

A Minimum Operator Approach to Control Design for Dynamical Systems



Thesis submitted in partial fulfillment
for the award of degree

Doctor of Philosophy

by

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2025

To
The
Supreme Personality
Lord Shiva
and
my beloved family

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Abstract

This thesis introduces a comprehensive framework of novel data-driven discrete-time sliding mode control (DSMC) techniques tailored for discrete-time systems (DTS) subjected to disturbances and uncertainties, with a focus on both theoretical rigor and practical applicability. The central methodology leverages compact form dynamic linearization (CFDL) to construct control-oriented models using only input-output (I/O) data, thereby eliminating the need for explicit state-space modeling or system identification.

The primary contribution lies in the development of the CFDL-DSMC strategy, which integrates CFDL with a Minimum Operator-Based Discrete Sliding Mode Control (MDSMC) scheme. This formulation enhances disturbance rejection and tracking accuracy while ensuring robust stability, without relying on traditional switching gains that typically induce chattering. Stability analysis confirms that the proposed control law guarantees finite-time convergence of the sliding variable despite system uncertainties.

To address prescribed performance constraints (PPC) in data-driven control, the thesis introduces a transformative error mapping function that converts constrained tracking errors into an unconstrained domain. Based on this transformation and using Full Form Dynamic Linearization (FFDL), new MDSMC laws are formulated to ensure that the original tracking error converges within an asymmetric, user-defined boundary in a pre-defined finite time. This controller maintains a simple structure and avoids the need for additional switching elements, distinguishing it from conventional DSMC approaches.

Furthermore, a Multi-Rate Output Feedback (MROF)-based DSMC is developed, employing fast output sampling and minimum-operator-based reaching laws (RL), relying solely on output measurements and eliminating the need for full-state feedback. This approach is particularly advantageous in practical settings where full state information is inaccessible. Additionally, a finite-time minimum-operator-based controller is proposed to enhance response speed and guarantee settling time, thereby supporting real-time im-

plementation.

An adaptive reaching law is also introduced, utilizing both minimum and ceiling operators, enabling dynamic adjustment of control gains in response to uncertainties and matched disturbances. This law ensures convergence to the sliding surface within a user-specified time frame and eliminates chattering in the sliding phase by employing a non-switching-type sliding motion definition.

The versatility and robustness of the proposed methods are demonstrated through a series of simulations and experimental validations on various nonlinear discrete-time systems:

- A two-tank water level control system demonstrating real-time tracking accuracy and implementation feasibility.
- A single-link robotic manipulator validating robustness and performance under nonlinear dynamic conditions.
- A two-degree-of-freedom (2-DOF) helicopter system illustrating finite-time attitude tracking for both pitch and yaw axes using recursive backstepping integrated with the minimum operator.
- A magnetic levitation system (MLS) highlighting the method’s capability to achieve precise finite-time stabilization while significantly mitigating chattering.

Collectively, these contributions offer robust, efficient, and practically implementable solutions to control challenges in uncertain nonlinear discrete-time systems. The approaches developed herein address key limitations of traditional sliding mode control—such as chattering, reliance on full-state information, and lack of settling time control—while ensuring theoretical guarantees and strong practical performance in real-world applications.

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Nomenclature

List of Greek and Roman Symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	n -dimensional Euclidean space
$\mathbb{R}_{\geq 0}$	Set of non-negative real numbers
$\mathbb{R}_{\geq k}$	Set of real numbers greater than or equal to k
$\aleph$	Sliding Variable
$\mathbb{R} \rightarrow \mathbb{R}$	Element-wise mapping

Abbreviations

DTS	Discrete Time System
CTS	Continuous Time System
FTS	Finite-Time Stability
VSC	Variable Structure Control
SMC	Sliding Mode Control
DSMC	Discrete Sliding mode Control
DDC	Data-Driven Control
MFAC	Model-Free Adaptive Control
FFDL	Full-Form Dynamic Linearization
CFDL	Compact-Form Dynamic Linearization
PPC	Prescribed Performance Control
MIMO	Multi-Input Multi-Output
min	Minimum
max	Maximum
sup	Supremum
MROF	Multirate Output Feedback
FOS	Fast Output Sampling
MLS	Magnetic Levitation System
DOF	Degree Of Freedom
RL	Reaching Law

Chapter 1

Introduction

Control system theory forms the cornerstone of modern methods for regulating and managing dynamic systems across a vast range of applications. It encompasses a structured set of principles, methodologies, and tools dedicated to achieving system stability, desired performance, and robustness, even in the presence of disturbances, uncertainties, and varying environmental conditions. Fundamentally, control theory is concerned with the modeling, analysis, and design of systems that can autonomously track target behaviors or maintain prescribed outputs.

Spanning fields such as engineering, physics, economics, and the biological sciences, control theory provides a systematic framework for understanding and optimizing the behavior of dynamic systems. Central to this framework is the concept of feedback, where the system's output is continuously monitored and compared to a reference, with corrective actions applied to minimize any deviation. This feedback-driven approach allows systems to dynamically adapt to changing conditions and maintain stability.

Control systems are generally categorized into continuous-time and discrete-time types. While continuous-time systems deal with signals that evolve smoothly over time, discrete-time systems process signals sampled at specific intervals. Both types have found widespread applications in industries including electrical engineering, mechanical systems, aerospace, and process control.

At the heart of control system theory lies mathematical modeling, where system dynamics are typically described by differential equations for continuous-time systems and by difference equations for discrete-time systems. Stability analysis is crucial, assessing whether a system can return to its equilibrium after experiencing a disturbance. Fur-

thermore, control design techniques are applied to shape the system’s behavior, ensuring compliance with performance criteria while safeguarding resilience against uncertainties.

The tuning of control parameters is essential for tailoring system behavior to meet specified performance objectives. Various control strategies—such as Proportional-Integral-Derivative (PID) control, state-space methods, and optimal control—have been developed to enhance the precision, adaptability, and reliability of modern control systems.

Control methodologies are often classified into conventional and non-conventional approaches. Conventional control, typically grounded in the continuous dynamics framework, uses tools like Bode plots and Nyquist diagrams for stability and performance evaluation. PID controllers are prevalent here, providing robust performance in linear time-invariant systems, particularly in domains like process control and automotive engineering.

In contrast, non-conventional control strategies, such as variable structure control (VSC), feature dynamic switching and structural discontinuities to address system uncertainties more effectively. A prominent technique within VSC is Sliding Mode Control (SMC), where the system’s state is driven onto a designated sliding surface and maintained there. The sliding motion grants the system remarkable robustness against model inaccuracies and external disturbances.

SMC is especially distinguished for its ability to handle nonlinearities and uncertainties. By forcing the system to operate on the sliding surface, it ensures that performance remains stable despite perturbations. Typically, SMC relies on discontinuous control laws that incorporate switching actions, enhancing resilience and guaranteeing stability. Formal stability proofs often utilize Lyapunov-based methods, providing rigorous guarantees of convergence even in uncertain settings.

The choice between conventional and variable structure control strategies depends heavily on system characteristics. Conventional methods are well-suited for precisely modeled, linear systems, while variable structure control techniques like SMC excel in handling nonlinear, time-varying, or highly uncertain systems—making them ideal for complex applications such as robotics, aerospace, and advanced manufacturing.

1.1 Motivation

- Nonlinear DTS often face disturbances and uncertainties, making precise control and stability a significant challenge. Moreover, the complete system model is not always available. Conventional control approaches typically depend on accurate system models; however, model inaccuracies and external disturbances can adversely affect their performance. In contrast, data-driven control (DDC) methods, which utilize input-output data through CFDL, offer a promising alternative by reducing or eliminating the reliance on precise system modeling.
- Controlling discrete-time nonlinear systems under PPC remains a significant challenge, particularly in the presence of uncertainties and disturbances. Traditional model-based methods often require precise system models and additional switching gains, complicating implementation and potentially degrading performance. To address these issues, our research proposes a minimum operator-based DDC strategy that eliminates reliance on explicit modeling and additional switching mechanisms. A FFDL approach constructs a data-driven model, while a transformative error mapping function converts constrained tracking errors into an unconstrained form, simplifying controller design. The resulting minimum operator-based discrete sliding mode control (MDSMC) ensures robust convergence within specified asymmetric bounds without introducing structural complexity.
- Uncertain systems pose significant challenges in control, especially when precise tracking and disturbance rejection are required under dynamic conditions. Traditional control methods often rely on full system state feedback, which may be infeasible due to the complexity of the system, measurement limitations, or the need for expensive sensors. Additionally, many conventional control strategies assume a constant, high sampling rate, which is not always practical in real-time systems with resource constraints. This can lead to reduced performance and increased implementation complexity. To address these challenges, this thesis introduces a novel approach that achieves quasi-sliding mode in uncertain systems using a fast output sampling control strategy combined with minimum operator-based DSMC. By operating solely on output samples, the proposed method eliminates the need for full system state feedback and reduces reliance on high sampling rates, making it

well-suited for systems with limited resources.

- In the control of uncertain DTS, challenges arise due to the presence of disturbances, model inaccuracies, and the need for precise tracking under varying conditions. Traditional control strategies, while effective in many scenarios, often struggle with ensuring robustness when system uncertainties or disturbances are present. Furthermore, many approaches are prone to undesirable phenomena such as chattering during the sliding phase, which can degrade performance or lead to instability. To address these issues, there is a growing need for adaptive and robust control methods that can handle uncertainties without relying on fixed parameters or switching gains. Moreover, it is crucial that these methods offer flexibility in controlling the reaching time for the sliding surface, allowing the system to meet predefined performance specifications while avoiding chattering. Our research aims to fill these gaps by introducing an adaptive reaching law that guarantees the system reaches the sliding surface within a predetermined time while maintaining robust performance. The proposed method eliminates chattering effects and adapts control gains to the system's evolving behavior, thus ensuring effective control even in the presence of disturbances and uncertainties.
- Real-world applications such as two-tank systems, robotic manipulators, magnetic levitation systems, and helicopter systems highlight the critical requirement for advanced control techniques capable of handling uncertainties and guaranteeing predefined performance levels.

1.2 Literature Review

- **Data-Driven Discrete Sliding Mode Control:** In recent years, there has been increasing demand for higher control precision, particularly in managing nonlinear DTS affected by uncertainties and disturbances. SMC has been widely applied due to its robustness and ease of implementation, although its performance in discrete systems often suffers as sampling times increase. DSMC has seen significant advancements in two primary areas: the inclusion of switching and non-switching [2, 4–7] elements in the RL. The concept of quasi-sliding mode (QSM) has

been developed to address the lack of ideal sliding mode behavior in DTS. However, the transition of VSC from continuous-time systems (CTS) to DTS has not been entirely straightforward, often requiring a model-based approach that assumes an accurate understanding of the system dynamics, which is not always feasible [8–12]. DDC methods have emerged as an alternative to model-based control, providing flexibility in real-world applications where model accuracy is challenging. DDC methods generally fall into two categories: one involves adjusting parameters based on measured data while maintaining a fixed controller structure [13], and the other utilizes function approximations, such as neural networks or Taylor series, to fine-tune controller parameters [14], an approach known as model-free adaptive control (MFAC). The integration of MFAC with CFDL data-driven modeling, where controllers are designed using only input-output data without relying on system dynamics, has shown promise in handling nonlinearities in systems. The fusion of CFDL with DSMC techniques has been explored to address these nonlinearities effectively. Early work by [15] introduced model-free adaptive sliding mode control but did not account for disturbances. More recent studies [16] have proposed adaptive DSMC using CFDL for discrete nonlinear systems, improving disturbance handling. Researchers have also combined DSMC with MFAC to tackle industrial applications, including robotic joint speed control [17], pressure control systems [18], and MIMO systems [19]. The concept of discrete finite-time stability (FTS), initially not central to DSMC, has gained attention, especially following the introduction of a minimum-based approach [20]. This concept has been extended to high-order systems [9], achieving finite-time convergence. Further work has focused on reducing chattering and achieving ideal sliding mode in the absence of disturbances [2]. While combining MDSMC with CFDL has the potential to enhance performance, this approach remains unexplored in the literature. Our research proposes a novel integration of MDSMC with CFDL for nonlinear DTS with perturbations, offering a promising direction for future research.

- **Prescribed Performance-Based Data-Driven Control Using Minimum Operator:** In recent years, the increasing demand for precision control in nonlinear DTS has heightened the focus on addressing perturbations and uncertainties while ensuring PPC are met. Many practical systems require PP [21], ensuring that

errors stay within predefined bounds. Although much of the work on PP control focuses on CTS [22], exploring this concept for DTS is crucial, given their widespread use in applications such as digital control and communication systems. The gap between continuous- and discrete-time control methods, exacerbated by sampling errors, complicates the direct application of CTS methods to DTS. While limited research on PP for DTS exists [23], a recent attempt to formulate MFAC using DSMC with features for PPC was made in [24], where the authors developed a data-driven adaptive sliding mode control (SMC) strategy based on the FFDL technique. Recently, minimum operator-based DSMC [2] has been developed. However, no prior attempt has been made to integrate the minimum operator-based DSMC approach with FFDL and PPC simultaneously, and the potential for performance improvement through this combination remains unexplored.

- Output Feedback Based Discrete-Time Sliding Mode Control Using Minima Operator:** The SMC theory, developed by Utkin for CTS, adapts the controller's structure to achieve desired outcomes despite disturbances, parameter uncertainties, and performance constraints [25]- [26]. However, implementing SMC in DTS often leads to performance degradation as sampling time increases [27]. Research on DSMC has evolved, addressing methods for both full-state feedback [28], [29] and output feedback [30]- [31], though the former assumes access to all plant states, which is often impractical. Data-driven methods leveraging output feedback have gained traction, especially with advancements in digital computing [32]- [33], fostering research in DTS and DSMC [4], [5], [34], [35]. Recent research extends DSMC to multirate systems, where varying dynamics across channels and hardware constraints make multiple rates necessary. This approach has become crucial in networked control systems with bandwidth limitations [31], [36], [37]. MROF, which utilizes output readings and prior input signals, has proven effective for maintaining performance with lower sampling frequencies. In [36], FOS is introduced, offering a flexible method for system pole allocation while stabilizing controlled dynamics. Periodic intermittent control combined with MROF is employed in [30] for robust stabilization, while [31] presents a rapid sampling strategy to achieve QSM without control switching, minimizing chattering.

- **Adaptive Discrete-Time Sliding Mode Control:** SMC is a robust nonlinear control strategy, particularly effective for uncertain systems, offering advantages in terms of stability and resilience to external perturbations and uncertainties. It has found widespread applications in diverse fields, including robotics, aerospace, and motion control [38]. DSMC, in contrast to continuous sliding mode control (CSMC), operates with a limited switching frequency, making it more practical for real-world applications [39]. However, DSMC faces challenges such as chattering and lengthy reaching times. Chattering is a key issue that has prompted extensive research on switching functions and boundary layers [40–42]. Additionally, the reduction of reaching time is crucial, especially in applications like missile control that require stringent time constraints [43]. To address these challenges, several strategies have been proposed, including modified RLs that aim to reduce chattering and improve convergence, as seen in [44]. Despite these improvements, DSMC methods still face limitations, particularly the need for prior knowledge of disturbance bounds, which is not always available. To overcome this, adaptive sliding mode control, which adjusts without relying on disturbance knowledge, has become a promising alternative [45–47]. However, issues related to over- or under-estimation of the adaptive switching gain persist [45].

1.3 Contributions

- **Development of DDC control using Minimum Operator-Based CFDL-DSMC Design Approach:** A novel DDC control strategy combining CFDL with minimum operator-based DSMC has been proposed to enhance tracking precision and disturbance rejection.
- **Development of DDC control using Minimum Operator-Based FFDL-DSMC Design Approach with prescribe performance feature:** A FFDL with minimum operator based DDC method has been introduced to achieve tracking within specified asymmetric ranges, utilizing a transformative error mapping technique that converts constrained errors into unconstrained forms without requiring additional switching gains.

- **Output Feedback Based Discrete-Time Sliding Mode Control Using Minimum Operator:** A quasi-sliding mode control strategy based on fast output sampling and minimum operators has been developed, requiring only output data for controller design and achieving finite-time control.
- **Adaptive Minimum Operator DSMC:** An adaptive control law with minimum operator and ceiling function-based gain adjustment has been proposed, allowing predetermined reaching time and mitigating chattering during sliding motion.
- **Finite-Time Attitude Tracking in Helicopter Systems:** A discrete finite-time backstepping control approach using minimum operators has been implemented and validated experimentally for a two-DOF helicopter system, ensuring robust finite-time tracking with settling time estimation.
- **Finite-Time Control of Magnetic Levitation System:** A discrete-time minimum operator-based sliding mode controller has been designed and validated for a magnetic levitation system, achieving improved stabilization and reducing chattering compared to conventional approaches.

1.4 Organization of the Thesis

This thesis is organized into eight chapters:

- **Chapter 1** introduces the work, providing a detailed literature survey, identifying the research gap, and outlining the contributions of the thesis.
- **Chapter 2** presents the notation and preliminaries used throughout the development of the thesis.
- **Chapter 3** discusses the methodology, focusing on the development of the DDC algorithms based on CFDL and minimum operator-based techniques.
- **Chapter 4** presents the methodology, focusing on the development of the DDC algorithms based on FFDL and minimum operator-based techniques with prescribed performance features.

- **Chapter 5** presents the development of the MROF-based DSMC control using minimum operators.
- **Chapter 6** provides an adaptive switching scheme design using a ceiling function for discrete-time sliding mode control.
- **Chapter 7** provides finite-time discrete control for a two-DOF helicopter system.
- **Chapter 8** presents a discrete-time sliding mode controller for a magnetic levitation system using minimum operators.
- **Chapter 9** concludes the thesis and suggests potential future research directions.

Chapter 2

Notation and Preliminaries

Throughout this thesis, standard mathematical notation is adopted. The symbol $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{R}_+$ refers to the set of positive real numbers, i.e., $\mathbb{R}_+ = \{z \in \mathbb{R} \mid z > 0\}$, and $\mathbb{R}_{\geq 0}$ indicates the set of non-negative real numbers. Likewise, $\mathbb{R}_{>k}$ and $\mathbb{R}_{\geq k}$ denote the sets of real numbers greater than k and greater than or equal to k , respectively, where k is a real constant. The set of non-negative integers is represented by $\mathbb{Z}_{\geq 0}$.

The notation $\mathbb{R}^n$ denotes the n -dimensional Euclidean space. For any $p \geq 1$, the ℓ_p -norm of a vector $y \in \mathbb{R}^n$ is defined as

$$\|y\|_p = \left(\sum_{i=1}^n |y_i|^p \right)^{1/p}.$$

If the norm type is not explicitly stated, it is assumed to be the Euclidean norm ($p = 2$), denoted by $|y|$ or $\|y\|_2$. For matrices, the transpose of a matrix A is denoted by $A^\top$. Given a vector $a = [a_1, a_2, \dots, a_n] \in \mathbb{R}^n$, the notation $\text{diag}(a)$ represents a diagonal matrix with diagonal entries a_i , where $i = 1, \dots, n$.

Given a set S , $\min\{S\}$ denotes the smallest element in S , and $\sup\{S\}$ denotes the supremum (least upper bound) of S . For functions $g_1 : S_1 \rightarrow S_2$ and $g_2 : S_2 \rightarrow S_3$, their composition is denoted as $g_2 \circ g_1 : S_1 \rightarrow S_3$, with $(g_2 \circ g_1)(\cdot) = g_2(g_1(\cdot))$.

An open n -dimensional ball of radius δ centered at a point $z_0 \in \mathbb{R}^n$ is denoted by

$$\mathbb{B}_\delta = \{z \in \mathbb{R}^n \mid \|z - z_0\| < \delta\}.$$

The absolute value function is represented as $|\cdot|$, and $\|\cdot\|_p$ is used for the p -norm of finite-dimensional vectors. The transpose of a vector is also written as $(\cdot)^\top$, and the

ceiling function is denoted by $\lceil z \rceil$, representing the smallest integer greater than or equal to z . The sign function is defined as

$$\text{sign}(z) = \begin{cases} z/|z|, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

The space ℓ_∞ refers to the set of all bounded sequences.

To comprehend the results presented in this thesis, a foundational understanding of stability concepts in discrete-time dynamical systems is essential. The definitions and mathematical tools introduced in this section form a critical basis for the subsequent chapters.

2.1 Finite-Time Stability of DTS

The subsequent outcomes will prove beneficial for comprehending the notion of finite-time stability of DTS. Let us consider the following forced nonlinear DTS

$$z(k+1) = f(z(k), u(z(k))), \quad z(0) = z_0, \quad k \in \mathbb{Z}^+ \quad (2.1)$$

where $z(k) \in \mathbb{D} \subseteq \mathbb{R}^n$ denotes the system state, $u(k) \in \mathbb{R}^m$ represents the control input and $\mathbb{D}$ is an open set which includes the origin. The function $f(\cdot, \cdot)$ is in $\mathbb{R}^n$, and $z(k) = 0$ is the zero solution of (2.1). An equation of this form signifies an actuated system with specific inputs. If the system lacks an explicit input $u(k)$, it is considered autonomous.

Definition 2.1 [20]: Consider the system represented by (2.1) for an autonomous system, the nonlinear function defined as $f : \mathbb{D} \rightarrow \mathbb{D}$. The solution $z(k) = 0$ for (2.1) is deemed finite time stable (FTS) if there exists, M containing origin such that $M \subseteq \mathbb{D}$ and an associated settling time function $K : M \setminus \{0\} \rightarrow \mathbb{Z}^+$ satisfying these conditions:

1. *Finite-time convergence:* For any initial condition $z(0)$ in $M \setminus \{0\}$ and $\forall k \geq K(z_0)$, the solution $z(k)$ lies in $M \cap \{0\}$;
2. *Lyapunov stability:* For any given $\epsilon > 0$, there exists $\delta > 0$ such that the ball of radius δ centered at the origin, $B_\delta(0)$, is a subset of M . Also, for any initial condition z_0 in $B_\delta(0) \setminus \{0\}$, the solution $z(k)$ resides in $B_\epsilon(0)$ for all time instances k in the set $\{0, \dots, K(z_0) - 1\}$.

If $z(k) = 0$ of (2.1) is FTS for $M = \mathbb{D} = \mathbb{R}^n$, it is considered globally finite-time stable.

Lemma 2.2 [20]: Consider the case of autonomous system for (2.1), if there exists a neighborhood $\mathbb{N} \subseteq \mathbb{D}$ such that $0 \in \mathbb{N}$ and a continuous function $V : \mathbb{D} \rightarrow \mathbb{R}$ such that $V(0) = 0$, $V(z) > 0$ for $z \in \mathbb{N} \setminus \{0\}$, and

$$\Delta V(z) \leq -c \min \left\{ \frac{V(z)}{c}, V(z)^\gamma \right\}, \quad z \in \mathbb{N} \setminus \{0\} \quad (2.2)$$

where $0 < \gamma < 1$ and $c > 0$ are real constants. Then, the zero solution $z(k) \equiv 0$ to (2.1) is FTS and the settling-time $K : M \rightarrow \mathbb{Z}^+$ can be represented as:

$$K(z_0) \leq \left\lceil \log_{[1-cV(z_0)^{\gamma-1}]} \frac{c^{\frac{1}{1-\gamma}}}{V(z_0)} \right\rceil + 1, \quad V(z_0) > c^{\frac{1}{1-\gamma}}$$

or

$$K(z_0) = 1, \quad |z_0| \leq c^{\frac{1}{1-\gamma}}, \quad V(z_0) \leq c^{\frac{1}{1-\gamma}}$$

where M is an open ball containing the origin. Moreover, if $V(\cdot)$ is expanding infinitely outward and (2.5) is satisfied for $\mathbb{D} = \mathbb{R}^n$, then the solution $z(k) \equiv 0$ is globally FTS.

Lemma 2.3 [20]: Consider the case of autonomous system for (2.1), if there exists a neighborhood $\mathbb{N} \subseteq \mathbb{D}$ such that $0 \in \mathbb{N}$ and a continuous function $V : \mathbb{D} \rightarrow \mathbb{R}$ such that $V(0) = 0$, $V(z) > 0$ for $z \in \mathbb{N} \setminus \{0\}$, and

$$\Delta V(z) \leq \min \{V(z), c\}, \quad z \in \mathbb{N} \setminus \{0\} \quad (2.3)$$

where $c > 0$ are real constants. Then, the zero solution $z(k) \equiv 0$ to (2.1) is FTS and the settling-time $K : M \rightarrow \mathbb{Z}^+$ can be represented as:

$$K(z_0) \leq \left\lceil \frac{V(z_0)}{c} \right\rceil + 1, \quad V(z_0) > c$$

or

$$K(z_0) = 1, \quad |V(z_0)| \leq c,$$

where M is an open ball containing the origin. Moreover, if $V(\cdot)$ is expanding infinitely outward and (2.5) is satisfied for $\mathbb{D} = \mathbb{R}^n$, then the solution $z(k) \equiv 0$ is globally FTS.

2.2 Comparison Functions

We begin by recalling several standard definitions of comparison functions that are pivotal in the stability analysis.

Definition 2.4 (Class $\mathcal{K}$ function [48]) *A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is continuous, strictly increasing, and satisfies $\alpha(0) = 0$. If additionally $\lim_{s \rightarrow \infty} \alpha(s) = \infty$, it is said to be a class $\mathcal{K}_\infty$ function.*

Definition 2.5 (Class $\mathcal{L}$ function [48]) *A function $\iota : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{L}$ if it is continuous, strictly decreasing, and $\lim_{s \rightarrow \infty} \iota(s) = 0$.*

Definition 2.6 (Class $\mathcal{KL}$ function [48]) *A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is of class $\mathcal{L}$.*

2.3 Minimum Operator Based Reaching Law for DSMC

In this subsection, we introduce two minima-based Reaching Law (RL) for DSMC [2] applicable to both unperturbed and perturbed DTS. These laws utilize a difference equation approach with minima to ensure the FTS of discrete dynamical systems [20].

2.3.1 RL1 for DTS

Considering the following system [2]:

$$\mathfrak{N}_{k+1}^i = \mathfrak{N}_k^i - \text{sign}[\mathfrak{N}_k^i] \min(|\mathfrak{N}_k^i|, \varsigma) + \delta_k^i \quad (2.4)$$

where $\varsigma \in \mathbb{R}^+$ is selected such that $\delta_L^i \leq \delta_k^i \leq \delta_U^i$ and $\varsigma \geq \max(\|\delta_L\|, \|\delta_U\|)$. We define $\max(\|\delta_L\|, \|\delta_U\|) = \delta_m$, where $\delta_m \in \mathbb{R}^+$. Furthermore, we define the following diagonal matrices: $\mathfrak{N}_{(k)} = \text{diag}\{\mathfrak{N}_{(k)}^1, \mathfrak{N}_{(k)}^2, \dots, \mathfrak{N}_{(k)}^i\}$, $\delta_{(k)} = \text{diag}\{\delta_{(k)}^1, \delta_{(k)}^2, \dots, \delta_{(k)}^i\}$, $\text{sign } \mathfrak{N}_{(k)} = \text{diag}\{\text{sign } \mathfrak{N}_{(k)}^1, \dots, \text{sign } \mathfrak{N}_{(k)}^i\}$, $i = 1, 2, \dots, n$. We define the switching function as

$$s_k := \alpha(z_k).$$

where, z_k is the state of system. We define the sliding manifold by the level set

$$\alpha^{-1}(0) := \{z \in D : \alpha(z) = 0\}.$$

Definition 1. We confirm the presence of QSM in the neighborhood of the set $\alpha^{-1}(0)$ when $|\aleph_k^i| \leq \omega_m = \delta_m$, where ω_m is QSM band.

Definition 2. The system described by equation (2.4) satisfies the QSM reachability in the vicinity of $\alpha^{-1}(0)$ if, for some $k \geq 0$, the following inequalities are satisfied:

$$\begin{aligned} |\aleph_k^i| > \varsigma &\Rightarrow |\aleph_{k+1}^i| < |\aleph_k^i| - \varepsilon, \\ |\aleph_k^i| \leq \varsigma &\Rightarrow 0 \leq |\aleph_{k+1}^i| \leq \omega_m. \end{aligned} \quad (2.5)$$

where $\varepsilon \in \mathbb{R}^+$ is minutely small.

2.3.2 RL2 for DTS

Consider the following RL for DTS [2]:

$$\aleph_{k+1}^i = \aleph_k^i - \varsigma \text{sign}[\aleph_k^i] \min \left(\frac{|\aleph_k^i|}{\varsigma}, |\aleph_k^i|^\beta \right) + \delta_k^i. \quad (2.6)$$

where $\varsigma \in \mathbb{R}^+$ as defined for RL1, and $\beta \in (0, 1)$.

Definition 3. The DTS (2.6) satisfies the QSM reaching condition in the vicinity of the set $\alpha^{-1}(0)$ if, for some $k \geq 0$, the following conditions hold:

$$\begin{aligned} \frac{|\aleph_k^i|}{\varsigma} > |\aleph_k^i|^\beta &\Rightarrow |\aleph_{k+1}^i| < |\aleph_k^i| - \varepsilon, \\ \frac{|\aleph_k^i|}{\varsigma} \leq |\aleph_k^i|^\beta &\Rightarrow 0 \leq |\aleph_{k+1}^i| \leq \delta_m. \end{aligned}$$

where $\varepsilon \in \mathbb{R}^+$ is arbitrarily small.

Remark 1. Definitions 2 and 3 resemble the ideal sliding mode definition for continuous-time systems in the case $\delta_k = 0$ as $\aleph_k$ precisely converges to $\alpha^{-1}(0)$ and remains there indefinitely.

Remark 2. The significance of the minimum operator in the control loop is to ensure finite-time reachability to the sliding surface. The role of minimum operator in the control loop can be further emphasized using the example given in [20] and further illustrated for DSMC in [2]. Consider scalar system given by $x(k+1) = x(k) - \text{sign}(x(k)) \min\{|x(k)|, c\}$, $x(0) = x_0$. The right-hand side of system is continuous everywhere and, for every initial condition in $\mathbb{R} \setminus \{0\}$, and it has a unique solution in forward time. Furthermore, if $|x(k)| \leq c$, $k \in \mathbb{Z}^+$, then $x(k+1) = 0$, which insures finite time reachability with settling time function given by $K(x_0) = \left\lceil \frac{|x_0|}{c} \right\rceil$.

It is now explicitly stated that the notions of FTS and the associated comparison functions are included solely to complete the formal definitions and notations used later; a comprehensive tutorial on these topics is beyond the scope of the thesis. The central contributions are: (i) a *data-driven* sliding-mode framework employing minimum-operator reaching laws, (ii) a *prescribed-performance data-driven* design ensuring predefined transient bounds, (iii) a *multirate* output-feedback control architecture, (iv) a finite-time discrete control strategy for a two-DOF helicopter system, and (v) an adaptive switching scheme using a ceiling function for discrete-time sliding-mode control. Each of these contributions addresses finite-time behavior through the proposed control laws and synthesis procedures, while the preliminaries on FTS and comparison functions serve only as foundational terminology for these results.

Chapter 3

Minimum Operator Based Data Driven Control

3.1 Introduction

Data-driven approaches have emerged as a popular research domain and have been applied to numerous real-world applications [49]. DDC methods can be broadly categorized into two main approaches. The first approach involves using a predefined transfer function for the controller, where its structure remains fixed while the numerical parameters are directly adjusted based on insights obtained from measured data [13]. The second approach focuses on designing a generic controller using function approximations, which may include techniques such as neural networks or Taylor series approximations. In this case, the controller parameters are fine-tuned by minimizing a specified performance criterion, utilizing both offline and online input-output data [14], a concept popularized as model-free adaptive control (MFAC). In this chapter, we have used a concept called the pseudo partial derivative (PPD) and pseudo gradient (PG) approaches to capture the dynamic behavior of the controlled plant, moving away from the traditional state-space representation. This framework includes three different dynamic linearization data models [14]: compact-form dynamic linearization (CFDL), partial-form dynamic linearization (PFDL), and full-form dynamic linearization (FFDL). These models offer equivalent representations of system dynamics, focusing on how changes in the control input and other influencing factors affect the system's output.

In [14], MFAC method utilizing CFDL data-driven modeling is presented. This ap-

proach designs controllers and analyzes closed-loop systems using only measured I&O, without relying on dynamics. Linearizing the nonlinear system using CFDL, a new non-affine system is established, which contains an unknown nonlinear term and a linear parametric term affine to the control input and preceding output data. Thus, it is necessary to address the influence of the unknown nonlinear term on the system effectively. The fusion of CFDL with a DSMC framework is effective in handling such nonlinearities. The concept of model-free adaptive sliding mode control, which combines MFAC with DSMC, was first introduced in [15], though it did not account for disturbances and system uncertainties. Building on this, [16] proposes an adaptive DSMC using CFDL for a class of discrete nonlinear systems. Recently, minimum operator approach has been extended to the discrete sliding mode domain, with notable advancements in DSMC [50], [2] as MDSMC. This chapter proposes the following novelties for nonlinear DTS with perturbations using DDC framework:

- Design of controllers by combining the CFDL approach with minimum-operator-based MDSMC.
- The proposed data-driven control laws ensure finite-time boundedness of the switching function and reduced quasi-sliding mode domain (QSMD), offering improved robust control compared to previous approaches.

3.2 CFDL Data Model & Disturbance Estimation

A MIMO discrete-time nonlinear system can be represented as:

$$y_{k+1} = f(y_k, y_{k-1}, \dots, y_{k-n_y}, u_k, u_{k-1}, \dots, u_{k-n_u}) + \delta_k \quad (3.1)$$

$y_k \in R^n$ and $u_k \in R^n$ denote the output and input of the dynamics, respectively; $n_y > 0$, $n_u > 0$ where $n_y, n_u \in \mathbb{Z}^+$ are unknown. δ_k denotes the constrained perturbations such that $\|\delta_k\| \leq \mathcal{D}$, where $\mathcal{D} > 0$. The function $f(\cdot)$ characterizes the DTS, such that:

$$z_{k+1} = f(y_k, y_{k-1}, \dots, y_{k-n_y}, u_k, u_{k-1}, \dots, u_{k-n_u}) \quad (3.2)$$

Accordingly, (3.1) can be expressed as

$$y_{k+1} = z_{k+1} + \delta_k \quad (3.3)$$

Assumption 1. The system described by (3.1) exhibits both controllability and observability. In particular, if the target signal y'_{k+1} remains bounded, it follows that the control input u_k will also be constrained, allowing y_{k+1} to accurately track the desired signal y'_{k+1} .

Assumption 2. The partial derivative $\frac{\partial f(\cdot)}{\partial u_k}$ is continuous.

Assumption 3. The DTS (3.1) satisfies a generalized Lipschitz condition, which is expressed as

$$|\Delta y_{k+1}| \leq b |\Delta u_k|$$

for any k where $|\Delta u_k| \neq 0$, with $b > 0$. Here, $\Delta y_{k+1} = y_{k+1} - y_k$ and $\Delta u_k = u_k - u_{k-1}$.

Remark 1. From a practical standpoint, Assumption 1 serves as a fundamental prerequisite for ensuring the system's controllability. Assumption 2 implies that a finite variation in the control input does not result in an infinite variation in the system output. From an energy perspective, Assumption 3 states that the rate of change of a controller's energy cannot approach infinity if its input energy changes are finite.

Lemma 1 [51]. Consider the nonlinear system (3.2). If $f(\cdot)$ satisfies Assumptions 1 and 2, then for $|\Delta u_k| \neq 0$, there exists a PPD ξ_k and CFDL data model can be expressed as:

$$\Delta z_{k+1} = \xi_k \Delta u_k \tag{3.4}$$

where

$$\xi_k = \begin{bmatrix} \phi_{11k} & \phi_{12k} & \cdots & \phi_{1n_k} \\ \phi_{21k} & \phi_{22k} & \cdots & \phi_{2n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1k} & \phi_{n2k} & \cdots & \phi_{nn_k} \end{bmatrix},$$

where $\|\xi_k\| \leq b$ is constrained $\forall k > 0$.

Rewriting (3.4) as:

$$z_{k+1} = z_k + \xi_k \Delta u_k \tag{3.5}$$

From (3.3), we obtain;

$$\Delta y_{k+1} = \Delta z_{k+1} + \Delta \delta_k \tag{3.6}$$

where $\Delta z_{k+1} = z_{k+1} - z_k$, and $\Delta \delta_k = \delta_k - \delta_{k-1}$. Using (3.4), (3.6) can be further written as:

$$y_{k+1} = y_k + \xi_k \Delta u_k + \Delta \delta_k \quad (3.7)$$

Let $\bar{\xi}_k =$

$$\begin{bmatrix} \phi_{11k} & 0 & 0 & \cdots & 0 \\ 0 & \phi_{22k} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_{nnk} \end{bmatrix}$$

Hence, (3.7) can be expressed as:

$$y_{k+1} = y_k + \bar{\xi}_k \Delta u_k + \Delta \bar{\delta}_k \quad (3.8)$$

where $\Delta \bar{\delta}_k = \sum_{j=1}^n \phi_{ij_k} \Delta u_{jk} + \Delta \delta_k$, $i \neq j$ and $i = 1, \dots, n$

Remark 2. The disturbance term $\Delta \bar{\delta}_k$, representing the coupling between control loops, disturbances, and unmodeled system dynamics, is treated as a generalized disturbance, estimated online via perturbation estimation techniques. In contrast, the dynamic linearization (DL) technique is applied to construct a virtual DL model at each sampling point, where the PPD implicitly estimates model uncertainties using I/O data.

Remark 3. It is important to note that the value of ξ_k in (3.4) may not always align with its variation range. In other words, ξ_k may satisfy, $\max \left| \frac{\xi_k - \xi_{k-1}}{t_s} \right| \leq \kappa$ where $\kappa > 0$ and t_s represents the sampling interval. Consequently, ξ_k can be expressed as a variable component ϑ_k combined with a constant component φ . It is important to note that ξ_k may take on very small values at certain sampling times, and the parameter φ is selected to maintain ξ_k within a suitable range.

Remark 4. The system represented by (3.8) can be approximated to quasi-linear dynamics in local operating regions. Specifically, systems where the non-linearity can be effectively captured by the PPD representation and the Lipschitz condition holds robustly.

3.2.1 Perturbation Estimation

The approach employs perturbation estimation methodology [1] also in [52] as one-step delayed estimation strategy for disturbance estimation, assuming slow varying disturbance. The perturbation component $\Delta \bar{\delta}_k$ of the DTS (3.8) is evaluated as

$$\begin{aligned}
y_{k+1} &= y_k + \bar{\xi}_k \Delta u_k + \Delta \bar{\delta}_k \implies y_k = y_{k-1} + \bar{\xi}_{k-1} \Delta u_{k-1} \\
&+ \Delta \bar{\delta}_{k-1} \implies \Delta \hat{\delta}_k = \Delta \bar{\delta}_{k-1} = y_k - y_{k-1} - \bar{\xi}_{k-1} \Delta u_{k-1},
\end{aligned} \tag{3.9}$$

which allows us to rewrite (3.8) as

$$y_{k+1} = y_k + \bar{\xi}_k \Delta u_k + \Delta \hat{\delta}_k - \Delta \tilde{\delta}_k, \tag{3.10}$$

where $\Delta \tilde{\delta}_k = \Delta \hat{\delta}_k - \Delta \bar{\delta}_k$ represents the perturbation assessment error. Thus,

$$\begin{aligned}
\Delta \tilde{\delta}_k &= \Delta \bar{\delta}_{k-1} - \Delta \bar{\delta}_k = [(y_k - y_{k-1}) - (\bar{\xi}_{k-1} \Delta u_{k-1}) \\
&- \bar{\xi}_k \Delta u_k].
\end{aligned}$$

Based on Assumption 1 and Lemma 1, it is established that y_k , u_k , and $\bar{\xi}_k$ are bounded. Consequently, it can be inferred that $\Delta \tilde{\delta}_k$ is also bounded.

Remark 5. It performs well when disturbances change linearly or predictably over time. Due to its simple architecture and exceptional performance, perturbation estimation has found widespread use in different control systems.

3.2.2 Pseudo Partial Derivative Assessment

The index function for the unknown PPD estimation, as given in [14], is used in this chapter as

$$J(\hat{\xi}_k) = \left\| y_k - y_{k-1} - \hat{\xi}_k \Delta u_{k-1} \right\|^2 + \nu \left\| \hat{\xi}_k - \hat{\xi}_{k-1} \right\|^2 \tag{3.11}$$

where $\hat{\xi}_k$ is the estimated value of $\bar{\xi}_k$. Setting $\frac{\partial J}{\partial \hat{\xi}_k} = 0$, we obtain

$$\begin{aligned}
\hat{\xi}_k &= \hat{\xi}_{k-1} + \frac{\Pi \Delta u_{k-1}^T}{\nu + \Delta u_{k-1}^2} \left[\Delta y_k - \hat{\xi}_{k-1} \Delta u_{k-1} \right], \\
\hat{\phi}_{ii_k} &= \hat{\phi}_{ii}(1), \text{ if } |\hat{\phi}_{ii_k}| \leq \epsilon.
\end{aligned} \tag{3.12}$$

$\nu > 0$, $\Pi \in (0, 1]$, and $\epsilon > 0$ is sufficiently small. Here $\hat{\phi}_{ii}(1) > 0$ is the initial value of $\hat{\phi}_{ii_k}$. Let us define $\hat{\xi}_k = \hat{\vartheta}_k + \varphi$. Then, (3.10) can be further written as

$$y_{k+1} = y_k + (\hat{\vartheta}_k + \varphi) \Delta u_k + \Delta \hat{\delta}_k - \Delta \tilde{\delta}_k \tag{3.13}$$

where $\hat{\vartheta}_k$ can be obtained using (3.12), and $\varphi = \text{diag}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$. The elements of φ are taken as a non-negative constant.

Remark 6. To guarantee the feasibility of the adaptive law (3.13), ν should be selected to be smaller than the magnitude of Δu_k .

Remark 7. The updating law (3.13) is designed based solely on the controlled system's input and output measurement data, without reliance on the model's structural and dynamic information.

Remark 8. At certain sampling instances, the diagonal element $\hat{\xi}_k$ may become exceedingly small. Consequently, the parameter φ is chosen to ensure that $\hat{\xi}_k$ remains within an acceptable range.

3.3 Minimum Operator Based CFDL-DSMC Design

This section outlines the controller design process of the MDSMC-CFDL scheme. A data-driven controller is derived from the CFDL data model and the MDSMC scheme, and the stability of the proposed control method is also demonstrated. Further development is presented for the SISO case; however, the MIMO case will follow the same approach.

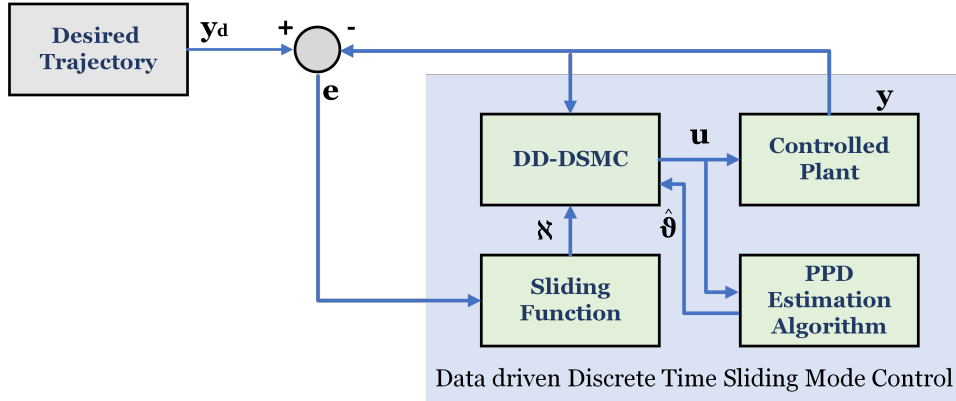


Figure 3.1: Flow chart of MDSMC-CFDL scheme.

