INTRODUCTION

The concept of ring theory was initiated in the 1870s by Richard Dedekind and was further developed by mathematicians such as Hilbert, Fraenkel, and Noether. The concept of a ring is the first generalization of the Dedekind domain that emerged in number theory and since then, ring theory has found widespread applications in various fields such as algebraic geometry, number theory, algebraic graph theory, coding theory etc.

There are two primary approaches to understanding the structure of rings. The first approach involves examining the inner conditions of a ring by studying its left and right ideals. The second approach is to investigate the outer conditions of a ring by studying the modules over them. In this thesis, the structure of rings is studies through the second approach, where the focus is on the properties of modules over the rings.

In 1940, R. Baer introduced the concept of injectivity [6], but many mathematicians were attracted to the study of injective modules only after Eckmann and Schopf [14] published their paper "Uber Injektive moduln". In 1953, they proved the existence of an injective hull of a module.

In 1958, Matlis [37] studied injective modules over Noetherian rings. He showed that the sum of two injective modules right R-modules is injective if and only if the ring is right hereditary. In 1961, Johnson and Wong generalized the idea of injective modules to quasi-injective modules [29]. They also proved that it is stable under all endomorphisms of its injective hull $E(N)$ of an R-module . Many mathematicians generalized the injective modules to continuous modules and quasi-continuous modules [44], etc.

The following implications are true

$Injective\ Modules \Rightarrow Quasi\ Injective\ Modules$

But the converse of the above implications need not be true. (see [32])

In 1964, Osofsky [50] studied the rings over which finitely generated modules are injective. She also proved that every finitely generated R-module is injective if and only if R is a semisimple Artinian ring. Also, in 1964, Faith and Utumi [16] characterized quasi-injective modules in terms of endomorphism rings of injective modules.

In 1965, Utumi [53] observed three conditions in the work on continuous rings, namely $C1$, $C2$, and $C3$ conditions, whenever the ring is self-injective. Later on, these three conditions were applied to modules. The theory of extending modules was developed by Harada and his school in Japan, Muller, and his collaborators [44] in Canada, Osofsky, Smith, Huynh, Dung, Wisbauer [13], and many more people in worldwide. The modules with $C1$ condition are a common generalization of injective modules and semisimple modules. The generalization of injective modules has been of interest for about three decades. Jermy [28], Mohamed and Bouhy [43] generalized the concept of continuity and quasi-continuity to modules. In 1990, Mohamed and Muller [44] studied continuous module ($C1$ and $C2$) modules, and quasi-continuous module ($C1$ and $C3$) modules.

The following implications are true

$Injective\ Modules \Rightarrow Quasi\ Injective\ Modules \Rightarrow Continuous\ Modules$

$\Rightarrow Quasi\ Continuous\ Modules \Rightarrow$ C1 $Modules.$

But the converse of the above implications need not be true. (see [44])

In 1976, Nicholson [47] introduced the notion of the direct-injective modules. The class of direct injective modules is the generalization of the class of quasi-injective modules. In 2003, Nicholson and Yousif [48] proved that the class of direct-injective modules is equivalent to the class of $C2$ modules. In 2018, Maurya and Gupta [39] introduced the notion of finite direct injective modules (or finite $C2$ modules) and showed that it is a strict generalization of direct injective modules and characterized semi-simple artinian rings, V- rings in [40]. Motivated by the above generalizations of injective modules and $C2$ modules to finite $C2$ modules. We introduce in Chapter 4, finite $C3$ modules as the generalization of finite $C2$ modules.

In 1959, P. M. Cohn [11] defined a submodule K of an R-module N to be a **pure submodule** of N, if and only if $0 \to K \otimes L \to N \otimes L$ is exact for every left R-module L. Further, an ideal I of a ring R is called pure if I is a pure submodule of R_R.

In 1969, Warfield [56] defined pure-injective modules as the generalization of injective modules. It was proved that pure-injective envelopes exist and the pure-injective modules are characterized as retracts of topologically compact modules. For this reason, the pure-injective modules are also called algebraically compact. In 1997, Thani [1] defined quasi-pure injective modules as a generalization of pure injective modules. Consider N and K be R-modules, then a module N is K-pure-injective if for every pure submodule L of K, a homomorphism $f : L \to N$ can be extended to $g : K \to N$. Further, N is considered as quasi-pure-injective if N is a N-pure-injective, while N is considered as a pure-injective if it is K-pure-injective for every R-module K. In 2022, Maurya et al. [38] generalized the class of $C2$ modules to the class of pure $C2$ modules.

