{33}(t)$
$(k_3x_{13}(t)+l_3x_{23}(t)+m_3x_{33}(t))*(n_3x_{43}(t)+o_3x_{53}(t))$
$(k_3x_{13}(t)+l_3x_{23}(t)+m_3x_{33}(t))*(n_3x_{43}(t)+o_3x_{53}(t))$

(c)



8
6
4
2
0
-2
-4
-6
0 1 2 3 4 5 6 7 8 9 10
t
$e_1(t)$, $e_2(t)$, $e_3(t)$
$e_1(t)$
$e_2(t)$
$e_3(t)$

(d)

Fig. 4.9. Triple compound synchronization among master systems (4.24)-(4.28) and response systems (4.29)-(4.31): (a) synchronization between the trajectories $a_1y_{11}+b_1y_{21}+c_1y_{31}$ and $(k_1x_{11}+l_1x_{21}+m_1x_{31})(n_1x_{41}+o_1x_{51})$; (b) synchronization between the trajectories $a_2y_{12}+b_2y_{22}+c_2y_{32}$ and $(k_2x_{12}+l_2x_{22}+m_2x_{32})(n_2x_{42}+o_2x_{52})$; (c) synchronization between the trajectories $a_3y_{13}+b_3y_{23}+c_3y_{33}$ and $(k_3x_{13}+l_3x_{23}+m_3x_{33})(n_3x_{43}+o_3x_{53})$; (d) plot of error functions $e_1(t)$, $e_2(t)$, $e_3(t)$.

4.6 Conclusion

This chapter has investigated a new type of synchronization among five master and three slave (response) chaotic systems with external disturbances, which is known as triple compound synchronization. The proposed synchronization has more complexity in error functions and due to this it has the better security of communication via signals. The signals will be very strong and have own stronger anti-attack and anti-translated ability. The nonlinear control functions are proposed using nonlinear control method with Lyapunov stability theory to achieve triple compound synchronizations. Numerical simulation exhibits the reliability and effectiveness of the proposed triple compound synchronization scheme towards predicting the accuracy of the theory.

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