3.3.1 Design of Controller Using CFDL-RL1 Scheme

Define the tracking error as follows:

$$e_k = y_k - y'_k \quad (3.14)$$

where y_k represents the system output signal and y'_k denotes the bounded desired input signal. Define the sliding function as

$$\aleph_k = \lambda_1 e_k \quad (3.15)$$

The CFDL-RL1-based control strategy can be obtained using [2] as shown below:

$$\begin{aligned} \aleph_{k+1} &= \lambda_1 e_{k+1} = \lambda_1 [y_{k+1} - y'_{k+1}] \\ \aleph_k - \text{sign}[\aleph_k] \min(|\aleph_k|, \varsigma) &= \lambda_1 [y_k + (\hat{\vartheta}_k + \varphi) \Delta u_k + \Delta \hat{\delta}_k - \Delta \tilde{\delta}_k - y'_{k+1}] \end{aligned}$$

The control can be obtained as

$$\Delta u_k = \frac{\eta^T}{\eta \eta^T} (\aleph_k - \text{sign}[\aleph_k] \min(|\aleph_k|, \varsigma) - \lambda_1 (y_k - \Delta \hat{\delta}_k + \Delta \delta_k + y'_{k+1}))$$

Here, $\eta = \lambda_1 (\hat{\vartheta}_k + \varphi)$, further to implement the controller, the future reference position y'_{k+1} is required. In practice, the desired trajectory is typically predefined within the control system, making y'_{k+1} available.

$$\Delta u_k = \frac{\eta^T}{\eta \eta^T} (\aleph_k - \text{sign}[\aleph_k] \min(|\aleph_k|, \varsigma) - \lambda_1 (y_k - \Delta \hat{\delta}_k + y'_{k+1})) \quad (3.16)$$

It is important to note that the control law in equation (3.16) is derived under the assumption that $\Delta \tilde{\delta}_k$ is accurately estimated, specifically $\Delta \tilde{\delta}_k = 0$. However, $\Delta \tilde{\delta}_k \neq 0$. To ensure robustness against errors in disturbance estimation, the parameter ς should satisfy the following condition:

$$\varsigma > \omega_m \geq \|\Delta \hat{\delta}_k\| \quad (3.17)$$

3.3.2 Design of Controller Using CFDL-RL2 Scheme

Consider the same tracking error and the sliding function as defined in (3.14) and (3.15), respectively. The CFDL-RL2-based control strategy can be obtained using [2] as shown below:

$$\begin{aligned} \aleph_{k+1} &= \lambda_1 e_{k+1} \\ \aleph_k - \varsigma \text{sign}[\aleph_k] \min\left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta\right) &= \lambda_1 [y_{k+1} - y'_{k+1}] \\ \aleph_k - \varsigma \text{sign}[\aleph_k] \min\left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta\right) &= \lambda_1 [y_k + (\hat{\vartheta}_k + \varphi) \Delta u_k + \Delta \hat{\delta}_k - \Delta \tilde{\delta}_k - y'_{k+1}] \end{aligned}$$

The control can be obtained as

$$\Delta u_k = \frac{\eta^T}{\eta\eta^T} (\aleph_k - \varsigma \text{sign}[\aleph_k] \min \left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta \right) - \lambda_1 (y_k - \Delta \hat{\delta}_k + y'_{k+1}) \quad (3.18)$$

Remark 11. In control law (3.18), setting ς to 1 and keeping β very close to zero results in behavior similar to control law (3.16) with $\varsigma = 1$. The interaction between ς and β influences the convergence of the switching function. Smaller values of ς and β lead to slower convergence, while keeping β close to 1 and ς large ensures rapid convergence of the switching function. Furthermore, the selection of ς and β for various applications can be tailored based on the actuator saturation limits of the respective systems.

3.3.3 Stability Analysis

Lemma 2. If the nonlinear system described in equation (3.1) satisfies Assumptions 1 to 3 and is controlled using the CFDL-RL1 algorithm using (3.16), with the desired signal y'_{k+1} being bounded, then the estimated value $\hat{\xi}_k$ will also be bounded.

Proof Considering the PPD estimation algorithm, we have two scenarios;

Case 1: if $\hat{\phi}_{iik} \leq \epsilon$, this will ensure $\hat{\phi}_{iik} = \hat{\phi}_{ii}(1)$, and $\hat{\xi}_k$ will be bounded.

Case 2: if $\hat{\phi}_{iik} > \epsilon$, using (3.12) we have:

$$\tilde{\xi}_k = \Delta \xi_k + \left(1 - \frac{\Pi \Delta u_{k-1}^2}{\nu + \Delta u_{k-1}^2} \right) \tilde{\xi}_{k-1}$$

Using Lemma 1, we have $\|\xi_k\| \leq b$, and

$$\|\tilde{\xi}_k\| < \left\| \left(1 - \frac{\Pi \Delta u_{k-1}^2}{\nu + \Delta u_{k-1}^2} \right) \right\| \|\tilde{\xi}_{k-1}\| + 2b$$

Considering (3.19), we have

$$\Pi \Delta u_{k-1}^2 < \Delta u_{k-1}^2 < \nu + \Delta u_{k-1}^2$$

Here, $\nu > 0$ and $\Pi \in (0, 1]$, Therefore, there exists a positive constant $\gamma \in (0, 1]$, leading to the following:

$$0 < 1 - \frac{\Pi \Delta u_{k-1}^2}{\nu + \Delta u_{k-1}^2} \leq \gamma < 1$$

Thus, we have:

$$\|\tilde{\xi}_k\| \leq \gamma \|\tilde{\xi}_{k-1}\| + 2b \leq \gamma^2 \|\tilde{\xi}_{k-2}\| + 2 + 2b$$

$$\leq \dots \leq \gamma^k \left\| \tilde{\xi}(0) \right\| + 2b \frac{(1 - \gamma^k)}{1 - \gamma}$$

it follows that $\tilde{\xi}_k$ is bounded. Consequently, $\hat{\xi}_k$ is also bounded.

Theorem 1. If Assumptions 1 to 3 are satisfied for the system (3.1) the controller described by (3.16) and (3.17) is used and the time-dependent term $\Delta\hat{\delta}_k$ and $\hat{\xi}_k$ are updated according to equations (3.10) and (3.12), then the QSM will converge in a finite number of steps.

Proof From (3.15) we have,

$$\aleph_{k+1} = \lambda_1 e_{k+1}$$

using (3.14) and (3.15), we have

$$\begin{aligned} \aleph_{k+1} &= \lambda_1 e_{k+1} = \lambda_1 [y_{k+1} - y'_{k+1}] \\ &= \lambda_1 [y_k + (\hat{v}_k + \varphi) \Delta u_k + \Delta\hat{\delta}_k - \Delta\tilde{\delta}_k \\ &\quad - y'_{k+1}] \end{aligned}$$

When applying control law (3.16), the system follows

$$\aleph_{k+1} = -\Delta\tilde{\delta}_k + \aleph_k - \text{sign}[\aleph_k] \min(|\aleph_k|, \varsigma)$$

From, (3.17) we have

$$-\omega_m < \Delta\tilde{\delta}_k < \omega_m \tag{3.19}$$

Hence, when $\aleph_0 \leq \varsigma$ using (3.17) we have the next step as $-\varsigma < \aleph_{k+1} < \varsigma$, Hence meeting definition 2 and for $k = \mathcal{T}(\aleph_0) = 1$, we have $\aleph_{k+1} \leq \omega_m$. Here ω_m is defined as (3.17) and $\mathcal{T}(\aleph_0)$ is a settling time function. Now, if we have the scenario, $\aleph_0 > \varsigma$

$$\begin{aligned} \aleph_{k+1} &= -\Delta\tilde{\delta}_k + \aleph_k - \varsigma \\ -2\varsigma &< \aleph_{k+1} - \aleph_k < 0 \end{aligned} \tag{3.20}$$

Hence, meeting definition 2. Also,

$$\begin{aligned} \aleph_{k+1} &\leq \aleph_k - \varsigma + \omega_m \\ \aleph_k &\leq \aleph_{k-1} - (\varsigma - \omega_m) \\ &\vdots \\ &\leq \aleph_0 - k(\varsigma - \omega_m) \end{aligned}$$

and for some $k = \mathcal{T}(\aleph_0) = \lceil \frac{\aleph_0 - \omega_m}{\varsigma - \omega_m} \rceil$, we have $\aleph_{k+1} \leq \omega_m$. Here, ω_m is defined as (3.17). Thus, it can be inferred that the modulus of the switching function reaches an ultimate bound of ω_m , attained within a finite number of steps.

Lemma 3. If the nonlinear system described in equation (3.1) satisfies Assumptions 1 to 3 and is controlled using the CFDL-RL2 algorithm using (3.18), with the desired signal y'_{k+1} being bounded, then the estimated value $\hat{\xi}_k$ will also be bounded.

Proof Using (3.18), the proof will follow a similar approach to that of Lemma 2.

Theorem 2. If Assumptions 1 to 3 are satisfied for the system (3.1) the controller described by (3.18) and (3.19) is used and the time-dependent term $\Delta\hat{\delta}_k$ and $\hat{\xi}_k$ are updated according to equations (3.10) and (3.12), then the QSM will converge in a finite number of steps.

Proof From (3.15) we have,

$$\aleph_{k+1} = \lambda_1 e_{k+1}.$$

Using (3.14) and (3.15), we have

$$\begin{aligned} \aleph_{k+1} &= \lambda_1 e_{k+1} = \lambda_1 [y_{k+1} - y'_{k+1}] \\ &= \lambda_1 [y_k + (\hat{v}_k + \varphi) \Delta u_k + \Delta\hat{\delta}_k - \Delta\tilde{\delta}_k - y'_{k+1}] \end{aligned}$$

When applying control law (3.18), the system follows

$$\aleph_{k+1} = -\Delta\tilde{\delta}_k + \aleph_k - \varsigma \text{sign}[\aleph_k] \min \left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta \right)$$

when $\frac{\aleph_0}{\varsigma} \leq |\aleph_0|^\beta$ using (3.17) we have the next step as $-\varsigma < \aleph_{k+1} < \varsigma$

Hence, meeting definition 3 and for $k = \mathcal{T}(\aleph_0) = 1$, we have $\aleph_{k+1} \leq \omega_m$. Here ω_m is defined as (3.17) and $\mathcal{T}(\aleph_0)$ is a settling time function. Now, if we have the scenario $\frac{\aleph_0}{\varsigma} > |\aleph_0|^\beta$

$$\begin{aligned} \aleph_{k+1} &= -\Delta\tilde{\delta}_k + \aleph_k - \varsigma \text{sign}(\aleph_k) |\aleph_k|^\beta \\ \Rightarrow |\aleph_{k+1}| &\leq |\aleph_k| - \varsigma |\aleph_k| |\aleph_k|^\beta + \omega_m \end{aligned}$$

proceeding ahead with $|\aleph_k|$, we have

$$\begin{aligned} |\aleph_k| &\leq |\aleph_{k-1}| (1 - \varsigma |\aleph_{k-1}| |\aleph_{k-1}|^{\beta-1} + \omega_m |\aleph_{k-1}|^{-1}) \\ &\vdots \\ &\leq |\aleph_0| (1 - \varsigma |\aleph_0| |\aleph_0|^{\beta-1} + \omega_m |\aleph_0|^{-1}) \dots \\ &\quad (1 - \varsigma |\aleph_{k-1}| |\aleph_{k-1}|^{\beta-1} + \omega_m |\aleph_{k-1}|^{-1}) \end{aligned}$$

$$|\aleph_k| \leq |\aleph_0|(1 - \varsigma|\aleph_0||\aleph_0|^{\beta-1} + \omega_m|\aleph_0|^{-1})^k \quad (3.21)$$

Now, at a certain k , we will have the scenario

$$|\aleph_0|(1 - \varsigma|\aleph_0|^{\beta-1} + \omega_m|\aleph_0|^{-1})^k \leq \varsigma|\aleph_k|^\beta.$$

Using (3.21), we have, $|\aleph_k| \leq \varsigma|\aleph_k|^\beta \implies |\aleph_{k+1}| \leq \omega_m$.

Therefore,

$$|\aleph_{k+1}| \leq \omega_m, \quad \forall k \geq \left\lceil \log_{[1-\varsigma|\aleph_0|^{-1}+\omega_m|\aleph_0|^{-1}]} \frac{\varsigma^{\frac{1}{1-\beta}}}{|\aleph_0|} \right\rceil,$$

and for some $k = \mathcal{T}(\aleph_0)$, we have

$$\mathcal{T}(\aleph_0) = \left\lceil \log_{[1-\varsigma|\aleph_0|^{-1}+\omega_m|\aleph_0|^{-1}]} \frac{\varsigma^{\frac{1}{1-\beta}}}{|\aleph_0|} \right\rceil.$$

Thus, it can be inferred that the modulus of the switching function reaches an ultimate bound of ω_m , attained within a finite number of steps.

3.4 Result and Discussion

To validate the performance of the CFDL-RL1 and CFDL-RL2 control schemes, this section presents numerical simulation using a nonlinear system example and experimental validation of the same on a coupled tank system test bench. Additionally, to highlight the characteristics of the proposed method, numerical comparisons are made with CFDL-DITSMC [1] for both proposed algorithms.

3.4.1 Numerical Simulation

Consider the SISO DTS given by

$$y_{k+1} = \frac{y_k y_{k-1} y_{k-2} u_{k-1} (y_{k-2} - 1) + u_k}{1 + y_{k-1}^2 + y_{k-2}^2} + d_k$$

where d_k represents the external perturbation, defined as

$$d_k = \left[0.5, 0.15 \sin \left(\frac{k}{30} \right) \right] [y_k, y_{k-1}]^T \quad (3.22)$$

The initial states are defined as $u(1 : 2) = 0$, $y(1 : 3) = 0$, $\hat{v}(1 : 2) = 2$, and $\epsilon = 10^{-5}$. The resetting value $\hat{v}(1)$ is set to 0.5, and the CFDL-DITSMC [1] terms are chosen as

$\nu = 0.5$, $\Pi = 0.3$, $\lambda_s = 0.0002$, $\lambda_1 = 0.3$, $\lambda_2 = 0.2$, $\alpha = \frac{5}{7}$, $\omega = 0.01$, $\Delta = 0.002$, $\delta = 3$, and $\epsilon = 10^{-5}$. The reference signal y'_k is given as

$$y_{rk} = \begin{cases} 0.5, & k < 500 \\ 0.8, & k \geq 500 \end{cases}$$

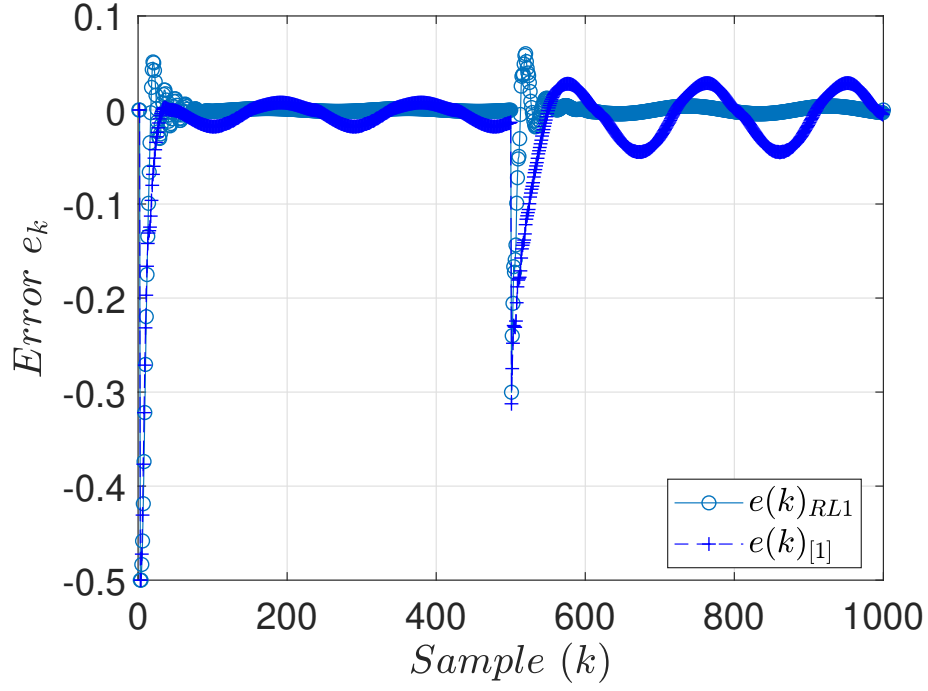
The coefficients λ_1 for RL1 and RL2 is considered as 1.1 and $\beta = 0.1$ for RL2. Fig. 3.2 presents the numerical simulation results using the RL1 scheme. Fig. 3.3b presents control input, and Fig. 3.3a represents disturbance estimation error, which is significantly lower than previous work. In Fig. 3.2a, the tracking error remains bounded and exhibits a reduced QSMD compared to the CFDL-DITSMC method. Additionally, Fig. 3.2b illustrates that the output successfully tracks the reference signal within a finite time.

Simulation results using RL2 scheme are shown in Fig. 3.3. The output successfully tracks the reference signal, and the error shows reduced QSMD, as shown in Fig. 3.4b and Fig. 3.4a, respectively. Consequently, the proposed method exhibits improved performance in both tracking accuracy and error reduction.

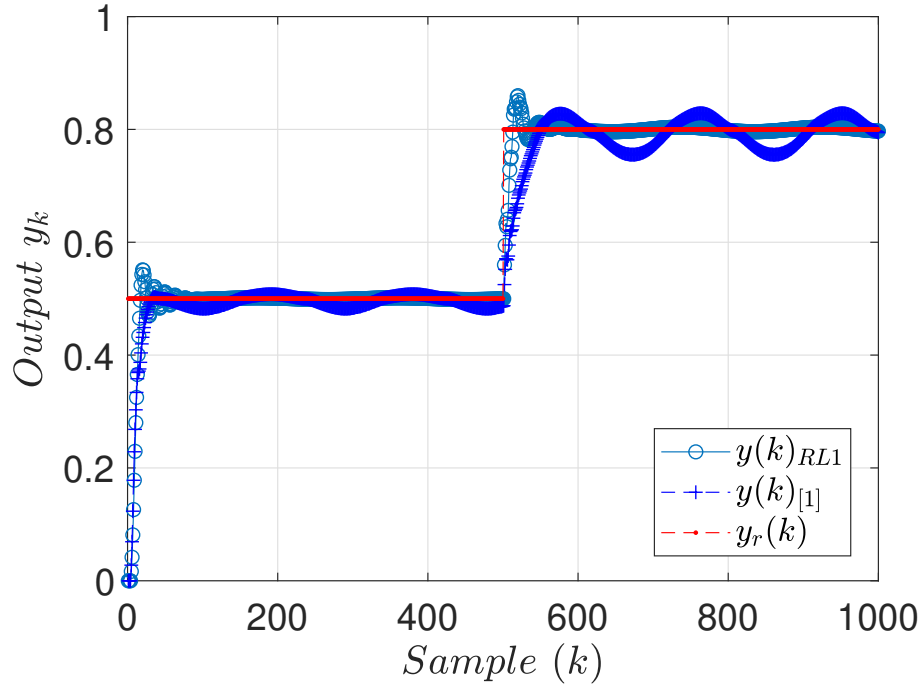
3.4.2 Experimental Validation

The coupled tank system is used as a test bench. The schematic in Fig. 3.6a illustrates the construction of the system, and Fig. 3.6 shows the actual test bench, where the control objective is to maintain the desired water level in tank 2. Configuration 2 of the coupled tank system [53] is chosen for the experiments. In this configuration, the control input is applied to the inlet of tank 1, and the outflow of tank 1 acts as a control for tank 2. This setup introduces a coupling effect, where the levels in both tanks are interdependent, adding complexity to the control process. The liquid level is measured by a pressure sensor at the bottom of each tank, providing a linear voltage output that increases with pressure. This signal is processed through a conditioning board and converted to a 0 to 5V DC output. The control parameters used for the experimental setup are $\beta = 0.1$, $\lambda_1 = 0.3$, $\lambda_2 = 0.3$ and $\varsigma = 1$. For the system parameters, reader may refer to [53]. The reference water level to track for MRL1 scheme is given as

$$y_r = \begin{cases} 4 \text{ cm}, & t < 400 \text{ sec} \\ 7 \text{ cm}, & t \geq 400 \text{ sec} \end{cases}$$

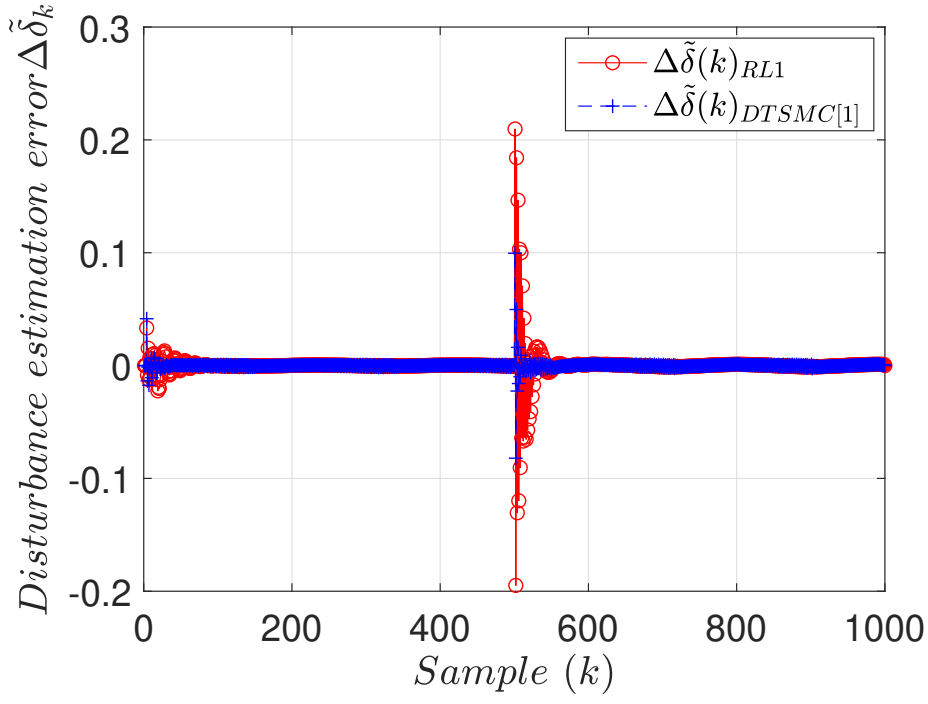


(a) CFDL-RL1 Trajectory error $e(k)$

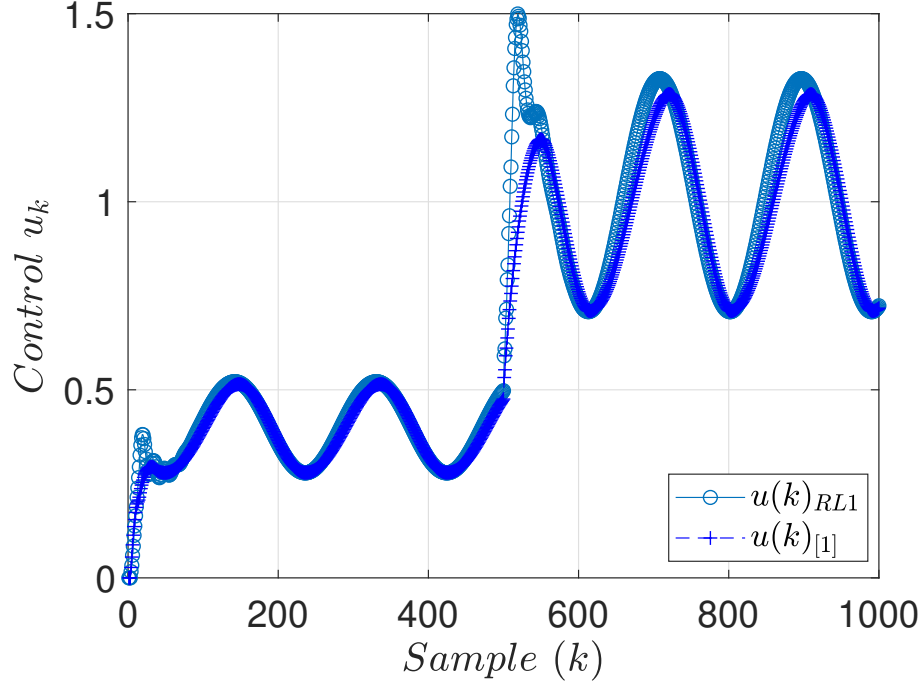


(b) CFDL-RL1 output

Figure 3.2: Comparison of trajectory error and output for CFDL-RL1 with [1] .

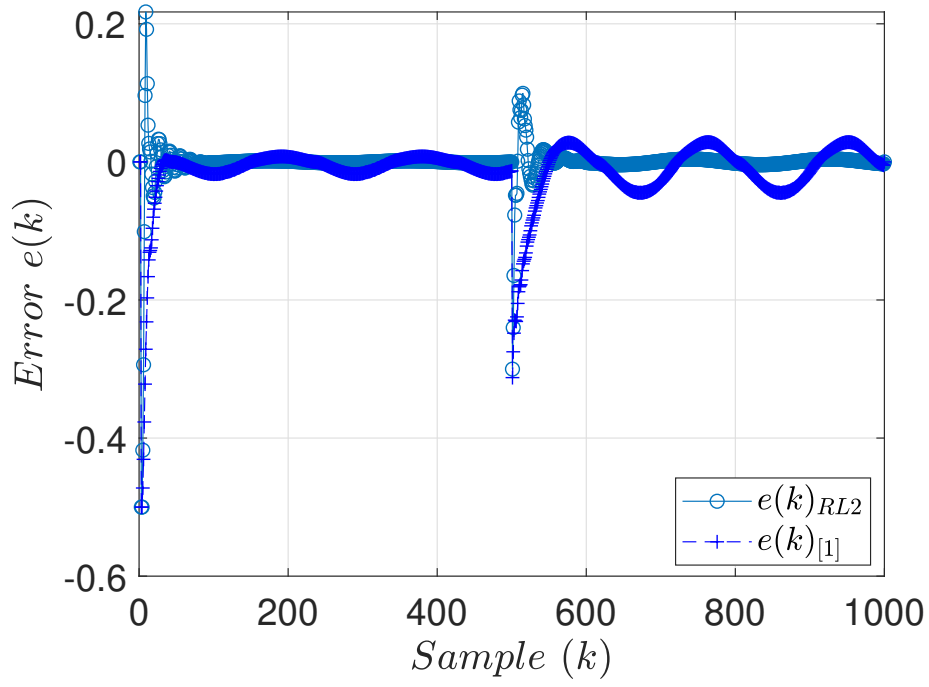


(a) CFDL-RL1 disturbance error estimation $\Delta\hat{\delta}_k$

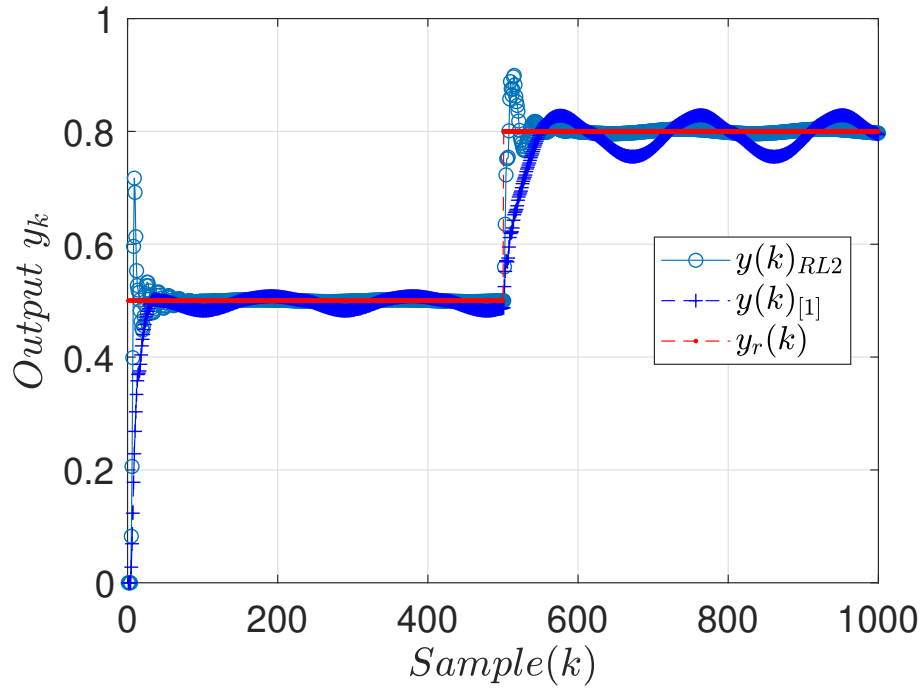


(b) CFDL-RL1 Control input $u(k)$

Figure 3.3: Comparison of control performance and disturbance estimation for CFDL-RL1 with [1].

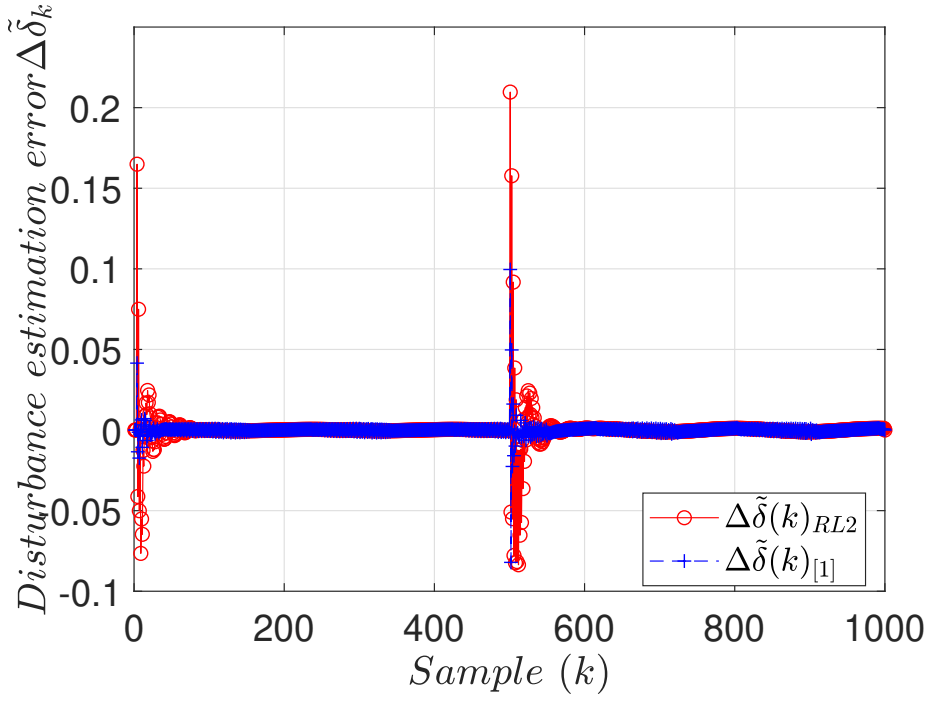


(a) CFDL-RL2 Trajectory error $e(k)$

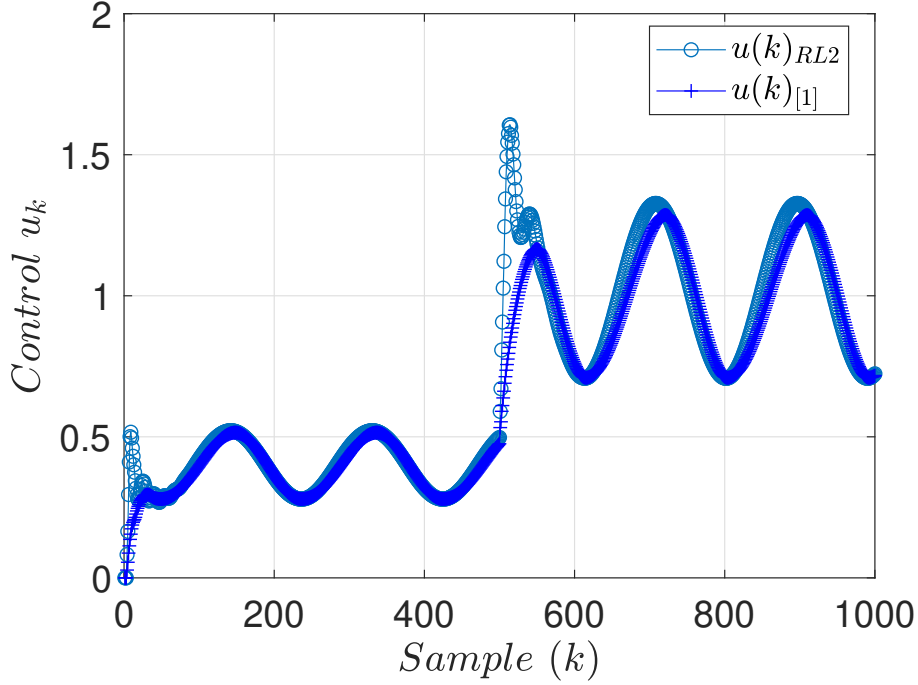


(b) CFDL-RL1 output

Figure 3.4: Comparison of trajectory error and output for CFDL-RL2 with [1].

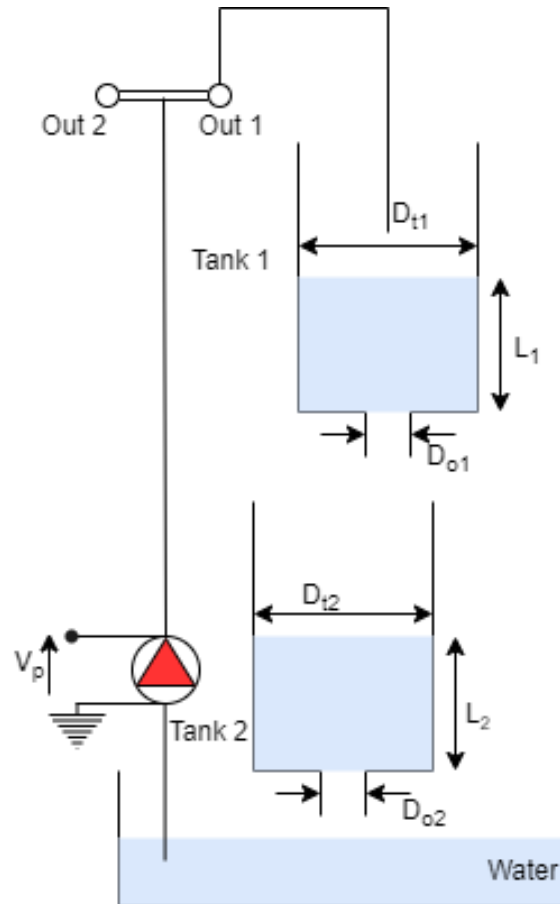


(a) CFDL-RL2 disturbance error estimation $\Delta\hat{\delta}_k$

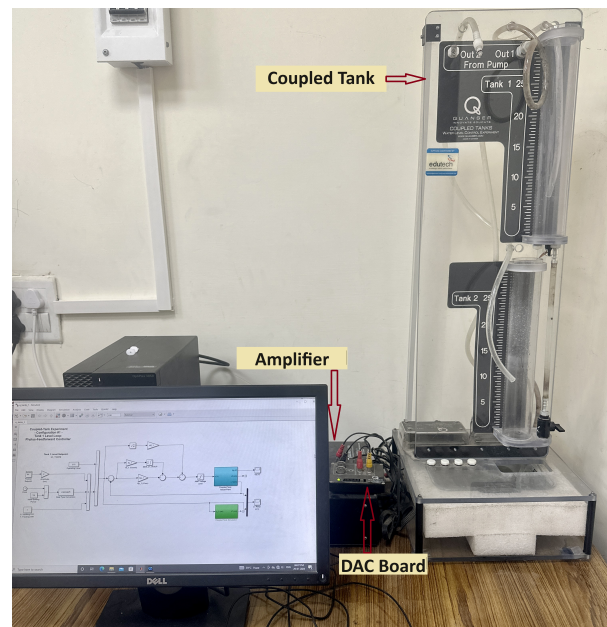


(b) CFDL-RL2 Control input $u(k)$

Figure 3.5: Comparison of control performance and disturbance estimation for CFDL-RL2 with [1].



(a) Coupled tank block diagram



(b) Coupled tank setup

Figure 3.6: Coupled tank system

Fig. 3.7a shows the experimental validation of the proposed MRL1 scheme where the reference levels are achieved in finite time. Fig. 3.7b shows the control input. Similarly, for MRL2 scheme, the reference level is set as

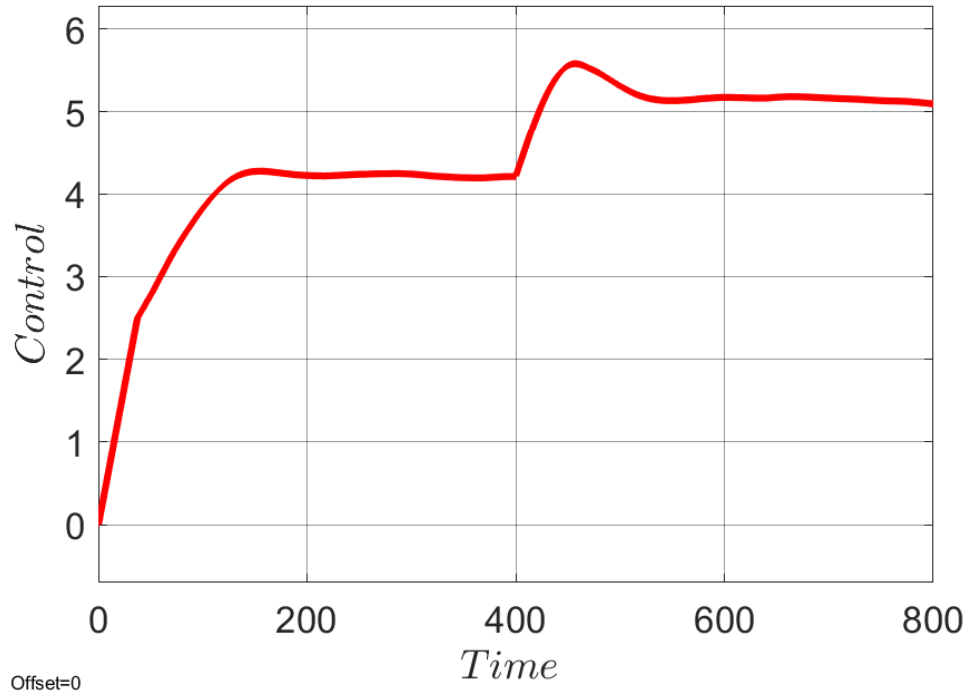
$$y_r = \begin{cases} 7 \text{ cm}, & t < 400 \text{ sec} \\ 8 \text{ cm}, & t \geq 400 \text{ sec} \end{cases}$$

Fig. 3.8a depicts the experimental validation of the proposed MRL2 scheme, and Fig. 3.8b shows the control input.

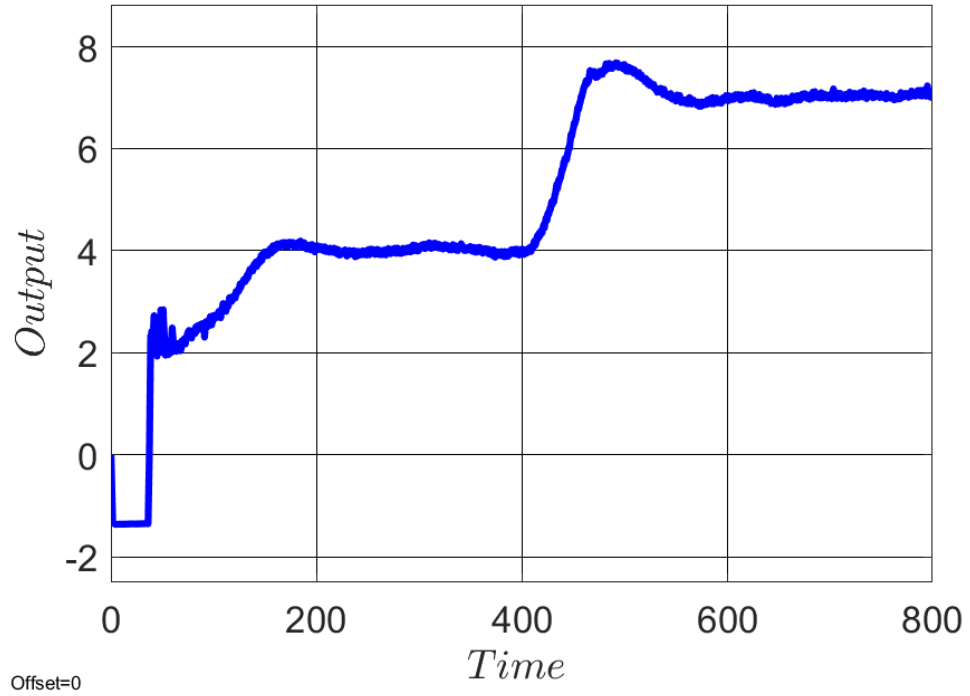
3.5 Conclusion

This chapter introduces two innovative DDC control schemes designed for discrete-time systems subjected to disturbances. By utilizing the CFDL technique and incorporating MDSMC methods, these schemes effectively mitigate QSMB and achieve finite-time tracking for model-free systems. Compared to existing methods, the proposed approaches are simpler and more robust due to their reliance on fewer tuning parameters. The effectiveness of the CFDL–MDSMC framework is demonstrated through numerical simulations and experimental validation on a liquid-level control system. Numerical results show improved disturbance rejection and finite-time convergence, while experimental findings confirm the practical feasibility and robustness of the proposed control schemes in real-world applications. The proposed DDC control schemes achieve finite-time convergence under disturbances while minimizing the required design parameters, making them highly suitable for complex, nonlinear systems, particularly in scenarios where system models are unavailable. Their successful implementation on a liquid-level control system highlights their potential for broader adoption across industrial and engineering applications.

As part of future work, we plan to extend this study to highly uncertain or weakly observable systems, thereby broadening the scope and applicability of the proposed framework. In addition, broader generalizability across different scenarios should be established with a focus on persistence of excitation, actuator wear, and the energy of the control signal.