As we know every submodule of a module N need not be a pure submodule (In fact no submodule of $\mathbb{Z}$ as $\mathbb{Z}$-module is a pure submodule). Motivated by this fact, the concept of purity and pure direct injective modules. We extend the theory of $C1$ modules, continuous modules, $C3$ modules, and quasi-continuous modules to pure $C1$ modules, pure continuous modules, pure $C3$ modules, and pure quasi-continuous modules respectively. We introduce in Chapter 2, pure $C1$ modules as the generalization of $C1$ modules and we introduce in Chapter 3, pure $C3$ modules as the generalization of $C3$ modules. The following implications are true but the converse need not be true.

$$\begin{array}{ccccccc}
Injective & \longrightarrow & Quasi-injec. & \longrightarrow & C2 & \longrightarrow & C3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
pure-injec. & \longrightarrow & Quasi-pure-injec. & \longrightarrow & Pure\,C2 & \longrightarrow & Pure\,C3
\end{array}$$

In 1960, H. Bass [7] introduced the concept of the projective cover of a module, dual to the injective envelope of a module and he proved that the projective cover of the module does not always exists, but whenever it exists, it is unique. The concept of projectivity is dual to the concept of injectivity, although they originated at the same time. Several generalizations of projectivity have been made so far. A generalization of projectivity can be obtained either by specialization of R-modules or by restricting the R-homomorphism. For instance, quasi-projectivity by Y. Miyashita [42] and M-projectivity by G. Azumaya [5] are due to the specialization of R-modules. In 1976, Nicholson [47] introduced the concept of direct-projective modules. He defined an R-module M is direct-projective if for any direct summand N of M with projection $p_N : M \longrightarrow N$ and any epimorphism $g : M \longrightarrow N$, there exists an $f \in End_R(M)$ such that $g \circ f = p_N$. In 1991, Wisbauer [58] defined an R-module M as direct-projective if for every direct summand X of M every epimorphism $M \longrightarrow X$ splits.

The following implications are true

Projective Modules $\Rightarrow$ Quasi-Projective Modules $\Rightarrow$ Direct Projective Modules

But the converse of the above implications need not be true (see [32], [44]).

In 1993, W. Xue [59] characterized semisimple and hereditary rings in terms of direct-projective and direct-injective modules. In 2003, Nicholson and Yousif [48] proved that the class of direct-projective modules is equivalent to the class of $D2$ modules. An R-module N is called a $D2$ module if $N/A \cong B \leq^{\oplus} N$ then $A \leq^{\oplus} N$, where A and B are submodules of N.

In 2015, Amin et al.[2] studied $C3$ modules and characterized several rings and also introduced $C3$ covers of an R module. In 2016, Atani et al. [4] discovered new properties of $C3$ modules and introduced the $C3$ envelopes of an R module.

In 2011, Lee et al. [33] introduced the notion of dual-Rickart modules, which is the dual of the class of Rickart modules. A module N is called dual-Rickart if the image of every endomorphism of N is a direct summand of N. These two classes of modules are closely connected with von Neumann regular rings, and the connection between them was first established by Rangaswamy [51] in 1967. The endomorphism ring $S = End_R(N)$ of a module N is von Neumann regular if and only if the kernel and image of every endomorphism of N are direct summands of N, i.e., N is both a Rickart and a dual-Rickart module. This result has important implications for the theory of rings and modules.

In this thesis, the author studies several rings, one of them a semisimple ring. The semisimple ring was introduced by Emmy Noether [49] in 1921. A semisimple ring is a type of ring that is particularly well-behaved from a module-theoretic perspective. A ring R is called semisimple if the module R_R is semisimple i.e. every R-module can be written as a direct sum of simple (irreducible) submodules. Another way to characterize a semisimple ring is in terms of short exact sequences of modules. A ring R is called semisimple if and only if any short exact sequence of R-modules splits.

In this thesis, the author also studies von Neumann regular rings. A ring R is called von Neumann regular if every element of R is regular [46]. An element a in a ring R is regular if an element x in R exists such that $axa = a$. A module is called Endoregular if its endomorphism ring is a von Neumann regular ring.

In 1958, Irving Kaplansky [30] defined hereditary rings as a ring in which every ideal is projective. Several characterizations of hereditary rings exist in the literature, including the fact that every submodule of a projective R-module is also projective, and every factor module of an injective R-module is injective.

In 1961, Shizuo Endo [15] generalized the idea of hereditary rings to semi-hereditary rings in his paper entitled- Semi-hereditary rings. Over a semi-hereditary ring, every finitely generated ideal is projective. Moreover, over a semi-hereditary ring, every finitely generated submodule of the projective module is also projective. In addition, every finitely generated factor module of an injective module is also injective.

