EXTENDED ABSTRACT

ON HERMITE EXPANSIONS

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ABSTRACT

For $a > 0$, let $g_a(x) = e^{-ax^2/2}$. For $a, b > 0$, let

$$E(a,b) = \{f \in L^1(\mathbb{R}) \mid |f(x)| \leq C g_a(x) \text{ and } \left|\hat{f}(\xi)\right| \leq C g_b(\xi) \quad \text{for some } C \in \mathbb{R}\}.$$

Then, Hardy's uncertainty principle (see [2, 4]) says that:

1. If $ab > 1$ then $E(a,b) = \{0\}$.

2. If $ab = 1$ then $E(a,b) = \mathbb{C}g_a$.

3. If $ab < 1$ then $\dim E(a,b) = \infty$.

In [5], Vemuri considered the case $ab < 1$, $a = b$, in greater detail, and proved that if $f \in E(a,a)$, then

$$\langle f, \varphi_n \rangle = O\left(n^{-1/4}\left(\frac{1-a}{1+a}\right)^{n/4}\right)$$

for $n = 1, 2, \ldots$, and this estimate is sharp.

This result has been generalized to the higher dimensions by Garg and Thangavelu (see [7]). In this thesis, we are interested in further improvement and generalization of Vemuri's result.

In **Chapter 1**, we give a brief introduction to the uncertainty principle, and discuss some useful tools such as Hermite functions, Bargmann transform, etc to prove our main results. We also provide proofs of some essential results about these objects. Other uncertainty principles in harmonic analysis can be seen in [1]. For more about the theory of Hermite expansions, see [3]. The Bargmann transform is discussed in

[6].

In **Chapter 2**, we show that under the same assumptions as in Vemuri's result, certain combinations of Hermite coefficients have a better rate of decay, namely, the following result. Let $t > 0$. If $f \in E(\tanh 2t, \tanh 2t)$ then

$$\langle f, \varphi_n \rangle + \frac{n(n+2)}{\sqrt{\prod_{j=1}^{4}(n+j)}} e^{4t} \langle f, \varphi_{n+4} \rangle = O\left(n^{-3/4} e^{-nt}\right),$$

for $n = 1, 2, \ldots$. If $t_0 > 0$ then this estimate is uniform for $t \in [t_0, \infty)$ (see [9]).

In **Chapter 3**, we generalize Vemuri's result, i.e, let $a, b \in (0, \infty)$ and suppose $ab < 1$. If $f \in E(a, b)$ then

$$\langle f, \varphi_n \rangle = O\left(n^{-\frac{1}{4}} \left(\frac{a+b-2ab}{a+b+2ab}\right)^{n/4}\right)$$

for $n = 1, 2, \ldots$, and this estimate is sharp (see [8]). As a byproduct of the proof of this result, we obtain the following strong Gaussian bound for the Bargmann transform of f. Let $a, b \in (0, \infty)$ and suppose $ab < 1$. Let $m = \min\{a, b\}$ and $f \in E(a, b)$. Then there exist unique numbers $A = A(a, b) \in (0, 1)$, $\theta_0 = \theta_0(a, b), \theta_1 = \theta_1(a, b) \in (0, \pi/2)$, and $\tau = \tau(a, b) \in (-\pi/4, \pi/4)$, and

$$|Bf(w)| \leq C\sqrt{\frac{2}{1+m}} \exp\left(A\frac{r^2}{4} \sin(2\theta - 2\tau)\right), \quad (w = re^{i\theta}), \tag{1}$$

for $\theta_0 \leq \theta \leq \theta_1$, $\theta_0 + \pi \leq \theta \leq \theta_1 + \pi$, and

$$|Bf(w)| \leq C\sqrt{\frac{2}{1+m}} \exp\left(A\frac{r^2}{4} \sin(-2\theta - 2\tau)\right), \quad (w = re^{i\theta}), \tag{2}$$

for $2\pi - \theta_1 \leq \theta \leq 2\pi - \theta_0$, and $\pi - \theta_1 \leq \theta \leq \pi - \theta_0$.

The proof of the last result is based on the Phragmén-Lindelöf argument.

References

[1] G. B. Folland and A. Sitaram. "The Uncertainty Principle: A Mathematical Survey", The Journal of Fourier Analysis and Applications, 1997

[2] G. H. Hardy, "A Theorem Concerning Fourier Transform", Journal of the London Mathematical Society, 1933.

[3] S. Thangavelu, "Lectures on Hermite and Laguerre expansions", volume 42 of Mathematical Notes. Princeton University Press, Princeton, NJ, 1993

[4] S. Thangavelu. "An introduction to the uncertainty principle: Hardy's theorem on Lie groups", Birkhauser, Boston, Basel, Berlin, 2003.

[5] M. K. Vemuri, "Hermite Expansion and Hardy's Theorem", https://arxiv.org/pdf/0801.2234.pdf, 2007.

[6] S. Thangavelu, "Hermite and Laguerre semigroups: some recent developments", In Orthogonal families and semigroups in analysis and probability, volume 25 of S´emin. Congr., pages 251–284. Soc. Math. France, Paris, 2012.

[7] R. Garg and S. Thangavelu, "On the Hermite expansions of functions from the Hardy class", Studia Math., 198(2):177–195, 2010.

[8] Manish Chaurasia, "A generalization of a result of Vemuri", Journal of the Ramanujan Mathematical Society, to appear.

[9] Manish Chaurasia, "On pair correlation of Hermite coefficients of functions from the Hardy class", The Journal of Analysis, published online.