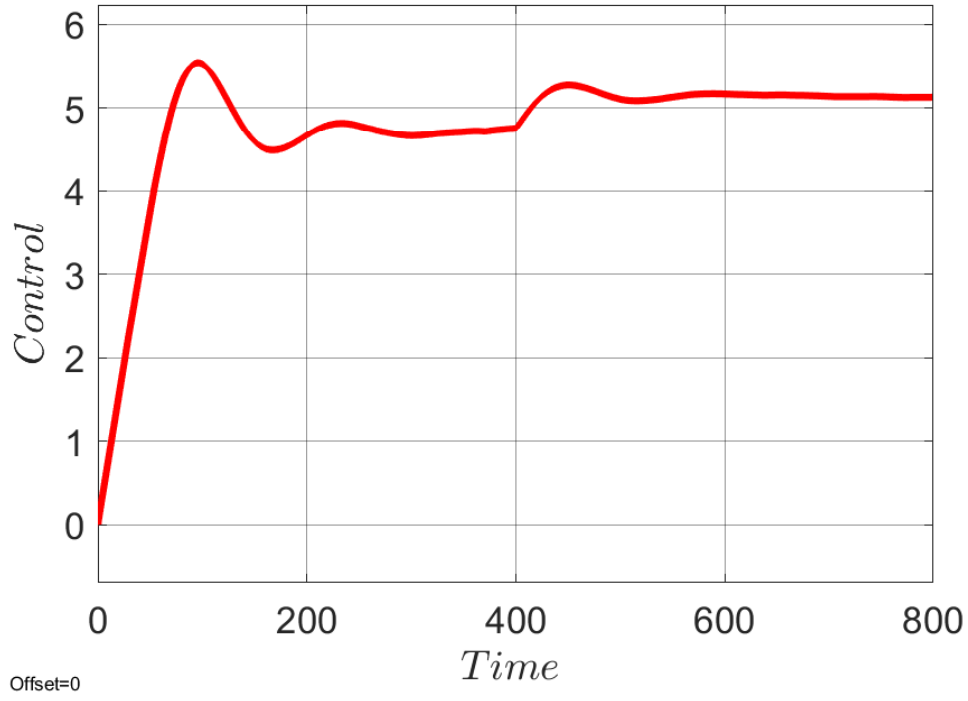


(a) Results of CFDL-RL1 scheme output,

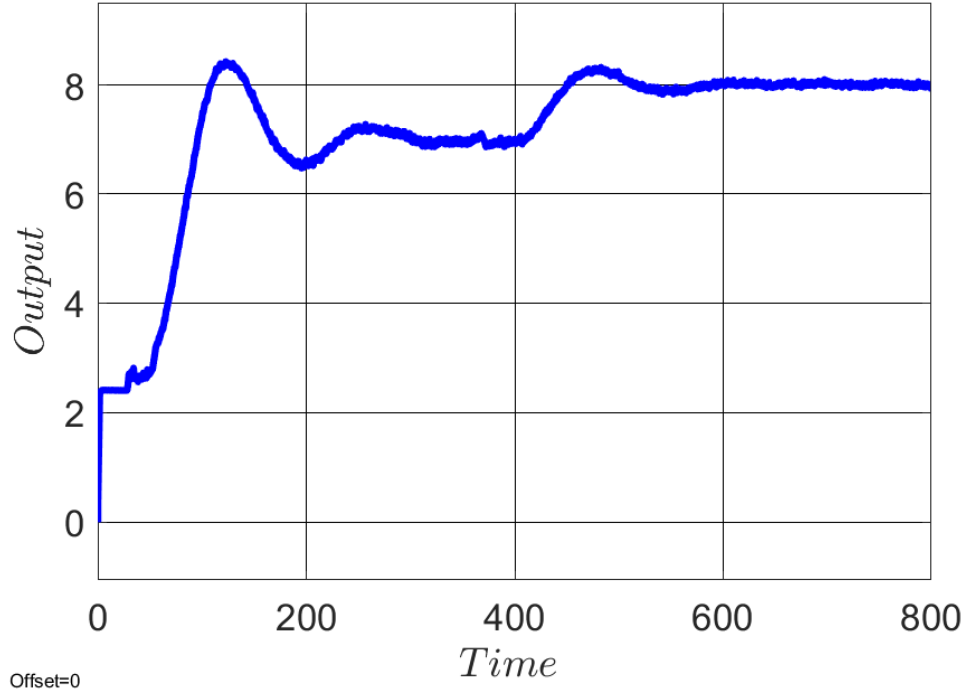


(b) Results of CFDL-RL1 scheme control input,

Figure 3.7: Overview of the experimental results for CFDL-RL1



(a) Results of CFDL-RL2 scheme output



(b) Results of CFDL-RL2 scheme control input.

Figure 3.8: Overview of the experimental results for CFDL-RL2

Chapter 4

Prescribed Performance-Based Data-Driven Control Using Minimum Operator For Discrete-Time Nonlinear Systems

4.1 Introduction

DDC approaches are gaining significant traction, as they rely solely on the input and output of the plant. In this chapter, we have used a concept called the pseudo partial derivative (PPD) and pseudo gradient (PG) approaches to capture the dynamic behavior of the controlled plant, moving away from the traditional state-space representation. This framework includes three different dynamic linearization data models [14] CFDL, PFDL, and FFDL. Among the three, FFDL is the most generalized form, and we have used it for our development purposes. Many practical systems are required to meet prescribed performance (PP) [21], ensuring that both dynamic and equilibrium errors are driven to predefined bounds. Building on the work in [21], various prescribed performance control (PPC) strategies have emerged [22]. However, these methods primarily focus on CTS. Unfortunately, discretizing control signals introduces additional complexity in controller design and stability analysis. Sampling errors further degrade system performance, creating a gap between continuous- and discrete-time control schemes. This gap hinders direct application of continuous-time methods to DTS, underscoring the need for dedicated re-

search in this area. While limited research exists on PPC for DTS [23], a recent attempt was made to formulate MFAC using DSMC with features for PPC [24], where the authors developed a data-driven adaptive SMC strategy based on the FFDL technique. Recently minimum operator based DSMC [2] are developed, however, no prior attempt has been made to integrate the minimum operator-based DSMC approach with FFDL and PPC simultaneously, and the potential for performance improvement through this combination remains unexplored. This paper proposes two FFDL-minimum operator-based DSMC schemes designed for nonlinear PPC.

This chapter introduces several novel contributions for nonlinear discrete-time systems (DTS) with perturbations:

- A controller design framework that integrates the FFDL approach with minima based DSMC.
- The proposed controllers leverage benefit of PPC methods, which is essential for many practical systems.
- We propose an ε -regularized logarithmic PPC that enforces asymmetric, adjustable funnels with a closed-form inverse, enabling direct next-step error set-pointing within an RL1/RL2 FFDL–DSMC framework while preserving numerical robustness and bounded control effort.
- The proposed data-driven control laws ensure finite-time boundedness of the switching function and reduced QSMC, offering improved robust control compared to previous approaches.
- Numerical simulations demonstrate that the PPC-FFDL-minima based DSMC algorithm significantly outperforms other methods in terms of tracking error reduction and disturbance rejection.

4.2 FFDL Data Model

A SISO discrete-time nonlinear system can be represented as:

$$y_{k+1} = f(y_k, y_{k-1}, \dots, y_{k-n_y}, u_k, u_{k-1}, \dots, u_{k-n_u}) + \delta_k \quad (4.1)$$

$y_k \in R$ and $u_k \in R$ denote the output and input of the dynamics, respectively; $n_y > 0, n_u > 0$, where $n_y, n_u \in \mathbb{Z}^+$, and $\mathbb{Z}^+$ denotes the set of positive integers. and we have considered $n_y = n_u = 1$, δ_k denotes the constrained perturbations such that $\|\delta_k\| \leq \mathcal{D}$, where $\mathcal{D} > 0$.

Assumption 1: The system described by (4.1) exhibits both controllability and observability. In particular, if the target signal y'_{k+1} remains bounded, it follows that the control input u_k will also be constrained, allowing y_{k+1} to accurately track the desired signal y'_{k+1} .

Assumption 2: The DTS (4.1) satisfies a generalized Lipschitz condition, which is expressed as $|\Delta y_{k+1}| \leq b\|\xi_k\|$ for any k where $\xi_k^T = [\Delta y_k, \Delta u_k]$, with $b > 0$. Here, $\Delta y_k = y_k - y_{k-1}$ and $\Delta u_k = u_k - u_{k-1}$.

Remark 1. From a practical standpoint, Assumption 1 serves as a fundamental prerequisite for ensuring the system's controllability. Assumption 2 implies that a finite variation in the control input does not result in an infinite variation in the system output. Assumptions 1–2 are also discussed in [54], and readers may refer to [54] for further details.

Lemma 1: Examine the nonlinear system (4.1). If $f(\cdot)$ satisfies Assumptions 1 and 2, then there exists PG ϕ_k such that system (4.1) can be expressed as:

$$y_{k+1} = y_k + \phi_{1,k}\Delta y_k + \phi_{2,k}\Delta u_k = y_k + \xi_k^T \phi_k + \delta_k \quad (4.2)$$

where $\phi_k = [\phi_{1,k}, \phi_{2,k}]^T$ is constrained $\forall k > 0$.

4.2.1 Pseudo Partial Derivative Assessment

Based on the approach in [14], an estimate of the unknown vector ϕ_k , denoted by $\hat{\phi}_k$, can be obtained by minimizing the following cost function: $J(\phi_k) = \|\Delta y_k - \zeta_{k-1}^T \phi_k\|^2 + \sigma \|\phi_k - \hat{\phi}_{k-1}\|^2$, where σ is a positive regularization parameter. To find the optimal $\hat{\phi}_k$, we set the derivative of J with respect to ϕ_k to zero, leading to the condition: $\sigma(\phi_k - \hat{\phi}_{k-1}) - \zeta_{k-1}^T(\Delta y_k - \zeta_{k-1}^T \phi_k) = 0$. From this condition, the recursive estimation algorithm is derived as:

$$\hat{\phi}_k = \hat{\phi}_{k-1} + \frac{\kappa \zeta_{k-1}(\Delta y_k - \zeta_{k-1}^T \hat{\phi}_{k-1})}{\sigma + \|\zeta_{k-1}\|^2}, \quad (4.3)$$

where $\kappa \in (0, 2)$ is a gain factor, and $\sigma > 0$ ensures stability.

Remark 1: The updating law (4.3) is designed based solely on the controlled system's input and output measurement data. The stability of this method are analyzed in

detail in [23].

Remark 2: The disturbance estimate $\hat{\delta}k$ of δ_k in (4.2) is obtained via perturbation estimation as $\hat{\delta}k = \delta_{k-1} = y_k - y_{k-1} - \xi_{k-1}^T \phi_{k-1}$. Now (4.2) can be written as:

$$y_{k+1} = y_k + \xi_k^T \hat{\phi}k + \hat{\delta}k - \tilde{\delta}k \quad (4.4)$$

where $\tilde{\delta}k = \delta_{k-1} - \delta_k$, and $\hat{\phi}k$ is estimate of PG.

4.3 Prescribed Performance Function

A discrete-time prescribed performance function (PPF) is adopted to specify a time-varying funnel within which the tracking error must evolve. The sequence $\{\rho_k\}$ is generated as a first-order filter:

$$\rho_{k+1} = (1 - \nu)\rho_k + \nu\rho_s, \quad \rho_0 > \rho_s > 0, \quad \nu \in (0, 1), \quad (4.5)$$

so that $\lim_{k \rightarrow \infty} \rho_k = \rho_s$. The tracking error (TE) $e_k := y_{d,k} - y_k$ is required to satisfy the asymmetric performance bounds

$$-a_k \rho_k < e_k < b_k \rho_k, \quad (4.6)$$

where the configurable frontier functions a_k and b_k tighten to one according to

$$a_{k+1} = (1 - \nu)a_k + \nu, \quad a_0 \geq 1, \quad b_{k+1} = (1 - \nu)b_k + \nu, \quad b_0 \geq 1. \quad (4.7)$$

To enforce (4.6) via a transformed tracking error (TTE), a monotone, bijective error-to-auxiliary map (error transfer function, ETF) is employed. Define the normalized error $\xi_k := e_k / \rho_k \in (-a_k, b_k)$. A *logarithmic barrier* ETF is proposed as:

$$\tau_k = \frac{1}{2} \ln \left(\frac{\xi_k + a_k + \varepsilon}{b_k - \xi_k + \varepsilon} \right), \quad \varepsilon > 0, \quad (4.8)$$

which, for a small $\varepsilon > 0$ maps the open interval $(-a_k, b_k)$ *bijectively and monotonically* onto $\mathbb{R}$. The inverse map is available in closed form,

$$\xi_k = \frac{(b_k + \varepsilon) e^{2\tau_k} - (a_k + \varepsilon)}{1 + e^{2\tau_k}}, \quad e_k = \rho_k \xi_k, \quad (4.9)$$

From (4.8) it follows that $\tau_k \rightarrow +\infty$ as $e_k \uparrow b_k \rho_k$ and $\tau_k \rightarrow -\infty$ as $e_k \downarrow -a_k \rho_k$, i.e., a barrier Lyapunov property is realized.

Remark 3. Compared with the tangent-based ETF and logarithmic function based ETF [24], the proposed logarithmic barrier ETF naturally enforces *asymmetric* bounds, respects the true limits $(-a_k, b_k)$, and realizes a strict barrier ($\tau_k \rightarrow \pm\infty$ at the bounds) with a closed-form inverse, while offering better numerical conditioning (low gain near $\xi_k \approx 0$, higher near the limits). Hence, the ε -regularized logarithmic barrier is preferable for PPC with asymmetric bounds; the tangent map may dilute the tighter side ($a_k \neq b_k$) and can overflow near its asymptotes.

4.4 Minimum Operator-Based PPC-FFDL-DSMC Design

This section outlines the controller design process of the minimum operator-based PPC-FFDL-DSMC scheme. Initially the controller is designed based on RL1 and RL2 and later the stability of the proposed controller is also demonstrated.

4.4.1 DESIGN OF CONTROLLER USING PPC-FFDL-RL1 SCHEME

Define the TE as

$$e_k = y'_k - y_k. \quad (4.10)$$

Using the FFDL model, the one-step-ahead error is written as

$$\begin{aligned} e_{k+1} &= y'_{k+1} - y_k - \phi_{1,k} \Delta y_k - \phi_{2,k} \Delta u_k - \hat{\delta}_k + \tilde{\delta}_k \\ &= \psi_k - \phi_{2,k} \Delta u_k, \end{aligned} \quad (4.11)$$

where $\Delta y_k := y_k - y_{k-1}$, $\Delta u_k := u_k - u_{k-1}$, and $\psi_k := y'_{k+1} - y_k - \phi_{1,k} \Delta y_k - \hat{\delta}_k$. The small residual $\tilde{\delta}_k$ is neglected.

The sliding function is defined as

$$\aleph_k = c \tau_k, \quad (4.12)$$

with $c \in \mathbb{R}$; for the SISO case, $c = 1$ is adopted (it may be tuned via level sets).

A boundary-layer version of RL1 (2.4) is imposed using (4.11):

$$\aleph_{k+1} = \tau_{k+1}, \quad \tau_{k+1} = \aleph_k - \text{sign}(\aleph_k) \min\{|\aleph_k|, \varsigma\},$$

where $\varsigma > 0$ denotes the boundary-layer width. Using (4.11) and the inverse ETF (4.9) at $k+1$, the target normalized error becomes

$$\xi_{k+1} = \frac{(b_{k+1} + \varepsilon) e^{2\chi_k} - (a_{k+1} + \varepsilon)}{1 + e^{2\chi_k}}, \quad (4.13)$$

The commanded control increment that enforces $e_{k+1} = \rho_{k+1}\xi_{k+1}$ is therefore

$$\Delta u_k^{\text{cmd}} = \frac{\psi_k - \rho_{k+1} \xi_{k+1}}{\phi_{2,k}}.$$

An adaptive/robust realization is employed by replacing $\phi_{2,k}$ with its estimate $\hat{\phi}_{2,k}$:

$$u_{1k}^r = \frac{\iota \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2} \left(\psi_k - \rho_{k+1} \Xi_1(k) \right), \quad (4.14)$$

where $\gamma > 0$, $\iota \in (0, 1]$,

$$\begin{aligned} \psi_k &:= y'_{k+1} - y_k - \phi_{1,k} \Delta y_k - \hat{\delta}_k, \\ \Xi_1(k) &:= \frac{(b_{k+1} + \varepsilon) e^{2\chi_1(k)} - (a_{k+1} + \varepsilon)}{1 + e^{2\chi_1(k)}}, \\ \chi_1(k) &:= \aleph_k - \text{sign}(\aleph_k) \min\{|\aleph_k|, \varsigma\}. \end{aligned} \quad (4.15)$$

The control input is updated as

$$u_{1k} = u_{1,k-1} + u_{1k}^r. \quad (4.16)$$

4.4.2 DESIGN OF CONTROLLER USING PPC-FFDL-RL2 SCHEME

Consider the same TE (4.10) and sliding function (4.12). The PPC-FFDL-RL2-based control law is obtained as follows. Impose the RL2 manifold with a boundary layer:

$$\begin{aligned} \aleph_{k+1} &= \tau_{k+1}, \\ \tau_{k+1} &= \aleph_k - \varsigma \text{sign}(\aleph_k) \min\left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta\right), \end{aligned} \quad (4.17)$$

Here, $\varsigma > 0$, $\beta \in (0, 1)$ together with (4.11) and the inverse ETF (4.9) at $k+1$, the target normalized error becomes

$$\begin{aligned} \xi_{k+1} &= \frac{(b_{k+1} + \varepsilon) e^{2\chi_k} - (a_{k+1} + \varepsilon)}{1 + e^{2\chi_k}}, \\ \chi_k &= \aleph_k - \varsigma \text{sign}(\aleph_k) \min\left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta\right). \end{aligned}$$

so that $e_{k+1} = \rho_{k+1}\xi_{k+1}$ and the commanded increment becomes

$$\Delta u_k^{\text{cmd}} = \frac{\psi_k - \rho_{k+1}\xi_{k+1}}{\phi_{2,k}}.$$

An adaptive/robust realization is

$$u_{2k}^r = \frac{\iota \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2} \left(\psi_k - \rho_{k+1} \Xi_2(k) \right) \quad (4.18)$$

with $\gamma > 0$ and $\iota \in (0, 1]$. The control input is updated as

$$u_{2k} = u_{2,k-1} + u_{2k}^r. \quad (4.19)$$

where,

$$\begin{aligned} \Xi_2(k) &= \frac{(b_{k+1} + \varepsilon) e^{2\chi_k} - (a_{k+1} + \varepsilon)}{1 + e^{2\chi_k}}, \\ \chi_k &= \aleph_k - \varsigma \operatorname{sign}(\aleph_k) \min\left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta\right). \end{aligned} \quad (4.20)$$

Remark 6. The proposed sliding function (4.12) features a simple structure that avoids the need for additional user-selected gains in designing the sliding plane, unlike the approach in [24]. Furthermore, the control methodology (4.16)–(4.19) exploits the reaching phase to reduce control effort compared with the equivalent control strategy in [24].

4.4.3 STABILITY ANALYSIS

Theorem 1. If Assumptions 1–2 hold for (4.1), the initial condition satisfies $-a_0\rho_0 < e_0 < b_0\rho_0$, the estimator update $\Delta\hat{\phi}_k$ and ETF τ_k are generated by (4.3) and (4.8), and the controller (4.14), (4.16) is applied with an RL1 step size ς chosen such that $\varsigma > \omega_m \geq \sup_k |\tilde{\delta}_k|$, then the tracking error (TE) remains in the prescribed performance funnel (4.6) for all k , the sliding signal $\aleph_k$ is ultimately bounded, and the control input u_{1k} is bounded (trivially so if actuator projection $u_{1k} \in [u_{\min}, u_{\max}]$ is used).

Proof.

Step 1 (Finite-time boundedness of the switching signal). From (2.4) and (4.12) we obtain

$$\tau_{k+1} = \aleph_{k+1} = \aleph_k - \operatorname{sign}(\aleph_k) \min\{|\aleph_k|, \varsigma\} + \tilde{\delta}_k. \quad (4.21)$$

Let $V(k) := |\aleph_k|$ and define $b_k := \aleph_k - \operatorname{sign}(\aleph_k) \min\{|\aleph_k|, \varsigma\}$, so $|b_k| = |\aleph_k| - \min\{|\aleph_k|, \varsigma\}$.

Using the triangle inequality,

$$\Delta V(k) = |b_k + \tilde{\delta}_k| - |\aleph_k| \leq -\min\{|\aleph_k|, \varsigma\} + |\tilde{\delta}_k|.$$

Hence, if $|\aleph_k| \geq \varsigma$ then $\Delta V(k) \leq -(\varsigma - |\tilde{\delta}_k|) \leq -(\varsigma - \omega_m) < 0$; if $|\aleph_k| \leq \varsigma$ then $\Delta V(k) \leq -(|\aleph_k| - \omega_m)$, so one more step renders $|\aleph_{k+1}| \leq \omega_m$. Therefore,

$$|\aleph_k| \leq \omega_m \quad \text{for all } k \geq \left\lceil \frac{|\aleph_0| - \omega_m}{\varsigma - \omega_m} \right\rceil.$$

Step 2 (Funnel invariance: ideal case). Substituting (4.14), (4.16) into (4.11) gives

$$e_{k+1} = \psi_k - \phi_{2,k} \frac{\iota \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2} (\psi_k - \rho_{k+1} \Xi_1(k)). \quad (4.22)$$

For the certainty–equivalence case $\hat{\phi}_{2,k} = \phi_{2,k}$, set $\kappa_k := \frac{\iota \phi_{2,k}^2}{\gamma + \phi_{2,k}^2} \in (0, 1)$ to obtain

$$e_{k+1} = (1 - \kappa_k) \psi_k + \kappa_k \rho_{k+1} \Xi_1(k).$$

With the commanded increment as in (4.14), $\psi_k \mapsto \rho_{k+1} \Xi_1(k)$ and $e_{k+1} = \rho_{k+1} \Xi_1(k)$. The ε –log–barrier ETF $\tau_k = \frac{1}{2} \ln \left(\frac{\xi_k + a_k + \varepsilon}{b_k - \xi_k + \varepsilon} \right)$ is strictly increasing and has a closed-form inverse mapping a *finite* interval in τ onto $\xi \in (-a_k, b_k)$. The RL1 update $\tau_{k+1} = \chi_k := \aleph_k - \text{sign}(\aleph_k) \min\{|\aleph_k|, \varsigma\}$ is a *contraction* in τ , so for any ς small enough that χ_k remains within the inverse’s domain, we have

$$\xi_{k+1} = \frac{(b_{k+1} + \varepsilon)e^{2\chi_k} - (a_{k+1} + \varepsilon)}{1 + e^{2\chi_k}} \in (-a_{k+1}, b_{k+1}), \quad (4.23)$$

and hence $-a_{k+1}\rho_{k+1} < e_{k+1} < b_{k+1}\rho_{k+1}$. Since $-a_0\rho_0 < e_0 < b_0\rho_0$ by hypothesis, induction yields

$$-a_k\rho_k < e_k < b_k\rho_k, \quad \forall k \in \mathbb{N}. \quad (4.24)$$

Step 3 (Funnel invariance: non-ideal estimate). Define the possibly mismatched contraction gain

$$\tilde{\kappa}_k := \frac{\iota \phi_{2,k} \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2},$$

so that

$$e_{k+1} - \rho_{k+1} \Xi_1(k) = (1 - \tilde{\kappa}_k) (\psi_k - \rho_{k+1} \Xi_1(k)). \quad (4.25)$$

Assume a standard projection $\hat{\phi}_{2,k} \in [\phi_{2,\min}, \phi_{2,\max}]$ with $\phi_{2,\min} > 0$. Using $0 \leq \frac{x}{\gamma + x^2} \leq \frac{1}{2\sqrt{\gamma}}$ for $x \geq 0$, we get $0 < \underline{\kappa} \leq \tilde{\kappa}_k \leq \bar{\kappa} < 1$ by suitable $\gamma > 0$ (and/or $\iota \in (0, 1]$). Thus (4.25) is a contraction toward $\rho_{k+1} \Xi_1(k)$, and the same induction as in Step 2 yields (4.24).

Step 4 (Boundedness of the input). From (4.14),

$$|\Delta u_k| = \left| \frac{\iota \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2} \right| |\psi_k - \rho_{k+1} \Xi_1(k)| \leq \frac{\iota}{2\sqrt{\gamma}} (|\psi_k| + \rho_{k+1} |\Xi_1(k)|),$$

and the right-hand side is bounded because of (4.24) and bounded regressors/disturbance (Assumptions 1–2). With actuator projection $u_{1k} \in [u_{\min}, u_{\max}]$, the input is trivially bounded; without projection, boundedness follows under the usual adaptive conditions, ensuring summability of the residual. $\square$

Remark 7. Choose $\varsigma > \omega_m$ to secure the finite-time bound in Step 1; pick $\varepsilon > 0$ to avoid singularities at the funnel edges and ensure a well-posed inverse; select $\gamma > 0$ (and $\iota \in (0, 1]$) so that the contraction gains satisfy $0 < \kappa_k, \tilde{\kappa}_k < 1$ uniformly by the bound $\sup_{x \geq 0} \frac{x}{\gamma + x^2} = \frac{1}{2\sqrt{\gamma}}$.

Theorem 2. If Assumptions 1–2 hold for (4.1), the initial condition satisfies $-a_0\rho_0 < e_0 < b_0\rho_0$, the estimator update $\Delta\hat{\phi}_k$ and the ETF τ_k are generated by (4.3) and (4.8), and the RL2 controller (4.18), (4.19) is applied with $0 < \beta < 1$ and $\varsigma > \omega_m \geq \sup_k |\tilde{\delta}_k|$, then the tracking error (TE) satisfies the prescribed performance bounds (4.6) for all k , while the sliding signal $\aleph_k$ and the input u_{2k} remain bounded.

Proof.

Step 1 (Ultimate boundedness of the switching signal). The RL2 manifold with boundary layer reads (cf. (4.17))

$$\tau_{k+1} = \aleph_{k+1} = \aleph_k - \varsigma \operatorname{sign}(\aleph_k) \min\left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta\right) + \tilde{\delta}_k. \quad (4.26)$$

Let $V(k) := |\aleph_k|$. Using the triangle inequality and $|x - \operatorname{sign}(x) \min(|x|/\varsigma, |x|^\beta)| = |x| - \min(|x|, \varsigma|x|^\beta)$, we get

$$\begin{aligned} \Delta V(k) &= |\aleph_{k+1}| - |\aleph_k| \\ &\leq -\min(|\aleph_k|, \varsigma|\aleph_k|^\beta) + |\tilde{\delta}_k| \\ &\leq -\min(|\aleph_k|, \varsigma|\aleph_k|^\beta) + \omega_m. \end{aligned}$$

Hence: (i) if $|\aleph_k| \geq r_* := (\omega_m/\varsigma)^{1/\beta}$, then $\Delta V(k) \leq -\varsigma|\aleph_k|^\beta + \omega_m \leq 0$ with strict decrease for $|\aleph_k| > r_*$; and (ii) if $|\aleph_k| \leq \varsigma$, then $\Delta V(k) \leq -|\aleph_k| + \omega_m \leq -(\varsigma - \omega_m)$ whenever $|\aleph_k| \geq \varsigma$. Therefore, $|\aleph_k|$ reaches in finite time the compact set $\mathcal{B}_\omega := \{|\aleph| \leq \bar{\omega}\}$ with $\bar{\omega} := \max\{\varsigma, (\omega_m/\varsigma)^{1/\beta}\}$, and stays there. In particular, $\aleph_k$ is ultimately bounded.

Step 2 (Funnel invariance via the $-\log$ barrier). Let $\xi_k := e_k/\rho_k$ and use the regularized log-barrier ETF $\tau_k = \frac{1}{2} \ln\left(\frac{\xi_k + a_k + \varepsilon}{b_k - \xi_k + \varepsilon}\right)$, $\varepsilon > 0$, with inverse

$$\mathcal{T}_{\log}^{-1}(\tau; a, b, \varepsilon) = \frac{(b + \varepsilon)e^{2\tau} - (a + \varepsilon)}{1 + e^{2\tau}}.$$

With the RL2 update (4.26) define

$$\chi_k := \aleph_k - \varsigma \operatorname{sign}(\aleph_k) \min\left(\frac{|\aleph_k|}{\varsigma}, |\aleph_k|^\beta\right),$$

so that $\tau_{k+1} = \chi_k + \tilde{\delta}_k$. Because $\aleph_k$ is ultimately bounded (Step 1) and $\tilde{\delta}_k$ is bounded, τ_{k+1} remains in a compact interval on which the inverse ETF is well defined. Therefore the *target* normalized error at $k+1$,

$$\xi_{k+1} = \mathcal{T}_{\log}^{-1}(\chi_k; a_{k+1}, b_{k+1}, \varepsilon) = \frac{(b_{k+1} + \varepsilon)e^{2\chi_k} - (a_{k+1} + \varepsilon)}{1 + e^{2\chi_k}},$$

satisfies $\xi_{k+1} \in (-a_{k+1}, b_{k+1})$, hence $-a_{k+1}\rho_{k+1} < \rho_{k+1}\xi_{k+1} < b_{k+1}\rho_{k+1}$.

Step 3 (Closed-loop one-step error and invariance). From (4.11) and the RL2 realization (4.18), (4.19),

$$e_{k+1} = \psi_k - \phi_{2,k} \frac{\iota \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2} \left(\psi_k - \rho_{k+1} \Xi_2(k) \right), \quad (4.27)$$

$$\Xi_2(k) := \mathcal{T}_{\log}^{-1}(\chi_k; a_{k+1}, b_{k+1}, \varepsilon).$$

(i) *Ideal estimate*) If $\hat{\phi}_{2,k} = \phi_{2,k}$, set $\kappa_k := \frac{\iota \phi_{2,k}^2}{\gamma + \phi_{2,k}^2} \in (0, 1)$ to obtain $e_{k+1} = (1 - \kappa_k)\psi_k + \kappa_k \rho_{k+1} \Xi_2(k)$. Choosing the commanded increment as in (4.18) makes $\psi_k \mapsto \rho_{k+1} \Xi_2(k)$, whence $e_{k+1} = \rho_{k+1} \Xi_2(k) \in (-a_{k+1}\rho_{k+1}, b_{k+1}\rho_{k+1})$. (ii) *Non-ideal estimate*) With the contraction gain $\tilde{\kappa}_k := \frac{\iota \phi_{2,k} \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2}$ and a standard projection $\hat{\phi}_{2,k} \in [\phi_{2,\min}, \phi_{2,\max}]$, one gets

$$|e_{k+1} - \rho_{k+1} \Xi_2(k)| \leq (1 - \underline{\kappa}) |\psi_k - \rho_{k+1} \Xi_2(k)|$$

for some $0 < \underline{\kappa} < 1$ (choose $\gamma > 0$, $\iota \in (0, 1]$ accordingly). Since $\Xi_2(k) \in (-a_{k+1}, b_{k+1})$ and ψ_k is bounded under Assumptions 1–2, the same induction as in Theorem 1 yields

$$-a_k \rho_k < e_k < b_k \rho_k, \quad \forall k \in \mathbb{N}. \quad (4.28)$$

Step 4 (Boundedness of the input). From (4.18),

$$|\Delta u_{2k}| = \left| \frac{\iota \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2} \right| |\psi_k - \rho_{k+1} \Xi_2(k)| \leq \frac{\iota}{2\sqrt{\gamma}} \left(|\psi_k| + \rho_{k+1} |\Xi_2(k)| \right),$$

which is uniformly bounded due to (4.28) and Assumptions 1–2. Thus u_{2k} is bounded (trivially so if projected onto $[u_{\min}, u_{\max}]$).

4.5 RESULTS AND DISCUSSION

To validate the performance of the PPC-FFDL-RL1 and PPC-FFDL-RL2 control schemes, this section presents numerical simulations using robotic manipulator. Additionally, to highlight the characteristics of the proposed method, comparisons are made with [24] for both proposed algorithms.

Table 4.1: MSE Comparison

Control Scheme	[24]	FFDL-RL1	FFDL-RL2
MSE 250	0.00492	0.00331	0.00335

4.5.1 NUMERICAL SIMULATION

Consider a robotic manipulator, whose dynamics are described as follows [24]:

$$J\ddot{\theta} + f\dot{\theta} + \left(\frac{1}{2}m + M\right)gl \sin(\theta) = u + d, \quad (4.29)$$

where, θ , $\dot{\theta}$, and $\ddot{\theta}$ represent the angular displacement, angular velocity, and angular acceleration of the manipulator, respectively. u is the driving torque. g is the gravitational acceleration. m , M , l , and f are the link mass, tip load, length, and damping coefficient of the manipulator, respectively. J is the joint moment of inertia, defined as $J = Ml^2 + \frac{1}{3}ml^2$. d represents an external disturbance. Define the state variables as $z_1 = \theta$ and $z_2 = \dot{\theta}$. Using these definitions, the system equation can be rewritten in state-space form:

$$\begin{aligned} \dot{z}_1 &= z_2, \quad \dot{z}_2 = \frac{1}{J} \left(-fz_2 - \left(\frac{1}{2}m + M\right)gl \sin(z_1) + u + d \right), \\ y &= z_1. \end{aligned}$$

To implement the system in a discrete-time framework, the equations are discretized, yielding:

$$\begin{aligned} y(t+1) &= f(y(t), y(t-1), \dots, y(0); u(t); d(t)) \\ &= \left(1 - \frac{hf}{J}\right) y(t) - \frac{h}{J} \left(\frac{1}{2}m + M\right) gl \sin(x_1(0) + E^T Y(t-1)) \\ &\quad + \frac{h}{J} u(t) + \frac{h}{J} d(t). \end{aligned}$$

where, h is the sampling period, $z_1(0)$ is the initial value of $z_1(t)$, $E = [1, \dots, 1]^\top$, $Y(t-1) = [y(0), \dots, y(t-1)]^\top$. The true parameter values of the system are specified as $m = 1 \text{ kg}$, $M = 2 \text{ kg}$, $l = 0.5 \text{ m}$, $f = 3 \text{ kg} \cdot \text{m}^2/\text{s}$, $g = 9.8 \text{ m/s}^2$. The angular velocity, $\dot{\theta}$, is considered as the output of the manipulator. For the simulation purpose, the initial values are set as $u_0 = z_0 = y_0 = 0$, and the sampling time is defined as $h = 0.004$ seconds. The parameters for the proposed control are chosen to be the same as those in [24] to ensure the results are comparable. $\kappa = 0.05$, $\sigma = 0.15$, $\lambda = 150$, $\iota = 0.9$, $\gamma = 0.00001$, $\tau_s = 0.001$, $\beta = 1.1$, $\varsigma = 0.1$. The initial estimate of $\hat{\phi}_k$ is specified as $\hat{\phi}_0 = [1.1, 0.004]^T$. The PP parameters are defined as: $a_0 = 1.3$, $b_0 = 1.1$, $\rho_0 = 1.5$, $\rho_\infty = 0.05$, $\nu = 0.08$.

To assess the tracking performance of the two proposed approaches mean squared errors (MSE) and maximum control input of these methods are compared with those presented in [24] in Table I. The MSE is computed using the formula $\text{MSE}_n = \frac{1}{n} \sum_{k=1}^n (y_{d,k} - y_k)^2$, where n represents the total number of time steps, which is set to 250 in this example. As illustrated in Table I, PPC-FFDL-RL1 and PPC-FFDL-RL2, demonstrate a reduction in MSE by 33.1% and 33.5%, respectively, compared to [24]. This further reinforces the effectiveness of the proposed approach. The simulation results indicate that the proposed method successfully tracks the reference signal, improving steady-state performance while reducing chattering phenomena. By incorporating DSMC, as presented in [2], and FFDL, the steady-state tracking performance is further enhanced. Additionally, with the inclusion of PPC, the method achieves improved settling times and reduced TE. The TE of the PPC-FFDL-RL1 strategy are compared with those in [24] in Figs. 1(a) and 1(b), illustrating both transient and steady-state performance. A non-symmetric performance constraint is plotted, demonstrating that the PPC-FFDL-RL1 keeps the TE within the performance constraint, as outlined in (4.6).

Furthermore, the TEs of the PPC-FFDL-RL2 strategy are compared with those in [24] in Figs. 1(e) and 1(f), showing both transient and steady-state performance. To evaluate the robustness of the proposed solution against measurement noise of varying magnitudes, a further simulation is conducted. In this case, the PP parameters are changed to $a_0 = b_0 = 2$, $\rho_0 = 1.5$, and $\rho_\infty = 0.25$, while keeping other parameters remain same. The measurement noise levels are considered as $n_{1,k} \in (0, 0.1)$, $n_{2,k} \in (0, 0.2)$, and $n_{3,k} \in (0, 0.3)$, respectively. Figs. 1(d) and 1(h) demonstrate that the proposed method PPC-FFDL-RL1 & RL2 effectively tracks the reference signal, even under varying noise

levels. When $\rho_\infty = 0.25$, and the high noise level $n_{3,k} \in (0, 0.3)$ is present, the control output successfully tracks the reference signal, ensuring that the tracking error (TE) remains within the predefined limits. In conclusion, the proposed strategy ensures the PP within a range of noise levels.

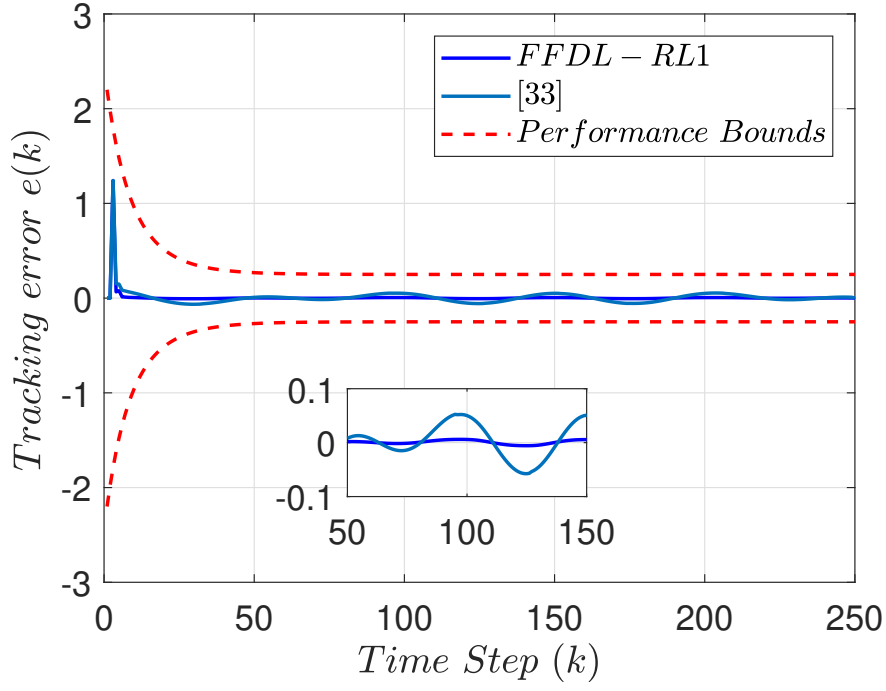
Important performance parameter observations

- κ, σ, γ : This parameter facilitates the convergence of the tracking error to zero at steady state, ensuring accurate final performance.
- β and ς : These parameters collectively influence the rate of control input.
- ν : A smaller value of ν reduces the convergence rate of the error boundaries, effectively controlling how quickly the boundary functions $a(k)$ and $b(k)$ adapt over time.
- ρ_0, a_0, b_0 : These parameters govern the initial bounds of the prescribed performance function. Together, they define the initial performance envelope for error evolution.

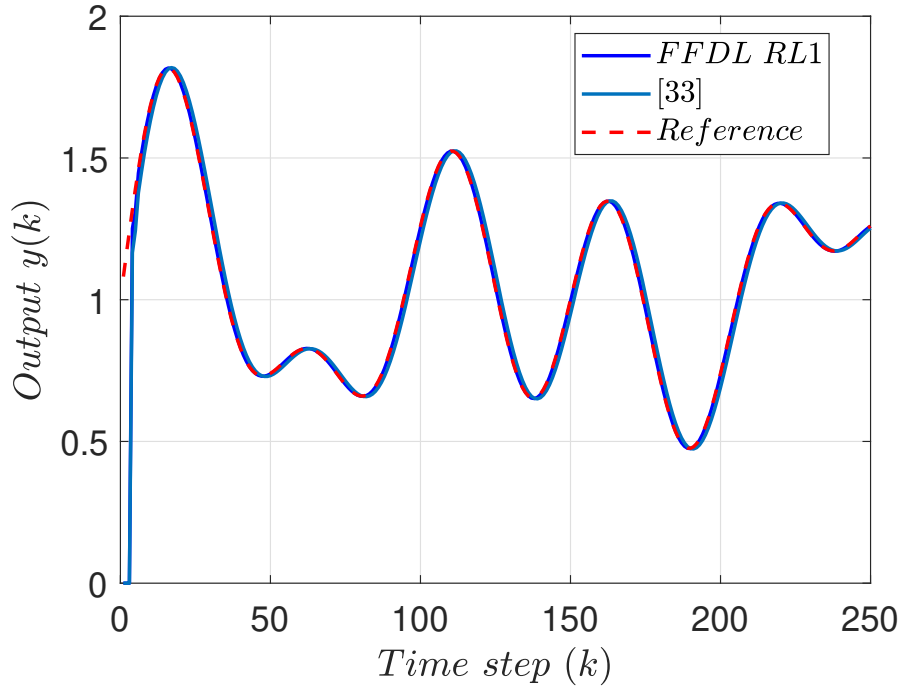
4.6 Conclusion

This work addressed prescribed-performance control (PPC) for nonlinear discrete-time systems with unknown dynamics. We proposed a data-driven adaptive DSMC framework built on FFDL and a minimum-operator sliding manifold, together with a regularized logarithmic-barrier error-transfer function (ETF) that maps the constrained tracking error to an unconstrained variable. Using this transformed error and the minimum operator, two PPC-FFDL-DSMC laws (RL1 and RL2) were derived. Compared with existing approaches, the proposed controllers retain a simple structure while achieving reduced mean squared error (MSE) with lower control effort. Rigorous analysis establishes forward invariance of the error within the prescribed asymmetric funnel and boundedness of all closed-loop signals. Simulation studies on a robotic manipulator corroborate the theoretical results.