In 1984, Hiremath [24] introduced the concept of copurity, as the dual counterpart to the concept of purity by employing co-finitely related modules. In an R-module N, a submodule L is classified as copure if, for any co-finitely related R-module K, any homomorphism from L to K can be extended to a homomorphism from N to K. In 1985, Hiremath [23] introduced the concept of cofinitely projective modules and studied cofinitely projective modules over the Dedekind domain. In 1988, Hiremath [19] also introduced Copure-projective modules and studied their properties. He proved that over an arbitrary ring R, the copure-injective R-modules are precisely the direct summands of direct products of cofinitely related R-modules. In 2022, Maurya et al. [41] generalized the concept of $C2$ modules to copure $C2$ modules. An R-module N is called a copure $C2$ module if every copure submodule of N is isomorphic to a direct summand of N is itself a direct summand.

Motivated by the above facts, the concept of copurity and Copure Direct Injective Module. In Chapter 5, we introduce the notion of copure $D2$, which is the generalization of $D2$ modules, dual of copure $C2$ modules and study their properties.

Now, we give a brief description of the present thesis. The thesis consists of the following five chapters with conclusions and future scope. **Chapter 1** is the collection of notations, definitions, and basic results which are used in the subsequent chapters.

In **Chapter 2**, we introduce the concept of Pure $C1$ modules. An R module N is called a pure $C1$ module if every pure submodule of N is essential in a direct summand of N. These modules are a proper generalization of extending modules. We study several properties of pure extending modules and characterize several rings i.e. von Neumann regular rings, semisimple rings, local rings, and PDS rings in terms of pure extending modules.

The following are the main results in Chapter 2;

1. The direct summand of a pure $C1$ module is pure $C1$ module.
2. A ring R is a von Neumann regular if and only if every pure $C1$ R-module is $C1$ module.
3. For a ring R, the following conditions are equivalent:
   (a) R is a von Neumann regular ring.
   (b) Every pure $C1$ R-module is flat.
4. For a ring R, the following conditions are equivalent:
   (a) R is a semisimple ring.
   (b) Every pure $C1$ R-module is projective or injective.

In **Chapter 3**, we introduce the concept of pure $C3$ modules, which is an extension of the concept of pure $C2$ modules. An R-module N is called a pure $C3$ module if the direct sum of two disjoint direct summands of N which is a pure submodule is also a direct summand of N. This class of modules generalizes the class of pure $C2$ modules as well as the class of $C3$ modules. Also, pure $C3$ modules are a strict generalization of $C3$ modules. We give new characterizations of many rings in respect of pure $C3$ modules, namely semisimple rings, pure-semisimple rings, von Neumann regular rings, Noetherian rings, pure-hereditary rings, pure-V-rings, etc. Moreover, we also discuss the pure $C3$ envelope and pure $C3$ cover of a module and introduce the pure quasi-continuous modules as the generalization of the quasi-continuous modules.

The following are the main results in Chapter 3;

1. The direct summand of a pure $C3$ module is a pure $C3$ module.
2. Any pure $C2$ module is pure $C3$ module.
3. For a ring R, the following statements are equivalent for a finitely generated module N:
   (a) Every R-module is dual-Rickart.
   (b) Every R-module is pure $C3$.
4. For a ring R, the following statements are equivalent :
   (a) R is a von-Neumann regular ring;
   (b) Every pure $C3$ R-module is $C3$ module.
5. For a ring R, the following conditions are equivalent:
   (a) R is a pure-semisimple ring;
   (b) Every R-module is pure $C3$.

In **Chapter 4**, we introduce the concept of finite $C3$ modules, which is an extension of the concept of finite $C2$ modules. An R-module N is called a finite $C3$ module if the direct sum of two disjoint direct summands of N which is a finitely generated submodule is also a direct summand of N. This class of modules generalizes the class of finite $C2$ modules as well as the class of $C3$ modules. Also, finite $C3$ modules are a strict generalization of finite $C2$ modules. The following are the main results in Chapter 4;

1. The direct summand of a finite $C3$ module is a finite $C3$ module.

2. Any finite $C2$ module is finite $C3$ module.

3. For a ring R, the following statements are equivalent for a finitely generated module N:

   (a) R is a hereditary;

   (b) The sum of injective modules is a finite $C3$ module.

4. For a ring R, the following conditions are equivalent:

   (a) R is a V-ring;

   (b) Every finitely cogenerated R-module is finite $C3$.

5. For a ring R, the following statements are equivalent :

   (a) R is a semi-simple Artinian;

   (b) Every R-module is finite $C3$ module.

In **Chapter 5**, we introduce the notion of copure $D2$ modules, which is a generalization of the class of $D2$ modules and the dual notion of copure $C2$ modules. In this chapter, we study the direct sums and direct summands of copure $D2$ modules. Further, we characterize copure semi-simple rings, copure hereditary rings and PDS rings in terms of copure $D2$ modules.

The following are the main results in Chapter 5;

1. The direct summand of a copure $D2$ module is a copure $D2$ module.

2. Let R be a copure hereditary ring then every copure injective R-module is copure $D2$ module.

3. For a ring R, the following statements are equivalent :

   (a) R is a copure semi-simple ring;

   (b) Every finitely generated R-module is copure $D2$ module.

