Chapter 4

Exponential synchronization of a class of fractional order complex chaotic systems and application through digital cryptography

4.1 Introduction

Chaos is one of the interesting phenomena which occurs in a deterministic dynamical system showing unpredictable behavior and sensitive dependence on initial conditions. Synchronization of chaotic dynamical system is a non-linear phenomenon that has been continuously studied during the last few decades by the researchers in the field of science, medical science, mathematics, and engineering from different parts of the world. The chaotic systems have also been used in some other areas of sciences such as information processing, computing, telecommunications and electrical engineering (Sprott (2000), Xu et al. (2009), Efimov(2006), Ge and Lin(2007)). Lorenz (1963) had established first a three-dimensional chaotic attractor of autonomous system, when he was studying the atmospheric convection. The chaotic system is a three dimensional or higher dimensional dynamical system that contains one positive Lyapunov exponent having additional complex and dynamics behaviors and also have the bounded and unstable dynamic behaviors. Chaos is found in different fields of physics, mathematics, engineering, secure communication, data encryption, biology, medicine, etc. Ling et al. (2007) introduced a new chaotic attractor in reverse butterfly-shaped type.

Fractional calculus has become an active topic among researchers in recent years, and it has potential applications in many areas such as image encryption, secure communication

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etc. The advantage of the fractional calculus is that the fractional order dynamical accumulates all information about the function in a certain time, and it is known as memory property. It also describes the hereditary property of the several processes and it allows the greater flexibility in the model. The fractional derivative is a nonlocal operator; it means that the future state depends upon the present state as well as the all history of its previous states. The realistic dynamic models related to real life are also dependent upon all history of past events or states. For this property the fractional derivative is becoming more popular. The fractional calculus is used in the areas of oscillator theory, energy field, control field etc. Modelling with fractional order derivative of a dynamical system can describe the exact behaviour of a real system and therefore it is essential to make possible improvements in its design and to control its stability. Numbers of chaotic systems in fractional order have been investigated (Pan et al. (2011), Golmankhaneh et al. (2013), Baleanu et al. (2015), Wu and Baleanu (2014), Wu and Baleanu (2015), Ott et al. (1990)). These types of the systems are explained in real variables while complex chaotic systems in fractional order are described by using the complex variables, so that the complex chaotic systems give us a best tool to explain the varieties of physical phenomena such as thermal convection of liquid flow, amplitudes of the electromagnetic field, laser system (Ning and Haken (1990), Roldan et al. (1993), Toronov and Derbov (1997), Luo and Wang (2013)). The complex chaotic systems in fractional order are continuously used to increase the security in secure communication via transmitting information signals. Therefore, it has become a significant and interesting topic among the researchers to study the dynamical behavior, chaos control, and synchronization of complex chaotic systems in fractional order.

The Complex systems have many nonlinear parts with feedback loops. Complex systems are weakly chaotic and more complicated. Vishal and Agrawal (2017) analyzed novel complex chaotic systems and also discussed several dynamics of the system. Synchronization between complex chaotic systems in fractional order in the presence of uncertainty has been studied by Singh et al. (2017). Umand and Srisuchinwong (2012) have discussed the highly complex chaotic system in the occurrence of piecewise nonlinearity with wide chaotic range. Chaotic synchronization and the basic properties of complex Lorenz system have been studied by Mahmoud and Kashif (2007). Votruba et al. (2009) visualized the path of astrophysical objects and show it as a chaotic complex system and experimentally discussed behaviors of the system. Peng et al. (2018) have investigated the parameter estimation in a complex chaotic system and have developed a parameter estimation method and a new algorithm. Zhang et al. (2018) have analyzed function projective synchronization with applications in communication with the help of chaotic systems in complex variables. Yu et al. (2014) discussed the concept of time-delay complex chaotic systems. Wei et al. (2015) proposed the feedback control method for synchronization with external disturbances and also analyzed the dynamical properties of a complex chaotic system.