Future research directions include: (i) extending the framework to multi-variable and multi-agent systems, (ii) investigating identifiability and consistency issues associated with performance funnels, (iii) integrating learning-based components (e.g., safe reinforcement

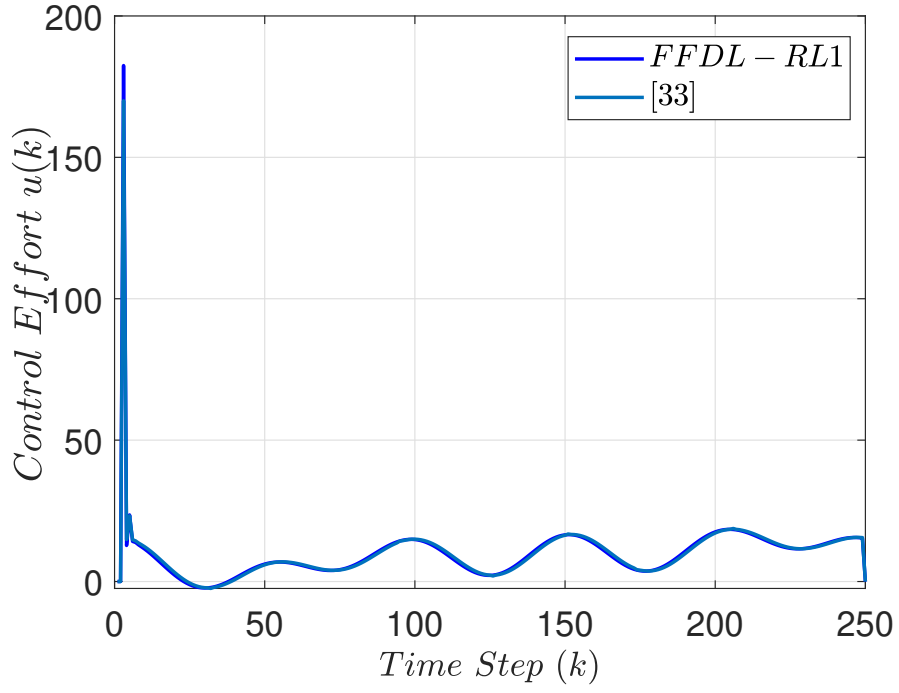


(a) Results of PPC-FFDL-RL1 scheme tracking error

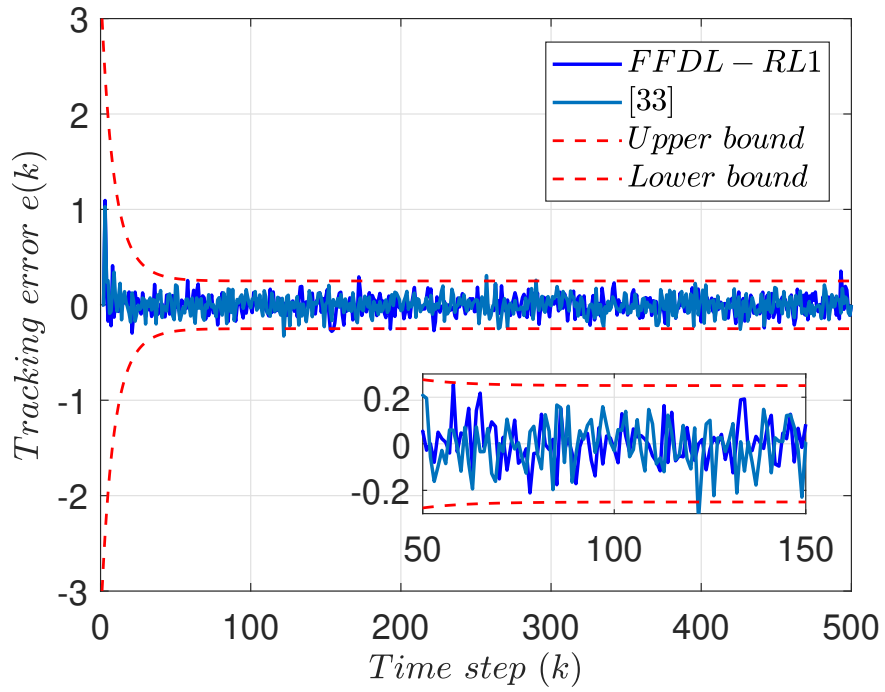


(b) Results of PPC-FFDL-RL1 scheme tracking performance

Figure 4.1: Overview of the PPC-FFDL-RL1 scheme results

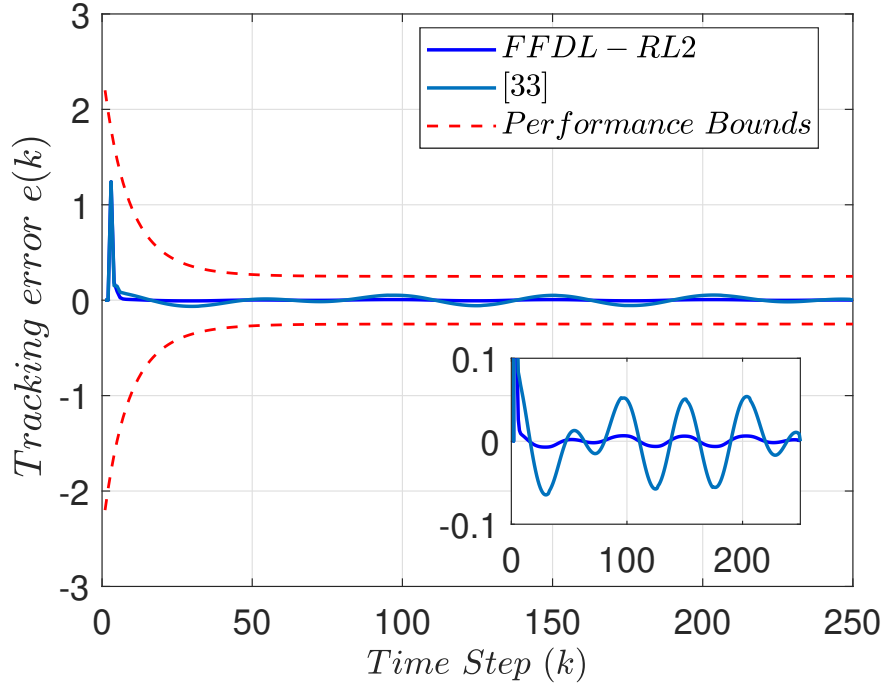


(a) Results of PPC-FFDL-RL1 control input

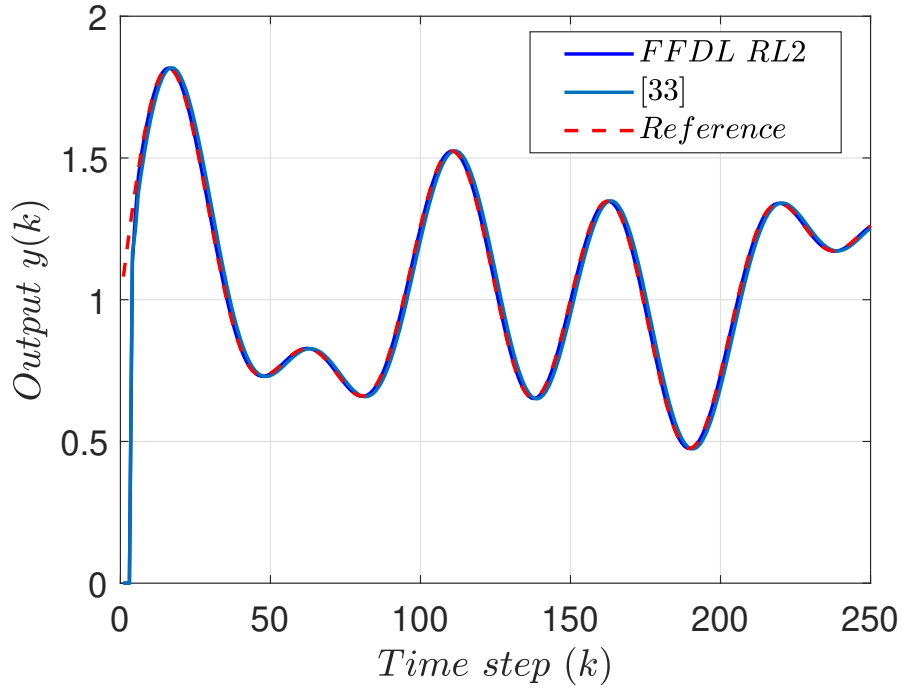


(b) Results of PPC-FFDL-RL1 scheme TE in varying noise level

Figure 4.2: Overview of the PPC-FFDL-RL1 scheme results

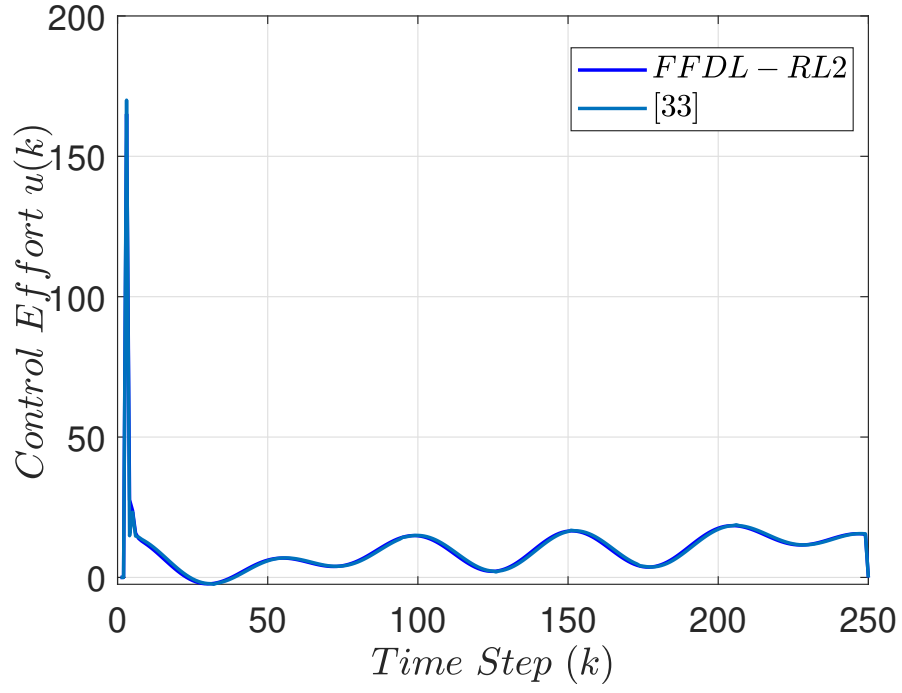


(a) Results of PPC-FFDL-RL2 scheme tracking error

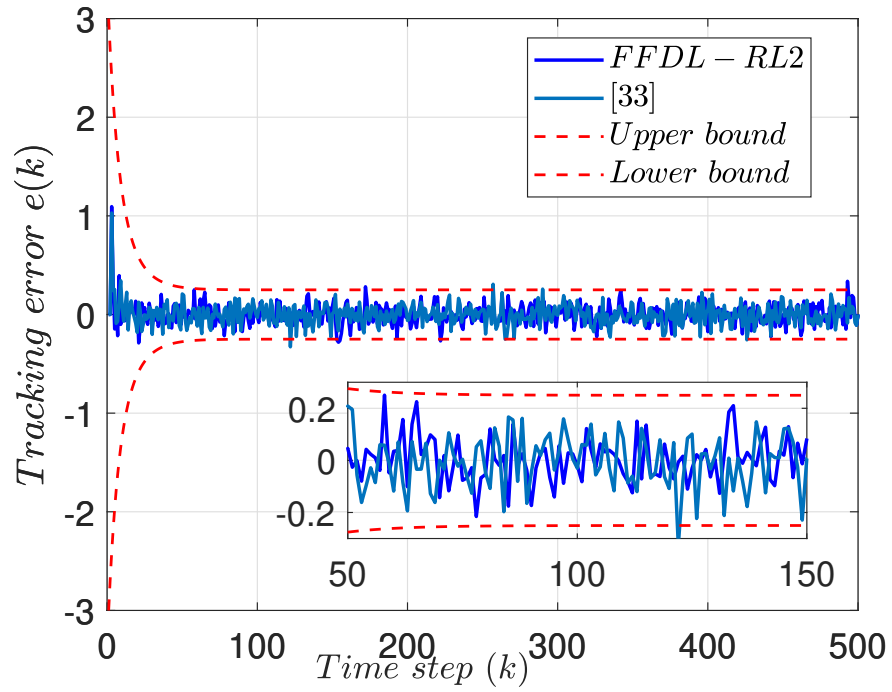


(b) Results of PPC-FFDL-RL2 scheme tracking performance

Figure 4.3: Overview of the PPC-FFDL-RL2 scheme results



(a) Results of PPC-FFDL-RL2 control input



(b) Results of PPC-FFDL-RL2 scheme TE in varying noise level

Figure 4.4: Overview of the PPC-FFDL-RL2 scheme results

learning or adaptive MPC) to enable online policy improvement under safety constraints and time-varying operating conditions, (iv) extending the framework to weakly excited systems and (v) addressing scenarios with limited excitation or constrained actuation.

Algorithm 1 PPC–FFDL–DSMC (RL1/RL2) Controller

Require: Initial (a_0, b_0, ρ_0) , $\hat{\phi}_0 = [\hat{\phi}_{1,0}, \hat{\phi}_{2,0}]^\top$, gains $(\nu, \kappa, \beta, \iota, \gamma, \varepsilon, \varsigma, \beta_{\text{RL2}})$

1: **for** $k = 0, 1, 2, \dots$ **do**

2: **PPF update:** $\rho_{k+1} = (1 - \nu)\rho_k + \nu\rho_s$, $a_{k+1} = (1 - \nu)a_k + \nu$, $b_{k+1} = (1 - \nu)b_k + \nu$.

3: **Measure/preview:** Acquire y_k and r_{k+1} (use $r_{k+1} \approx r_k$ if ZOH).

4: **Error & ETF:** $e_k = r_k - y_k$, $\xi_k = \frac{e_k}{\rho_k}$,

$$\tau_k = \frac{1}{2} \ln \left(\frac{\xi_k + a_k + \varepsilon}{b_k - \xi_k + \varepsilon} \right).$$

5: **Sliding update:**

6: **if** RL1 **then**

7: $\chi_k = \tau_k - \text{sign}(\tau_k) \min\{|\tau_k|, \varsigma\}$

8: **else if** RL2 **then**

9: $\chi_k = \tau_k - \varsigma \text{sign}(\tau_k) \min\left(\frac{|\tau_k|}{\varsigma}, |\tau_k|^{\beta_{\text{RL2}}}\right)$

10: **end if**

11: **Inverse ETF target:**

$$\Xi(k) = \frac{(b_{k+1} + \varepsilon)e^{2\chi_k} - (a_{k+1} + \varepsilon)}{1 + e^{2\chi_k}}, \quad e_{k+1}^* = \rho_{k+1} \Xi(k).$$

12: **FFDL update (normalized):**

13: $\zeta_{k-1} = \begin{bmatrix} \Delta y_{k-1} \\ \Delta u_{k-1} \end{bmatrix} = \begin{bmatrix} y_k - y_{k-1} \\ u_k - u_{k-1} \end{bmatrix}$, $\Delta y_k = y_{k+1} - y_k$ (or its measured/estimated proxy).

14: $\hat{\phi}_k = \hat{\phi}_{k-1} + \kappa \frac{\zeta_{k-1}(\Delta y_k - \zeta_{k-1}^\top \hat{\phi}_{k-1})}{\beta + \|\zeta_{k-1}\|^2}$.

15: **Predictor:** $\psi_k = r_{k+1} - y_k - \hat{\phi}_{1,k} \Delta y_k$.

16: **Control increment:** $u_k^r = \frac{\iota \hat{\phi}_{2,k}}{\gamma + \hat{\phi}_{2,k}^2} (\psi_k - e_{k+1}^*)$.

17: **Integrate/saturate:** $u_k = u_{k-1} + u_k^r$; apply actuator limits if needed.

18: **end for**

Chapter 5

Output Feedback Based Discrete-Time Sliding Mode Control Using Minima Operator

5.1 Introduction

Often in practical applications, variations in dynamics across different channels, along with hardware constraints on specific input-output pairs, often necessitate the use of multiple rates. Multirate systems provide the advantage of maintaining performance while operating at lower sampling frequencies compared to single-rate systems. This characteristic is particularly crucial in networked control systems, where bandwidth limitations impose constraints on sampling rates. Extensive research has explored the challenges associated with static output feedback and fast output sampling (FOS) [31], [36,37]. MROF presents an alternative approach that relies solely on measured output readings and prior input signals, offering a practical and effective solution.

In [36], FOS feedback is introduced, which encompasses the characteristics of static output feedback while offering the flexibility to arbitrarily allocate system poles and ensure the stabilization of controlled dynamics. In the FOS method, every measurement duration $\mathfrak{t}$ is partitioned into $\mathbb{M}$ subintervals $\mathfrak{t}_o = \mathfrak{t}/\mathbb{M}$. The value of $\mathbb{M}$ should be selected such that $\mathbb{M} \geq O$, where O is the observability index. The latest $\mathbb{M}$ output measurements are recorded at time interval $t = l\mathfrak{t}_o$, where, $l = 0, 1, \dots$, and a constant control signal is maintained throughout $\mathfrak{t}$. This input signal is formed as a linear aggregate of the last $\mathbb{M}$

output measurements. In [30], authors utilizing a periodic intermittent control combined with the MROF technique aim to robustly stabilize discretized linear time-invariant systems, thereby redefining the concept of discrete-time sliding mode. In [31], a proposed method for attaining quasi-sliding mode (QSM) based on Gao's law in uncertain systems employs a rapid output sampling control strategy that eliminates control switching and consequently minimizes chattering.

The objective is to implement DSMC for the system by driving its states toward the designated sliding surface. Upon reaching the vicinity of this surface, the focus shifts to ensuring that the system states remain confined within the neighborhood of the sliding surface. A solution to this problem using state feedback has been proposed in [2]. However, since the states are seldom available for feedback, output feedback control is typically required. Consequently, the problem now involves achieving QSMC using only the system's output. As noted in [55], a static output feedback solution is not always feasible. In this paper, we propose two discrete-time output-feedback SMC algorithms for systems with bounded uncertainty, incorporating the following contributions:

- The proposed algorithms are based on Minima operator-Based Reaching Laws and FOS based feedback.
- The design integrates features of multi-rate output systems and ensures finite-time reachability to sliding surface.
- The algorithm operates on the measured system output, with the control input computed solely from previous output measurements and the most recent control signal.
- The settling time can be explicitly estimated based on initial conditions.

Consider a DTS of order n , with sampling time $\mathfrak{t}$ seconds:

$$z(k+1) = A_{\mathfrak{t}}z(k) + \Gamma_{\mathfrak{t}}u(k) + \tilde{\delta}(k) \quad y(k) = Cz(k) \quad (5.1)$$

Here, the state $z(k) \in \mathbb{R}^n$, $\tilde{\delta}(k)$ represents a vector that captures unaccounted dynamics and extrinsic disturbances within the system, $y(k)$ represents the output of the system and $k \in \mathbb{Z}$ is an integer. It is considered that $\tilde{\delta}(k) = 0$ for all $k < 0$. The matrices $A_{\mathfrak{t}}$, $\Gamma_{\mathfrak{t}}$,

and C are properly sized, ensuring compatibility, where $(A_{\mathbf{t}}, \Gamma_{\mathbf{t}})$ exhibits controllability, and $(A_{\mathbf{t}}, C)$ showcases observability.

We propose the sliding function as $\aleph(k) := \alpha(z(k))$, and corresponding sliding surface by the contour set $\alpha^{-1}(0) := \{z \in D : \alpha(z) = 0\}$. For our discussion purpose, switching function is characterized as:

$$\aleph(k) = b^T z(k), \quad (5.2)$$

where $b \in \mathbb{R}^{n \times 1}$ with $b^T \Gamma_{\mathbf{t}} \neq 0$. Consider bounded matched-type perturbation such that $b^T \tilde{\delta}(k) = \delta(k)$ and $\delta_l \leq \delta(k) \leq \delta_u$, where $\delta_l, \delta_u \in \mathbb{R}$ are known a priori bounds. Additionally, the following relationships hold: $\delta_0 = \frac{\delta_l + \delta_u}{2}$ represents the mean value of $\delta(k)$, and $\delta_d = \frac{\delta_u - \delta_l}{2}$ denotes the variation from the mean value of $\delta(k)$.

5.2 State and Output Relation for Multirate Uncertain System

Examine the system outlined in equation (5.1) captured with a sampling period $\mathbf{t}_o = \mathbf{t}/\mathbb{M}$, where $\mathbb{M} \in \mathbb{Z} \geq O$. Consider this system described by the matrices A , Γ , and C . Suppose the system's output is recorded in every $\mathbf{t}_o$ seconds, while the input is applied in every $\mathbf{t}$ seconds. Consider that the disturbance vector in the system described by equation (5.1) influences the dynamics of the $\mathbf{t}_o$ system as follows:

$$\begin{aligned} z(k\mathbf{t} + (i+1)\mathbf{t}_o) &= Az(k\mathbf{t} + i\mathbf{t}_o) + \Gamma u(k\mathbf{t}) + \delta'(k) \\ y(k\mathbf{t} + i\mathbf{t}_o) &= Cz(k\mathbf{t} + i\mathbf{t}_o), \end{aligned} \quad (5.3)$$

where $\delta'(k)$ represents the perturbation in the $\mathbf{t}_o$ system resulting from $\tilde{\delta}(k)$. By applying equations (5.3), the state vector at $i = 0, 1, \dots, \mathbb{M} - 1$ can be expressed in terms of $z(k\mathbf{t})$ as shown below:

$$\begin{aligned} z(k\mathbf{t} + \mathbf{t}_0) &= Az(k\mathbf{t}) + \Gamma u(k\mathbf{t}) + \delta'(k). \\ z(k\mathbf{t} + 2\mathbf{t}_0) &= Az(k\mathbf{t} + \mathbf{t}_0) + \Gamma u(k\mathbf{t}) + \delta'(k). \\ &= A^2 z(k\mathbf{t}) + (A\Gamma + \Gamma)u(k\mathbf{t}) \\ &\quad + A\delta'(k) + \delta'(k). \\ &\vdots \\ z((k+1)\mathbf{t} - \mathbf{t}_0) &= A^{\mathbb{M}-1} z(k\mathbf{t}) + \sum_{i=0}^{\mathbb{M}-2} A^i \Gamma u(k\mathbf{t}) + \sum_{i=0}^{\mathbb{M}-2} A^i \delta'(k). \end{aligned}$$

$$z((k+1)\mathbf{t}) = A^{\mathbb{M}}z(k\mathbf{t}) + \sum_{i=0}^{\mathbb{M}-1} A^i \Gamma u(k\mathbf{t}) + \sum_{i=0}^{\mathbb{M}-1} A^i \delta'(k). \quad (5.4)$$

By harnessing the features of Linear time invariant systems in discrete time and juxtaposing equation (5.4) to equation (5.1), we derive the following:

$$\delta'(k) = \left(\sum_{i=0}^{\mathbb{M}-1} A^i \right)^{-1} \tilde{\delta}(k) \quad (5.5)$$

This leads to the relationship between the system states $z(k)$ and the lifted output y_{k+1} as follows:

$$\begin{aligned} z((k+1)\mathbf{t}) &= A_{\mathbf{t}}z(k\mathbf{t}) + \Gamma_{\mathbf{t}}u(k\mathbf{t}) + \tilde{\delta}(k) \\ y_{k+1} &= C_0z(k) + D_0u(k) + C_{\delta}\tilde{\delta}(k). \end{aligned} \quad (5.6)$$

where,

$$\begin{aligned} y_k &= \begin{bmatrix} y((k-1)\mathbf{t}) \\ y((k-1)\mathbf{t} + \mathbf{t}_o) \\ \vdots \\ y(k\mathbf{t} - \mathbf{t}_o) \end{bmatrix}, C_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\mathbb{M}-1} \end{bmatrix}, \\ D_0 &= \begin{bmatrix} 0 \\ C\Gamma \\ C(A\Gamma + \Gamma) \\ \vdots \\ \sum_{i=0}^{\mathbb{M}-2} CA^i\Gamma \end{bmatrix}, C_{\delta} = \begin{bmatrix} 0 \\ C \\ CA + C \\ \vdots \\ \sum_{i=0}^{\mathbb{M}-2} CA^i \end{bmatrix}. \end{aligned}$$

Remark 5.1 Equation (5.6) represents the relationship between the lifted output and system states using (5.1), (5.4), and (5.5) [36].

Thus, we have the relation:

$$\mathbf{C}_0^T \mathbf{y}_{t+1} = \mathbf{C}_0^T (\mathbf{C}_0 z(k) + \mathbf{D}_0 u(k) + \mathbf{C}_{\delta} \tilde{\delta}(k)). \quad (5.7)$$

Hence, using (5.6), the state $z(k+1)$ can be inferred as:

$$z(k+1) = B_y \mathbf{y}_{k+1} + B_u u(k) + B_{\delta} \tilde{\delta}(k) \quad (5.8)$$

where:

$$\begin{aligned} B_y &= A_{\mathbf{t}} (\mathbf{C}_0^T \mathbf{C}_0)^{-1} \mathbf{C}_0^T, \\ B_u &= \Gamma_{\mathbf{t}} - A_{\mathbf{t}} (\mathbf{C}_0^T \mathbf{C}_0)^{-1} \mathbf{C}_0^T \mathbf{D}_0, \\ B_{\delta} &= I - A_{\mathbf{t}} (\mathbf{C}_0^T \mathbf{C}_0)^{-1} \mathbf{C}_0^T \mathbf{C}_{\delta}. \end{aligned} \quad (5.9)$$

Thus, the state $z(k)$ can be expressed using the lifted output vector $\mathbf{y}_k$ as:

$$z(k) = B_y \mathbf{y}_k + B_u u(k-1) + B_{\delta} \tilde{\delta}(k-1). \quad (5.10)$$

5.3 Multirate Output Control Scheme Based on Modified RL

Consider the system represented by equations (5.6). Let's introduce a new variable, $\eta(k)$, defined as follows

$$\eta(k) = b^T A_{\mathbf{t}} B_{\delta} \tilde{\delta}(k) \quad (5.11)$$

Since the perturbation is bounded, we also have $\eta_l \leq \eta(k) \leq \eta_u$ and $\eta_m = \max(|\eta_l|, |\eta_u|)$. Also define the QSM band as $\omega_m = \delta_m + \eta_m$, here, δ_m as defined as $\delta_l < \delta(k) < \delta_u$ and $\max\{|\delta_l|, |\delta_u|\} = \delta_m$.

Motivated by RL1, we defined modified RL (MRL1) as

$$\aleph(k+1) = \aleph(k) - \varsigma \text{sign}[\aleph(k)] \min \left(\frac{|\aleph(k)|}{\varsigma}, |\aleph(k)|^{\beta} \right) + \delta(k) + \eta(k-1). \quad (5.12)$$

Theorem 5.2 Consider the system described by (5.6) and using MRL (5.12) and

$$u(k) = -(b^T \Gamma_{\mathbf{t}})^{-1} \left[b^T A_{\mathbf{t}} (B_y \mathbf{y}_k + B_u u(k-1) - \aleph(k) + \varsigma \text{sign}[\aleph(k)] \min \left(\frac{|\aleph(k)|}{\varsigma}, |\aleph(k)|^{\beta} \right)) \right] \quad (5.13)$$

using control input (5.13) the system's switching function magnitude is ultimately bounded by ω_m . Also, it enters in ultimate band for some $k \leq I(\aleph(0))$

where $I(\aleph(0)) \leq \left\lceil \log_{[1-\varsigma|\aleph(0)|^{\beta-1} + \omega_m|\aleph(0)|^{-1}]} \frac{\varsigma^{\frac{1}{1-\beta}}}{|\aleph(0)|} \right\rceil + 1$

Proof

Consider the switching function (5.2),

$$\aleph(k+1) = b^T z(k+1),$$

Consider the system described by (5.6) and MRL (5.12).

$$\begin{aligned}
b^T(A_{\mathfrak{t}}z(k) + \Gamma_{\mathfrak{t}}u(k) + \tilde{\delta}(k)) &= \mathfrak{N}(k) - \varsigma \text{sign}[\mathfrak{N}(k)] \min\left(\frac{|\mathfrak{N}(k)|}{\varsigma}, |\mathfrak{N}(k)|^\beta\right) + \delta(k) + \eta(k-1) \\
b^T A_{\mathfrak{t}}z(k) + b^T \Gamma_{\mathfrak{t}}u(k) + b^T \tilde{\delta}(k) &= \mathfrak{N}(k) - \varsigma \text{sign}[\mathfrak{N}(k)] \min\left(\frac{|\mathfrak{N}(k)|}{\varsigma}, |\mathfrak{N}(k)|^\beta\right) + \delta(k) + \eta(k-1)
\end{aligned} \tag{5.14}$$

Using equation (5.8)

$$\begin{aligned}
b^T A_{\mathfrak{t}}(B_y \mathbf{y}_k + B_u u(k-1) + B_d \tilde{\delta}(k-1)) + b^T \Gamma_{\mathfrak{t}}u(k) &= \mathfrak{N}(k) - \varsigma \text{sign}[\mathfrak{N}(k)] \min\left(\frac{|\mathfrak{N}(k)|}{\varsigma}, |\mathfrak{N}(k)|^\beta\right) \\
&+ \eta(k-1) \\
b^T A_{\mathfrak{t}}(B_y \mathbf{y}_k + B_u u(k-1)) + b^T \Gamma_{\mathfrak{t}}u(k) &= \mathfrak{N}(k) - \varsigma \text{sign}[\mathfrak{N}(k)] \min\left(\frac{|\mathfrak{N}(k)|}{\varsigma}, |\mathfrak{N}(k)|^\beta\right)
\end{aligned}$$

At this point, the control input can be determined as follows:

$$u(k) = -(b^T \Gamma_{\mathfrak{t}})^{-1} \left[b^T A_{\mathfrak{t}}(B_y \mathbf{y}_k + B_u u(k-1)) - \mathfrak{N}(k) + \varsigma \text{sign}[\mathfrak{N}(k)] \min\left(\frac{|\mathfrak{N}(k)|}{\varsigma}, |\mathfrak{N}(k)|^\beta\right) \right] \tag{5.15}$$

Thus, the control input can be determined utilizing previous output samples along with the most recent input signal. However, at $k = 0$, no preceding outputs are available for control purposes. Therefore, referring to equation (5.15), and assuming the system is unaffected by disturbances before $k = 0$, and with initial state $z(0)$ and the corresponding $\mathfrak{N}(0)$ are considered.

$$u(0) = -(b^T \Gamma_{\mathfrak{t}})^{-1} \left[b^T A_{\mathfrak{t}}(z(0)) - \mathfrak{N}(0) + \varsigma \text{sign}[\mathfrak{N}(0)] \min\left(\frac{|\mathfrak{N}(0)|}{\varsigma}, |\mathfrak{N}(0)|^\beta\right) \right] \tag{5.16}$$

When the control input obtained from equation (5.15) is implemented within the system (5.6), the system adheres to the MRL1:

$$\mathfrak{N}(k+1) = \mathfrak{N}(k) - \varsigma \text{sign}[\mathfrak{N}(k)] \min\left(\frac{|\mathfrak{N}(k)|}{\varsigma}, |\mathfrak{N}(k)|^\beta\right) + \delta(k) + \eta(k-1).$$

Considering the trivial scenario, if $\frac{|\mathfrak{N}(k)|}{\varsigma} \leq |\mathfrak{N}(k)|^\beta$, then we will have in next step

$$\begin{aligned}
\mathfrak{N}(k+1) &= \mathfrak{N}(k) - \varsigma \text{sign}[\mathfrak{N}(k)] |\mathfrak{N}(k)| + \delta(k) + \eta(k-1) \\
\Rightarrow \mathfrak{N}(k+1) &= \delta(k) + \eta(k-1)
\end{aligned}$$

Therefore, we derive

$|\aleph(k+1)| \leq \omega_m$, , with $I(\aleph(0)) = 1$. Moreover, for all $k \geq I(\aleph(0))$, $|\aleph(k)| \leq \omega_m$, demonstrating the presence of QSM. The next scenario, if $\frac{|\aleph(k)|}{\varsigma} > |\aleph(k)|^\beta$, then

$$\aleph(k+1) = \aleph(k) - \varsigma \text{sign}[\aleph(k)] |\aleph(k)|^\beta + \delta(k) + \eta(k-1).$$

We can further write,

$$|\aleph(k+1)| = |\aleph(k) - \varsigma \text{sign}[\aleph(k)] |\aleph(k)|^\beta + \delta(k) + \eta(k-1)|.$$

Since $|\aleph(k)| > |\aleph(k)|^\beta$, it follows that,

$$|\aleph(k+1)| \leq |\aleph(k)| - \varsigma |\aleph(k)|^\beta + \omega_m.$$

Additionally, it can be expressed as

$$\begin{aligned} |\aleph(k)| &\leq |\aleph(k-1)|(1 - \varsigma |\aleph(k-1)|^{\beta-1} + \omega_m |\aleph(k-1)|^{-1}) \\ &\vdots \\ &\leq |\aleph(0)|(1 - \varsigma |\aleph(0)|^{\beta-1} + \omega_m |\aleph(0)|^{-1}) \dots \\ &\quad (1 - \varsigma |\aleph(k-1)|^{\beta-1} + \omega_m |\aleph(k-1)|^{-1}) \\ &\vdots \\ |\aleph(k)| &\leq |\aleph(0)|(1 - \varsigma |\aleph(0)|^{\beta-1} + \omega_m |\aleph(0)|^{-1})^k \end{aligned} \tag{5.17}$$

If $||\aleph(0)|(1 - \varsigma |\aleph(0)|^{\beta-1} + \omega_m |\aleph(0)|^{-1})^k| \leq \varsigma |\aleph(k)|^\beta$, then from (5.18), $|\aleph(k)| \leq \varsigma |\aleph(k)|^\beta \Rightarrow |\aleph(k+1)| \leq \omega_m, \forall k \geq \left\lceil \log_{[1 - \varsigma |\aleph(0)|^{\beta-1} + \omega_m |\aleph(0)|^{-1}]} \frac{\varsigma |\aleph(0)|^{\frac{1}{1-\beta}}}{|\aleph(0)|} \right\rceil + 1$. Moreover, the settling time expression is given by

$$I(\aleph(0)) \leq \left\lceil \log_{[1 - \varsigma |\aleph(0)|^{\beta-1} + \omega_m |\aleph(0)|^{-1}]} \frac{\varsigma |\aleph(0)|^{\frac{1}{1-\beta}}}{|\aleph(0)|} \right\rceil + 1$$

Therefore, it is concluded that the modulus of the switching function has an ultimate bound ω_m , which is achieved in a finite number of time steps.

Remark 5.3 *The computation of the control input (5.15) relies on past output samples and the most recent input signal. Additionally, the use of the lifted output signal effectively eliminates the dependence on state information, as outlined in (5.10).*

Corollary 5.4 *The DTS (5.1) satisfies the QSM reaching condition in the neighborhood of the set $\alpha^{-1}(0)$ if, for some $k \geq 0$, the following conditions hold:*

$$\begin{aligned} \frac{|\aleph(k)|}{\varsigma} > |\aleph(k)|^\beta &\Rightarrow |\aleph(k+1)| < |\aleph(k)| - \varepsilon \\ \frac{|\aleph(k)|}{\varsigma} \leq |\aleph(k)|^\beta &\Rightarrow 0 \leq |\aleph(k+1)| \leq \omega_m \end{aligned}$$

where $\varepsilon \in \mathbb{R}^+$ is arbitrarily small.

In the following section, we will focus on the development of the controller and conduct a detailed analysis for the special case of the reaching law RL1, which was previously defined as RL2. We have proposed a new modified reaching law, MRL2, based on RL2 by integrating its features with the MROF. Furthermore, we have outlined the conditions under which the results of MRL2 converge with those of MRL1. Consider the MRL2 defined as

$$\aleph(k+1) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma) + \delta(k) + \eta(k-1) \quad (5.18)$$

Here $\varsigma \in \mathbb{R}^+$ is selected s.t. $\varsigma \geq \omega_m$.

Theorem 5.5 *Consider the system described by (5.6) and using MRL (5.18), and*

$$u(k) = -(b^T \Gamma_t)^{-1} \left[b^T A_t (B_y \mathbf{y}_k + B_u u(k-1)) - \aleph(k) + \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma) \right] \quad (5.19)$$

When applying the control input given in equation (5.19), the switching function of the system is ultimately bounded by $\omega_m = \delta_m + \eta_m$. Also, it enters in ultimate band for some $k \leq I(\aleph(0))$ where

$$I(\aleph(0)) \leq \left\lceil \frac{|\aleph(0)| - \omega_m}{\varsigma - \omega_m} \right\rceil.$$

Proof

Using switching function (5.2),

$$\aleph(k+1) = b^T z(k+1),$$

Consider the system described by (5.6) and MRL2 (5.18).

$$b^T(A_{\mathfrak{t}}z(k) + \Gamma_{\mathfrak{t}}u(k) + \tilde{\delta}(k)) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma) + \delta(k) + \eta(k-1)$$

$$b^T A_{\mathfrak{t}}z(k) + b^T \Gamma_{\mathfrak{t}}u(k) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma) + \eta(k-1)$$

Using equation (5.8)

$$b^T A_{\mathfrak{t}}(B_y \mathbf{y}_k + B_u u(k-1) + L_d \tilde{\delta}(k-1)) + b^T \Gamma_{\mathfrak{t}}u(k) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma) + \eta(k-1)$$

$$b^T A_{\mathfrak{t}}(B_y \mathbf{y}_k + B_u u(k-1)) + b^T \Gamma_{\mathfrak{t}}u(k) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma)$$

At this point, the control input can be determined as follows:

$$u(k) = -(b^T \Gamma_{\mathfrak{t}})^{-1} \left[b^T A_{\mathfrak{t}}(B_y \mathbf{y}_k + B_u u(k-1)) - \aleph(k) + \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma) \right] \quad (5.20)$$

Thus, the control input can be determined utilizing previous output samples along with the most recent input signal. However, at $k = 0$, no preceding outputs are available for control purposes. Consequently, using equation (5.18) and based on the assumption that no disturbances before $k = 0$ influence the system, and by assuming an initial state $z(0)$ and corresponding $\aleph(0)$ we have,

$$u(0) = -(b^T \Gamma_{\mathfrak{t}})^{-1} \left[b^T A_{\mathfrak{t}}(z(0)) - \aleph(0) + \text{sign}[\aleph(0)] \min(|\aleph(0)|, \varsigma) \right]$$

Now, when the control input obtained from equation (5.20) is implemented within the system (5.6), the system adheres to the MRL2.

$$\aleph(k+1) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma) + \delta(k) + \eta(k-1)$$

Consider the scenario where $|\aleph(k)| \leq \varsigma$, then $\aleph(k+1) = \delta(k) + \eta(k-1)$ implies in the next step $|\aleph(k+1)| \leq \delta(k) + \eta(k-2)$. Consequently, $|\aleph(k+1)| \leq \omega_m$, with $I(\aleph(0)) = 1$. Moreover, for all $k \geq I(\aleph(0))$, $|\aleph(k)| \leq \omega_m$, demonstrating the existence of QSM.

For another scenario, if $|\aleph(k)| > \varsigma$, then $|\aleph(k+1)| \leq |\aleph(k) - \text{sign}[\aleph(k)]\varsigma| + \delta(k) + \eta(k-1)$. As $|\aleph(k)| > \varsigma$, it follows that $|\aleph(k+1)| \leq |\aleph(k)| - \varsigma + \omega_m$. Additionally, it can

be deduced as:

$$\begin{aligned}
|\aleph(k)| &\leq |\aleph(k-1)| - \varsigma + \omega_m \\
&\leq |\aleph(k-2)| - 2(\varsigma - \omega_m) \\
&\leq \\
&\vdots \\
&\leq |\aleph(0)| - k(\varsigma - \omega_m)
\end{aligned}$$

If $(|\aleph(0)| - k(\varsigma - \omega_m)) \leq \varsigma$, then $|\aleph(k)| \leq \varsigma$, implying $|\aleph(k+1)| \leq \omega_m$ for all $k \geq \frac{|\aleph(0)| - \omega_m}{\varsigma - \omega_m}$. The settling time function is $I(\aleph(0)) = \left\lceil \frac{|\aleph(0)| - \omega_m}{\varsigma - \omega_m} \right\rceil$. Therefore, it is concluded that $|\aleph(k)| \leq \omega_m$ within finite time steps.

Corollary 5.6 *The QSM in the vicinity of the set $\alpha^{-1}(0)$ when $|\aleph(k)| \leq \omega_m = \delta_m + \eta_m$.*

Corollary 5.7 *The system described by equation (5.1) satisfies the QSM reaching condition in the vicinity of $\alpha^{-1}(0)$ if, for some $k \geq 0$, the following conditions are met:*

$$\begin{aligned}
|\aleph(k)| > \varsigma &\Rightarrow |\aleph(k+1)| < |\aleph(k)| - \varepsilon; \\
|\aleph(k)| \leq \varsigma &\Rightarrow 0 \leq |\aleph(k+1)| \leq \omega_m,
\end{aligned} \tag{5.21}$$

where $\varepsilon \in \mathbb{R}^+$ is arbitrarily small.

Remark 5.8 *In control law (5.15), setting ς to 1 and keeping β very close to zero results in behavior similar to control law (5.20) with $\varsigma = 1$. The interaction between ς and β influences the convergence of the switching function. Smaller values of ς and β lead to slower convergence, while keeping β close to 1 and ς large ensures rapid convergence of the switching function. Furthermore, the selection of ς and β for various applications can be tailored based on the actuator saturation limits of the respective systems.*

Remark 5.9 *Corollary 1 aligns with the ideal sliding mode definition for continuous-time systems when $\delta(k) = 0$, as $\aleph(k)$ converges exactly to $\alpha^{-1}(0)$ and remains there indefinitely.*

Remark 5.10 *The control input (5.20) can be computed using past output samples in conjunction with the most recent input signal. Moreover, by utilizing the lifted output signal, we effectively mitigate the need for state information as described in (5.10).*

5.4 Result and Discussion

The effectiveness of the proposed control strategy is evaluated by analyzing its performance in a given example. The obtained results are then bench-marked against two existing approaches: the state feedback-based method outlined in [2] and the MROF methodology detailed in [3]. This comparative assessment provides insights into the relative advantages and potential improvements offered by the proposed strategy.

Example 5.11 [56] *Consider the DTS of smart-structure beams:*

$$z(k+1) = \begin{pmatrix} 0 & 1 \\ 0.4 & -0.3 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) + \begin{pmatrix} 0 \\ \sin\left(\frac{k}{2}\right) \exp\left(-\frac{k}{5}\right) \end{pmatrix};$$

$$y(k) = [1 \ 0]z(k).$$

5.4.1 Comparative Study with State Feedback

In [2], the authors presented two RL-based DSMC methods, as mentioned in Section II. For the sake of comparison, we have used the state feedback control law derived from these algorithms to compare with the proposed fast output sampling sliding mode control laws.

State Feedback Control Based on RL1

A control signal was created using RL1, and the resulting state feedback control follows this structure:

$$u_1(k) = (b^T \Gamma_t)^{-1} \left[-b^T A_t z(k) + \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \varsigma) \right] \quad (5.22)$$

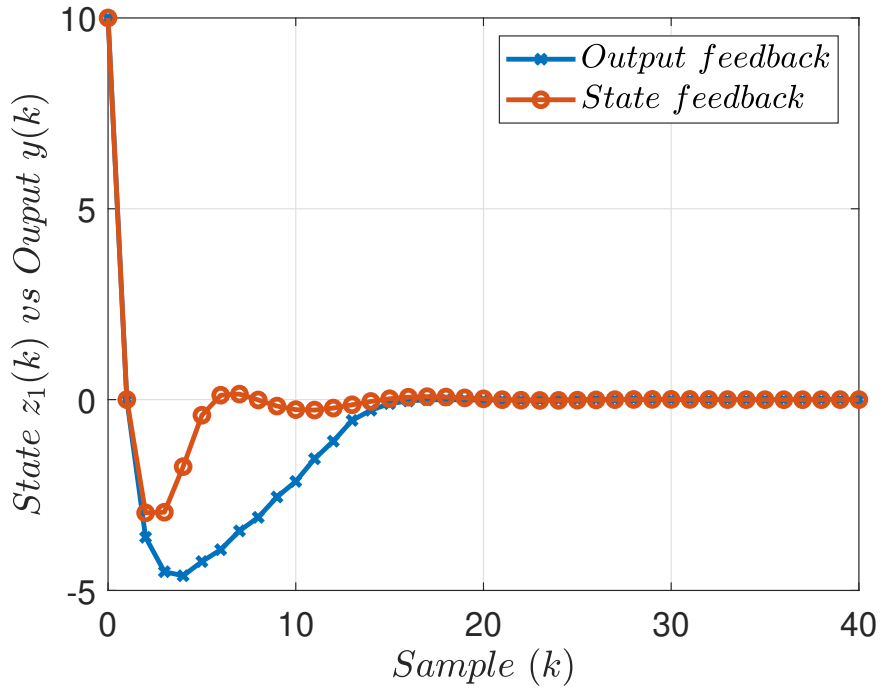
The system input is sampled at $T = 0.2$ sec, and the resulting state feedback-based control utilizes a sliding function defined as $\aleph(k) = b^T z(k)$, where $b^T = [-0.4 \ 1]$. Additionally, the control gain $\varsigma = 0.8$ is used.

State Feedback Control Based on RL2

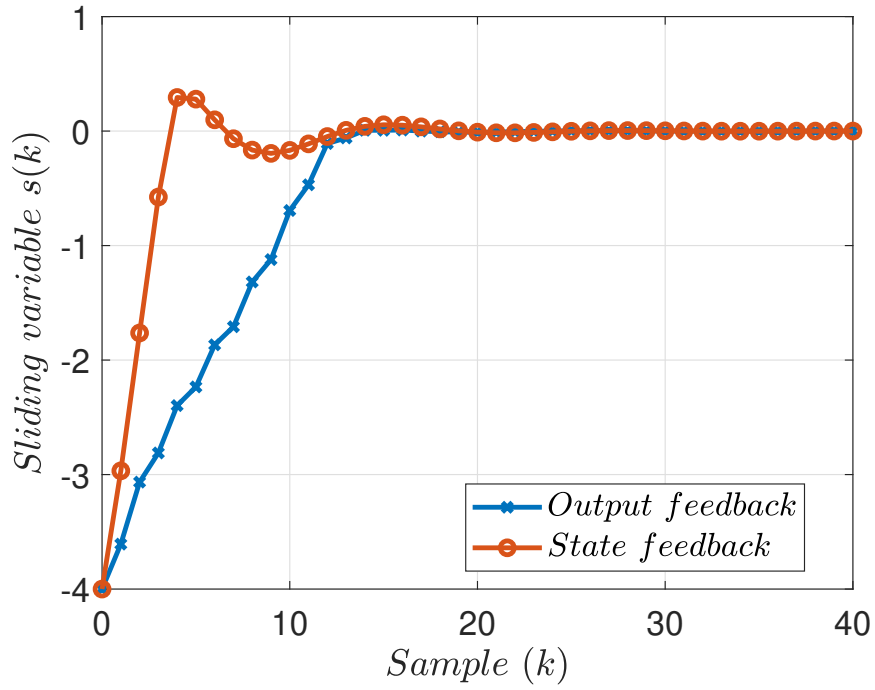
Additionally, another control signal was crafted using RL2, and the resulting state feedback control law follows this structure:

$$u_2(k) = (b^T \Gamma_t)^{-1} \left[-b^T A_t z(k) + \aleph(k) - \varsigma \text{sign}[\aleph(k)] \min\left(\frac{|\aleph(k)|}{\varsigma}, |\aleph(k)|^\beta\right) \right] \quad (5.23)$$

The sampling interval and sliding function are the same as those used in the state feedback control $u_1(k)$, while other parameters are set to $\varsigma = 0.8$ and $\beta = 0.1$.