4. For a classical ring R, the following statements are equivalent:

   (a) R is a copure semi-simple ring.

   (b) R is a PDS ring.

   (c) Every R-module is copure $D2$.

   (d) Every R-module is pure $D2$.

Chapter 1

Preliminaries

This chapter is mainly devoted to the collection of definitions and basic results, which are used in the subsequent chapters of the thesis. Throughout the thesis, unless otherwise stated, all modules over a ring R will be understood to be right R-modules, a ring R, we shall always mean an associative ring with unity and all modules are unital R-modules.

Rings, Modules and Module Homomorphisms

Definition 1.0.1. *An algebraic structure $(R, +, .)$, where R is a non-empty set together with two binary operations $+$ and $.$ is said to be a ring if the following conditions are satisfied.*

(1) $(R, +)$ *is an abelian group.*

(2) $(R, .)$ *is a semi-group.*

(3) The binary operation $.$ distributes over $+$ from the left as well as from the right, i.e., $\forall\ r, s, t \in R$

(i) $r.(s+t) = r.s + r.t$

(ii) $(r+s).t = r.t + s.t$

A ring $(R, +, .)$ is called commutative if $r.s = s.r \ \forall \ r, s \in R$. Further, R is said to be a ring with unity if there exists $1 \in R$ such that $1.r = r.1 = r \ \forall \ r \in R$.

Definition 1.0.2. *(1) Let R be a ring not necessarily containing identity element. An additive abelian group $(N, +)$ is called a right R-module if there exists a mapping from $N \times R$ to N defined by $(n, r) \to nr$, $\forall \ n \in N$, $r \in R$ satisfying the following conditions:*

(i) $(m+n)r = mr + nr \ \forall m, n \in N$ *and* $\forall r \in R$.

(ii) $n(r+s) = nr + ns \forall n \in N$ *and* $\forall \ r, s \in R$.

(iii) $n(rs) = (nr)s \forall n \in N$ *and* $\forall \ r, s \in R$.

A left R-module can be defined by taking action of the ring R from left.

(2) Further, if $n.1 = n$, $\forall n \in N$, where 1 is the unity of R, then N is called an unital R-module.

(3) A non-empty subset K of an R-module N is called a submodule of N if K is also an R-module and we denote it by $K \leq N$.

Definition 1.0.3. *Let M and N be R-modules:*

(1) A mapping $\varphi : M \to N$ is a called module homomorphism if and only if $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ and $\varphi(mr) = \varphi(m)r$, $\forall m_1, m_2 \in M$. and $r \in R$.

The set of all R-homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$.

(2) For any $\varphi \in \mathrm{Hom}_R(M, N)$, the kernel *and the* image *of φ defined as follows*

$\mathrm{Ker}(\varphi) = \{m \in M : \varphi(m) = 0\}$ *and* $\mathrm{Im}(\varphi) = \{\varphi(m) \in N : m \in M\}$.

(3) An R-homomorphism from N to N is called an endomorphism and the set of all endomorphisms of N is denoted by $End_R(N)$.

Theorem 1.0.4. (Fundamental theorem of module homomorphism) *Let M and N be R-modules. If $\varphi : M \to N$ is any R-homomorphism, then $\varphi(M) \cong M/\mathrm{Ker}(\varphi)$.*

Proposition 1.0.5. *Let M and N be R-modules.*

(i) An onto R-homomorphism $f : M \to N$ splits if there exists an R-homomorphism $g : N \longrightarrow M$ with $f \circ g = I_N$. In this case, g is called a splitting map for f.

(ii) A one-one R-homomorphism $f : N \to M$ splits if there exists an R-homomorphism $g : M \longrightarrow N$ with $g \circ f = I_N$. In this case, g is called a splitting map for f.

Direct Sums, Direct Products and Direct Summands

Definition 1.0.6. *Let $\{N_i\}_{i\in I}$ be a family of R-modules, where I is an arbitrary index set. Then*

(i) $\prod_{i\in I} N_i = \{(n_1, n_2, \ldots, n_i, \ldots) : n_i \in N_i \, \forall \, i \in I\}$ denotes the direct product of the family of R-modules $\{N_i\}_{i\in I}$.

(ii) $\oplus_{i\in I} N_i = \{(n_1, n_2, \ldots, n_i, \ldots) : n_i \in N_i \}$ and finitely many $n_i's$ are nonzero denotes the direct sum of the family of R-modules $\{N_i\}_{i\in I}$.