Naderi and Kheiri (2016) have introduced the concept of exponential synchronization in secure communication. Huang et al. (2009) obtained exponential synchronization using delayed feedback gain matrix. Li and Bohner (2010) studied the exponential synchronization and impulsive effects in mixed delays chaotic neural networks by using output coupling with delay feedback method. Zhao and Wang (2013) discussed exponential synchronization and a new impulsive time delayed complex networks by using output coupling. Cai et al. (2011) considered exponential synchronization between time varying delayed chaotic systems. Liu et al. (2008) also discussed the application of exponential synchronization using general topology. Mathiyalagan et al. (2014) analyzed exponential synchronization of chaotic systems with mixed uncertainties for fractional order case. Zhang et al. (2017) discussed exponential synchronization between time-varying delays complex-valued complex networks and also considered the stochastic perturbations. Gui and Wang (2017) applied exponential synchronization in Hopfield neural networks and employed the Lyapunov functional method in their study. Cai et al. (2012) have discussed global exponential synchronization on chaotic networks in the presence of double delayed coupling and also discussed chaotic neural network without time delayed. Yogambigai and Ali (2017) have discussed the exponential synchronization with switched time-varying delayed complex networks using intermittent control method. Hu et al. (2011) have studied exponential synchronization between the complex networks using the periodically intermittent method. Exponential synchronization of complex networks with sample data, probabilistic time-varying delays with mode dependent in the presence of Markovian jumping parameters had been studied by Rakkiyappan et al. (2014). Wang and Xue (2013) also discussed locally exponential synchronization for the complex network. Sun et al. (2007) have studied the exponential synchronization for a class of stochastic perturbed chaotic delayed neural networks. Recently, Lv et al. (2018) have discussed the exponential synchronization in neural networks via feedback control.

The synchronization in secure communication of the chaotic systems has been the focused by many researchers and scientists, and they are working on it since the last few years (Kocarev and Parlitz (1995)). Digital cryptography scheme is based on the synchronization of chaotic systems, and it's a useful technique to generate secret keys during secure communication (Mitra and Banerjee (2011), Banerjee and Chowdhary

(2008)). Many cryptography schemes have been studied and developed for communication of chaotic signals or periodic for the digital messages using with coupled chaotic systems or time-delayed chaotic system (Banerjee et al. (2008)) in identical or non-identical cases. Mitra and Banerjee (2011) have studied the digital cryptography with feedback synchronization scheme of chaotic systems and they developed digital cryptography scheme through secret keys using the synchronized systems. Zhang et al. (2013) have studied a new synchronization scheme named general projective chaos synchronization and applied it in secure communication.

In fractional order complex systems, the study of exponential synchronization is not easy because of exponential stability theory. During exponential synchronization, exponential stability theorem is used, and it is given for integer order systems. Dadras et al. (2017) gave the exponential stability theorem on fractional order nonlinear systems.

The chapter is designed as follows. Some preliminaries and lemmas have been introduced in section 4.2. Descriptions of fractional order complex chaotic systems are given in section 4.3. The illustration of exponential synchronization between fractional order complex chaotic systems and numerical simulation results are carried out in section 4.4. The application through digital cryptography is discussed in section 4.5. The overall conclusion of the work is given in section 4.6.

4.2 Some preliminaries and lemma

4.2.1 Fractional order dynamical system

Consider the fractional order system with the initial condition $x(t_0)$ as

$$D^{\alpha} x(t) = f(t, x(t)),\ 0 < \alpha < 1, \tag{4.1}$$

which is asymptotically stable at its equilibrium points if

$$\left|\arg(eig(J))\right| = \left|\arg(\lambda_i)\right| > \frac{\pi\alpha}{2},\ i = 1, 2, ..., n, \tag{4.2}$$

is satisfied for all eigenvalues $\lambda_i,\, i = 1, 2, \ldots, n$ of the Jacobian matrix $J = \frac{\partial f}{\partial x}$.

where $f = [f_1, f_2, ..., f_n]^T$, (4.3)

is evaluated at the equilibrium points of the system (4.1).