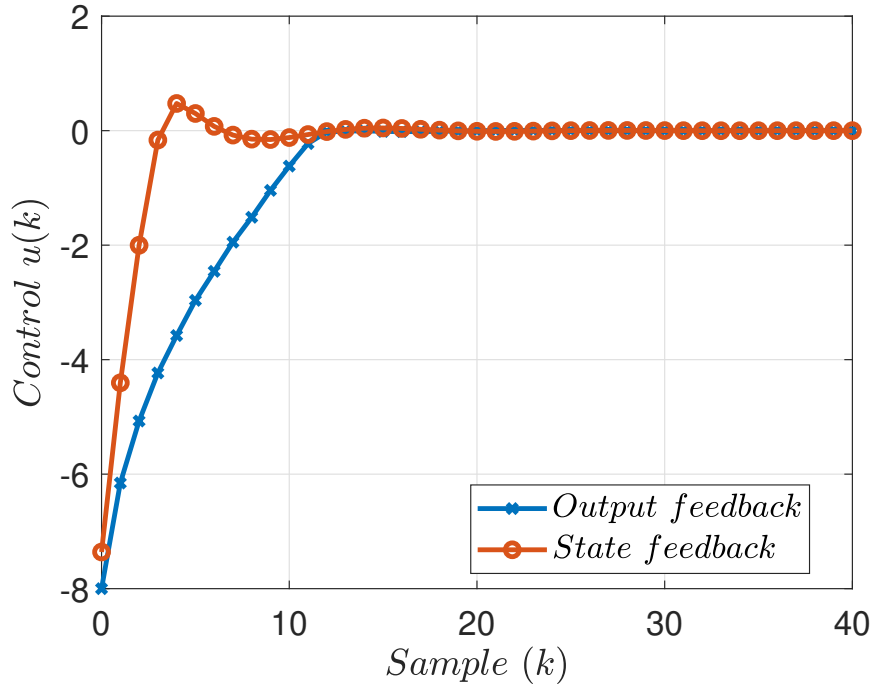


(a) MRL1 simulation result vs RL 1 [2]



(b) MRL1 vs RL1 sliding variable [2]

Figure 5.1: MRL1 simulation result vs RL1 [2]



(a) Control input $u(k)$: MRL1 vs RL1 [2]

Figure 5.2: Comparison of control performance for MRL1 vs RL1 [2].

Proposed Multirate Output Feedback Based Control Law MRL1 and MRL2

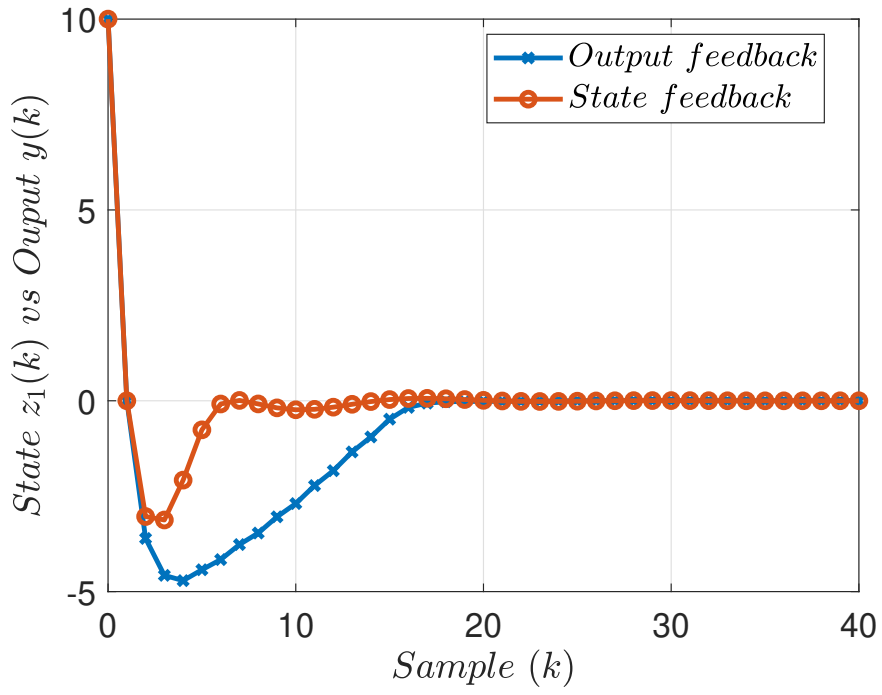
The developed control laws use sliding line, same as above for the sake of comparison.

Computation of the parameters:

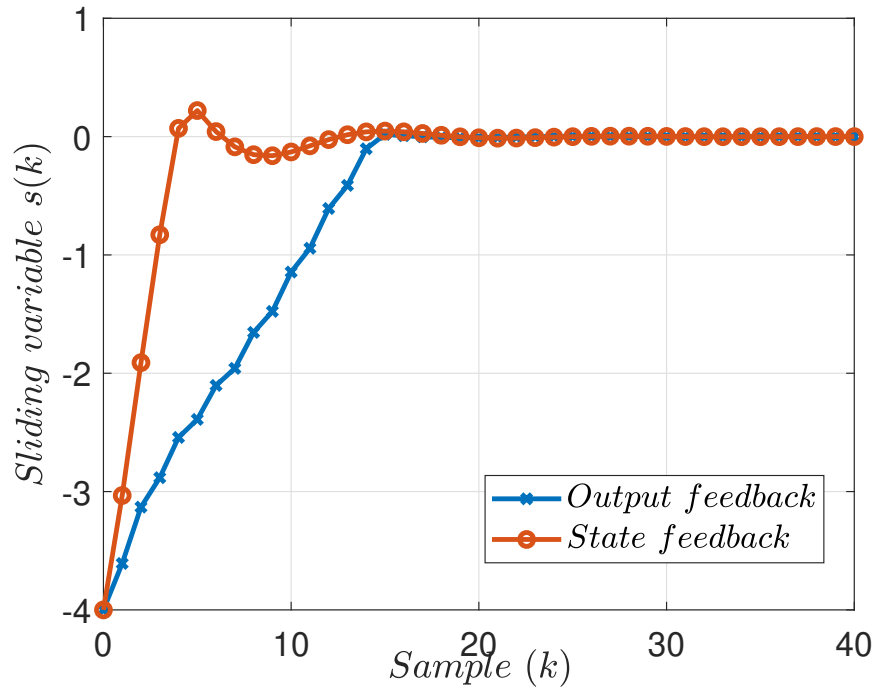
$$B_y = \begin{pmatrix} 5.407 & -6.94 \\ -1.222 & 2.082 \end{pmatrix}, \quad B_u = \begin{pmatrix} 2.814 \\ 0.156 \end{pmatrix}, \quad B_d = \begin{pmatrix} 7.94 & 0 \\ -2.082 & 1 \end{pmatrix}$$

The system output is sampled at a frequency twice that of the input sampling rate. The sliding surfaces are designed such that one of them coincides with the sliding surface used in the state feedback approach, ensuring a meaningful basis for comparison. For consistency, the parameters ς and β are set identical to those in the state feedback control law.

Fig. 5.1a, Fig. 5.1b and Fig. 5.2a presents the simulation results for a time-varying disturbance applied to MRL1, depicting the evolution of the sliding function $\aleph(k)$, output $y(k)$, and control input $u(k)$. This figure facilitates a comparative analysis between the proposed MRL1-based controller and the state feedback sliding mode control method outlined in [2]. The results show that the state trajectory progressively converges toward

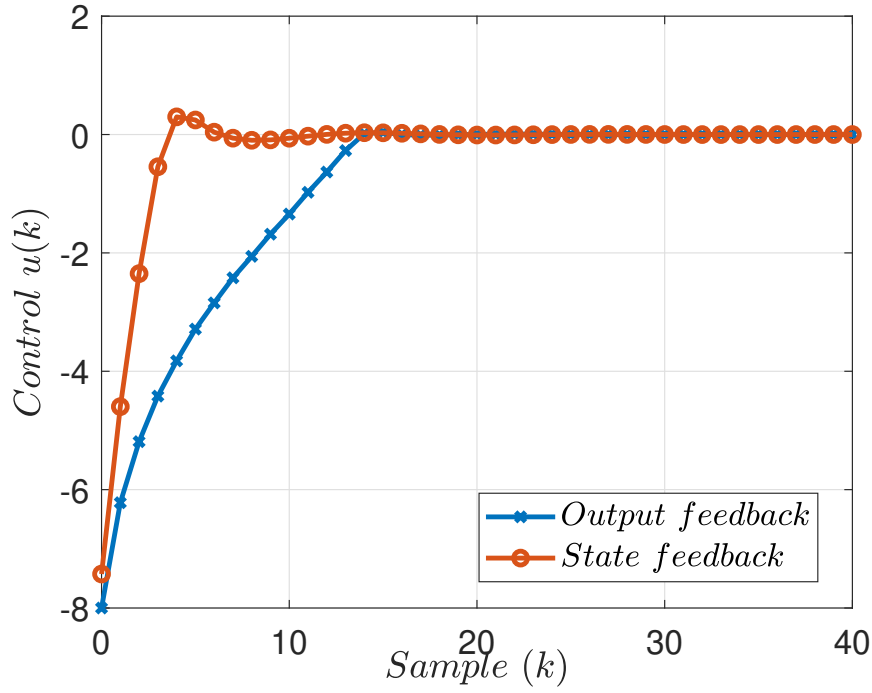


(a) MRL2 simulation result vs RL 2 [2]



(b) MRL2 vs RL2 sliding variable [2]

Figure 5.3: MRL1 simulation result vs RL1 [2]



(a) Control input $u(k)$: MRL2 vs RL2 [2]

Figure 5.4: Comparison of control performance for MRL2 vs RL2 [2].

the quasi-sliding mode band.

Similarly, Fig. 5.3a, Fig. 5.3b and Fig. 5.4a illustrates the behavior of the sliding function $\mathfrak{N}(k)$, output $y(k)$, and control input $u(k)$ for MRL2, offering a comparative assessment with the state feedback sliding mode control approach described in [2]. The findings indicate that while the proposed controller exhibits a longer settling time than the state feedback sliding mode control—primarily due to the absence of full state information—it significantly reduces the quasi-sliding mode band.

5.4.2 Comparative Study with MROF

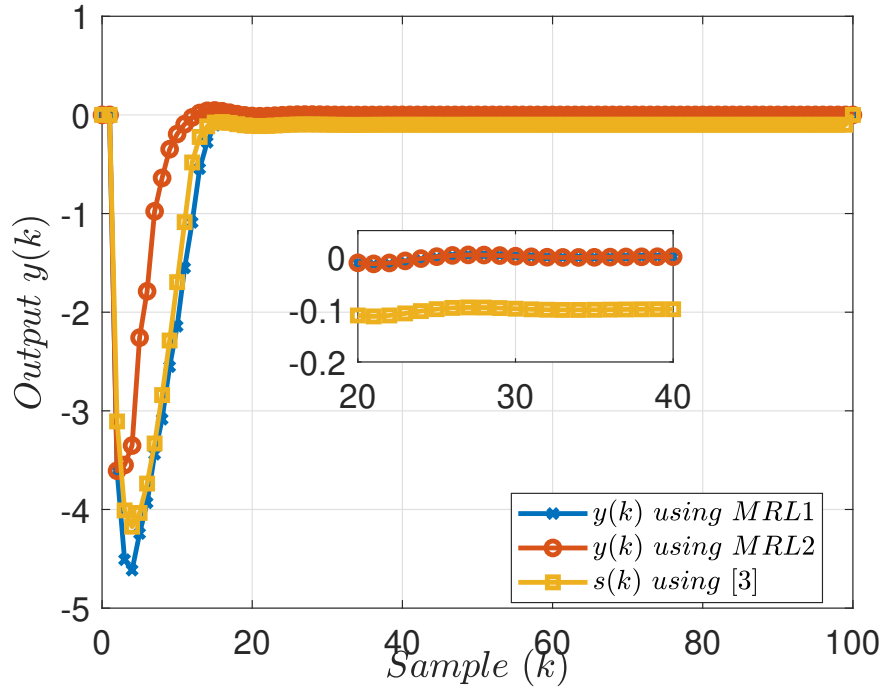
In [3], the authors introduced MROF-based DSMC methods. To further validate these results, we compare the output feedback control law derived from these algorithms with the two proposed fast output sampling sliding mode control laws. The MROF output control law used in [3] is given by

$$u(k) = \begin{bmatrix} -3.019 & 4.233 \end{bmatrix} y(k) - 1.017u(k-1) + s_d(k+1) - 0.06.$$

Here, $s_d(k)$ is a priori known; for further details, readers may refer to [3]. Fig. 5.5a, Fig. 5.6a and Fig. 5.7a & table 1 presents a comparison between MRL1 and MRL2 with the results in [3] for $k_1 = 10$, demonstrating that both proposed algorithms exhibit nearly identical settling times. This comparison clearly highlights the reduced quasi-sliding mode band.

Table 5.1

Control Methods	Control input steady state	Settling time	Sliding variable
MRL1	0.001	11	0.0009
MRL2	0.001	6	0.0009
[28]	-0.1	10	-0.08

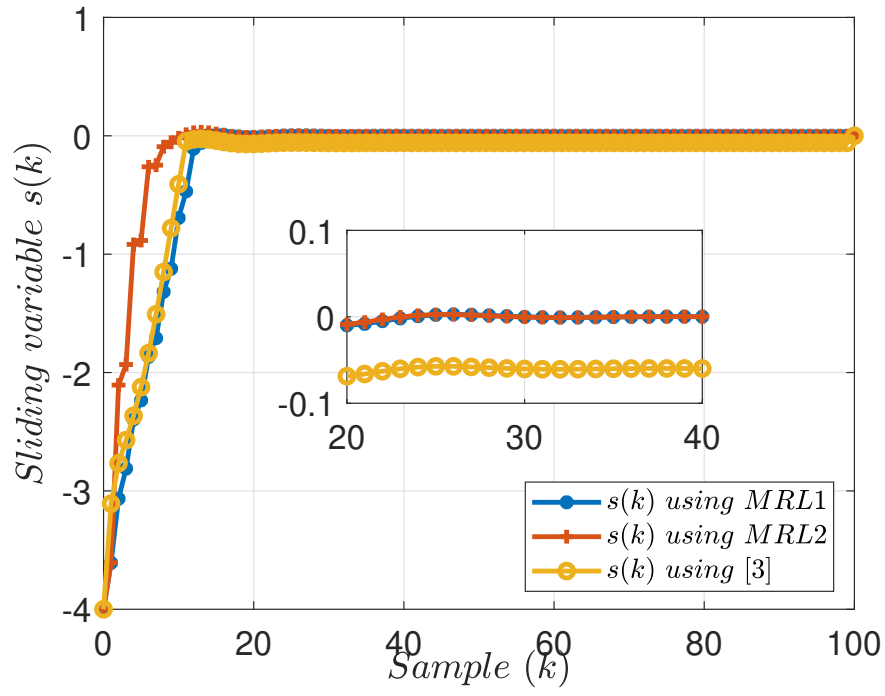


(a) Output comparison

Figure 5.5: Comparative Study of MRL1 , MRL2 and [3]

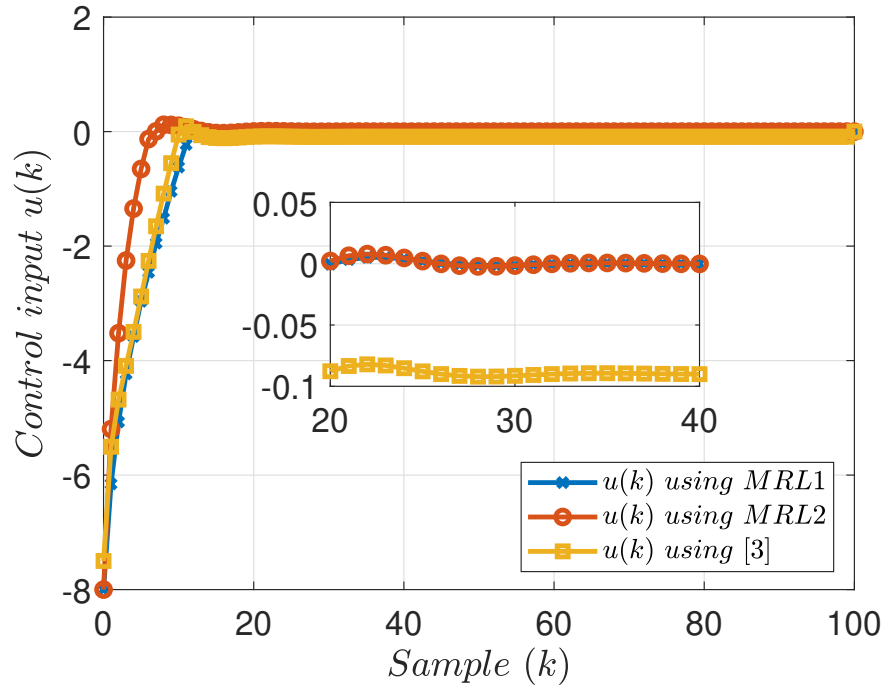
5.4.3 Experimental Validation

The coupled tank system is used as a test bench. The schematic in Fig. 5.8 illustrates the construction of the system, and Fig. 5.9 shows the actual test bench,



(a) Sliding variable comparison

Figure 5.6: Comparative Study of MRL1 , MRL2 and [3]



(a) Control input comparison

Figure 5.7: Comparative Study of MRL1 , MRL2 and [3]

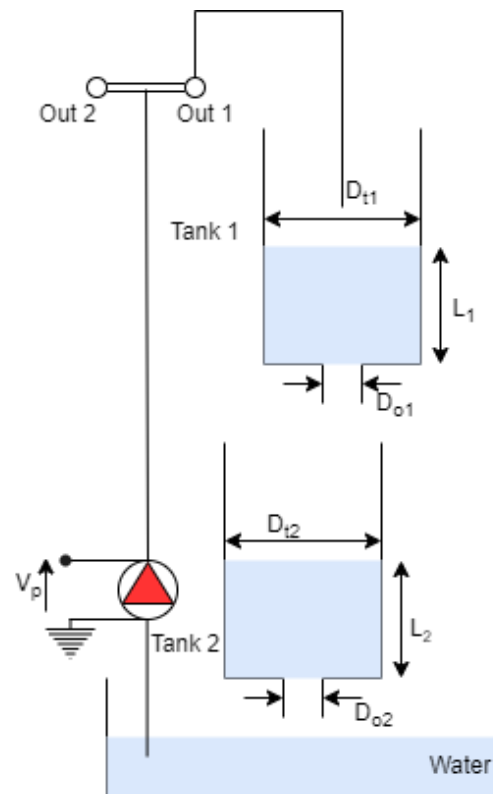


Figure 5.8: Coupled tank block diagram,

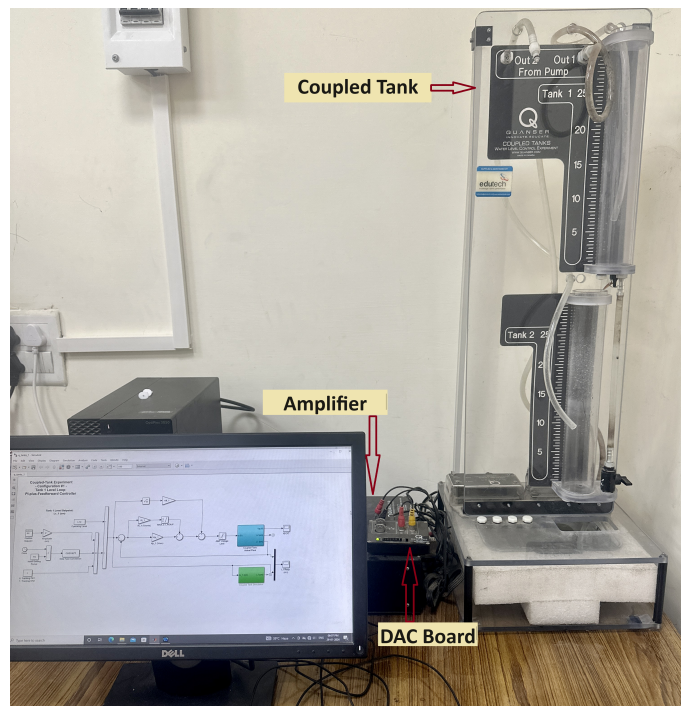


Figure 5.9: Experimental setup coupled tank

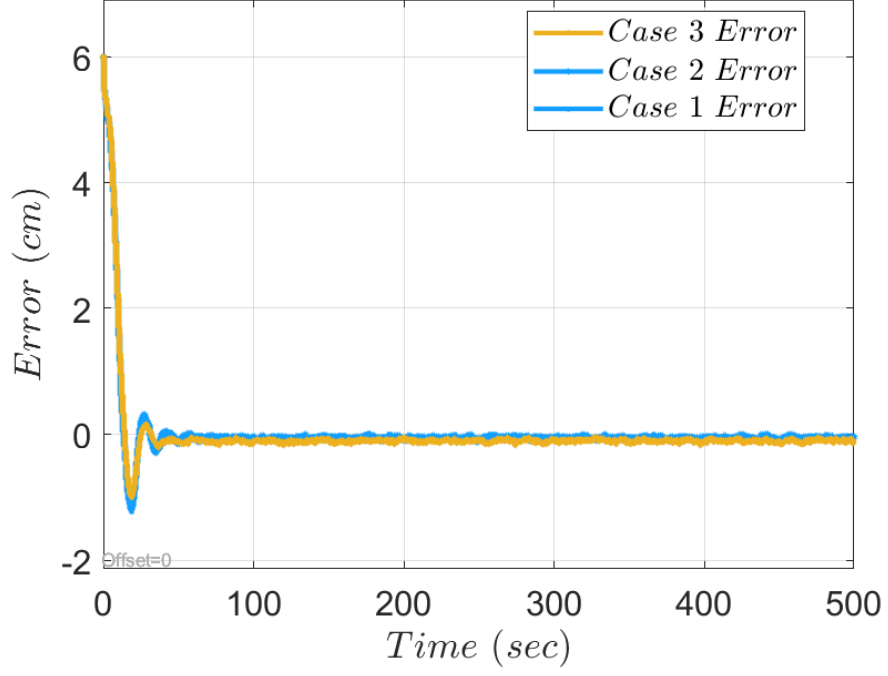


Figure 5.10: Overview of the experimental results of MRL1 scheme for constant hight tracking: Tracking error

Table 5.2: Control parameter for MRL1

Control Parameter	MRL1
β	0.1
t_o	0.1
ς	1.1

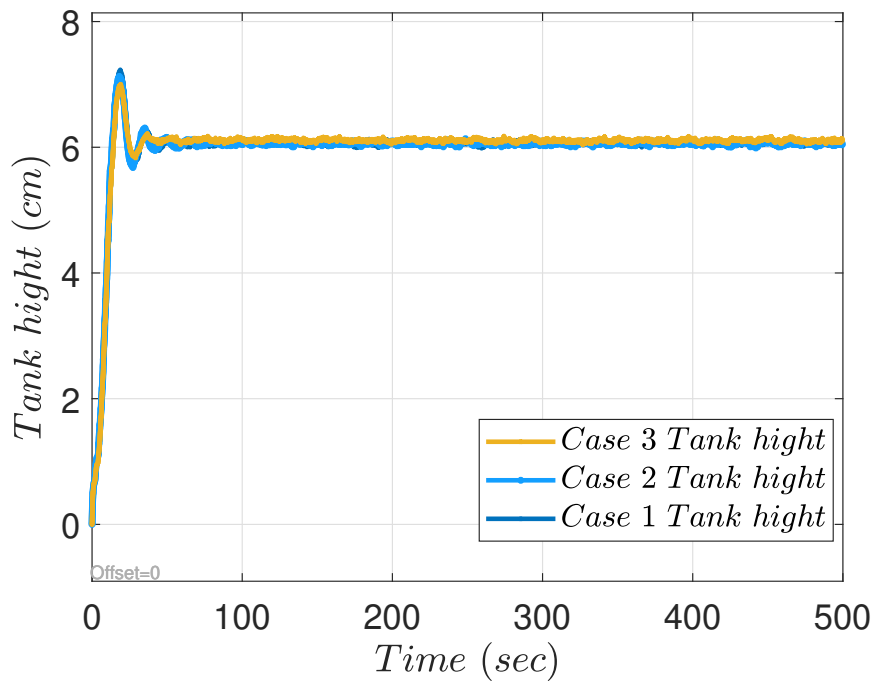


Figure 5.11: Overview of the experimental results of MRL1 scheme for constant height tracking: Output (tank 2 height)

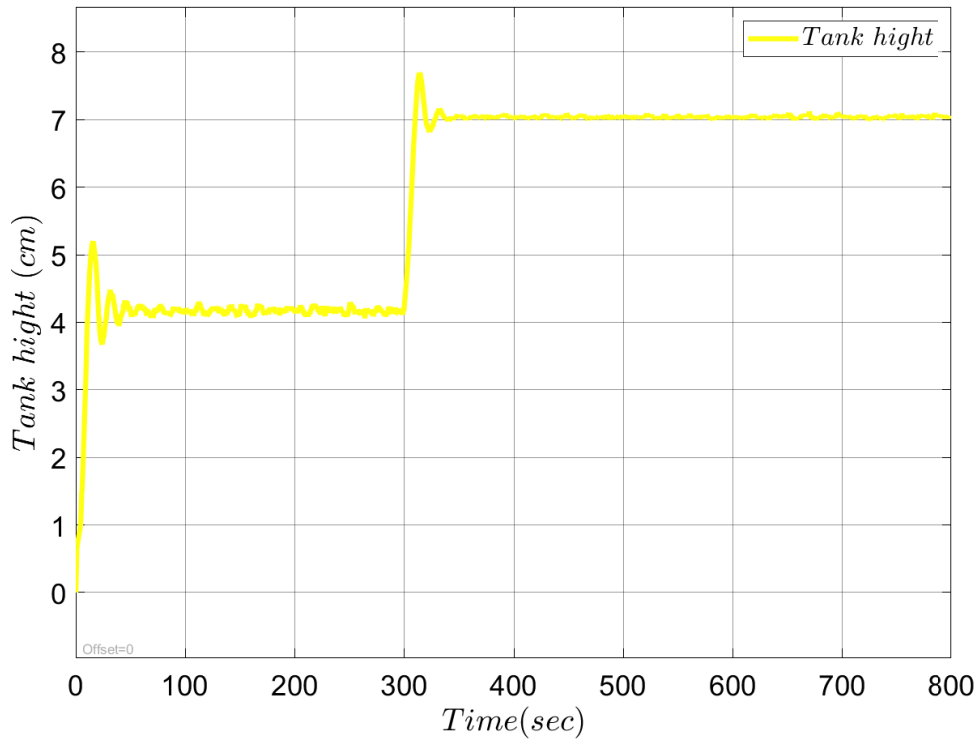


Figure 5.12: Overview of the experimental results of MRL2 scheme for step signal tracking: Tracking error

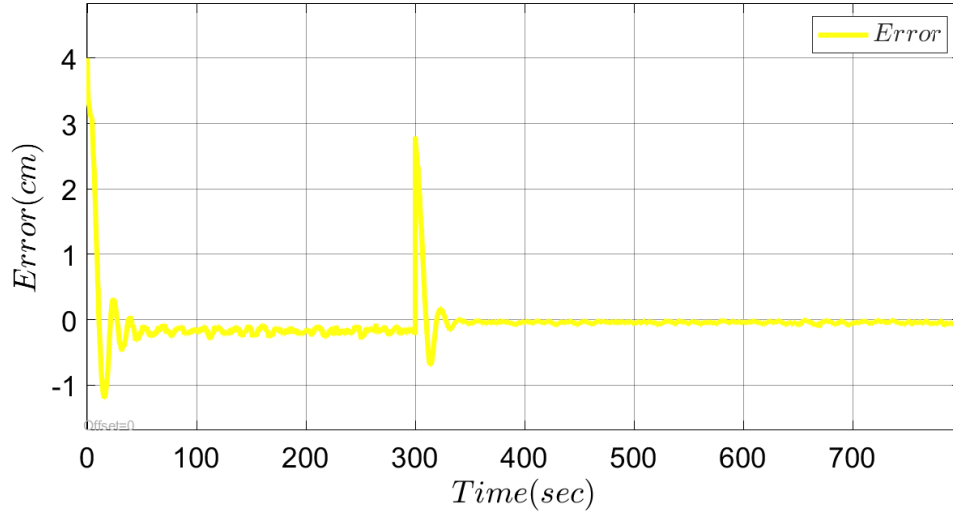


Figure 5.13: Overview of the experimental results of MRL2 scheme for step signal tracking: Output (tank 2 height)

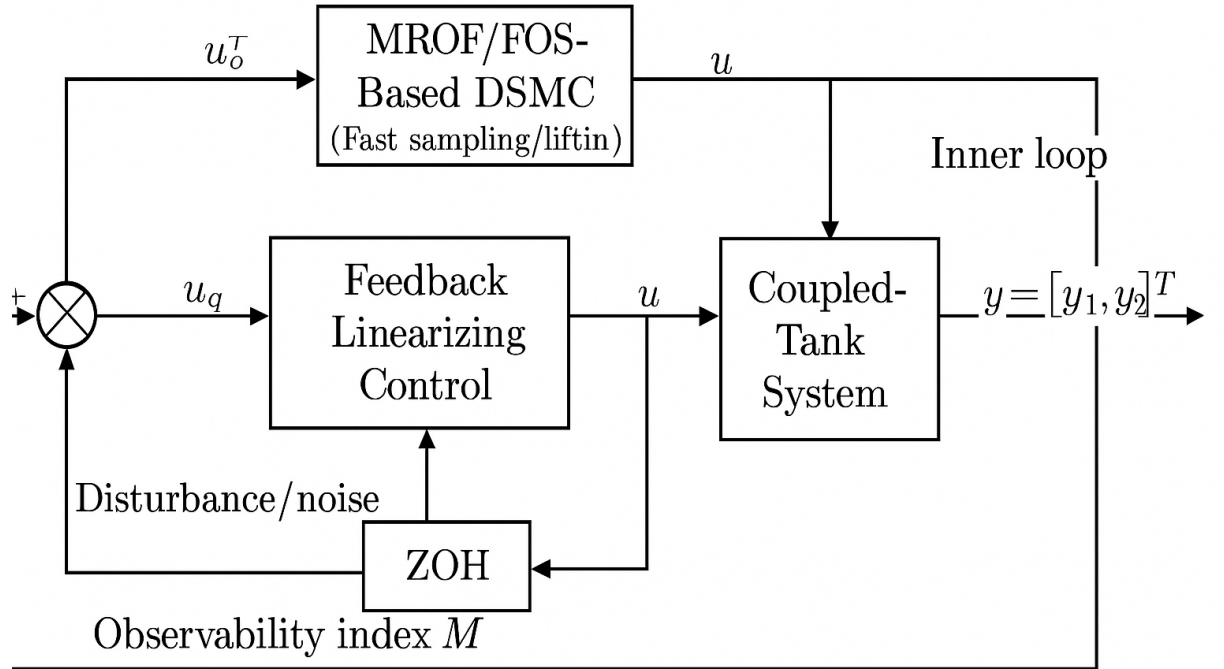


Figure 5.14: Control block diagram for experimental setup

where the control objective is to maintain the desired water level in tank 2 using framework as described in Fig. 5.14. Configuration 2 of the coupled tank system [53] is chosen for the experiments. In this configuration, the control input is applied to the inlet of tank 1, and the outflow of tank 1 acts as a control for tank 2. This setup introduces a coupling effect, where the levels in both tanks are interdependent, adding complexity to the control process. The liquid level is measured by a pressure sensor at the bottom of each tank, providing a linear voltage output that increases with pressure. This signal is processed through a conditioning board and converted to a 0 to 5V DC output. The control parameters used for the experimental setup are listed in Table 5.2. For the system parameters reader may refer to [53].

Fig. 5.10 and Fig. 5.11 present the experimental validation of the proposed MRL1 scheme for constant-height tracking, in which the reference levels are achieved within a finite time. In this experiment, the desired height is maintained at 6 cm. Fig. 5.10 illustrates the system error, whereas Fig. 5.11 depicts the desired signal tracking performance for three different cases, each corresponding to a distinct past output sampling conditions:

- Case 1: $M = 2$,
- Case 2: $M = 3$,
- Case 3: $M = 4$.

It is observed that the tracking dynamics vary with the past output sampling conditions, demonstrating the adaptability and robustness of the proposed scheme.

Furthermore, Fig. 5.12 and Fig. 5.13 illustrate the tracking performance for a step reference signal $y_{r,t}$ defined as

$$y_{r,t} = \begin{cases} 4, & t < 300, \\ 7, & t \geq 300. \end{cases}$$

The results confirm that the proposed method ensures smooth and finite-time convergence for both step levels, thereby validating its effectiveness in handling abrupt reference changes.

5.5 Conclusions

A novel control methodology for discrete-time sliding mode control (DSMC) is proposed, integrating a minimum-operator-based reduced-learning sequence (RLs) approach with a multi-rate output feedback (MROF) technique. This hybrid control framework eliminates the need for a switching control input, thereby mitigating the undesirable chattering effects that are typically observed in conventional DSMCs. In contrast to traditional DSMC schemes that depend on full-state feedback, the proposed method utilizes only past output measurements and previous control inputs. This characteristic enhances its suitability for practical applications, where full-state information is often unavailable or challenging to obtain. The incorporation of the multi-rate output feedback mechanism further improves control efficacy and facilitates practical implementation, resulting in enhanced system response characteristics. Comparative analysis with an existing MROF-based output feedback strategy demonstrates that the proposed MRLs approach yields a significant reduction in the quasi-sliding mode band compared with the control algorithm presented in [3], leading to improved precision and robustness. To assess the effectiveness of the proposed scheme, numerical simulations were conducted and benchmarked against the methods reported in [3] and [2]. These simulations evaluate the system behavior under the specified MRLs, showing that the proposed method achieves superior performance in terms of settling time, steady-state error, and robustness against disturbances. Experimental validation was carried out on a coupled-tank system by implementing the developed MRL1 scheme for three different observability indices. The results clearly reveal the influence of the past output sampling conditions on the tracking error. Furthermore, experiments were performed for two distinct scenarios—step reference tracking and constant-height tracking—which further reinforce the proposed method’s effectiveness.

Future work will focus on extending the methodology to systems with more pronounced nonlinearities, higher-dimensional dynamics and lightly excited systems.

Chapter 6

Adaptive Switching Scheme Design Using Ceiling Function for Discrete-Time Sliding Mode Control

6.1 Introduction

The Chattering and reaching time, the main issues associated with its real-world application, have been actively researched in DSMC. To reduce chattering, extensive studies on switching functions and boundary layers have been conducted by researchers [35]. Reducing reaching time is also considered a challenging problem, as numerous applications require strict time constraints, as in the case of missile applications [43].

Most DSMC-based RLs are designed based on the switching function, leading to the undesirable chattering phenomenon. On the other hand, the DSMC-based equivalent control strategy lacks an approaching process, which results in an inevitably considerable control effort. Both methods result in the sliding mode state being unable to remain on the sliding surface to reach the origin. Consequently, a quasi-sliding-mode domain is formed, and chattering occurs around the origin. To address these problems, various DSMC strategies have been proposed by researchers worldwide [47, 57, 58]. Also, in [44], a modified RL was proposed by the authors, which achieves system state convergence, and the time steps for the switching function to reach the QSMD and the range of the sliding manifold for the closed-loop control system were calculated and simulated.

However, obtaining prior knowledge of the disturbance bounds, which is necessitated

by these algorithms, is not always feasible. The adaptive sliding mode is served as a fitting design approach, seeking to harness the advantages of sliding mode design without relying on prior knowledge of disturbance bounds. The technique in [45–47] is the most widely recognized among reaching law-based DSMs employing adaptive gain techniques. Nevertheless, the issues of both over- and under-estimation of the adaptive switching gain are noted in [45]. Motivated by the problem of gain estimation in the absence of a bound of disturbance, a minimum operator and ceiling function-based adaptive RL are proposed.

Let us consider a perturbed discrete-time linear time-invariant system:

$$z(k+1) = Az(k) + bu(k) + \xi(k), \quad k \in \mathbb{Z}^+ \quad (6.1)$$

In the context of system (6.1), the symbol z represents the state vector, which belongs to $\mathbb{R}^n$. The control input, denoted by u , belongs to $\mathbb{R}$, while A is a matrix and b is a vector of suitable dimension. The disturbance vector, $\xi(k)$, belongs to $\mathbb{R}^n$ and represents the external disturbance impact on the system and satisfies the matching condition. The sliding variable is described as

$$\mathfrak{N}(k) = c^T z(k), \quad (6.2)$$

where the vector $c^T \in \mathbb{R}^{n \times 1}$ is chosen in a way that ensures $c^T b \neq 0$ and (6.2) is intended to guarantee the stability of the system dynamics when confined to the sliding surface $\mathfrak{N}(k) = 0$.

Definition 6.1 *If the motion of system (6.1) satisfies the inequality $|\mathfrak{N}(k)| \leq \Delta W_b$ for all $k \geq [k_f + k_0]$ where $k_f \in \mathbb{R}^+$ is the user-specified time and $k_0 \in \mathbb{R}$ is the initial time, then the system is said to be in a quasi sliding mode (QSM) within the ΔW_b neighborhood of the specified $\mathfrak{N}(k) = 0$. The quasi-sliding-mode domain (QSMD) is the region in which the quasi-sliding mode occurs, and its width is denoted by ΔW_b .*

Definition 6.2 *For a discrete-time system (6.1) to satisfy the condition of reachability, it must fulfil the following conditions for all time steps $k \geq 0$*

$$|\mathfrak{N}(k+1)| < |\mathfrak{N}(k)| - \delta, \text{ if } |\mathfrak{N}(k)| > \Delta W_b$$

$$|\mathfrak{N}(k+1)| \leq \Delta W_b, \text{ if } |\mathfrak{N}(k)| \leq \Delta W_b$$

where $\delta > 0$ is a small constant.

The disturbance fulfils the matching condition. The sliding function (6.2) is affected by the disturbance vector $\xi(k)$, and this effect can be described as

$$d(k) = c^T \xi(k). \quad (6.3)$$

Although the disturbance is unknown, it is assumed to be bounded in magnitude. There is a positive constant $\bar{\xi}$ that satisfies

$$|d(k)| \leq \bar{\xi} \quad \forall k \quad (6.4)$$

In [6], authors proposed an RL given by

$$\mathfrak{N}(k+1) = (1 - \phi)\mathfrak{N}(k) - \gamma \text{sgn}(\mathfrak{N}(k)) + d(k), \quad (6.5)$$

where $\phi \in (0, 1)$ and $\gamma > 0$. Provided that the gain γ is selected in a manner such that (s.t.)

$$\gamma > \frac{1 + \phi}{1 - \phi} \bar{\xi}, \quad (6.6)$$

in such a case, the RL (6.5) ensures QSM within a range $\Delta W_b = \gamma + \bar{\xi}$. The width of the band ΔW_b depends on the switching gain γ

Thus, the challenge is to design γ to effectively handle uncertainties and produce acceptable controller accuracy while minimizing control input consumption.

6.2 DSMC with Adaptive Switching Gain

In [45], authors proposed a discrete sliding mode approach with an adaptive switching gain when bound on uncertainty is unknown. The uncertain system given by (6.1) and the sliding function defined by (6.2) along with

$$\mathfrak{N}(k+1) = (1 - \phi)\mathfrak{N}(k) - \gamma(k)\text{sgn}(\mathfrak{N}(k)) + d(k), \quad (6.7)$$

where $\gamma(k)$ is given by

$$\gamma(k) = \gamma(k-1) + \eta \text{sgn}(\mathfrak{N}(k))\text{sgn}(\mathfrak{N}(k-1)), \quad \eta > 0 \quad (6.8)$$

When using the adaptation law (AL) (6.8) in conjunction with the RL (6.7), there is always an issue of overestimating and underestimating control input [47]. An over-estimated switching gain leads to unnecessarily large control input, while an under-estimated gain

reduces controller accuracy due to the applied gain being lower than required. Also, (6.8) exhibits a constant rate of adaptation.

The limitation of constant gain adaptation in (6.8) was addressed in [46] by implementing an adaptive gain strategy, however, the underestimation problem still prevails, as the adaptive gain needs to account for the sliding function. In [47], the authors proposed a solution using an adaptive gain strategy to overcome the problem of overestimation and underestimation with a time-varying gain strategy having a gain function based on the sliding function.

6.3 Main Result

A novel AL is proposed to achieve better convergence and simultaneously alleviate chattering. The RL (6.7), along with AL (6.9), aim to efficiently guide the system toward the desired state while reducing oscillations or chattering effects. Consider the following RL

$$\aleph(k+1) = (1 - \varphi)\aleph(k) - \gamma(k)\text{sgn}(\aleph(k)) + d(k)$$

We have proposed a new gain AL for RL (6.7) as

$$\begin{aligned} \gamma(k) &= \min \left\{ |\aleph(k)|^\beta, \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \right\}, \\ \text{where } k_f &\in \mathbb{R}^+ > 1, \quad k_0 \in \mathbb{R}, \quad (1 - \varphi) \in (0, 1), \\ \beta &< 0 \quad \text{if } s(k_0) \in B(\aleph(k), 1), \text{ otherwise } \beta > 1 \\ \beta &= 1 + \frac{\log(1 - \varphi)}{\omega} \quad \text{if } k = \lceil k_f + k_0 \rceil, \\ \omega &= \max\{\log(|N(k)|/s_0), \underline{\omega}\}, \end{aligned} \tag{6.9}$$

where $s_0 \in \mathbb{R}^+$ is a design parameter expressed in the same units as the sliding variable and $\underline{\omega} \in \mathbb{R}^+$ small design parameter.

The motivation behind the formation of $\gamma(k)$ is to balance the control action between ensuring the desired time to reach the sliding surface and handling the problem of overestimation and underestimation.

Theorem 6.3 *Consider the DTS (6.1) with the sliding surface defined as (6.2) and if we choose RL (6.7) along with AL (6.9), irrespective of the initial state of the system, it is ensured that the sliding variable becomes zero in the case of unperturbed system at*

$k^* \geq \lceil k_f + k_0 \rceil$ also under the disturbance condition as given in (6.3) with Assumption 1, $\aleph(k)$ will enter the region ΔW_b at $k^* \geq \lceil k_f + k_0 \rceil$ steps.

Proof:

Unperturbed Case: We will first analyze the positive half of state space, such that $0 < s(k_0)$.

We will further break the analysis into two segments, firstly considering $0 < s(k_0) < 1$. Using RL (6.7) we have

$$\aleph(k+1) = (1 - \varphi)\aleph(k) - \gamma(k). \quad (6.10)$$

For $0 < s(k_0) < 1$, we have $\beta < 0$ and $k_f > 1$, proposed AL (6.9) will ensure that at $k_0 + 1$ instance,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^\beta. \quad (6.11)$$

For $s(k_0) \geq 1$, we have $\beta > 1$ and $k_f > 1$, proposed AL (6.9) will again ensure that at $k_0 + 1$ instance,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^\beta. \quad (6.11)$$

using (6.10) and (6.11) we have two scenarios at $k_0 + 1$ instance. If $(1 - \varphi)\aleph(k) = \gamma(k)$, then $\aleph(k+1) = 0$, and if $(1 - \varphi)\aleph(k) > \gamma(k)$, then $\aleph(k)$ will continue to decrease upto $k \leq \lceil k_f + k_0 \rceil - \lfloor (1 - \varphi)^{-1} \rfloor = \lceil k_f + k_0 \rceil - 1$.