Definition 1.0.7. *A submodule K of a module N is called* **direct summand** *of N if there exists a submodule K' of N such that $N = K \oplus K'$, and we denote it by $K \leq^{\oplus} N$. In this case $N = K \oplus K'$ implies $N = K + K'$ and $K \cap K' = 0$.*

Exact Sequences

Definition 1.0.8. *Let N_n be an R-module $\forall n$ and φ_n be a homomorphism from N_n to N_{n-1} $\forall n \in \mathbb{N}$. Then a sequence $\cdots \to N_{n+1} \xrightarrow{\varphi_{n+1}} N_n \xrightarrow{\varphi_n} N_{n-1} \to \ldots$ is called an exact sequence at N_n if $\operatorname{Ker}(\varphi_n) = \operatorname{Im}(\varphi_{n+1})$, while this sequence is called an exact sequence if it is exact at N_n for each n.*

Definition 1.0.9. *Let L, M and N be R-modules. Then,*

(i) The exact sequence $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is called a short exact sequence.

(ii) A sequence $0 \to L \xrightarrow{\varphi} M$ is exact if and only if φ is a monomorphism.

(iii) A sequence $M \xrightarrow{\psi} N \to 0$ is exact if and only if ψ is an epimorphism.

Definition 1.0.10. *[8] For R-modules L, M and N, the following conditions are equivalent:*

(i) An exact sequence $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ splits;

(ii) There exists a homomorphism $\varphi' : M \to L$ such that $\varphi' o \varphi = I_L$, where I_L is an identity map on L;

(iii) There exists a homomorphism $\psi' : N \to M$ such that $\psi o \psi' = I_N$, where I_N is an identity map on N.

Finitely Generated Modules, Cyclic Modules, Free Modules, Finitely Related Modules, Finitely Presented Modules, Coherent Modules, Small Submodules, Minimal Epimorphism and Essential Submodules

Definition 1.0.11. *[32] A module N is called* **finitely generated** *if there exist $n_1, n_2, \dots, n_r \in N$ such that $N = \sum_{i=1}^{r} n_i R$. The set $\{n_1, n_2, \dots, n_r\}$ is called the set of generators of N. A module generated by a single element is called a* **cyclic** *module. Further, a submodule is called cyclic if a single element generates it.*

Definition 1.0.12. *[32] An R-module N is called a* **free module** *if it has a basis, i.e., there exists a subset $B \subseteq N$ such that each element $n \in N$ can be uniquely expressed as a finite sum, $n = \sum_{i=1}^{s} n_i r_i$ for some $r_1, r_2, \dots, r_n \in R$ and $n_1, n_2, \dots, n_s \in B$.*

Definition 1.0.13. *[32] A module N is called* **finitely related** *if there exists an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, where M is a free module (of arbitrary rank) and L is finitely generated.*

Definition 1.0.14. *[32] A module N is called* **finitely presented** *if there exists an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, where M is a free module (of finite rank) and L is finitely generated (or equivalently, there exists an exact sequence $R^m \to R^n \to N \to 0$ with $m, n \in \mathbb{N}$).*

Definition 1.0.15. *[32] A finitely generated R-module N is called* **coherent** *if every finitely generated submodule of N is finitely presented. A ring R is called coherent if R_R is a coherent R-module.*

Definition 1.0.16. *[32] A submodule N of a module M is called a small (or superfluous) submodule in M, abbreviated as $N << M$ in case $N + L = M \implies L = M$ for any submodule L of M.*

Definition 1.0.17. *[32] An epimorphism $f : M \longrightarrow N$ is called minimal epimorphism if $Ker(f)$ is small submodule in M.*

Definition 1.0.18. *[32] A submodule K of a module N is called* **essential submodule** *denoted as $K \leq^e N$ if $K \cap L \neq 0$ for each non-zero submodule L of N. In this case, N is called an essential extension of K.*

Artinian and Noetherian Modules and Rings

Definition 1.0.19. *A module N is called* **Noetherian** *if it satisfies the ascending chain condition on its submodules, i.e., every ascending chain $N_1 \leq N_2 \leq \cdots \leq N_n \leq \ldots$ of submodules of N becomes stationary after finitely many steps. A ring R is called right (left) Noetherian if the R-module R_R ($_RR$) is Noetherian.*

Theorem 1.0.20. *For a module N, the following conditions are equivalent:*

(i) N is Noetherian;

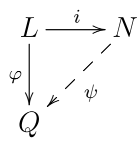
(ii) Every submodule of N is finitely generated;

(iii) Every non-empty set A of submodules of N has a maximal element.

Definition 1.0.21. *A module M is called* **Artinian** *if it satisfies the descending chain condition on its submodules, i.e., every descending chain $N_1 \geq N_2 \geq \cdots \geq N_n \geq \ldots$ of submodules of N becomes stationary after finitely many steps. A ring R is called right (left) Artinian if the R-module R_R ($_RR$) is Artinian.*

Injective Modules, Quasi-injective Modules, and Injective Envelopes

Definition 1.0.22. *Let Q and N be R-modules, then* *(i) A module Q is said to be an N-injective module if for every submodule L of N with a monomorphism $i: L \to N$ and for any homomorphism $\varphi : L \to Q$, there exists a homomorphism*



$\psi : N \to Q$ such that the above diagram is commutative, i.e., $\varphi = \psi \circ i$.

