PREFACE

Fractional calculus (a mathematical concept of arbitrary order integrals and derivatives) is an old topic as it was started Since 1695, but it has been continuously evolving and advancing up to the present. Fractional calculus has demonstrated its utility in modeling and controlling innovative applications across numerous emerging research fields such as engineering and science [5, 6, 7, 8, 9]. Fractional order derivatives possess a nonlocal property, which makes them particularly suitable for introducing memory into complex phenomena and assist in understanding the behavior of phenomenas characterized by non-locality and long-term memory.

In the last few decades, there has been a notable presence of research papers in the literature focusing on numerical solutions of time-fractional partial differential equations (PDEs). In this thesis, we are mainly concerned on numerical solution of one and two-dimensional time-fractional diffusion-wave equation with weak initial singularity. This model helps to understand the intermediate process between diffusion and wave phenomenas. This problem is obtained from the standard diffusion and wave equation by replacing the first and second-order time derivatives with fractional order derivative of order $\alpha > 0$. Mainardi and Paradisi [10] generalized the diffusion equation by choosing derivative of order $1 < \alpha < 2$. This generalized problem is exhibited to govern the propagation of stress waves in viscoelastic solids.

In this thesis, we develop approximation techniques for solving linear/nonlinear time-fractional diffusion-wave equations with weak initial singularity. In recent years, many researchers have focused on studying the solution for fractional differential equations through various techniques. Finding exact solutions of fractional differential equations is a difficult task. However, discretization offers a productive approach to manage numerous time-fractional PDEs by developing an effective numerical scheme that balances high accuracy with minimal computational time.

The content of this thesis covers the behavior of solution for the linear/nonlinear time-fractional mixed diffusion and diffusion-wave equations near the weak initial singularity because such types of studies remain to be addressed in the literature.

Chapter 2 explores two efficient difference schemes to solve the two-dimensional time-fractional diffusion-wave equation with initial singularity at $t = 0$. To overcome the low accuracy caused by the presence of singularity in Caputo fractional derivative of order $\alpha \in (1, 2)$, we discretize approximation formula $L1$ and Crank-Nicolson $L1 - 2$ on nonuniform temporal meshes. After that, the Alternating Direction Implicit (ADI) approach is developed by using nonuniform $L1$ and $L1 - 2$ methods and central difference formula for space derivative to obtain the system of equations for considered problem. The established nonuniform $L1$ difference scheme has convergence rate $3 - \alpha$ in time and second-order in space, whereas the Crank-Nicolson $L1 - 2$ scheme has second-order accuracy in both space and time direction. We provide two numerical examples for non-smooth exact solutions with initial singularity at $t = 0$ to demonstrate the benefits of the nonuniform difference methods.

In Chapter 3, we consider a mathematical model, two-dimensional time-fractional mixed diffusion and diffusion-wave equation with a weak singularity at $t = 0$. We used linear polynomials on nonuniform time meshes to approximate the Caputo fractional derivatives of order $\beta \in (0, 1)$ and $\alpha \in (1, 2)$. To calculate the numerical solution of a two-dimensional considered model, the ADI approach is utilized to get an equivalent system of equations. The discrete difference scheme was established by using the nonuniform $L1$ method in time and central difference for space direction discretization. Finally, numerical examples are given to validate the theory.

In Chapter 4, we use a nonlinear model with variable coefficients instead of the linear problem that is discussed in Chapter 2. An efficient linearized difference scheme is studied by using Taylor series expansion for nonlinear term. This chapter deals with nonsmooth exact solutions as well as discontinuous initial data for the nonlinear time-fractional diffusion-wave equation with variable coefficients. Also, we

discussed a difference scheme for multi-term nonlinear time-fractional diffusion-wave equation. At the end, numerical experiments are provided to check the efficiency of the established difference schemes.

Chapter 5 deals with the high-order accurate difference scheme based on the cubic interpolation polynomials in solving the generalized time-fractional advection-diffusion equation. The time-fractional derivative is defined in generalized Caputo sense [11]. The convergence order of the provided difference scheme is $(4-\beta)$, $0<\beta<1$ in time direction and it is second order accurate in space. The numerical results are given at the end to validate the theory. Finally, the thesis is concluded in Chapter 6.