At $k = \lceil k_f + k_0 \rceil$,

$$\gamma(k) = |\aleph(k)|^\beta \leq \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \text{ since } \beta = 1 + \frac{\log(1 - \varphi)}{\omega} \quad (6.12)$$

using (6.10) and (6.12), we have

$$\aleph(k+1) = 0 \text{ for } k^* \geq \lceil k_f + k_0 \rceil$$

Similar proof holds for the negative half of state space $s(k_0) < 0$;

Consider $s(k_0) = \aleph(k_0) < 0$. The RL (6.7) yields

$$\aleph(k+1) = (1 - \varphi)\aleph(k) - \gamma(k)\text{sgn}(\aleph(k)) = (1 - \varphi)\aleph(k) + \gamma(k), \quad (6.13)$$

since $\text{sgn}(\aleph(k)) = -1$ for $\aleph(k) < 0$. As before, two regimes are analyzed.

if $-1 < \aleph(k_0) < 0$, then $\beta < 0$ and $k_f > 1$. The AL (6.9) ensures at $k_0 + 1$ that

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^\beta. \quad (6.14)$$

Using (6.13)–(6.19), two possibilities exist at $k_0 + 1$. If $\gamma(k) = (1 - \varphi) |\aleph(k)|$, then $\aleph(k + 1) = 0$. If $\gamma(k) < (1 - \varphi) |\aleph(k)|$, then $\aleph(k + 1) = -((1 - \varphi) |\aleph(k)| - \gamma(k)) < 0$, i.e., $\aleph(k)$ remains negative with strictly reduced magnitude and continues to move toward zero for $k \leq \lceil k_f + k_0 \rceil - 1$.

if $\aleph(k_0) \leq -1$, then $\beta > 1$ and $k_f > 1$, and again at $k_0 + 1$

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^\beta. \quad (6.19)$$

The same dichotomy follows from (6.13): equality gives $\aleph(k + 1) = 0$; otherwise $\aleph(k + 1) < 0$ with a strictly smaller $|\aleph(k + 1)|$, and the trajectory keeps approaching zero up to $k \leq \lceil k_f + k_0 \rceil - 1$.

At $k = \lceil k_f + k_0 \rceil$, the adaptive law sets

$$\gamma(k) = |\aleph(k)|^\beta \leq \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \quad \text{since} \quad \beta = 1 + \frac{\log(1 - \varphi)}{\omega}, \quad \omega = \log \frac{|\aleph(k)|}{s_0}, \quad (6.15)$$

and substituting (6.15) into (6.13) gives

$$\aleph(k + 1) = 0 \quad \text{for} \quad k^* \geq \lceil k_f + k_0 \rceil.$$

Thus, for all initial conditions with $\aleph(k_0) < 0$, the sliding variable reaches the origin no later than $k^* = \lceil k_f + k_0 \rceil$, completing the symmetry of the argument for the negative half of the state space.

Perturbed Case: Now consider again positive half of state space $s(k_0) > 0$. Using RL(6.7) and disturbance condition (6.4)

$$\aleph(k + 1) \leq (1 - \varphi)\aleph(k) - \gamma(k) + \bar{\xi}. \quad (6.16)$$

For $0 < s(k_0) < 1$, we have $\beta < 0$ and $k_f > 1$, proposed AL (6.9) will ensure that at $k_0 + 1$ instance,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^\beta \quad (6.11)$$

Also, for $s(k_0) \geq 1$, we have $\beta > 1$ and $k_f > 1$, proposed AL (6.9) will again ensure that at $k_0 + 1$ instance,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^\beta$$

using (6.16) and (6.11) we have two scenarios. If $(1 - \varphi)\aleph(k) = \gamma(k)$, then $\aleph(k + 1) \leq \bar{\xi}$, and if $(1 - \varphi)\aleph(k) > \gamma(k)$, then $\aleph(k)$ will continue to decrease upto $k < \lceil k_f + k_0 \rceil - 1$.

For $k \leq \lceil k_f + k_0 \rceil$

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^\beta, \quad (6.17)$$

using (6.16) and (6.17) we have

$$\aleph(k+1) \leq \bar{\xi} \text{ at } k^* \leq \lceil k_f + k_0 \rceil.$$

The similar proof is for $s(k_0) < 0$;

Let $s(k_0) = \aleph(k_0) < 0$. With the RL (6.7) and disturbance $d(k)$ satisfying the bound in (6.4), i.e., $|d(k)| \leq \bar{\xi}$ (Assumption 1), we have for $\aleph(k) < 0$ (thus $\text{sgn}(\aleph(k)) = -1$)

$$\aleph(k+1) = (1 - \varphi)\aleph(k) + \gamma(k) + d(k) \Rightarrow \aleph(k+1) \geq (1 - \varphi)\aleph(k) + \gamma(k) - \bar{\xi}. \quad (6.18)$$

As in the unperturbed analysis, consider two magnitude regimes, noting that $k_f > 1$. if $-1 < \aleph(k_0) < 0$, then $\beta < 0$ and the AL (6.9) gives, at $k_0 + 1$,

$$\gamma(k) = \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} < |\aleph(k)|^\beta. \quad (6.19)$$

There are two possibilities. If $\gamma(k) = (1 - \varphi)|\aleph(k)|$, then from (6.18)

$$\aleph(k+1) \geq -(1 - \varphi)|\aleph(k)| + (1 - \varphi)|\aleph(k)| - \bar{\xi} = -\bar{\xi}.$$

Otherwise, if $\gamma(k) < (1 - \varphi)|\aleph(k)|$, then

$$\aleph(k+1) = -((1 - \varphi)|\aleph(k)| - \gamma(k)) + d(k) \geq -((1 - \varphi)|\aleph(k)| - \gamma(k)) - \bar{\xi},$$

so $\aleph(k)$ remains negative with strictly decreasing $|\aleph(k)|$ and keeps moving toward 0 for $k \leq \lceil k_f + k_0 \rceil - 1$.

if $\aleph(k_0) \leq -1$, then $\beta > 1$ and again at $k_0 + 1$ (6.19) holds. The same dichotomy as above implies either immediate entry $\aleph(k+1) \geq -\bar{\xi}$ (when $\gamma(k) = (1 - \varphi)|\aleph(k)|$) or a strict reduction in $|\aleph(k)|$ while remaining negative up to $k \leq \lceil k_f + k_0 \rceil - 1$.

Finally, at $k = \lceil k_f + k_0 \rceil$, by the AL (6.9),

$$\gamma(k) = |\aleph(k)|^\beta \leq \frac{|\aleph(k)|}{\lceil k_f + k_0 \rceil - k} \text{ since } \beta = 1 + \frac{\log(1 - \varphi)}{\omega}, \quad \omega = \log \frac{|\aleph(k)|}{s_0}. \quad (6.20)$$

Substituting (6.20) into (6.18) yields

$$\aleph(k+1) \geq -\bar{\xi} \text{ for } k^* \leq \lceil k_f + k_0 \rceil.$$

Hence, for all initial conditions with $\aleph(k_0) < 0$, the trajectory enters the band $\Delta W_b = \{\aleph : |\aleph| \leq \bar{\xi}\}$ no later than $k^* = \lceil k_f + k_0 \rceil$, completing the perturbed-case proof for the negative half of the state space.

Remark 1 Here, φ determines how the control effort is distributed among the sampling instances and the maximum control input needed. Moreover, $\lceil k_f + k_0 \rceil$ is the maximum instance at which the sliding variable enters boundary layer ΔW_b , which will be decided by the user a priori and where $k_f > 1$. Once the sliding variable changes its sign, it remains on the surface in the case of an unperturbed system, and in the case of a perturbed system, it remains in the vicinity of the sliding surface. This adaptation helps to reduce chattering, as this AL does not necessitate that the sliding function will cross the sliding plane in every instance.

Further boundedness of $\gamma(k)$ is studied through lemma 1.

Lemma 6.4 *Consider the RL given by (6.7) and the AL given by (6.9), then there exists $\gamma^* \in \mathbb{R}^+$ such that:*

$$|\gamma(k)| \leq \gamma^* \quad \forall \quad k_0 \geq 0.$$

Proof:

The AL (6.9) states that if $|\aleph(k)| \leq |s(k-1)|$, then the switching gain $\gamma(k)$ decreases. Therefore, whenever $|\aleph(k)| \leq |s(k-1)|$ then $\gamma(k) \leq \gamma^*$, ensuring boundedness of $\gamma(k)$. When $|\aleph(k)| > |\aleph(k-1)|$, the switching gain γ increases. To guarantee that $\gamma(k)$ remains bounded, we will examine two scenarios.

Case 1: $\text{sign}(\aleph(k)) = \text{sign}(\aleph(k-1))$, $\aleph(k) > 0$ and $|\aleph(k)| > |\aleph(k-1)|$. In this scenario, the sliding function deviates from the sliding surface. Let k_m instance indicate the beginning of Case 1, and using (6.9) and assuming $\bar{\xi}$ in the positive half of state space, we have $\aleph(k+1) \leq (1-\varphi)\aleph(k) - \gamma(k)\text{sgn}(\aleph(k)) + \bar{\xi}$. Considering $\bar{\xi}$ remains fixed and $\gamma(k)$ rises with each sample, there exists a sampling instance k_n at which $\gamma(k) - \bar{\xi} > 0$. Since $\aleph(k_n) > 0$ and $0 < (1-\varphi) < 1$. Hence, $\aleph(k_n+1) < (1-\varphi)\aleph(k_n) < |\aleph(k_n)|$. Now, we will investigate two scenarios

Case 1a: If $\aleph(k_n+1) > 0$, then we have $|\aleph(k)| < |\aleph(k-1)|$ at $k = k_n+1$. In this scenario, $\gamma(k)$ will decrease. Let $\gamma_1 \in \mathbb{R}^+$ such that $\gamma(k) < \gamma_1 \quad \forall \quad k \in [k_0, k_n+1]$.

Case 1b: If $\aleph(k_n+1) < 0$, then at $k = k_n+1$ we have two possibilities

- If $|\aleph(k)| < |\aleph(k-1)|$, then as mentioned in case 1a, $\gamma(k)$ decreases, and our assumption that $\gamma(k) < \gamma_1$ remains valid as long as the sliding variable continues to decrease under the condition of $|\aleph(k)| < |\aleph(k-1)|$.

- If $|\aleph(k)| > |\aleph(k-1)|$, then $\aleph(k) < 0, \aleph(k-1) > 0$. The behavior of s and the limitation of γ in this scenario have been examined in case (2).

Case 2: $\text{sign}(\aleph(k)) \neq \text{sign}(\aleph(k-1)), \aleph(k-1) > 0$

$\aleph(k) < 0$ and $|\aleph(k)| > |\aleph(k-1)|$. In this scenario, using (6.9) and (6.7), the sliding function will be as follows: $\aleph(k+1) = -(1-\varphi)\aleph(k) + \gamma(k) - d(k) \rightarrow \gamma(k) < |\aleph(k)|$. Hence, from case (1a), $\gamma(k)$ will decrease, and it will be bounded as $\gamma(k) < \gamma_1 \forall k$.

Case 3: $\text{sign}(\aleph(k)) = \text{sign}(\aleph(k-1)), \aleph(k) < 0$ and $|\aleph(k)| > |\aleph(k-1)|$.

Assuming $\bar{\xi}$ in the negative half of sliding manifold, $\aleph(k+1) \leq -(1-\varphi)\aleph(k) + \gamma(k)\text{sgn}(\aleph(k)) - \bar{\xi}$, let k_p indicate the beginning of Case 3, Since $|\aleph(k)| > |\aleph(k-1)|$, $\gamma(k)$ will increase at $k = k_p$ and will continue to increase upto condition $|\aleph(k)| > |\aleph(k-1)|$ holds. considering $\bar{\xi}$ remains fixed and $\gamma(k)$ rises until $k = k_q$, at which $\gamma(k) - \bar{\xi} > 0$ Since $\aleph(k_q) < 0$ as in Case 1 we will have $\aleph(k_q+1) > -(1-\varphi)\aleph(k_q) > -|\aleph(k_q)|$ and $k = k_q + 1$, the sliding variable stops increasing in the negative side of the sliding surface. This leads us to two scenarios,

Case 3a: if $\aleph(k_q+1) < 0$, then we have $|\aleph(k)| < |\aleph(k-1)|$ at $k = k_q + 1$. In this scenario, $\gamma(k)$ decreases. Let us assume there exists $\gamma_2 \in \mathbb{R}^+$ such that $\gamma(k) < \gamma_2 \forall k \in [k_p, k_q + 1]$

Case 3b: if $\aleph(k_q+1) > 0$, then at $k = k_q + 1$ we have two possibilities

- If $|\aleph(k)| \leq |\aleph(k-1)|$, then, as mentioned in case 3a $\gamma(k)$ decreases, and our assumption that $\gamma(k) < \gamma_2$ remains valid as long as the sliding variable continues to decrease under the condition of $|\aleph(k)| \leq |\aleph(k-1)|$.
- If $|\aleph(k)| > |\aleph(k-1)|$, then $\aleph(k) > 0, \aleph(k-1) < 0$. The behavior of s and the limitation of γ in this scenario have been examined in case (4).

Case 4: $\text{sign}(\aleph(k)) \neq \text{sign}(\aleph(k-1)), \aleph(k-1) < 0, \aleph(k) > 0$ and $|\aleph(k)| > |\aleph(k-1)|$.

In this scenario, the sliding function will be $\aleph(k+1) = (1-\varphi)\aleph(k) - \gamma(k) + d(k)$, and the analysis will be similar to that in case 2, and therefore, $\gamma(k)$ is bounded. for Case (3) and Case (4), let $\gamma_2 \in \mathbb{R}^+$ s.t. $\gamma(k) < \gamma_2$ also, $\gamma(k) \leq (\gamma_1, \gamma_2) = \max(\gamma_1, \gamma_2)$.

Now, using the above results, we can further state the boundness results for the sliding variable.

Theorem 6.5 *The combination of the RL (6.7) and the gain AL (6.9) results in a solution that is ultimately confined within a global bound, with the switching function being ultimately bounded within a band ΔW_b .*

$$|\aleph(k)| = \Delta W_b < (\gamma^* + \bar{\xi}). \quad (6.21)$$

Proof: Consider the Lyapunov candidate function.

$$V(k) = \mu |\aleph(k)|. \quad (6.22)$$

Here, $\mu > 0$. We have:

$$\begin{aligned} \Delta V(k) &= V(k+1) - V(k) \\ &= \mu |\aleph(k+1)| - \mu |\aleph(k)| \end{aligned} \quad (6.23)$$

Consider

$$j(k) = -\gamma(k) \text{sgn}(\aleph(k)) + d(k) \quad (6.24)$$

Using (6.23) and (6.7), we have

$$\Delta V(k) = \mu |(1 - \varphi) \aleph(k) + j(k)| - \mu |\aleph(k)| \quad (6.25)$$

Now, using (6.25)

$$\begin{aligned} \Delta V(k) &\leq \mu |\aleph(k)| - \varphi \mu |\aleph(k)| + \mu |j(k)| - \mu |\aleph(k)| \\ &\leq -\varphi \mu |\aleph(k)| + \mu |j(k)| \\ &\leq -\varphi V(k) + \mu |j(k)| \end{aligned} \quad (6.26)$$

Now using lemma 1 and from (6.4) we have

$$|j(k)| \leq -\gamma(k) \text{sgn}(\aleph(k)) + \bar{\xi} \leq \gamma^* + \bar{\xi} \quad (6.27)$$

Now using (6.26) and (6.27), we have

$$\Delta V(k) \leq -\varphi V(k) + \mu(\gamma^* + \bar{\xi}) \quad (6.28)$$

Hence, The solution of the RL (6.7) is Globally Ultimately Bounded if,

$$\Delta W_b \leq \frac{(\gamma^* + \bar{\xi})}{\varphi} < (\gamma^* + \bar{\xi}) \quad (6.29)$$

Also, the ultimate band $|\aleph(k)| \leq \Delta W_b$

Corollary 6.6 *Consider the DTS (6.1) under Assumption 1. If we choose the sliding surface (6.2) under the disturbance condition as (6.3) with RL (6.7) and AL (6.9)*

- *It is guaranteed that the state $\aleph(k)$ will enter the region ΔW_b within a maximum of k^* steps, regardless of the initial state of the system.*
- *Once the state $\aleph(k)$ becomes a part of the region ΔW_b , it remains confined within it for the future.*

Proof: Our first aim is to demonstrate that the trajectories of the system will inevitably converge to the region,

$$\Delta W_b = \aleph(k) : |\aleph(k)| < \gamma^* + \bar{\xi}$$

regardless of the initial state.

Consider $\aleph(k) > 0$,

$$\aleph(k+1) = (1 - \varphi)\aleph(k) - \gamma(k)\text{sgn}(\aleph(k)) + d(k).$$

using (6.29) and (6.4), we have

$$\aleph(k+1) \leq \aleph(k) - \gamma^* + \bar{\xi}. \quad (6.30)$$

Using (6.17), equation (6.30) reduces to

$$\aleph(k+1) \leq \aleph(k).$$

This indicates that the sequence $\aleph(k)$ undergoes a decreasing trend until it intersects the switching function. Let us assume the existence of an instant k_t s.t.

$$\aleph(k_t) > 0 \text{ and } \aleph(k_t + 1) < 0.$$

then

$$|\aleph(k_t + 1)| \leq |(1 - \varphi)\aleph(k_t) - \gamma(k)\text{sgn}(\aleph(k_t)) + \xi(k_t)|$$

$$|\aleph(k_t + 1)| \leq -\gamma^* - \bar{\xi}$$

Hence, $\aleph(k_t + 1) \in \Delta W_b$ at the instance $k^* = k_t$, and for the case $\aleph(k) < 0$, the proof is the same.

Now our aim is to prove that when $\aleph(k) \in \Delta W_b$, it inevitably follows that $\aleph(k+1) \in \Delta W_b$, provided that $0 < \aleph(k) \leq (\gamma^* + \bar{\xi})$.

Consider $0 < \aleph(k) < (\gamma^* + \bar{\xi})$, then,

$$\begin{aligned}
|\aleph(k+1)| &= |(1-\varphi)\aleph(k) - \gamma(k)\text{sgn}(\aleph(k)) + d(k)| \\
&\leq \aleph(k) - \gamma^* + d(k) \\
&\leq \gamma^* + \xi - (\gamma^* - \xi) \\
&\leq (\gamma^* + \xi).
\end{aligned}$$

a Similar way, we can prove for the condition $-(\gamma^* + \bar{\xi}) < \aleph(k) \leq 0$.

6.4 Simulation

In this section, the validation of the proposed results is showcased through simulation using two examples and comparing with previous results [46].

Example 6.7 *Consider a second-order spring mass damper system*

$$z(k+1) = \begin{pmatrix} 0 & 1 \\ -1 & -0.13 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) + d(k) \quad (6.31)$$

Here, $z(k)$ represents the state of the system, and $d(k)$ represents the disturbance signal.

$$d(k) = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \times \left(\sin(k) + \sin\left(\frac{k}{3}\right) \right) \quad (6.32)$$

In (6.2), the sliding surface is designed with the parameter $c^T = [1 \ 2]^T$ and the initial conditions as specified in Table I. Fig.6.4 and Fig.6.4 displays the outcome of the simulation and its comparison with [46] using parameter $w = 0.8, v = 0.6$. Table II provides numerical evaluations for the absolute sum of control input $u(k)$ and the absolute sum of switching gain $\gamma(k)$ concerning different k_f . Table II shows that the switching gain $\Sigma|\gamma(k)|$ and control input $\Sigma|u(k)|$ concerning k_f decreases and increases, respectively. Also from Fig 1, it is clear that the proposed AL reduces QSMD concerning [46].

Example 6.8 *Consider the discrete dynamical system*

$$z(k+1) = \begin{pmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 1 \\ 0 & -0.5 & 1 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k) + d(k). \quad (6.33)$$

Table 6.1

S.No	Initial condition / Parameter	Value	k_0	k_f	$\sum_{k=1}^{k_f} u(k) $	$\sum_{k=1}^{k_f} \gamma(k) $
1	k_f	16	2	4	10.71	9.18
2	k_0	2	2	8	41.97	6.80
3	$z_1(0)$	-5	2	12	43.06	4.93
4	$z_2(0)$	20	2	16	43.43	3.86

Table 6.2

Here, $z_1(k)$, $z_2(k)$, $z_3(k)$ represent the states of the system. $d(k)$ represent discretized disturbances signals.

$$d(k) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times (0.2(-1)^{\lfloor k/10 \rfloor}). \quad (6.34)$$

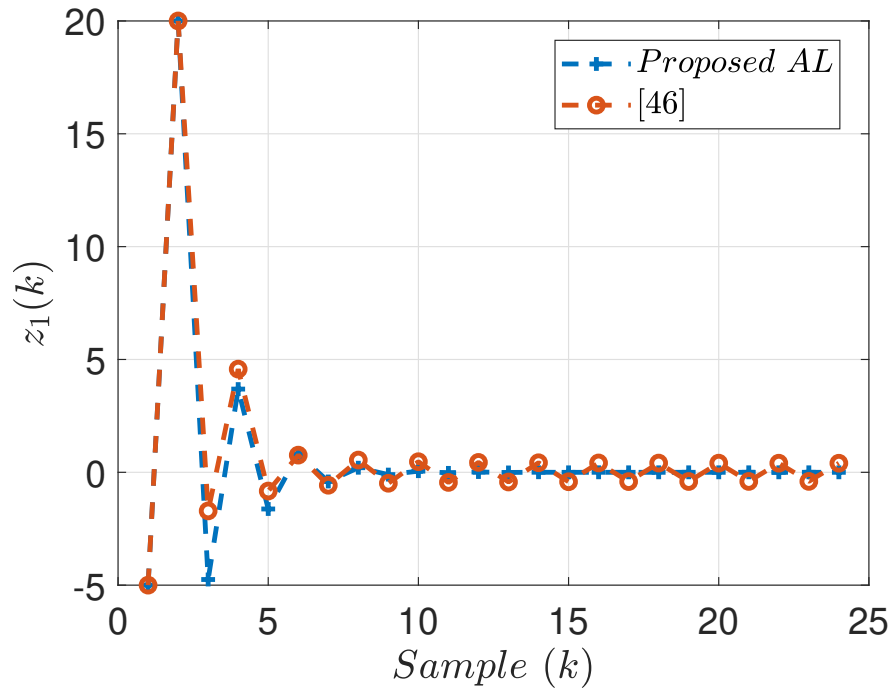
The considered system is representative of a third-order discrete-time system with matched uncertainty. In equation (6.2), the sliding surface is constructed using the parameter $c^T = [1 \ 1.5 \ 1]^T$ and initial condition as per Table III. The simulation results and its comparison with [46] using the same parameter as defined above, are depicted in Fig.6.4 and Fig.6.4. Tables IV provide a numerical assessment for the absolute sum of the control input $u(k)$ and the absolute sum of the switching gain $\gamma(k)$ at different values of k_f . Obtained results show that the system state, when utilizing the proposed algorithm, stays near the origin. As a result, the proposed adaptive sliding mode control algorithm ensures reduced control efforts, and reduces QSMC concerning [46].

Table 6.3

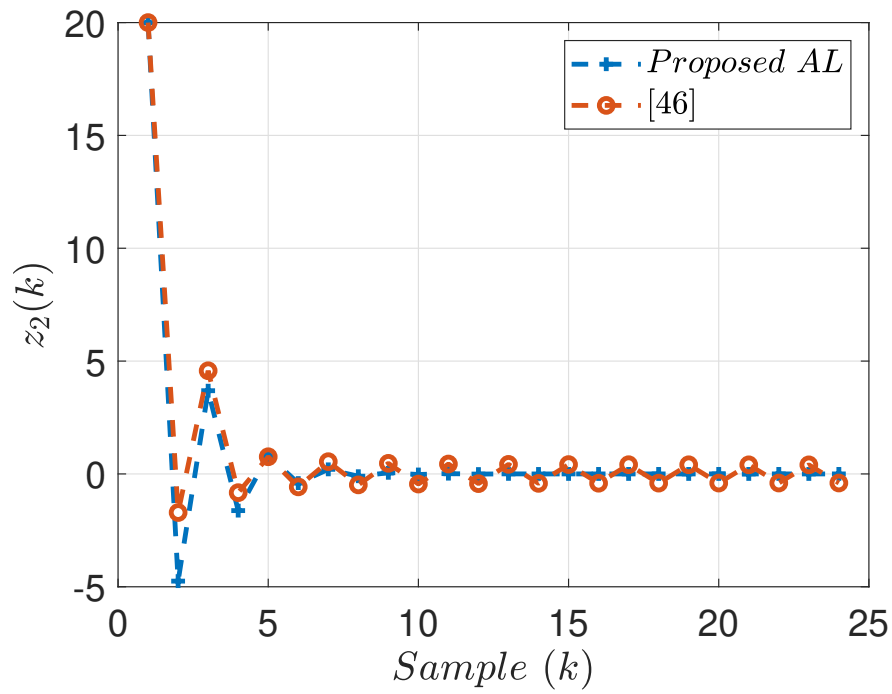
S.No	Initial condition / Parameter	Value
1	k_f	16
2	k_0	1
3	$z_1(0)$	-5
4	$z_2(0)$	20
5	$z_3(0)$	20

Table 6.4

k_0	k_f	$\sum_{k=1}^{k_f+1} u(k) $	$\sum_{k=1}^{k_f} \gamma(k) $
1	4	246.33	15.93
1	8	224.46	9.65
1	10	217.23	6.81
1	16	214.05	5.25



(a) Proposed AL vs [47] for state $z_1(k)$



(b) Proposed AL vs [47] for state $z_2(k)$

Figure 6.1: Proposed AL vs [47]

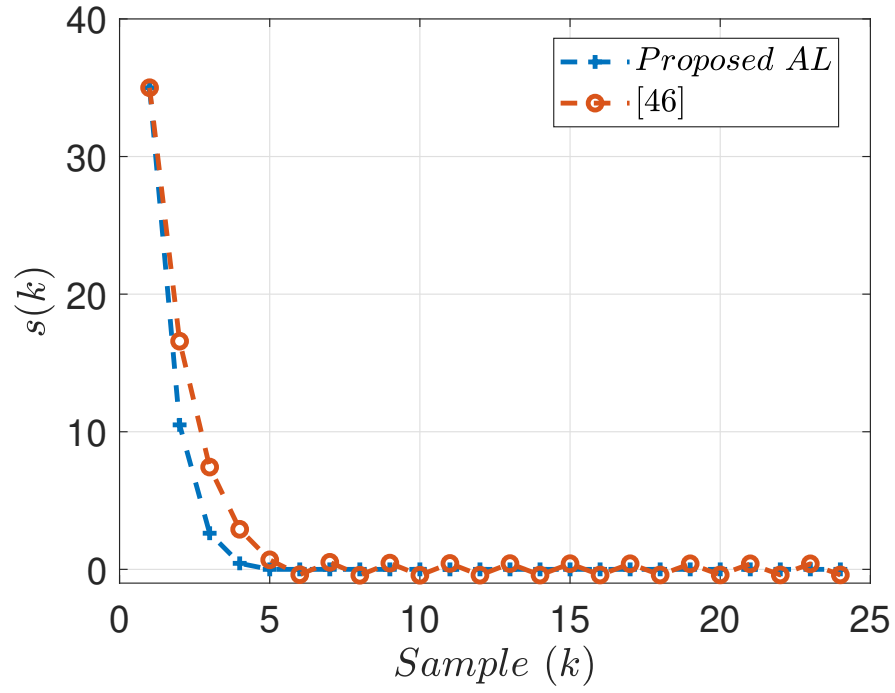


Figure 6.2: Comparison of sliding variable for Proposed AL vs [47]

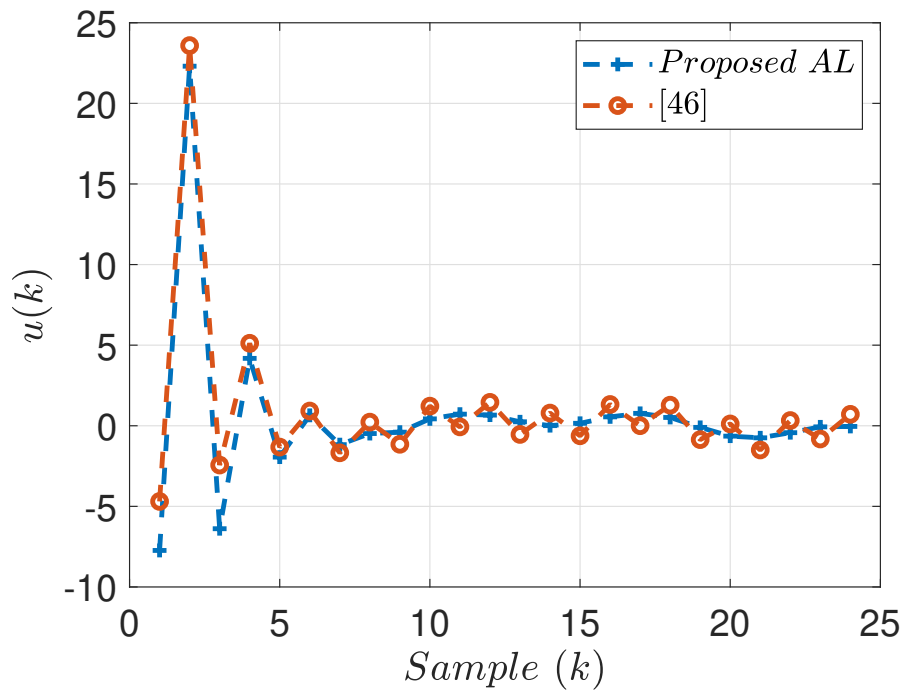
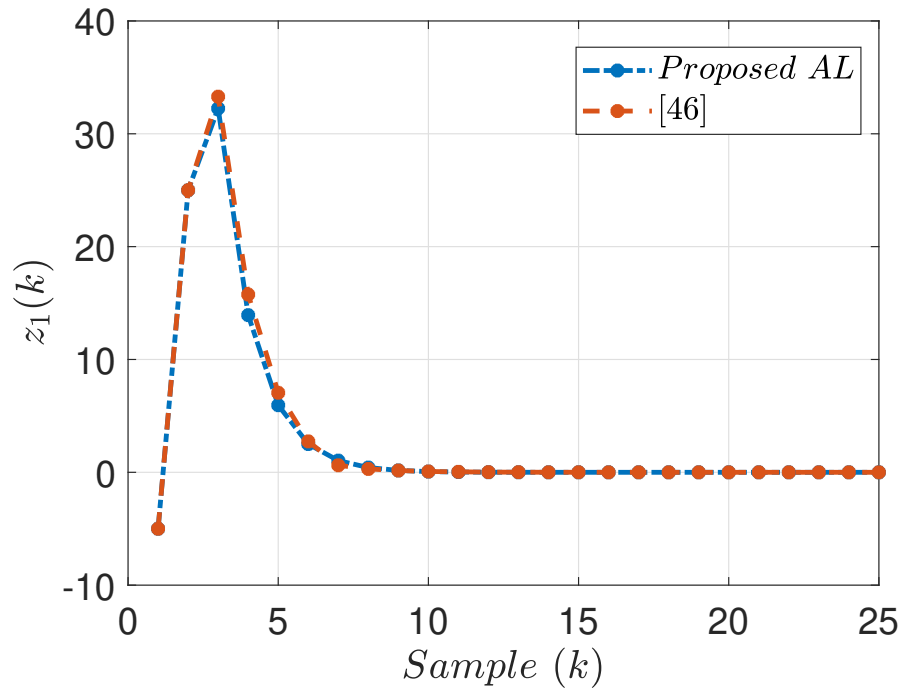
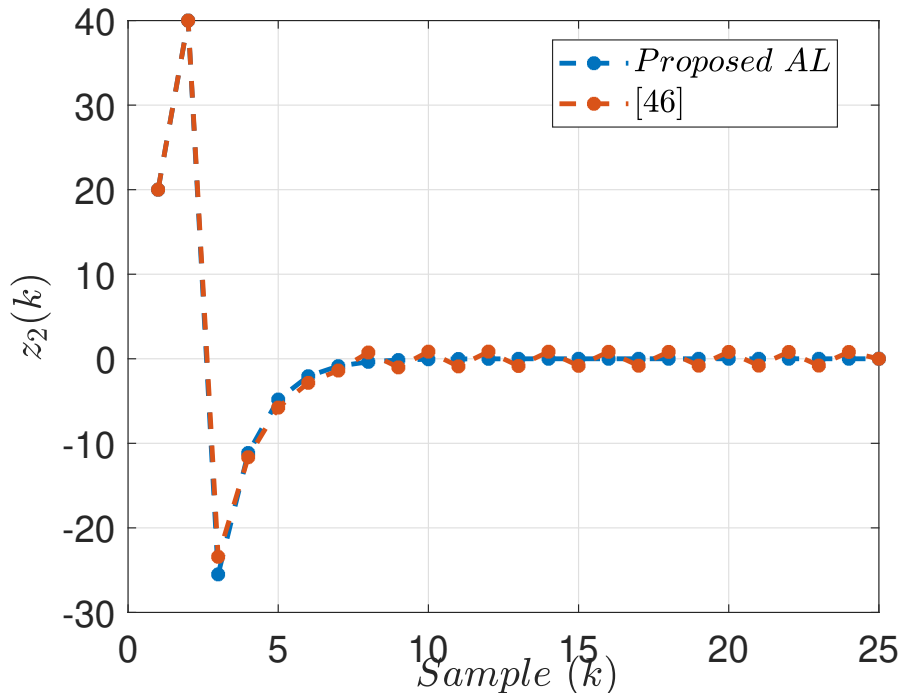


Figure 6.3: Comparison of control performance for Proposed AL vs [47]



(a) Proposed AL vs [47] for state $z_1(k)$



(b) Proposed AL vs [47] for state $z_1(k)$

Figure 6.4: Proposed AL vs [47]

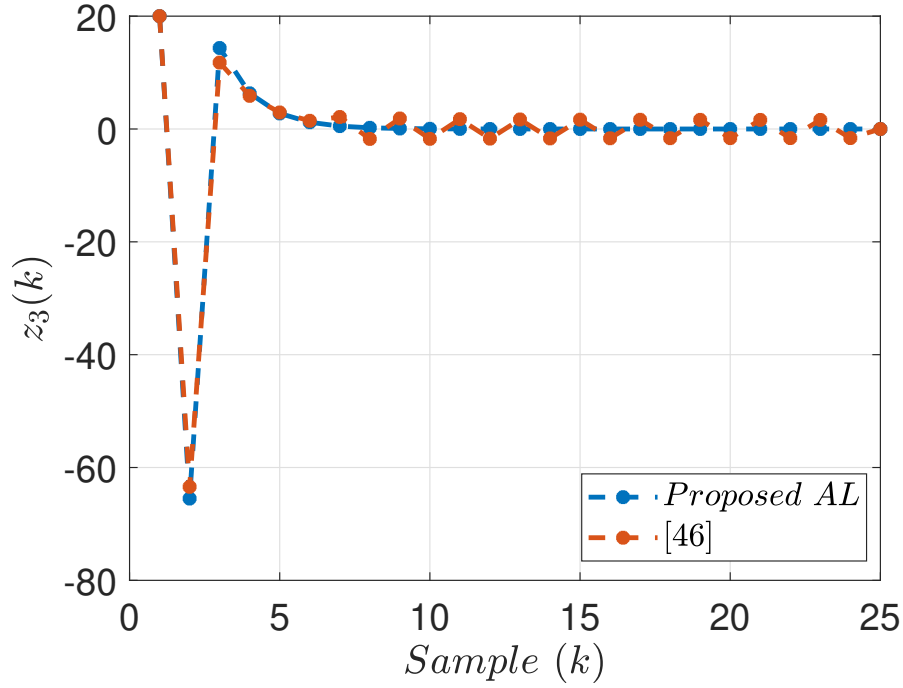


Figure 6.5: Proposed AL vs [47] for state $z_1(k)$

6.5 Conclusions

This chapter presents the design and verification of a discrete-time sliding-mode control system based on an adaptive RL. By incorporating a minimum operator and ceiling function in AL, the proposed adaptive RL gives better control in terms of convergence rate and reaching time. This chapter introduces better AL for gain adaptation concerning [46] in adaptive discrete-time sliding mode. The study also includes a theoretical analysis of the reaching steps and convergence properties under disturbance. Theoretical analysis and simulation results demonstrate that the proposed approach offers reduced QSMD and a better selection of reaching time based on available control effort. This new method also avoids the issues of chattering.

Future work will focus on extending the methodology to systems with more pronounced nonlinearities, higher-dimensional dynamics, lightly excited systems, and unmatched uncertainties. In addition, comparisons with a broader range of state-of-the-art approaches will be conducted to demonstrate wider applicability.

Chapter 7

Finite-Time Discrete Control for Two-DOF Helicopter System

7.1 Introduction

In the past few decades, unmanned aerial vehicles and helicopters have seen a surge in popularity due to their demand in various sectors, such as trans-shipment, defense, emergency response, police operations, and drone imagery [59]. The discrete control methodologies, extensively explored in a range of articles [60–62]. The discrete-time approach emerges as notably superior to continuous-time emulation in terms of both stability and achievable performances. Most of the available techniques for feedback stabilization of discrete-time systems (DTS) lead to closed-loop systems that guarantee exponential convergence with an infinite settling time [10, 63]. While these methods demonstrate efficacy, exploring finite-time discrete algorithms holds promise for enhanced performance. Recently, in [64] and [20], authors have established finite-time stabilization for discrete autonomous systems based on a minimum operator, and later its application was developed in discrete sliding mode control [65, 66]. Expanding on the concepts outlined in [20], the researchers in [11] investigate finite-time control strategies for high-order discrete-time nonlinear systems. They develop a method for designing finite-time controllers using a recursive computation approach grounded in backstepping principles.

Certainly, the realm of control design grounded in discrete finite-time stability has entered a new frontier, particularly in the domain of rated convergence for discrete-time systems (DTS). However, there are no results pertaining to discrete finite-time stabiliza-

tion of two-DOF helicopters. Based on the above observations, we study the backstepping-based discrete finite-time control of a two-DOF helicopter for desired reference tracking.

It is crucial to note that employing backstepping procedures for finite-time stabilization of two-DOF helicopters introduces future state information of the virtual control signal, raising concerns about causality violation and the potential failure to achieve the designed control scheme. To address the causality problem inherent in backstepping procedures for finite-time stabilization, a new finite-time backstepping control technique is proposed in [11] where the finite-time stability analysis is given from the last subsystem to the first subsystem, and the settling time can be estimated.

Expanding upon the principles explored in [11, 20], this chapter presents a discrete finite-time control approach aimed at accomplishing finite-time trajectory tracking for a two-degree-of-freedom (2-DOF) helicopter system. The primary contributions of this research are summarized as follows:

1. Departing from conventional continuous-time and asymptotically convergent control methods applied to nonlinear two-DOF helicopter systems [10, 63, 67, 68], this study marks the first substantial effort in investigating discrete finite-time control of two-DOF helicopter system. The approach leverages the finite-time Lyapunov stability theorem, introducing a paradigm shift in addressing the challenges associated with these complex systems.
2. In alignment with the methodology proposed in [11] and [10], the causality contradiction is effectively managed through a recursive backstepping approach, enabling the acquisition of future information.
3. In contrast to conventional control methods, this study introduces a technique for estimating the settling time of the two-degree-of-freedom (2-DOF) helicopter system by examining finite-time stability, starting from the last subsystem and progressing to the first.

7.2 Two-DOF Helicopter System Description

In this study, we focus on the two-DOF helicopter system. Utilizing the Euler-Lagrangian equation, the subsequent continuous-time mathematical representation is expressed as



Figure 7.1: Experimental setup of two-DOF helicopter.

follows:

$$\ddot{\theta}_1(t) = -\frac{B_p \dot{\theta}_1(t)}{J_{pp} + m_{heli} l^2} + \frac{m_{heli} l^2 \dot{\theta}_2^2(t) \sin(\theta_1) \cos(\theta_1)}{J_{pp} + m_{heli} l^2} - \frac{m_{heli} g l \cos(\theta_1)}{J_{pp} + m_{heli} l^2} + \frac{K_{pp} V_p}{J_p + m_{heli} l^2} + \frac{K_{py} V_y}{J_{pp} + m_{heli} l^2} \quad (7.1)$$

$$\ddot{\theta}_2(t) = \frac{2m_{heli} l^2 \dot{\theta}_2(t) \sin(\theta_1) \cos(\theta_1) \dot{\theta}_1(t)}{J_{yy}} - \frac{B_y \dot{\theta}_2(t)}{J_{yy}} + \frac{K_{yp} V_p}{J_{yy}} + \frac{K_{yy} V_y}{J_{yy}} \quad (7.2)$$

Here, $\theta_1(t)$, $\dot{\theta}_1(t)$, $\theta_2(t)$, and $\dot{\theta}_2(t)$ correspond to the pitch angle, pitch velocity, yaw angle, and yaw velocity, respectively. The input voltages applied to the pitch and yaw motors are represented as V_p and V_y . Additional details regarding the system parameters can be accessed in [68].

The mathematical model presented above for the two-DOF helicopter system is described in continuous time [68]. However, in the current study, we adopt a discrete-time approach. Consequently, we propose the discretization of the helicopter model using

the Euler discretization technique [69].

$$x_1(k+1) = x_1(k) + x_2(k)h \quad (7.3)$$

$$\begin{aligned} x_2(k+1) = & x_2(k) - c_1 \cos(x_1(k))h - c_2 x_2(k)h \\ & + c_4 h V_p(k) - c_3 \sin(x_1(k)) \cos(x_1(k)) x_4^2(k)h \\ & + c_5 h V_y(k) \end{aligned} \quad (7.4)$$

$$x_3(k+1) = x_3(k) + x_4(k)h \quad (7.5)$$

$$\begin{aligned} x_4(k+1) = & x_4(k) - c_6 x_4(k)h + c_8 h V_p(k) + c_9 h V_y(k) \\ & + c_7 \sin(x_1(k)) \cos(x_1(k)) x_2(k) x_4(k)h. \end{aligned} \quad (7.6)$$

where h is the sampling period and the constant parameters c_i , $i = 1, \dots, 9$ are expressed as,

$$\begin{aligned} c_1 &= \frac{m_{\text{heli}} g l_{cm}}{J_{pp} + m_{\text{heli}} l_{cm}^2}, & c_2 &= \frac{B_p}{J_{pp} + m_{\text{heli}} l_{cm}^2}, \\ c_3 &= \frac{m_{\text{heli}} l_{cm}^2}{J_{pp} + m_{\text{heli}} l_{cm}^2}, & c_4 &= \frac{K_{pp}}{J_{pp} + m_{\text{heli}} l_{cm}^2}, \\ c_5 &= \frac{K_{py}}{J_{pp} + m_{\text{heli}} l_{cm}^2}, & c_6 &= \frac{B_y}{J_{yy}}, \\ c_7 &= \frac{2m_{\text{heli}} l_{cm}^2}{J_{yy}}, & c_8 &= \frac{K_{yp}}{J_{yy}}, & c_9 &= \frac{K_{yy}}{J_{yy}}. \end{aligned}$$

In this configuration, $x_1(k)$ denotes the angle on the pitch axis, while $x_3(k)$ signifies the angle on the yaw axis. A positive pitch position corresponds to the helicopter's nose elevating, and a positive yaw position indicates a clockwise rotation. The velocities in the pitch and yaw axes are represented by $x_2(k)$ and $x_4(k)$, respectively. The voltage input for the pitch motor is given by $V_p(k)$, and for the yaw motor, it is $V_y(k)$. In order to simplify further analysis, we set $h = 1$.