(ii) A module Q is called injective if Q is N-injective for every right R-module N.

(iii) A module Q is called quasi-injective if Q is Q-injective.

Definition 1.0.23. *An R-module E is called the injective hull (envelope) of an R-module N if E is the minimal injective module containing N. The injective hull of a module N is denoted by $E(N)$. In general, every module has an injective hull.*

Theorem 1.0.24. *(Bass-Papp Theorem, [52, Theorem 4.1]) A ring R is right Noetherian if and only if every direct sum of injective right R-modules is injective.*

$C1$-Modules, $C2$-Modules, $C3$-Modules, Extending Modules Continuous Modules and Quasi-continuous Modules

Consider the following conditions for a right R-module N introduced by Jeremy [28], Mohammed and Muller [44].

C1 : Every submodule of N is essential in a direct summand of N.

C2 : Every submodule of N that is isomorphic to a direct summand of N is itself a direct summand of N.

C3 : If L and K are direct summands of N with $L \cap K = 0$. Then $L \oplus K$ is also a direct summand of N.

Definition 1.0.25. *(i) A module with $C1$ condition is called a $C1$ module. A $C1$ module is also known as **extending module** or **CS module**.*

*(ii) A module with $C2$ condition is called a $C2$ module. A $C2$ module is also known as **direct injective module***

(iii) A module with $C3$ condition is called a $C3$ module.

*(iv) A module with $C1$ and $C2$ conditions is called a **continuous module**.*

*(v) A module with $C1$ and $C3$ conditions is called a **quasi-continuous module**.*

Proposition 1.0.26 (Proposition 2.1, [44])**.** *Any quasi-injective module satisfies $C1$ and $C2$ conditions.*

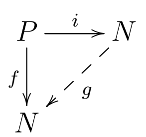
Remark 1.0.27. *The following implications are true for a module ([44]),*

$Injective \Rightarrow Quasi{-}injective \Rightarrow Continuous \Rightarrow Quasi{-}continuous \Rightarrow Extending$

But the converse of these implications need not be true, in general ([34]).

Direct-injective Modules, Finite $C2$ Modules

Definition 1.0.28. *An R-module N is called direct-injective if given any direct summand P of N with injection $i : P \to N$ and any monomorphism $f : P \to N$ there exists $g \in End_R(N)$ such that $g \circ i = f$.*



Note : In 2003, Nicholson and Yousif [48] proved that the class of direct-injective modules is equivalent to the class of $C2$ modules.

Definition 1.0.29. *An R-module N is considered as a finite $C2$ module (finite-direct-injective) if A is a finitely generated submodule of N such that $A \cong B \leq^{\oplus} N$ then $A \leq^{\oplus} N$*

Pure Submodules, Flat Modules, Pure Simple Modules, and Pure essential Submodules

Definition 1.0.30. *A short exact sequence $0 \to N_1 \xrightarrow{\phi} N_2 \to N_3 \to 0$ of right R-modules is called pure exact if $0 \to N_1 \otimes F \to N_2 \otimes F \to N_3 \otimes F \to 0$ is an exact sequence for any left R-module F [32].*

According to P.M. Cohn [11], a submodule K of an R-module N is said to be a **pure submodule** *of N, if and only if $0 \to K \otimes L \to N \otimes L$ is exact for every left R-module L. Further, an ideal I of a ring R is said to be pure if I is a pure submodule of R_R.*

Definition 1.0.31. *An R-module N is called* **flat** *if $0 \to N \otimes N_1 \to N \otimes N_2$ is exact whenever $0 \to N_1 \to N_2$ is exact for left R-modules N_1 and N_2.*

Lemma 1.0.32. *[Proposition 8.1, [17]]. The following conditions hold:*

(i) Let K be a submodule of an R module N. If N/K is flat, then K is a pure submodule of N. Moreover, for a flat R module N, K is a pure submodule of N if and only if N/K is flat.

(ii) If K is a submodule of N such that every finitely generated submodule of K is a pure submodule of N, then K is a pure submodule of N.

Lemma 1.0.33. *[Proposition 7.2, [17]]. Suppose $L \subseteq N \subseteq M$ be R modules. Then*

(i) If $L \leq^p N$ and $N \leq^p M$, then $L \leq^p M$.

(ii) If $L \leq^p M$, then $L \leq^p N$.

(iii) If $L \leq^p N$, then $N/L \leq^p M/L$.

(iv) If $L \leq^p M$ and $N/L \leq^p M/L$, then $N \leq^p M$.