The goal is to formulate the control laws for $u_p(k)$ and $u_y(k)$ such that the outputs $x_1(k)$ and $x_3(k)$ align with the desired reference signals $x_{1d}(k)$ and $x_{3d}(k)$, respectively, within a finite time. Rewriting equations (7.3)-(7.6) into the state space form, we get

$$x_1(k+1) = x_1(k) + x_2(k) \quad (7.7)$$

$$x_2(k+1) = x_2(k) + \phi_1(x(k))^T \eta_1 + u_p(k) \quad (7.8)$$

$$x_3(k+1) = x_3(k) + x_4(k) \quad (7.9)$$

$$x_4(k+1) = x_4(k) + \phi_2(x(k))^T \eta_2 + u_y(k) \quad (7.10)$$

where

$$\phi_1(x(k)) = \begin{bmatrix} -\cos(x_1(k)) \\ -x_2(k) \\ -\cos(x_1(k)) \sin(x_1(k)) x_4^2(k) \end{bmatrix}, \eta_1 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

$$\phi_2(x(k)) = \begin{bmatrix} -x_4(k) \\ x_2(k)x_4(k) \cos(x_1(k)) \sin(x_1(k)) \end{bmatrix}, \eta_2 = \begin{bmatrix} c_6 \\ c_7 \end{bmatrix}.$$

and $u_p(k)$ and $u_y(k)$ as the control inputs for the pitch and yaw motor, respectively, defined as:

$$u_p(k) = c_4 V_p(k) + c_5 V_y(k) \quad (7.11)$$

$$u_y(k) = c_8 V_p(k) + c_9 V_y(k) \quad (7.12)$$

The future information of the reference signal $x_{di}(k)$ is assumed to be known, bounded, and piecewise continuous.

Remark 7.1 *Please note that the future reference signal $x_{di}(k+d)$ is required to implement the controller. In practice, the trajectory to be tracked is usually predefined in tracking applications. Thus, $x_{di}(k+d)$ is known a priori.*

7.3 Finite-Time Discrete Control Design

We will employ the recursive backstepping technique to design a finite-time feedback control law for the two-DOF helicopter system (7.7)-(7.10). To initiate this process, we will first apply a change of coordinates to transform the state variables as follows:

$$z_1(k) = x_1(k) - x_{1d}(k) \quad (7.13)$$

$$z_2(k) = x_2(k) - \alpha_1(k) \quad (7.14)$$

$$z_3(k) = x_3(k) - x_{3d}(k) \quad (7.15)$$

$$z_4(k) = x_4(k) - \alpha_2(k) \quad (7.16)$$

where $\alpha_1(k)$ and $\alpha_2(k)$ are virtual control signals. Choose $\alpha_1(k)$ and $\alpha_2(k)$ as

$$\alpha_1(k) = f_1(z_1(k)) = -x_{1d}(k) + x_{1d}(k+1) - c \min \left(\frac{|z_1(k)|}{c}, |z_1(k)|^\gamma \right) \quad (7.17)$$

$$\alpha_2(k) = f_2(z_3(k)) = -x_{3d}(k) + x_{3d}(k+1) - c \min \left(\frac{|z_3(k)|}{c}, |z_3(k)|^\gamma \right) \quad (7.18)$$

where $c > 0$ and $\gamma \in (0, 1)$ be positive design constants. With these, the transformed dynamics can be expressed as:

$$z_1(k+1) = z_1(k) + z_2(k) + x_{1d}(k) - x_{1d}(k+1) + \alpha_1(k) \quad (7.19)$$

$$z_2(k+1) = z_2(k) + f_1(z_1(k)) - f_3(z_1(k), z_2(k)) + u_p(k) + \phi_1(x(k))\eta_1 \quad (7.20)$$

$$z_3(k+1) = z_3(k) + z_4(k) + x_{3d}(k) - x_{3d}(k+1) + \alpha_2(k) \quad (7.21)$$

$$z_4(k+1) = z_4(k) + f_2(z_3(k)) - f_4(z_3(k), z_4(k)) + u_y(k) + \phi_2(x(k))\eta_2 \quad (7.22)$$

where in order to overcome the causality we have denoted $z_1(k+1) = f_3(z_1(k), z_2(k))$ and $z_3(k+1) = f_4(z_3(k), z_4(k))$.

Also considering (7.17) and (7.18), we can express $\alpha_1(k+1) = f_1(f_3(z_1(k), z_2(k)))$ and $\alpha_2(k+1) = f_2(f_4(z_3(k), z_4(k)))$.

Finally, We introduce the finite-time control law for the transformed system (7.19)-(7.22)

$$u_p(k) = -f_1(z_1(k)) - \phi_1(x(k))\eta_1 + f_1(f_3(z_1(k), z_2(k))) - \min\left(\frac{|z_2(k)|}{c}, |z_2(k)|^\gamma\right) \quad (7.23)$$

$$u_y(k) = -f_2(z_3(k)) - \phi_2(x(k))\eta_2 + f_2(f_4(z_3(k), z_4(k))) - \min\left(\frac{|z_4(k)|}{c}, |z_4(k)|^\gamma\right). \quad (7.24)$$

Remark 7.2 The controllers $u_p(k)$ and $u_y(k)$ in equations (7.23) and (7.24) have been developed by representing the error $z_1(k)$ as $f_1(\cdot)$ and $f_2(\cdot)$, thereby expressing them as functions of all system states through a recursive process.

The following theorem is presented to succinctly summarize our main result.

Theorem 7.3 For the discrete-time nonlinear two-DOF helicopter system described by equations (7.19)-(7.22) under Assumptions 1, the application of the proposed control laws (7.23) and (7.24), in conjunction with virtual control laws (7.17) and (7.18), guarantees that the system's output trajectory will track the desired reference signal within a finite time.

Proof: Utilizing the closed-loop system formulated in (7.19)-(7.22), a finite-time stability analysis will be performed, progressing from the last component to the first component of the subsystem. Furthermore, an assessment of the settling time will be offered. In this instance, the entire system will be divided into two cascade subsystems.

7.3.1 Subsystem 1

Consider signals (7.19) and (7.20) from subsystem 1. Using control law (7.23), the last component (7.20) of the first subsystem is reduced to

$$z_2(k+1) = z_2(k) - \min\left(\frac{|z_2(k)|}{c}, |z_2(k)|^\gamma\right). \quad (7.25)$$

Consider the Lyapunov function as $V_1(z) = z_2^2(k)$. Given that $|z_2(0)| > c^{\frac{1}{1-\gamma}}$ (If $|z_2(0)| \leq c^{\frac{1}{1-\gamma}}$ and $z_2(0) \neq 0$, then $z_2(k) = 0$ will lead to (7.25) FTS with $K(z_{20}) = 1$).

Then, we have:

$$\Delta V_1(z) = \left[z_2(k) - \min\left(\frac{|z_2(k)|}{c}, |z_2(k)|^\gamma\right) \right]^2 - z_2^2(k) \leq -c \min\left\{\frac{V_1(k)}{c}, V_1(k)^\gamma\right\}.$$

one can obtain the settling time function as [20]

$$K(z_{20}) \leq \left\lceil \log_{[1-cV(z_{20})^{\gamma-1}]} \frac{c^{\frac{1}{1-\gamma}}}{V(z_{20})} \right\rceil + 1,$$

such that $k > K(z_{20})$, $z_2(k) = 0$ and second component (7.19) of the first subsystem will reduce to

$$z_1(k+1) = z_1(k) - \min\left(\frac{|z_1(k)|}{c}, |z_1(k)|^\gamma\right). \quad (7.26)$$

Similar to the first component, consider $V_2(k) = z_1^2(k)$. By performing a similar analysis as above, it is evident that if $|z_1(K(z_{20}))| \leq c^{\frac{1}{1-\gamma}}$, $K(z_{10}) = K(z_{20}) + 1$ and if $|z_1(K(z_{20}))| > c^{\frac{1}{1-\gamma}}$

$$K(z_{10}) \leq \left\lceil \log_{[1-cV(z_{10})^{\gamma-1}]} \frac{c^{\frac{1}{1-\gamma}}}{V(z_{10})} \right\rceil + 1 + K(z_{20}),$$

will ensure that at $k > K(z_{10})$, $z_1(k) = 0$ and $z_2(k) = 0$. Using (7.13), we got $x_1(k) = x_{1d}(k) \forall k > K(z_{10})$, which shows finite-time tracking of reference signal $x_{1d}(k)$.

7.3.2 Subsystem 2

Consider signals (7.21) and (7.22) from subsystem 2. Using control law (7.24), the last component (7.22) of the second subsystem is reduced to

$$z_4(k+1) = z_4(k) - \min\left(\frac{|z_4(k)|}{c}, |z_4(k)|^\gamma\right). \quad (7.27)$$

Similar to subsystem 1, using $V_3(k) = z_4^2(k)$ shows that if $|z_4(0)| \leq c^{\frac{1}{1-\gamma}}$ and $z_4(0) \neq 0$, then using [20] $z_4(k) = 0$ will lead to (7.27) FTS with $K(z_{40}) = 1$ for $|z_4(0)| \leq c^{\frac{1}{1-\gamma}}$. Otherwise, if $|z_4(0)| > c^{\frac{1}{1-\gamma}}$ then

$$K(z_{40}) \leq \left\lceil \log_{[1-cV(z_{40})^{\gamma-1}]} \frac{c^{\frac{1}{1-\gamma}}}{V(z_{40})} \right\rceil + 1,$$

such that $z_4(k) = 0 \forall k > K(z_{40})$ and (7.21) will reduce to

$$z_3(k+1) = z_3(k) - \min\left(\frac{|z_3(k)|}{c}, |z_3(k)|^\gamma\right). \quad (7.28)$$

Also, consider $V_4(k) = z_3^2(k)$. Similar to subsystem 1 if $|z_3(K(z_{40}))| \leq c^{\frac{1}{1-\gamma}}$ will lead to $K(z_{30}) = K(z_{40}) + 1$ otherwise $K(z_{30}) \leq \left\lceil \log_{[1-cV(z_{30})^{\gamma-1}]} \frac{c^{\frac{1}{1-\gamma}}}{V(z_{30})} \right\rceil + K(z_{40}) + 1$ such that $\forall k > K(z_{30})$, $z_4(k) = 0$ and using (7.15) we have $\forall k > K(z_{30})$, $x_3(k) = x_{3d}(k)$ and $x_1(k) = x_{1d}(k)$ insures finite-time of tracking of reference signals. Hence, the overall system settling time will be $K(z_0) = \text{Max}[K(z_{30}), K(z_{10})]$. Therefore, $z(k)=0$ is finite-time stable, which completes the proof.

Remark 7.4 *This paper introduces a finite-time recursive backstepping control framework for a two-DOF helicopter system, differentiating itself from previous methods using NN-based approximation [69]. Our approach, utilizing variable substitution techniques, establishes a direct connection between the system states $x_i(k)$ and the controlled errors $z_i(k)$, overcoming challenges associated with causality.*

7.4 Result Discussion

A set of simulations and experiments was undertaken to evaluate the efficacy of the finite-time backstepping feedback control strategy devised in this study. The system parameters for the two-degree-of-freedom (2-DOF) helicopter system are derived from [68]. In this study, we initialized the two-DOF helicopter with $x_1(0) = -0.72\text{rad}$ and $x_3(0) = 0.5\text{rad}$. The desired reference trajectory is defined as $x_{1d}(k) = 0.2 \sin(2\pi 0.05k)$ and $x_{3d}(k) = 0.5 \sin(2\pi 0.05k)$, adhering to Assumption 1. The control design parameters are set to $\gamma = 0.8$ and $c = 0.5$ with a sampling period of $h = 0.05$.

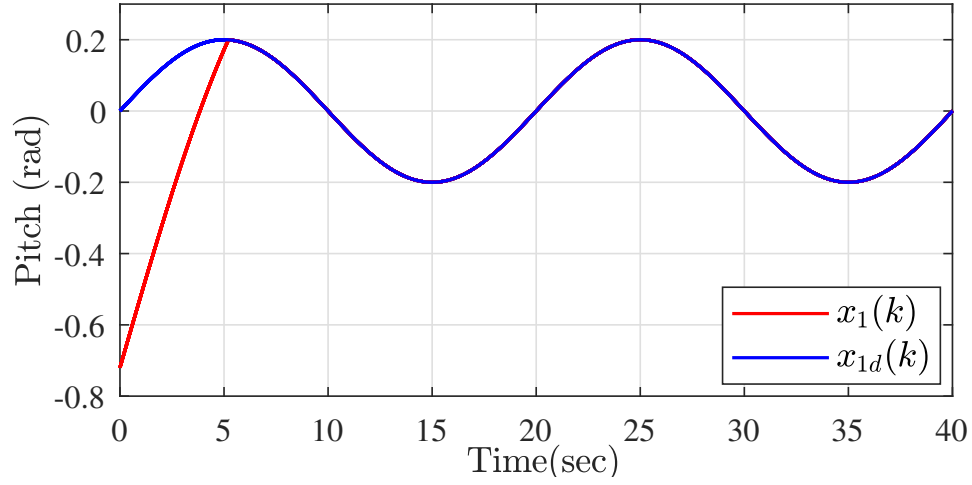
The simulation results with the proposed finite-time backstepping feedback control strategy are depicted in Fig. 7.2, showcasing the successful attainment of the desired pitch and yaw angle tracking within a finite time. In Fig. 7.2a and Fig. 7.2b, the tracking

of pitch angle $x_1(k)$ and yaw angle $x_3(k)$ is illustrated, respectively. The pitch tracking error $z_1(k)$ and yaw tracking error $z_3(k)$ are presented in Fig. 7.3a and Fig. 7.3b Fig. 7.4a and Fig. 7.4b depict the required control input. To reinforce this concept, we conducted a comparison with a control strategy developed in [10], as illustrated in Fig. 7.5 In Fig. 7.5a and Fig. 7.5b comparison of the proposed control scheme with [10] for the desired pitch and yaw angle tracking is presented. The results obtained from the comparison highlight the superiority of the proposed approach.

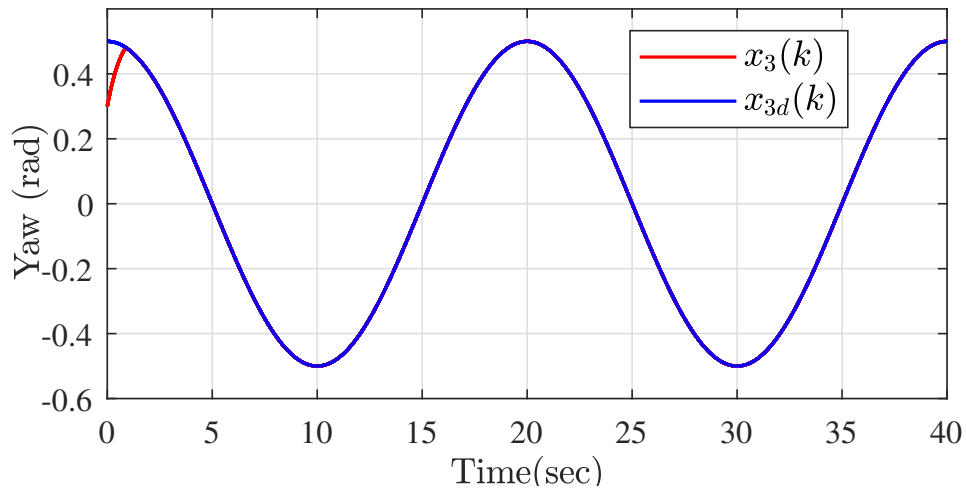
To validate both the practical feasibility and effectiveness of the proposed control strategy, we also conduct experimental validation using Quanser’s two DOF helicopter test beds, building upon the insights gained from the simulation results. The experimental setup is depicted in Fig.7.1 Employing identical reference signals and control parameter values as used in the simulations, we execute the experiments, and the corresponding experimental results presented in Fig.7.6 illustrate the successful achievement of the desired pitch and yaw angle tracking. The corresponding pitch angle and yaw angle tracking errors are displayed in 7.6a and 7.6b, respectively.

7.5 Conclusion

In this concise presentation, we introduce a discrete-time recursive backstepping controller tailored for a 2-DOF helicopter, ensuring the tracking of pitch and yaw angle reference trajectories within a finite time frame. We consider the mathematical representation of the 2-DOF helicopter system, based on Euler-Lagrange equations, along with its Euler-based discretizations model. The introduced discrete-time recursive backstepping controller provides a systematic approach to achieve finite-time tracking. Simulations, along with experimental demonstrations on a Quanser 2-DOF helicopter, highlight the effectiveness and control potential of our proposed controller. As part of future work, the oscillatory behavior observed during the transient phase will be specifically addressed, with the aim of reducing transient oscillations to further enhance the overall performance.

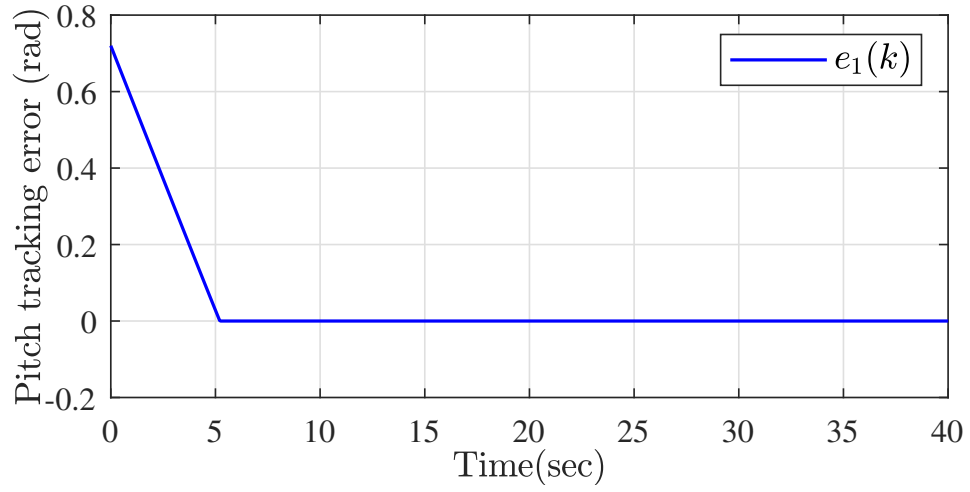


(a) Simulation results pitch angle.

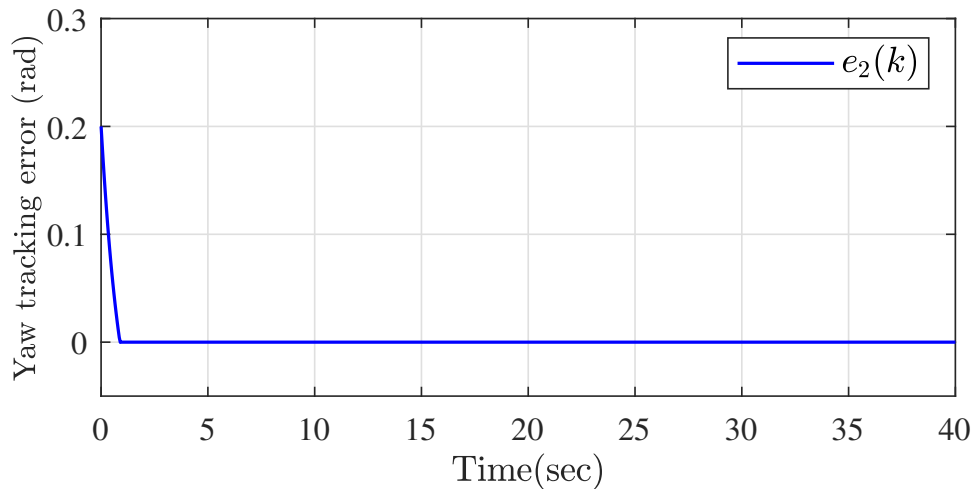


(b) Simulation results Yaw angle

Figure 7.2: Simulation result 2 DOF system

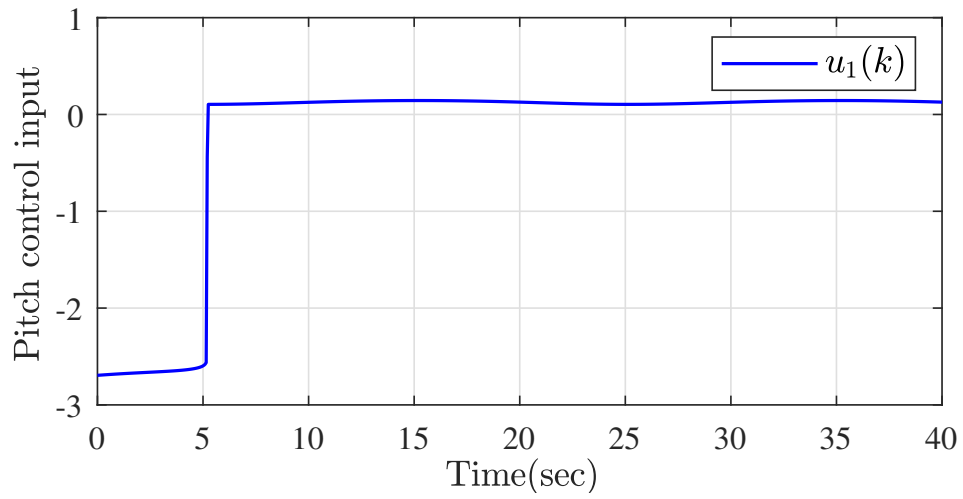


(a) Simulation result pitch error

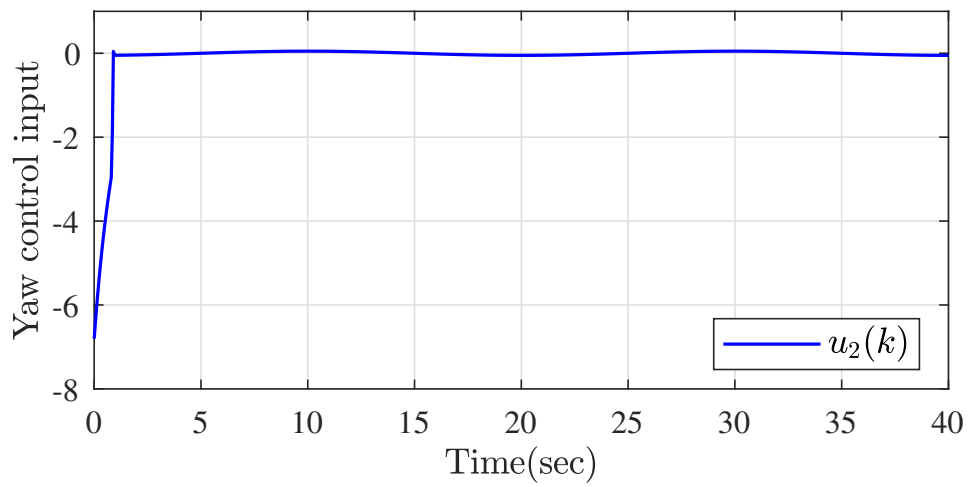


(b) Simulation result yaw error

Figure 7.3: Simulation result 2 DOF system

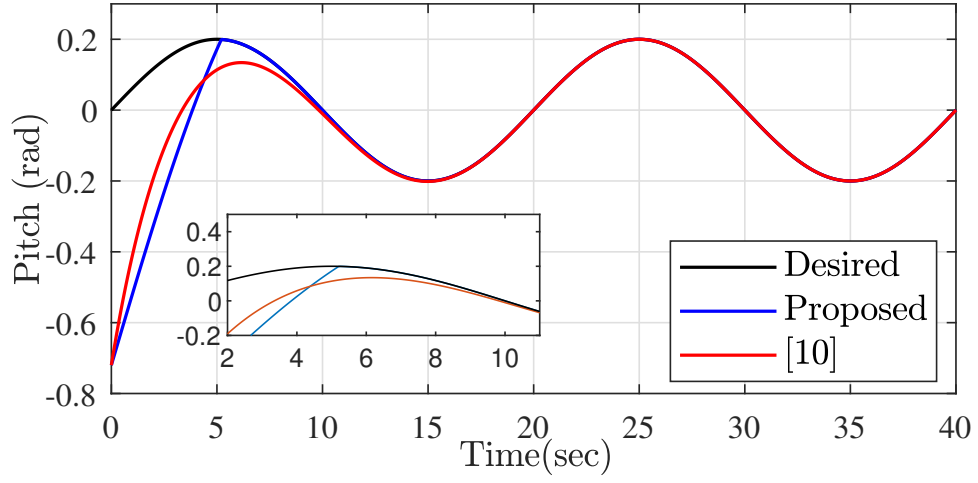


(a) Simulation result pitch control input

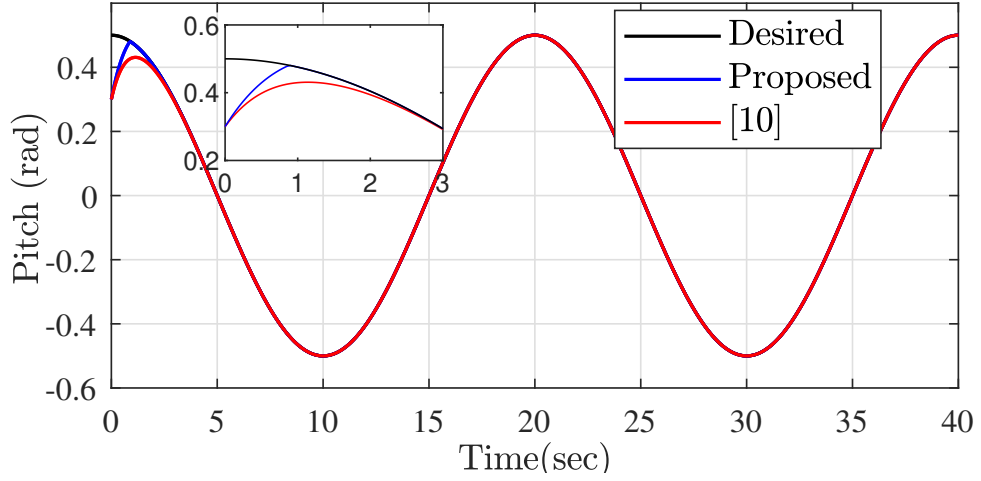


(b) Simulation result yaw control input

Figure 7.4: Simulation result 2 DOF system

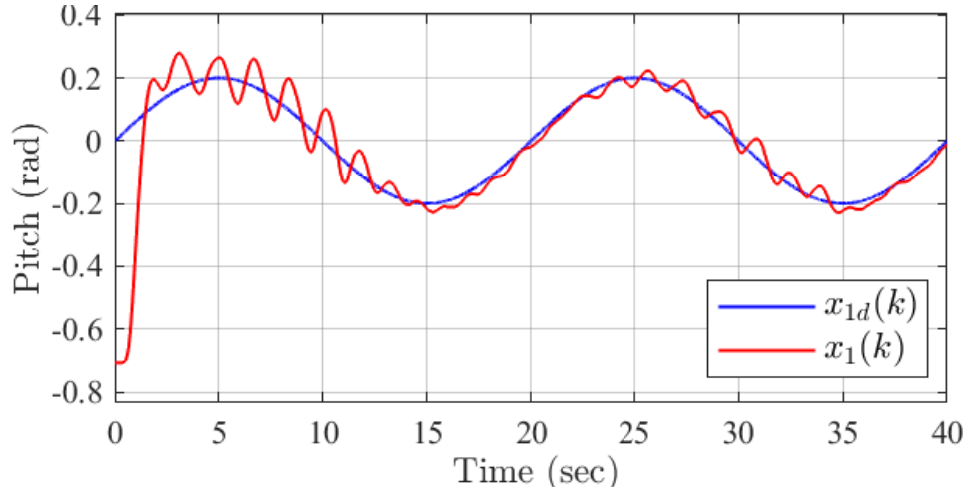


(a) Performance comparison for pitch angle between developed control and [10]

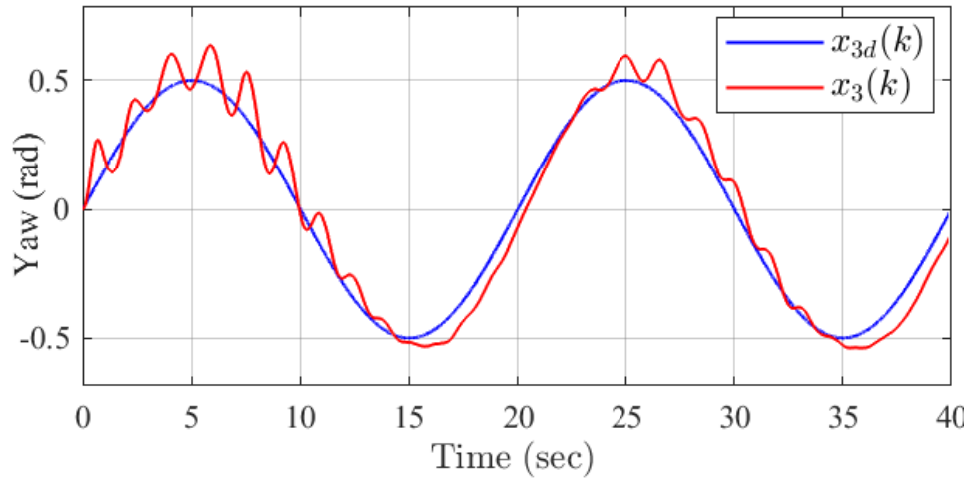


(b) Performance comparison for yaw angle between developed control and [10]

Figure 7.5: Simulation result comparison for 2 DOF system [10]



(a) Experimental result tracking pitch angle



(b) Experimental result tracking yaw angle

Figure 7.6: Experimental result tracking 2 DOF system

Chapter 8

Discrete-Time Sliding Mode Controller For Magnetic Levitation System Using Minima Operator

8.1 Introduction

Magnetic levitation systems (MLS) are versatile in their engineering applications, spanning a wide range of industries and technologies. Magnetic levitation offers an intriguing scenario for exploring nonlinear unstable dynamics and crafting corresponding control strategies. The system aims to elevate objects to a specific height by magnetizing the coil, employing the principle of non-contact. Previous studies have explored the control design using feedback linearization techniques [70] and [71], PID control [52], adaptive control [72], robust control [73] and using neural networks based control [74]. Additionally, continuous sliding mode control (CSMC) strategies, known for their robustness compared to other feedback techniques, have been implemented in MLS [75]- [76].

The levitation dynamics display inherent instability characterized by open-loop non-linear behavior. The objective of the control law introduced in [77] was to minimize the power consumption of the magnetic suspension system. Research in [78] has investigated magnetically levitated conveyors. Additionally, [79] outlines the design of an estimator employing variable structure systems principles. The majority of controllers developed thus far for MLS, including CSMCs, have been in continuous time. very few works [76], [80]- [81], have been reported for discrete control of MLS.

In this chapter, we have developed a DSMC for MLS utilizing RLs [2]. Subsequently, we compared the developed results with previous results. This approach proves more practical for MLS compared to previous methods, chattering associated with quasi sliding mode domain reduces its practical implementation. Our simulation results validate the effectiveness of the minimum operator-based DSMC for magnetic levitation system control.

8.2 System Description

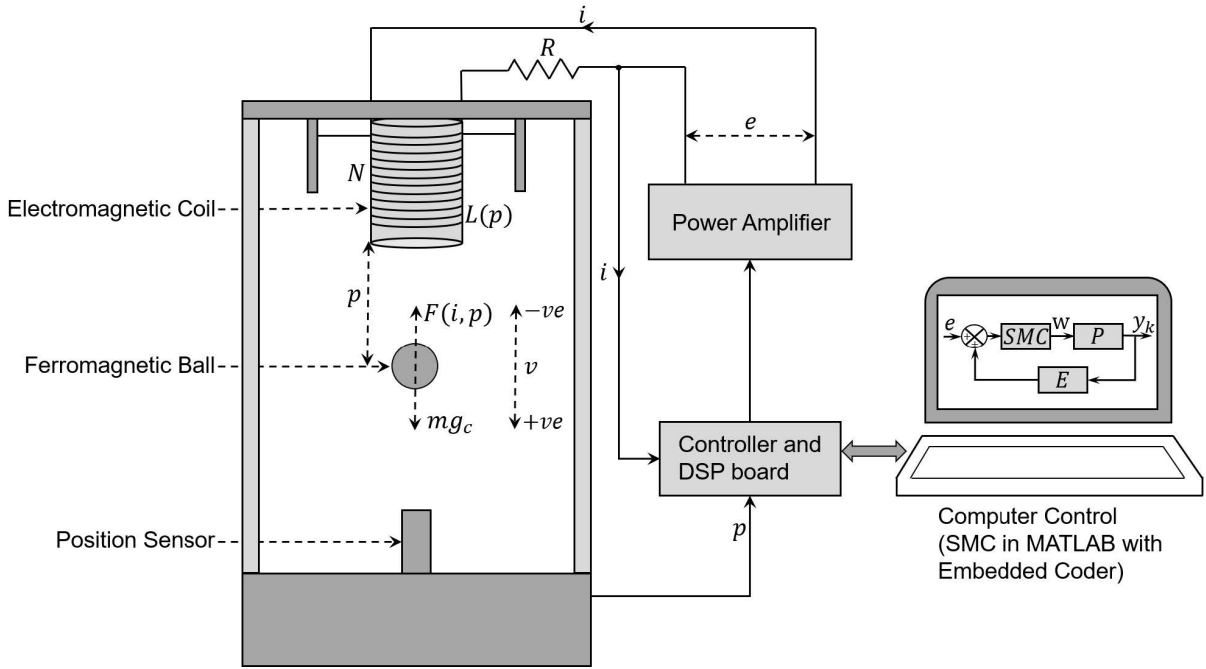


Figure 8.1: Schematic of a MLS

The illustrated MLS, depicted in Fig 8.1, consists of a sphere exhibiting ferromagnetic properties, which is suspended within a magnetic field generated by an electromagnet. This configuration allows the electromagnet to exert an attractive force on the ferromagnetic sphere, counteracting the force of gravity and thus keeping the sphere at a predetermined elevation. The governing equation of this system is delineated below.

$$\begin{aligned} \frac{dp}{dt} &= v, Ri + \frac{d(L(p)i)}{dt} = e, m \frac{dv}{dt} = mg_c - F(i, p) \\ F(i, p) &= Q \left(\frac{i}{p} \right)^2, Q = \frac{\mu_0 AN^2}{4}, L(p) = L_1 + \frac{2Q}{p} \end{aligned} \quad (8.1)$$

The following variables are assigned as: $p = x_1$ (representing the position of the ball),

$v = x_2$ (representing the velocity of the ball), $i = x_3$ (denoting the current in the coil), R (indicating the resistance of the coil), L (referring to the inductance of the coil), $g_c =$ gravitational constant, $Q =$ magnetic force constant, and $m =$ mass of the ball also, L_1 denotes the parameter related to electromagnetic coil inductance.

Therefore, the dynamic formulation of the MLS model (8.1) can be expressed in the guise of nonlinear dynamics as follows:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{8.2}$$

Considering the state vector x as $[x_1, x_2, x_3]^T$ where, $x_1 = p, x_2 = v, x_3 = i$, input as u and output y , the state-space model of the MLS will be expressed as :

$$\begin{aligned}\frac{dx_1}{dt} &= x_2, \frac{dx_2}{dt} = g - \frac{Q}{m} \left(\frac{x_3}{x_1}\right)^2 \\ \frac{dx_3}{dt} &= -\frac{R}{L}x_3 + \frac{2Q}{L} \left(x_2 \frac{x_3}{x_1^2}\right) + \frac{u}{L}\end{aligned}\tag{8.3}$$

The control strategies aim to guide the variables x_1 , x_2 , and x_3 towards their target equilibrium values x_{1d} , x_{2d} , and x_{3d} , correspondingly where x_{3d} satisfies $x_{3d} = x_{1d} \sqrt{\frac{gm}{Q}}$.

8.2.1 Discrete Linear Model of the MLS

The nonlinear representation of the MLS provided in equation (8.2) undergoes precise linearization through a feedback linearizing transformation.

Consider the transformation $z = (z_1, z_2, z_3)^T$ as:

$$z = T(x) = \begin{bmatrix} x_1 - x_{1d} \\ x_2 \\ g - \frac{Q/m \cdot (x_3/x_1)^2}{1} \end{bmatrix}\tag{8.4}$$

The system in the form of state space will be as follows:

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \alpha(z) + \theta(z)u \end{bmatrix}\tag{8.5}$$

where,

$$\begin{aligned}\alpha(z) &= 2(g - z_3) \left(\left(1 - \frac{2Q}{L(z_1 + x_{1d})} \right) \frac{z_2}{z_1 + x_{1d}} + \frac{R}{L} \right) \\ \theta(z) &= -\frac{2Q}{L(z_1 + x_{1d})} \left(\sqrt{\frac{Q}{m}} (g - z_3) \right).\end{aligned}\tag{8.6}$$

Table 8.1: Parameters of the MLS.

Parameter	Symbol	Value
Coil resistance	R	28.7 Ω
Coil inductance	L1	0.65 H
Gravitational constant	g	9.81 m/sec ²
Magnetic force constant	Q	1.8×10^{-4}
Levitated object mass	m	$11.87 \times 10^{-3} Kg$
Plant initial condition	x1	0.0255 m
	x2	0 m/sec
	x3	1.1 A
Desired steady state values		
	x1d	0.01 m
	x2d	0 m/sec
	x3d	0.2884 A

To nullify the nonlinear elements in equation (8.5), the design of the outer-loop feedback controller u takes the following form:

$$u = -\theta^{-1}(z)(\alpha(z) + v) \quad (8.7)$$

where v represents the inner-loop feedback controller. Therefore, the magnetic levitation system's nonlinear model (8.2) can be expressed as after the application of control input (8.7),

$$\dot{z} = Az + Bu \quad y = Cz; \quad (8.8)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

The linear model undergoes discretization with a sampling time of $\tau = 0.1$ seconds. Consequently, the continuous-time system described in (8.8) is transformed into a discrete-time system (DTS) is as follows:

$$z(k+1) = \Phi z(k) + \Gamma u(k) \quad y(k) = C\Phi z(k); \quad (8.9)$$

where the state $z(k) \in \mathbb{R}^3$, the input $u(k) \in \mathbb{R}$, and the output $y(k) \in \mathbb{R}$. Moreover, $\Phi = e^{A\tau} \in \mathbb{R}^{3 \times 3}$, $\Gamma = A^{-1}(e^{A\tau} - I) \in \mathbb{R}^{3 \times 1}$ and $C \in \mathbb{R}^{1 \times 3}$. the DTS matrices are as follows:

$$\Phi = \begin{pmatrix} 1 & 0.1 & 0.005 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \Gamma = \begin{pmatrix} 0.0002 \\ 0.005 \\ 0.1 \end{pmatrix}.$$

After the discretization of the system model.

8.3 DSMC For MLS

The switching function is defined as $\aleph(k) := \rho(z(k))$, and the corresponding sliding manifold is defined by the level set $\rho^{-1}(0) := \{z \in D : \rho(z) = 0\}$. For analytical purposes, the switching function is defined as follows:

$$\aleph(k) = c^T z(k), \tag{8.10}$$

where $c \in \mathbb{R}^{n \times 1}$ with $c^T \Gamma \neq 0$.

RL1 for DTS

$$\aleph(k+1) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \omega) \tag{8.11}$$

RL2 for DTS

$$\aleph(k+1) = \aleph(k) - \omega \text{sign}[\aleph(k)] \min\left(\frac{|\aleph(k)|}{\omega}, |\aleph(k)|^\beta\right) \tag{8.12}$$

8.3.1 Controller Design Based on RL1

Using switching function (8.10),

$$\aleph(k+1) = c^T z(k+1),$$

consider the system described by (8.9) and RL1 (8.11).

$$c^T (\Phi z(k) + \Gamma u(k)) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \omega)$$

Now, the control input will be expressed as

$$u(k) = -(c^T \Gamma)^{-1} (c^T \Phi z(k) - \aleph(k) + \text{sign}[\aleph(k)] \min(|\aleph(k)|, \omega)) \quad (8.13)$$

Theorem 8.1 *Consider system (8.9) and RL (8.11), using control input (8.13) the absolute value of the switching function of the system goes to zero for some $t \geq I(\aleph_0)$ where $I(\aleph_0) = \frac{|\aleph_0|}{\omega}$ is the settling time function (STF).*

Proof When the control input obtained from equation (8.13) is implemented within the system (8.9), the system adheres to the RL1.

$$\aleph(k+1) = \aleph(k) - \text{sign}[\aleph(k)] \min(|\aleph(k)|, \omega)$$

Consider the scenario where $|\aleph(0)| \leq \omega$, then $\aleph(k+1) = 0$ implies in the next step $|\aleph(k+1)| = 0$. Consequently, $|\aleph(k+1)| = 0$, with $I(\aleph_0) = 1$. Moreover, for all $k \geq I(\aleph_0)$, $|\aleph(k)| = 0$, demonstrating the existence of QSM. For another scenario, if $|\aleph(0)| > \omega$, then $|\aleph(k+1)| = |\aleph(k) - \omega \text{sign}[\aleph(k)]|$. As $|\aleph(k)| > \omega$, it follows that $|\aleph(k+1)| \leq |\aleph(k)| - \omega$. Additionally, it can be deduced as:

$$\begin{aligned} |\aleph(k)| &\leq |\aleph(k-1)| - \omega \\ &\leq |\aleph(k-2)| - 2\omega \\ &\leq \\ &\vdots \\ &\leq |\aleph(0)| - t\omega \end{aligned}$$

If $(|\aleph(0)| - t\omega) \leq \omega$, then $|\aleph(k)| \leq \omega$, implying $|\aleph(k+1)| = 0$ for all $t \geq \frac{|\aleph_0| - \omega}{\omega}$. The STF is $I(\aleph_0) = \frac{|\aleph_0|}{\omega}$. Therefore, $|\aleph(k)| = 0$ assures within finite time steps.

8.3.2 Controller Design Based on RL2

Using switching function (8.10),

$$\aleph(k+1) = c^T z(k+1),$$

consider the system described by (8.9) and RL2 (8.12).

$$c^T (\Phi z(k) + \Gamma u(k)) = \aleph(k) - \gamma_1 \text{sign}[\aleph(k)] \min \left(\frac{|\aleph(k)|}{\gamma_1}, |\aleph(k)|^\beta \right)$$

Now, the control input will be expressed as

$$u(k) = -(c^T \Gamma)^{-1} \left(c^T \Phi z(k) - \aleph(k) + \gamma_1 \text{sign}[\aleph(k)] \min \left(\frac{|\aleph(k)|}{\gamma_1}, |\aleph(k)|^\beta \right) \right) \quad (8.14)$$

Theorem 8.2 Consider system (8.9) and RL (8.12). If $\frac{|\aleph(0)|}{\gamma_1} \leq |\aleph(0)|^\beta$ with $\gamma_1 \in (0, 1)$ and $\beta > 0$, using control input (8.14), the absolute value of the switching function of the system goes to zero for some $t \geq I(\aleph_0)$, where $I(\aleph_0) \leq \left\lceil \log_{[1-\gamma_1|\aleph_0|^{\beta-1}]} \frac{\gamma_1^{\frac{1}{1-\beta}}}{|\aleph_0|} \right\rceil + 1$.

Proof When the control input obtained from equation (8.14) is implemented within the system (8.9), the system adheres to the RL2:

$$\aleph(k+1) = \aleph(k) - \gamma_1 \text{sign}[\aleph(k)] \min \left(\frac{|\aleph(k)|}{\gamma_1}, |\aleph(k)|^\beta \right).$$

Given the simplistic scenario, if $\frac{|\aleph(0)|}{\gamma_1} \leq |\aleph(0)|^\beta$, then

$$\begin{aligned} \aleph(k+1) &= \aleph(k) - \text{sign}[\aleph(k)] |\aleph(k)| \\ \Rightarrow \aleph(k+1) &= 0 \end{aligned}$$

In the subsequent time step. Hence, we obtain

$|\aleph(k+1)| = 0$, with $I(\aleph_0) = 1$. Also, for all $t \geq I(\aleph_0)$, $|\aleph(k+1)| = 0$, showcasing the sliding reaching existence. Taking into account the following scenario, if $\frac{|\aleph(0)|}{\gamma_1} > |\aleph(0)|^\beta$, then

$$\aleph(k+1) = \aleph(k) - \gamma_1 \text{sign}[\aleph(k)] |\aleph(k)|^\beta.$$

We can express it as,

$$|\aleph(k+1)| = |\aleph(k) - \gamma_1 \text{sign}[\aleph(k)] |\aleph(k)|^\beta|.$$

Since $|\aleph(k)| > |\aleph(k)|^\beta$, it follows that,

$$|\aleph(k+1)| \leq |\aleph(k)| - \gamma_1 |\aleph(k)|^\beta.$$

Further, it can be written as

$$\begin{aligned} |\aleph(k)| &\leq |\aleph(k-1)| (1 - \gamma_1 |\aleph(k-1)|^{\beta-1}) \\ &\vdots \\ &\leq |\aleph(0)| (1 - \gamma_1 |\aleph(0)|^{\beta-1}) \dots \\ &\quad (1 - \gamma_1 |\aleph(k-1)|^{\beta-1}) \\ &\vdots \end{aligned}$$

$$|\aleph(k)| \leq |\aleph(0)|(1 - \gamma_1|\aleph(0)|^{\beta-1})^k \quad (8.15)$$

If $|\aleph(0)|(1 - \gamma_1|\aleph(0)|^{\beta-1}) \leq \gamma_1|\aleph(k)|^\beta$, then from (8.15), $|\aleph(k)| \leq \gamma_1|\aleph(k)|^\beta \Rightarrow |\aleph(k+1)| = 0, \forall k \geq \left\lceil \log_{[1-\gamma_1|\aleph(0)|^{\beta-1}]} \frac{\gamma_1^{\frac{1}{1-\beta}}}{|\aleph(0)|} \right\rceil + 1$. Moreover, the settling time expression is given by $I(\aleph_0) \leq \left\lceil \log_{[1-\gamma_1|\aleph(0)|^{\beta-1}]} \frac{\gamma_1^{\frac{1}{1-\beta}}}{|\aleph(0)|} \right\rceil + 1$. Thus, it can be inferred that the absolute value of the switching function approaches zero within a finite number of time steps.

8.4 Results and Discussion

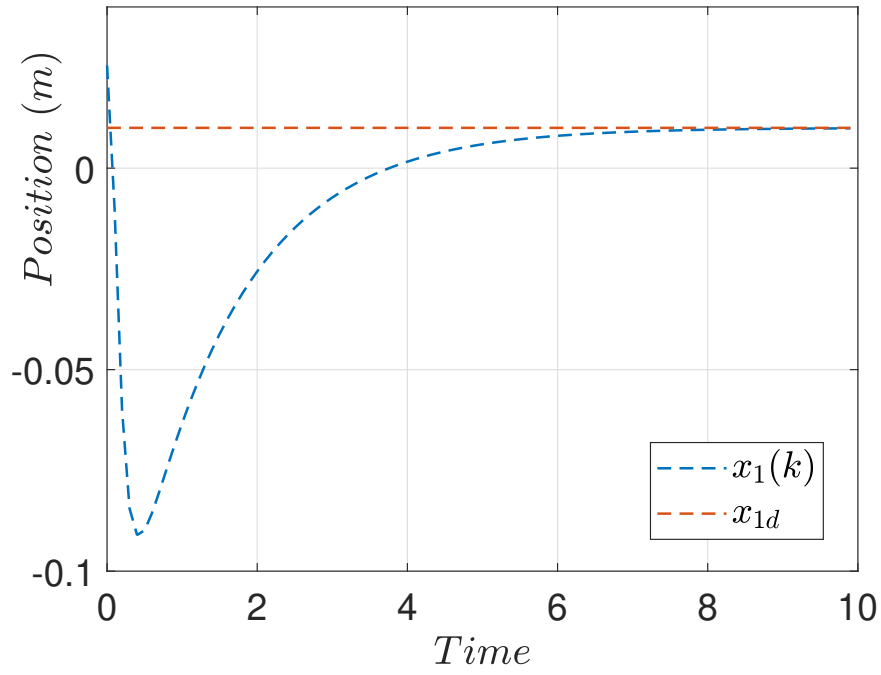
The simulation outcomes demonstrate the efficacy of two devised DSMC controllers. Employing parameters $\omega = 0.5$, $\gamma_1 = 0.5$, $\beta = 0.1$ and $c = [0.66; 1; 0.12]^T$, along with initial conditions and simulation parameters detailed in Table I. Fig 8.2a, Fig 8.2b, Fig 8.3a, Fig 8.3b and Fig 8.4 portrays the cumulative results for the RL1-based DSMC controller, while Fig 8.5a, Fig 8.5b, Fig 8.6a, Fig 8.6b and Fig 8.7 illustrates the cumulative results for the RL2-based DSMC controller. Additionally, Fig 8.8a, Fig 8.8b, Fig 8.9a, Fig 8.9b and Fig 8.10 presents a comparison between RL1, RL2, and prior work by Bandal et al. [76]. Notably, Bandal et al. using Gao's reaching law utilized quasi-sliding mode (QSM) with a band of 0.0625, depicted in Fig 8.3b and Fig 8.6b, whereas both RL1 and RL2 controllers exhibit an ideal sliding mode akin to CSMC. Fig 8.2a and Fig 8.5a represent the position versus time plot, showcasing precise attainment of the desired state x_{1d} (0.01m) for RL1 and RL2 within a finite time frame, respectively.

Furthermore, Fig 8.2b, Fig 8.3a and Fig 8.3b respectively demonstrate velocity, switching surface and current control input versus time plots for a RL1 based discrete-time sliding mode controller. Also, Fig 8.5b, Fig 8.6a and Fig 8.6b respectively demonstrate velocity, switching surface and current control input versus time plots for an RL2 based discrete-time sliding mode controller. The ferromagnetic ball achieves its desired steady-state position of 0.01 m in approximately 6.5 seconds, with the current stabilizing around 0.285 Amperes, near the targeted value of 0.2884 A with an RL1-based controller. Also with RL2-based controller ensures the desired steady-state position of 0.01 m in approximately 6.2 seconds, with the current stabilizing around 0.281 Amperes. The previous work shows minor oscillations around the desired values attributed to the quasi-sliding motion of the state trajectory, whereas RL1 and RL2 based controller shows ideal sliding

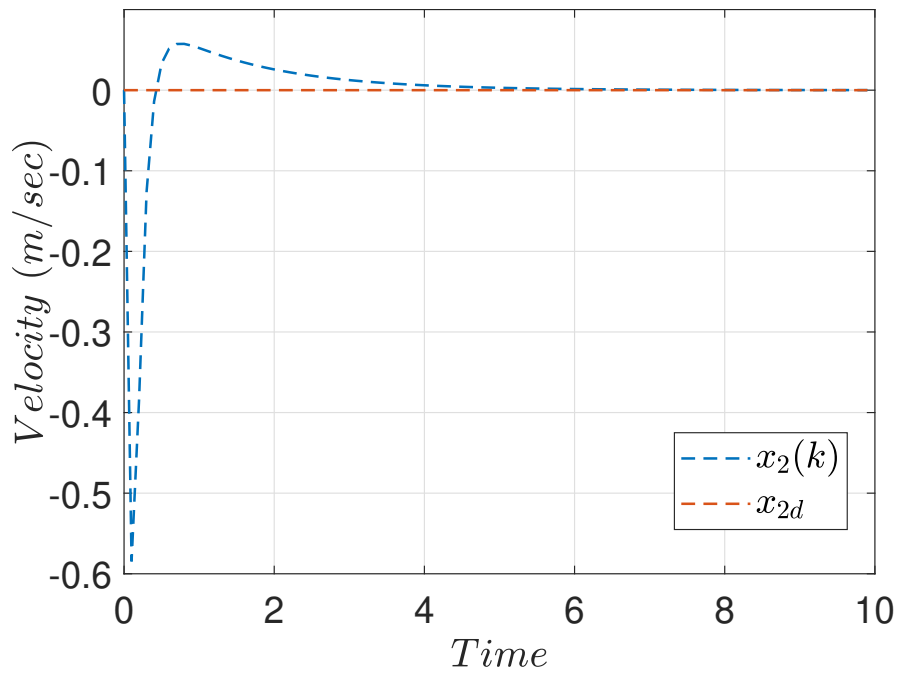
motion along the surface.

8.5 Conclusions

This chapter introduces two DSMC control strategies for a magnetic levitation system. Its fast convergence rate characterizes DSMC based on RL1 and RL2 as compared to Gao's based control. These regulations tackle the constraints linked with Gao's reaching technique and Utkin's equivalent control method. With the proposed DSMC, the trajectory reaches a sliding surface, closely resembling its continuous-time counterpart without disturbances, thereby enhancing robustness. Moreover, DSMC proves more practical for magnetic levitation systems compared to previous DSMC-based control methods. The proposed laws for MLS successfully eliminate chattering while ensuring minimal control effort. Their effectiveness is supported by simulation results. Moving forward, the prospect of experimental validation using real-world setups presents an exciting opportunity for future research.

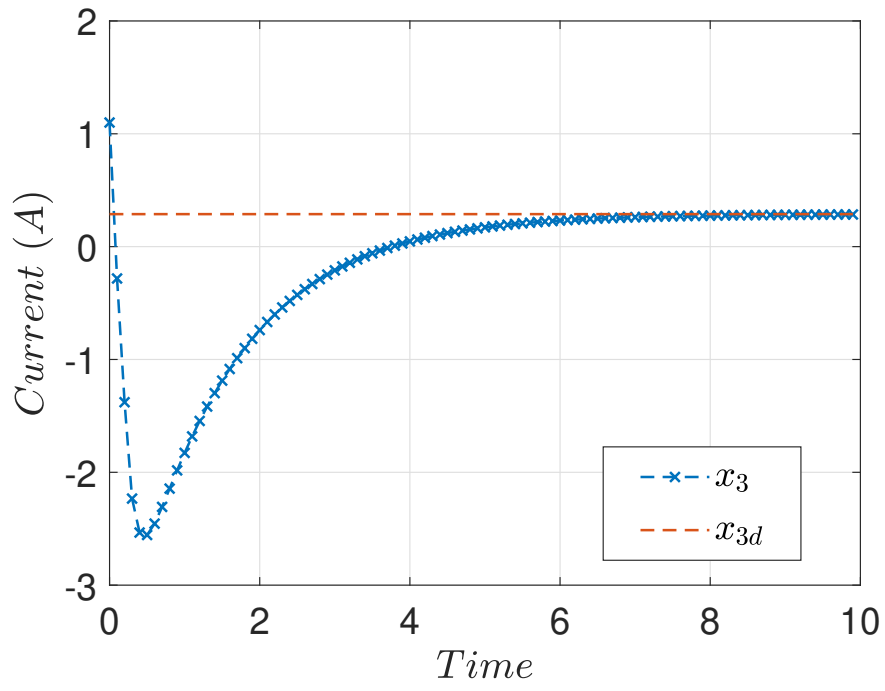


(a) RL1 state variable $x_1(k)$

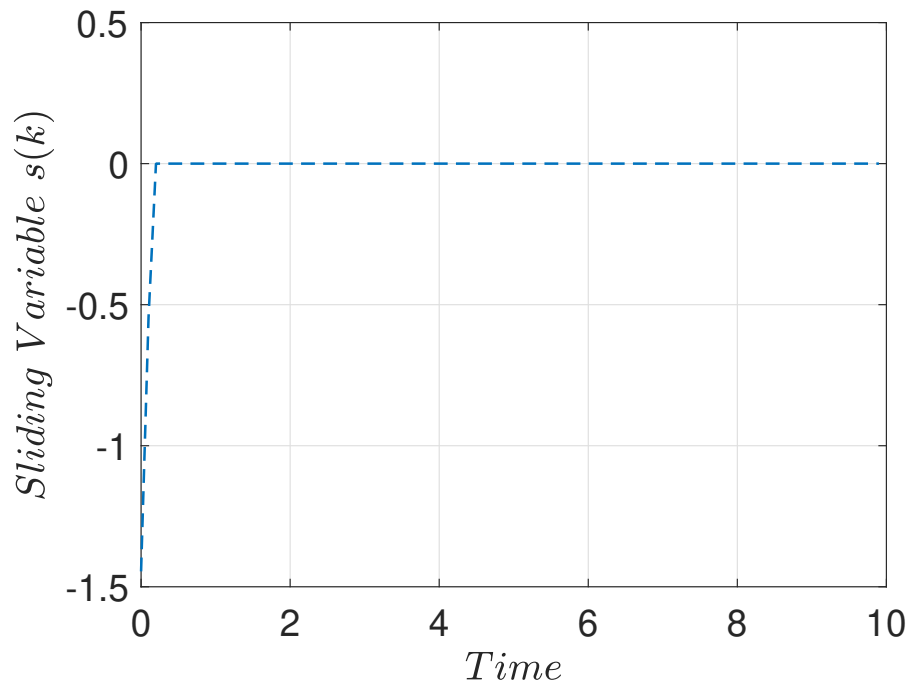


(b) RL1 state variable $x_2(k)$

Figure 8.2: RL1 State Variables



(a) RL1 state variable $x_3(k)$



(b) RL1 sliding variable

Figure 8.3: RL1 State Variables and Sliding variable

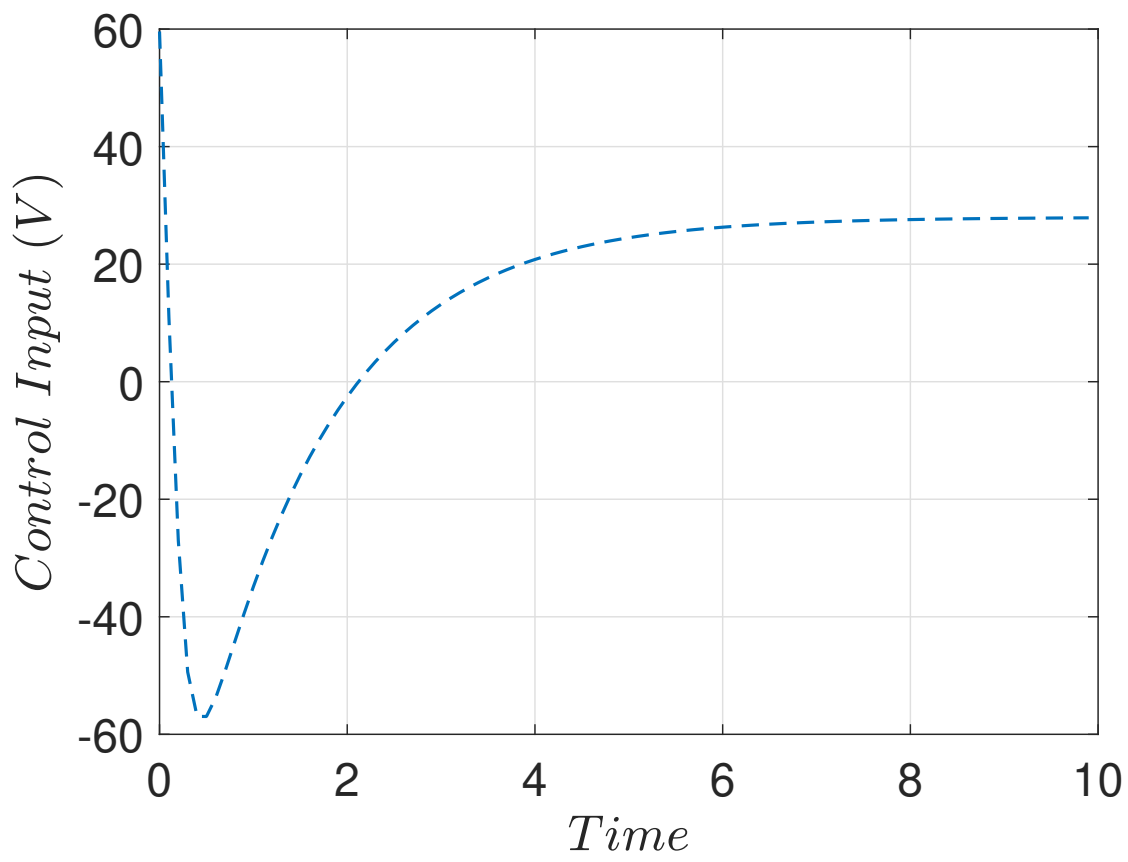
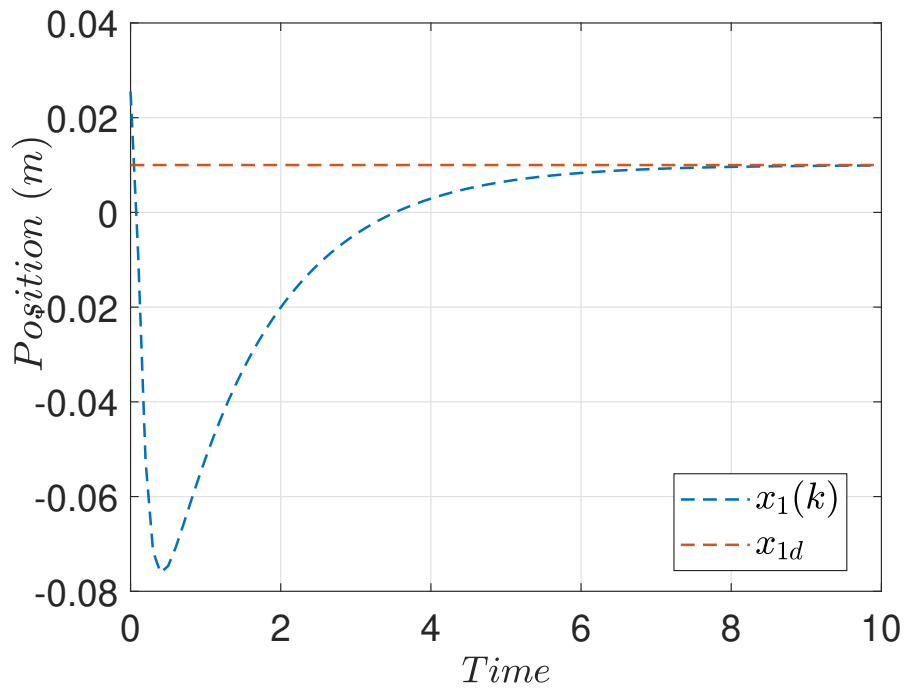
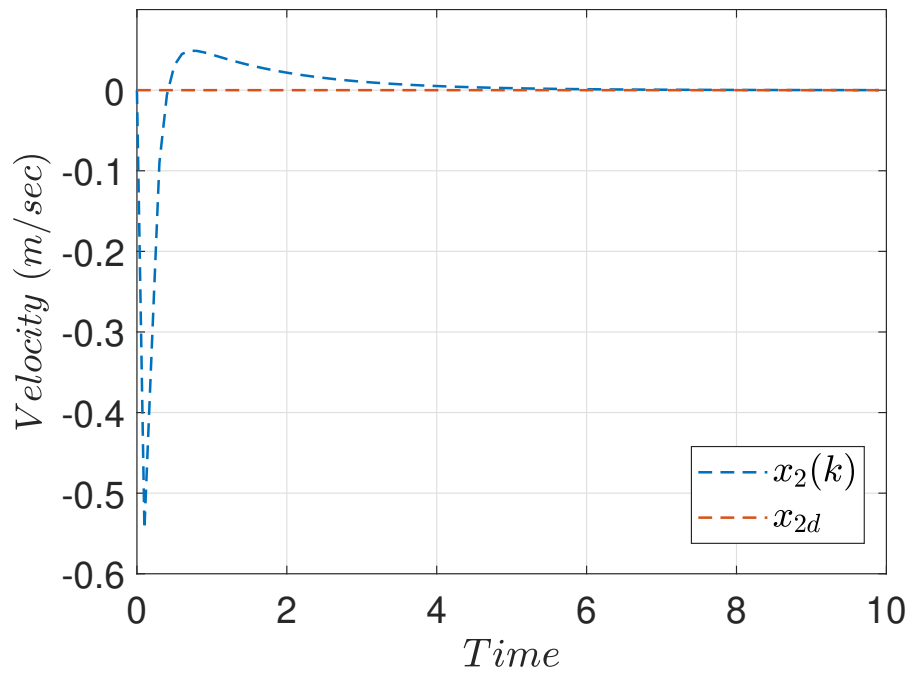


Figure 8.4: RL1 Control Input

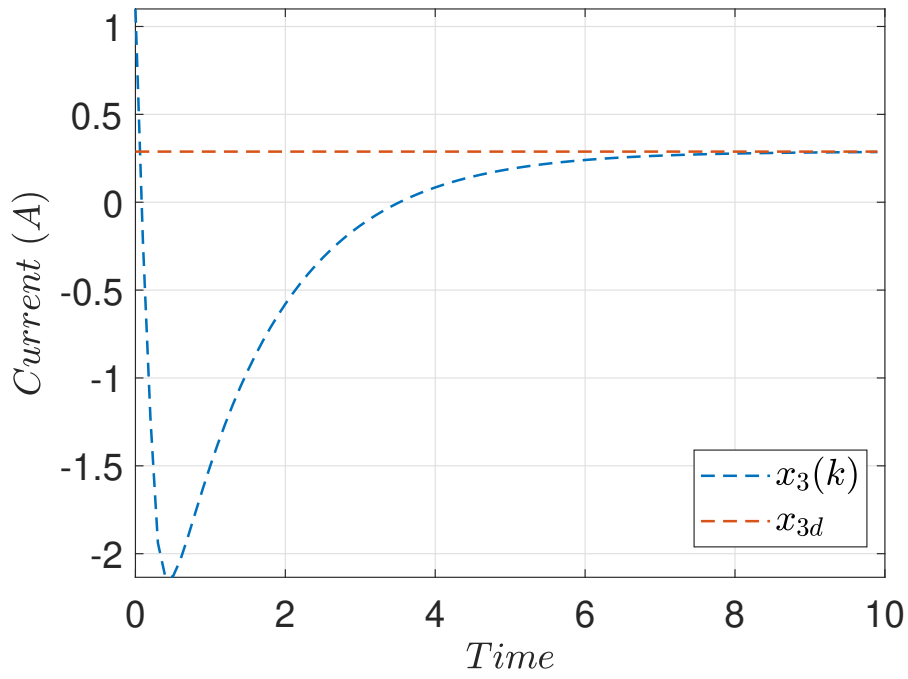


(a) RL2 state variable $x_1(k)$

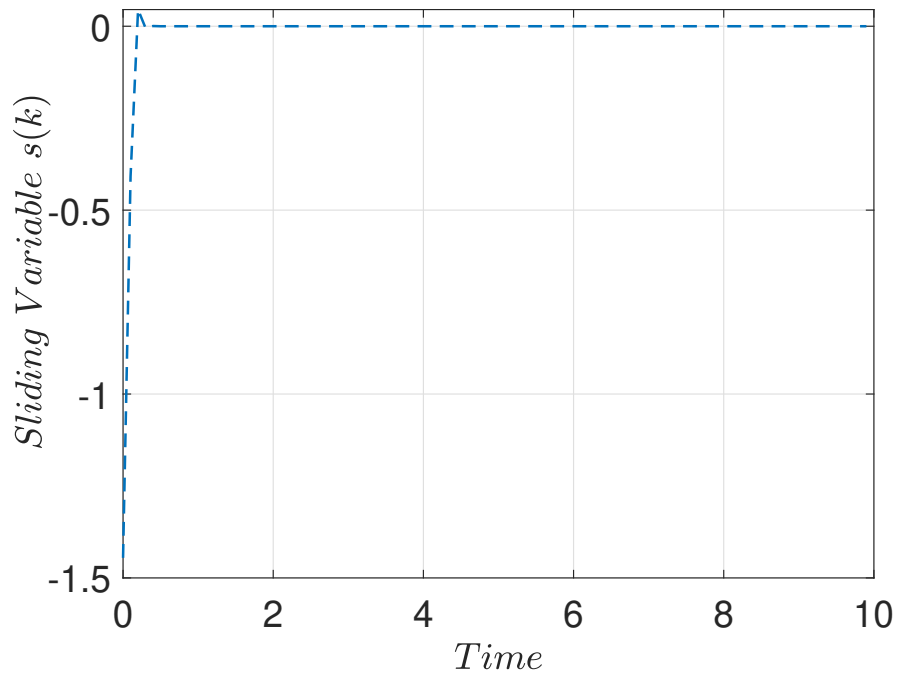


(b) RL2 state variable $x_2(k)$

Figure 8.5: RL2 State Variables



(a) RL2 state variable $x_3(k)$



(b) RL2 sliding variable

Figure 8.6: RL2 State Variables and Sliding variable

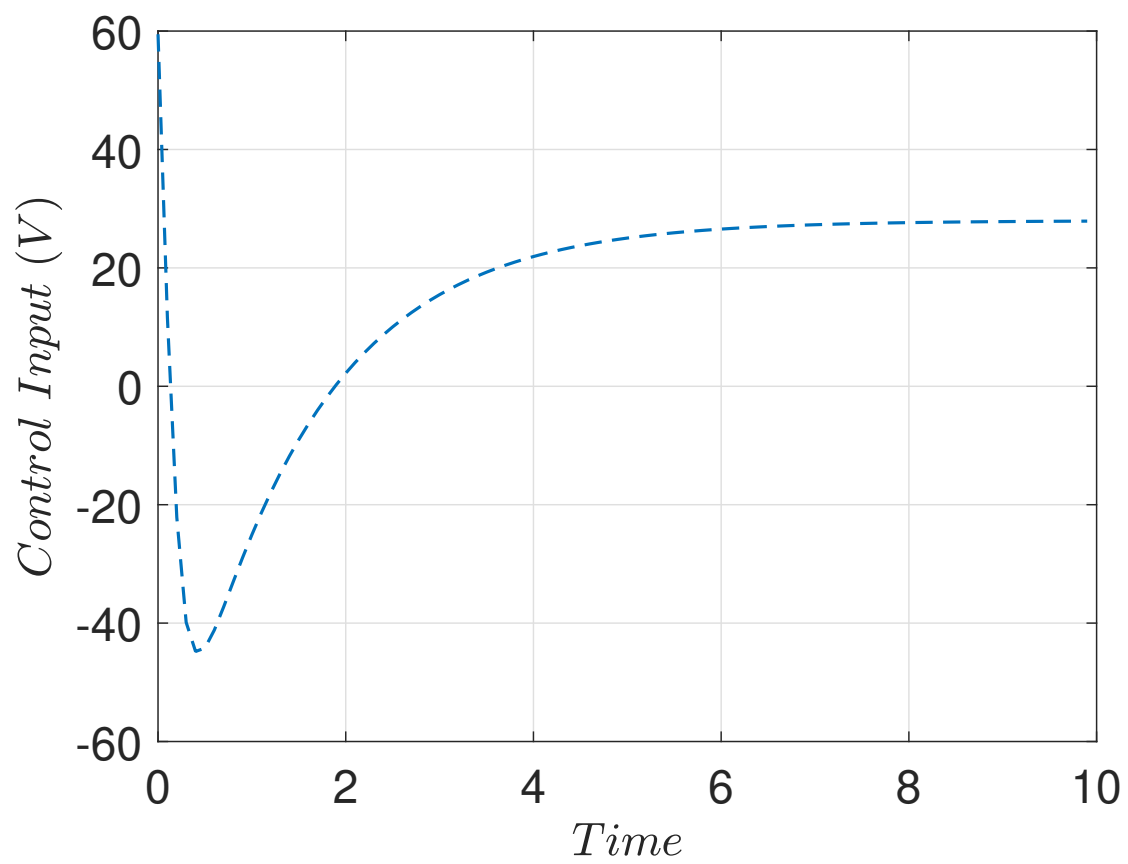
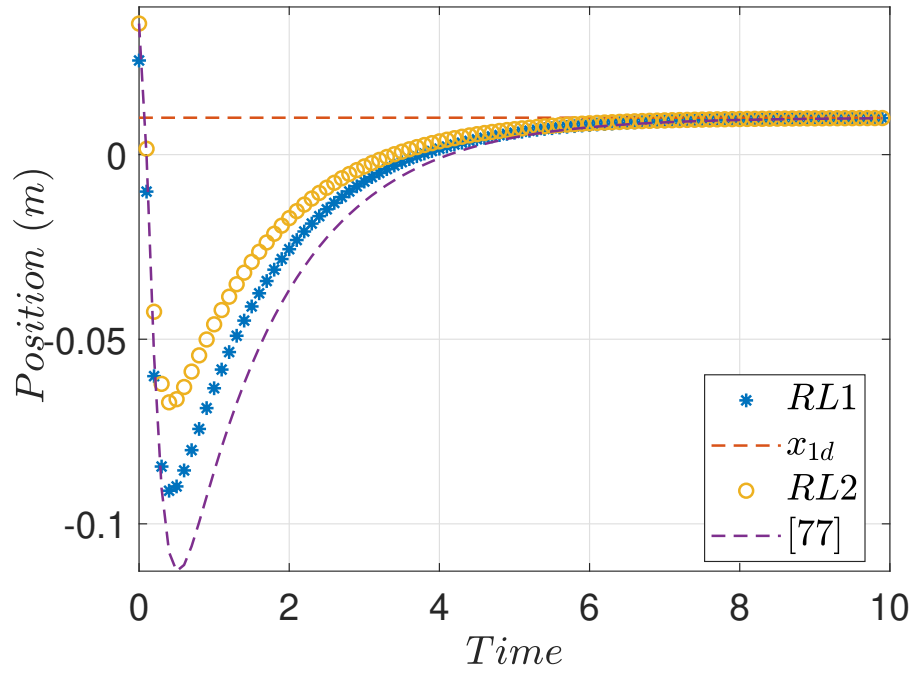
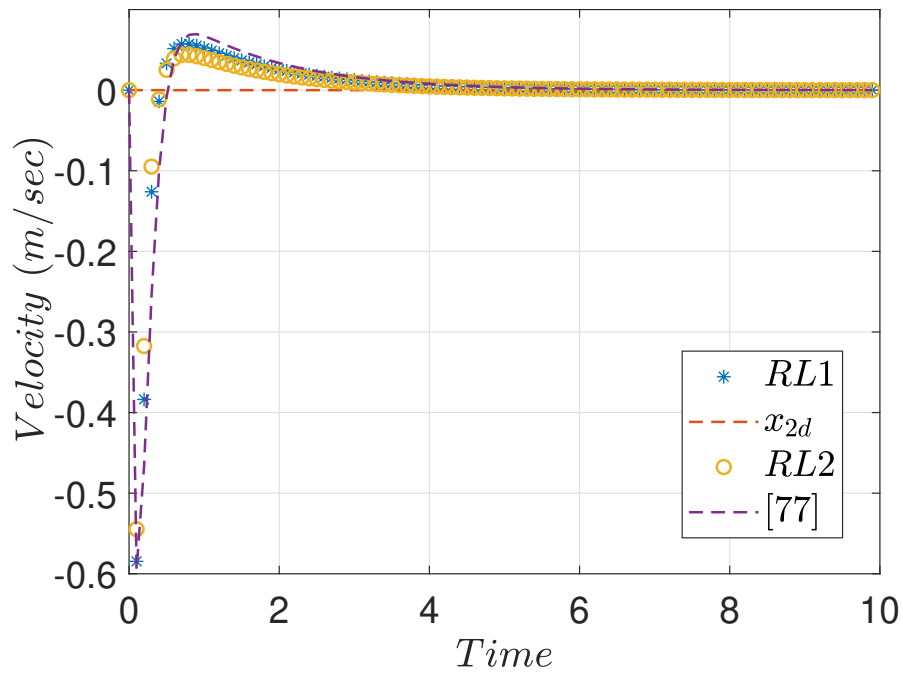


Figure 8.7: RL2 Control Input

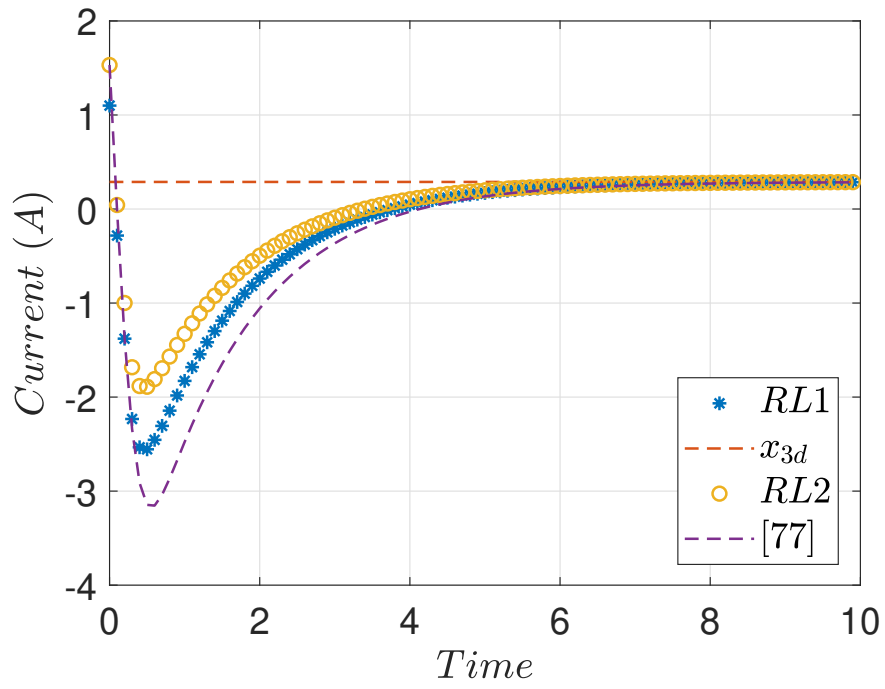


(a) Comparison state variable $x_1(k)$

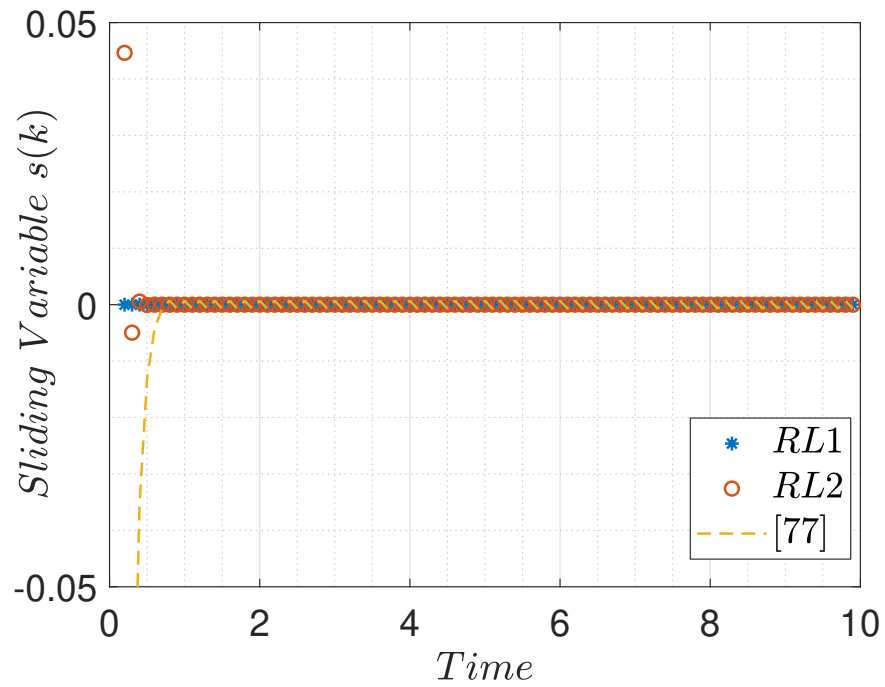


(b) Comparison state variable $x_2(k)$

Figure 8.8: RL1, RL2 and [77] Comparison



(a) Comparison state variable $x_3(k)$



(b) Comparison sliding variable

Figure 8.9: RL1, RL2 and [77] Comparison

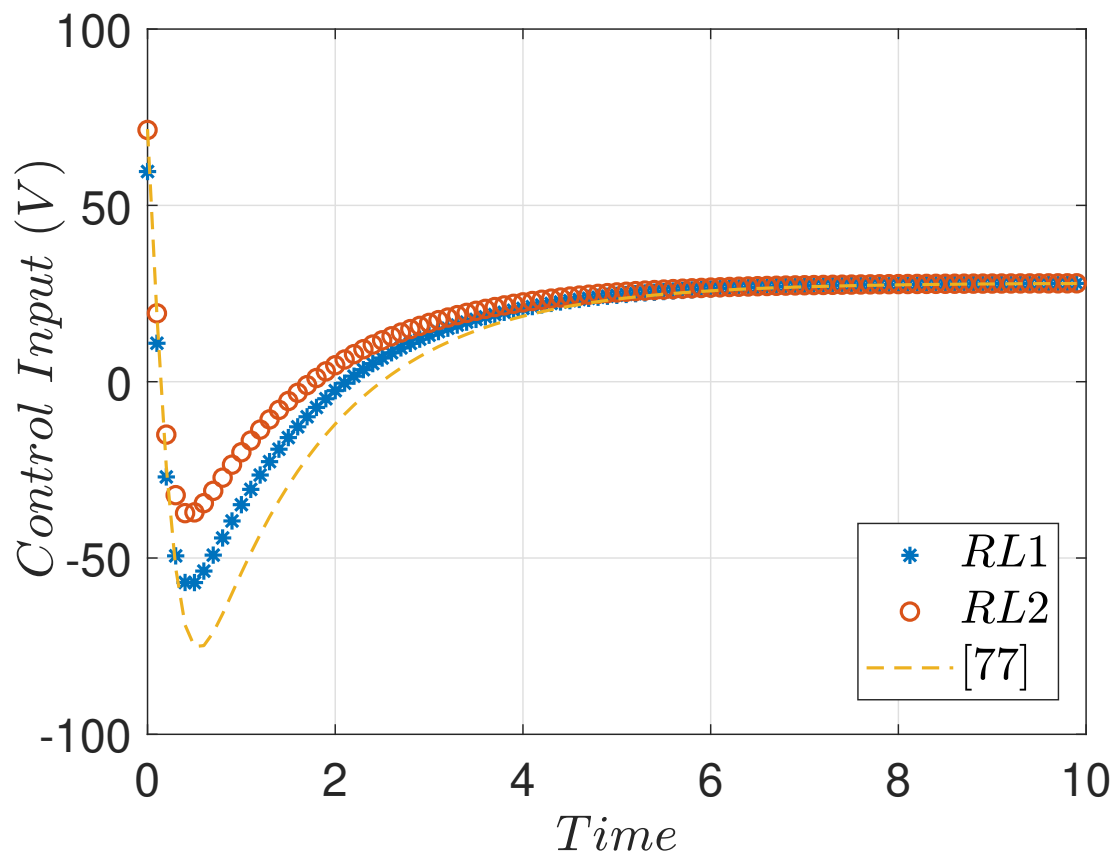


Figure 8.10: Comparison control input for RL1, RL2 and [77]

Chapter 9

Summary and Future Perspectives

In the brief, we summarize the work done in the thesis by pointing out the main results, limitations and proposed future investigations.

9.1 Research Summary

This thesis presents two innovative DDC schemes based on DSMC tailored for DTS operating under disturbances and unknown dynamics. The primary contributions lie in developing robust, MFAC strategies that achieve finite-time convergence while minimizing design complexity and control effort.

A central theme of the work is the use of the CFDL and minimum-operator-based DSMC techniques, which together form a simplified yet powerful DDC framework. These approaches exhibit improved disturbance rejection and reduced QSDM without relying on detailed system models, making them suitable for complex, nonlinear environments. Numerical simulations and experimental validation on a liquid-level control system confirm the practical feasibility of the CFDL-MDSMC controllers.

In addressing prescribed performance control (PPC) challenges, the thesis proposes an adaptive DSMC design using the Full Form Dynamic Linearization (FFDL) method and an error transformation function (ETF) to handle constrained tracking errors. The resulting schemes outperform traditional methods by maintaining errors within predefined asymmetric bounds with reduced control input, as demonstrated through simulations on a single-link robotic manipulator.

A novel contribution integrates minimum-operator-based reaching laws (RLs) with

a multi-rate output feedback (MROF) mechanism to eliminate switching control inputs, thereby mitigating chattering and reducing dependency on full-state feedback. Comparative results show significant improvement over prior art, including reduced steady-state error, enhanced precision, and robustness.

Further, the study introduces an adaptive RL-based DSMC with improved gain adaptation using ceiling functions and minimum operators, offering enhanced convergence rates and better reaching time estimation, while avoiding chattering. Theoretical analysis and simulations underscore its advantages under bounded disturbances.

The thesis also demonstrates a discrete-time recursive backstepping control for a two-degree-of-freedom (2-DOF) helicopter system. By leveraging an Euler-discretized model, the controller achieves finite-time trajectory tracking of pitch and yaw angles, validated through both simulations and physical experiments.

Finally, two novel DSMC laws are proposed for a magnetic levitation system, addressing limitations in conventional DSMC techniques. These strategies ensure fast convergence, reduced chattering, and practical applicability, paving the way for future experimental validations.

Overall, the thesis significantly advances the field of discrete sliding mode and data-driven control by offering low-complexity, high-performance strategies suitable for real-world DTS with unknown or partially known dynamics.

9.2 Limitations and Proposed Future Investigation

- Future research of DDC could explore extending the developed methods to more complex, multi-variable systems and integrating them with adaptive or learning-based control strategies to enhance performance in dynamically changing environments. For instance, incorporating reinforcement learning (RL) or model predictive control (MPC) could enable the continuous updating of control policies based on real-time interactions with the system.
- Future research of PPC based DDC will focus on extending this approach to multi-agent systems and addressing the consistency challenges associated with PP.
- Additionally, real-time experimental work will be prioritized to validate theoretic

cal models and algorithms developed for DDC-PPC based framework in practical scenarios.

- Future research will also focus on theoretical developments for multiple PP and their associated performance metrics.
- Systems with more pronounced nonlinearities and higher-dimensional dynamics are also planned as part of future work for the proposed MROF-DSMC technique, along with the implementation of the same on a test bench or real system.

Appendix A

List of Publications

A.1 Journal papers

- V. Pandey, S. Kamal and S. Ghosh, "Finite-Time Discrete Control for Two-DOF Helicopter System," in IEEE Transactions on Circuits and Systems II: Express Briefs, vol. 71, no. 8, pp. 3800-3804, Aug. 2024, doi: 10.1109/TCSII.2024.3369681.
- Pandey, V.; Taslima, E.; Bartoszewicz, A.; Kamal, S.; Ghosh, S. Minimum Operator Based Data Driven Control. IEEE Transaction of Industrial Electronics.(Accepted)
- Pandey, V.; Taslima, E.; Bartoszewicz, A.; Kamal, S.; Ghosh, S. Prescribed Performance-Based Data-Driven Control Using Minimum Operator For Discrete-Time Nonlinear Systems. (under review)
- Pandey, V.; Bandyopadhyay, B.; Kamal, S.; Ghosh, S.; Yu, X. Output Feedback Based Discrete-Time Sliding Mode Control Using Minima Operator. IEEE Transactions on Systems, Man, and Cybernetics (Under revision)

A.2 Conference papers

- Pandey, V., Kamal, S., & Ghosh, S. (2024, October). Discrete-Time Sliding Mode Controller For Magnetic Levitation System Using Minima Operator. In 2024 17th International Workshop on Variable Structure Systems (VSS) (pp. 69-74). IEEE.
- Pandey, V., Singh, V. K., Taslima, E., Kamal, S., & Ghosh, S. (2024, October).

Adaptive Switching Scheme Design Using Ceiling Function for Discrete-Time Sliding Mode Control. In 2024 17th International Workshop on Variable Structure Systems (VSS) (pp. 75-80). IEEE.

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