Definition 1.0.34. *A module N is called* **pure simple** *if it contains no non-trivial pure submodule of N.*

Definition 1.0.35. *[21] A submodule K of a module N is said to be* **pure essential** *in N if K is pure in N and for any non-zero submodules L of N either $K \cap L \neq 0$ or $(K \oplus L)/L$ is not pure in N/L.*

Pure-Injective Modules, Quasi Pure-Injective Modules and Pure $C2$ Modules

Definition 1.0.36. *Consider N and K be R-modules, then a module N is K-pure-injective if for every pure submodule L of K, a homomorphism $f : L \to N$ can be extended to a homomorphism $g : K \to N$. Then N is called a pure-injective if it is K-pure-injective for every R-module K and N is called as quasi-pure-injective if N is a N-pure-injective.*

For a detailed study of the pure-injective modules, quasi-pure-injective modules, and concept of purity, refer to [17],[21],[57].

Definition 1.0.37. *[38] An R-module N is called as a pure $C2$ (pure-direct-injective) module if A is a pure submodule of N and B is some submodule of N such that $A \cong B \leq^{\oplus} N$ then $A \leq^{\oplus} N$.*

Pure Semisimple Rings, Pure Hereditary Rings, PDS Rings

Definition 1.0.38. *[17] A ring R is called a* **pure semisimple** *ring if every pure submodule of an R-module N is a direct summand of N.*

Lemma 1.0.39. *If R is a Noetherian ring and N is a finitely generated R-module, then each pure submodule of N is a direct summand of N.*

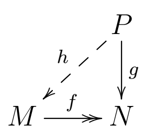
Definition 1.0.40. *A ring R is called a* **pure hereditary** *if every surjective image of an injective R-module is pure-injective.*

Definition 1.0.41. *A ring R is called* **Xu***-ring if every cotorsion R-module is pure-injective.*

Definition 1.0.42. *A ring R is called* **PDS** *ring if every pure submodule of an R-module is a direct summand of N.*

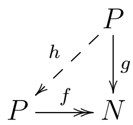
Projective Modules, Quasi-Projective module, Projective Cover of a Module

Definition 1.0.43. *(i) An R-module P is projective if given any R-epimorphim $f : M \to N$ and a module homomorphism $g : P \to N$, there exists a module homomorphism $h : P \to M$ such that $f \circ h = g$.*



(ii) An R-module P is called quasi-projective if given any R-homomorphism $g : P \to A$, and an epiomorphism $f : P \to N$, there is an $h \in End_R(P)$ such that

the diagram below

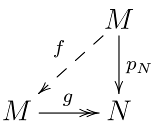


commutes i.e. $f \circ h = g$.

(iii) *A projective module* P *is called a projective cover of a module* N *if there exists a minimal epimorphism* $f : P \to N$.

Direct Projective Modules and Direct Projective Covers of a Modules

Definition 1.0.44. *An* R*-module* M *is called direct-projective if for any direct summand* N *of* M *with projection* $p_N : M \longrightarrow N$ *and any epimorphism* $g : M \longrightarrow N$, *there exists a* $f \in End_R(M)$ *such that* $g \circ f = p_N$.



Definition 1.0.45. *A direct projective module* P *is called a direct projective cover of a module* N *if there exists a minimal epimorphism* $f : P \longrightarrow N$.

Hereditary, Semi-Hereditary Rings and von Neumann Regular Rings, V-Rings, SI-Rings

Definition 1.0.46. (i) *A ring* R *is called a* **hereditary** *if each ideal of* R *is projective as an* R*-module.*

(ii) *A ring R is called a* **semi-hereditary** *if each finitely generated ideal of R is projective as an R-module.*

(iii) *A ring R is called a V-ring if every simple R-module is injective.*

(iv) *A ring R is called a semisimple Artinian ring if it is the direct sum of a finite number of simple rings.*

(v) *A ring is called a SI-ring if each singular R- module is injective.*

Theorem 1.0.47. *[9, Theorem 5.4] The following statements are equivalent for a ring R:*

(i) *R is a hereditary ring;*

(ii) *Every submodule of a projective R-module is projective;*

(iii) *Every quotient module of an injective R-module is injective.*

Definition 1.0.48. *A ring R is called* **von Neumann regular** *if, for each $a \in R$, there exists $b \in R$ such that $a = aba$.*

Definition 1.0.49. *[35] A module N is called an* **endoregular** *if the endomorphism ring of N is a von Neumann regular.*

$D1$-Modules, $D2$-Modules, and $D3$-Modules

Consider the following conditions defined in [12] and [44] for an R-module N.

$D1$**:** For every submodule K of N, there is a decomposition $N = N_1 \oplus N_2$ such that $N_1 \leq K$ and $K \cap N_2 << N$.

$D2$**:** If K is a submodule of N and N/K is isomorphic to a direct summand of N, then K is also a direct summand of N.

$D3$**:** If L and K are direct summands of N with $N = L + K$, then $L \cap K$ is a direct summand of N.

Definition 1.0.50. *A module with Di-condition is called a Di-module for every* $i = 1, 2, 3$.

Note : In 2003, Nicholson and Yousif [48] proved that the class of direct-projective modules is equivalent to the class of $D2$ modules.

Simple Modules, Indecomposable Modules, Endoregular Modules, Semisimple Modules, Semisimple Rings

Definition 1.0.51. *Let N be an R-module. Then*

(i) The module N is called a **simple module** *if it contains no non-trivial proper submodule.*

(ii) The module N is called an **indecomposable module** *if it can not be written as a direct sum of two proper direct summands of N.*

(iii) The module N is called an endoregular if $End_R(N)$ is a von Neumann regular ring.

Definition 1.0.52. *A non-zero module N is called* **semisimple** *if it is expressible as a sum of simple submodules. A ring R is called semisimple if the R-module R_R is a semisimple module.*

Proposition 1.0.53. *The following conditions are equivalent for a ring R:*

(i) R is semisimple;

(ii) Every R-module is semisimple;

(iii) Every R-module is injective;

(iv) Every R-module is projective;

(v) Every ideal of R is a direct summand of R.

SSP Modules, SIP Modules

Definition 1.0.54. *An R-module N is said to have summand sum property* (**SSP**) *if the sum of any two direct summands of N is a direct summand of N. A module N is called an SSP module if N has SSP.*

Definition 1.0.55. *An R-module N is said to have summand intersection property* (**SIP**) *if the intersection of any two direct summands of N is a direct summand of N. A module N is called an SIP module if N has SIP.*

Rickart Modules, Dual Rickart Modules

Definition 1.0.56. *Let N be an R-module and* $S = End_R(N)$*. Then N is called a Rickart module if and only if for every* $\phi \in S$ *and* $r \in R$*, the annihilator of* ϕ *in N is generated by an idempotent of S, which is equivalent to* $rN = Ker(\phi) \leq^{\oplus} N$*.*

Definition 1.0.57. *Let N be an R-module and* $S = End_R(N)$*, then N is a dual Rickart module if and only if for every* $\phi \in S$*, the image of* ϕ *in N is a direct summand of N, which is equivalent to* $\phi N = Img(\phi) = eN$ *for some* $e^2 = e \in S$*.*

Pure Rickart Modules, Pure Dual Rickart Modules

Definition 1.0.58. *An R-module N is called Pure Rickart if, for every $f \in End_R(N)$, $Ker(f)$ is a pure submodule of N [3]. A ring R is called a Pure Rickart Ring if R_R is a Pure Rickart module.*

Definition 1.0.59. *An R-module N is called Pure Dual Rickart if, for every $f \in End_R(N)$, $Img(f)$ is a pure submodule of N [3]. A ring R is called a Pure Dual Rickart Ring if R_R is a Pure Dual Rickart module.*

Co-finitely generated Modules, Cofree Modules, Co-finitely related Modules, Copure short exact sequence, Copure Submodules

Definition 1.0.60. *An R-module N is called co-finitely generated if its injective envelope $E(N)$ can be expressed as $E(N) = E(S_1) \oplus E(S_2) \oplus \ldots \oplus E(S_n)$, where S_i for $i = 1, 2, \ldots, n$ represents a simple R-module ([26], [54]).*

Definition 1.0.61. *A cofree module is an R-module N which is expressed as the direct product, denoted as $\Pi_{\alpha \in I} E(S_\alpha)$, where I is an index set and S_α represent simple R-modules, then N is isomorphic to that direct product. ([23])*

Definition 1.0.62. *An R-module N is considered co-finitely related if there exists an exact sequence $0 \to N \to K \to L \to 0$ of R-modules, where K is generated co-finitely, cofree and L is generated co-finitely ([23]).*

Definition 1.0.63. *A short exact sequence of R modules*

$$0 \to N \to K \to L \to 0$$

is called a copure short exact sequence if every co-finitely related R-module is injective with respect to this sequence. So a submodule S of an R-module N is said to be copure in N if the canonical short exact sequence

$$0 \to S \to N \to N/S \to 0$$

is copure.

Definition 1.0.64. *In an R-module N, a submodule L is called as copure if, for any co-finitely related R-module K, any homomorphism from L to K can be extended to a homomorphism from N to K.*

Definition 1.0.65. *A module is said to possess the copure intersection property when the intersection of any two copure submodules remains copure.*

Definition 1.0.66. *A ring R is called copure semisimple if every copure submodule of an R-module N is a summand of N.*

Definition 1.0.67. *A ring is called copure hereditary if every factor module of a copure injective module over the ring R is copure injective.*