Property 4.1: (Podlubny (1999)) The fractional derivative $D_t^\alpha f(t)$ is integrable as

$$I_t^\alpha \left(D_t^\alpha f(t)\right) = f(t) - \sum_{j=1}^{n}\left[D_t^{\alpha-j} f(t)\right]_{t=0} \frac{t^{\alpha-j}}{\Gamma(\alpha - j + 1)}. \tag{4.4}$$

Lemma 4.1: (Kilbas et al. (2006)) The fractional integration operator ${}_aI_t^\alpha$ with $\lfloor \alpha \rfloor > 0$ and $-\infty < \alpha < +\infty$ is bounded in $L_p(a,b)$ i.e.,

$$\left\|I^\alpha x\right\| \le k\left\|x\right\|_p,\quad 1 \le p < \infty. \tag{4.5}$$

Theorem 4.1: (Li et al. (2010)) Let us consider $x = 0$ be equilibrium point of the system (4.1) and let a continuous Lyapunov function $V(t,\ x(t)) : [0,\ \infty) \times D \to R$ defined on a domain $D \subset R^n$ contains that equilibrium point. Let $B_r \subset D,\ r > 0$. Then there exit $\gamma_1, \gamma_2 \in K$ in $[0, a]$ such that

$$\gamma_1(\|x\|) \le V(t,\ x(t)) \le \gamma_2(\|x\|),\quad \forall\, x \in B_r \tag{4.6}$$

and $D^\alpha V(t,\ x(t)) \le -\gamma_3\left(\|x\|\right)$. (4.7)

The system (4.1) will be asymptotically stable, if $D = R^n$, γ_1 and γ_2 are defined on $[0,\infty)$. If $V(t,\ x(t))$ is radially unbounded, then $\gamma_1, \gamma_2 \in K_\infty$.

Theorem 4.2: (Dadras et al. (2017)) Consider $x = 0$ as the equilibrium point of the system (4.1) and let $V(t,\, x(t)) : [0,\, \infty) \times D \to R$ in a domain $D \subset R^n$ is containing $x = 0$ and be locally Lipschitz w.r.to x such that

$$\gamma_1 \|x\| \le V(t,\, x(t)) \le \gamma_2 \|x\|, \tag{4.8}$$

and $D^{\alpha} V(t,\, x(t)) \le 0$, (4.9)

and γ_1, γ_2 are arbitrary positive constants. Then $x = 0$ is asymptotically stable for any $\|x_0\| \le \gamma$.

Proof: Taking integration on both sides of the equation (4.9), we obtain

$$V(t,\, x(t)) - \left[D^{\alpha-1} V(t,\, x(t)) \right]_{t=0} \frac{t^{\alpha-1}}{\Gamma\alpha} \le 0. \tag{4.10}$$

From equations (4.8) and (4.10), we get

$$\gamma_1 \|x\| \le \left[D^{\alpha-1} V(t,\, x(t)) \right]_{t=0} \frac{t^{\alpha-1}}{\Gamma\alpha}. \tag{4.11}$$

Using Lemma (4.1), we get

$$\begin{aligned} \left[D^{\alpha-1} V(t,\, x(t)) \right]_{t=0} &\le \left\| D^{\alpha-1} V(t,\, x(t)) \right\|_{t=0} \\ &= \left\| I^{1-\alpha} V(t,\, x(t)) \right\|_{t=0} \\ &\le K \left\| V(t,\, x(t)) \right\|_{t=0} \\ &\le K\gamma_2 \left\| (t,\, x(t)) \right\|_{t=0} \le M\gamma_2\gamma = \phi. \end{aligned} \tag{4.12}$$

From equations (4.11) and (4.12), we get

$$\|x(t)\| \le \left[D^{\alpha-1} V(t,\, x(t)) \right]_{t=0} \frac{t^{\alpha-1}}{\gamma_1 \Gamma\alpha} \le \frac{\psi}{\gamma_1 \Gamma\alpha} t^{\alpha-1}. \tag{4.13}$$

Therefore, $\|x(t)\| \to 0$ as $t \to \infty$, which concludes the asymptotical stability of the system (4.1).

Theorem 4.3: (Dadras et al. (2017)) Let $x = 0$ be an equilibrium point of the system (4.1) and let a continuously differentiable function $V(t, x(t)) : [0, \infty) \times D \to R$ in the domain $D \subset R^n$ is containing $x = 0$ and be locally Lipschitz w.r. to x such that

$$\gamma_1 \|x\| \le V(t, x(t)) \le \gamma_2 \|x\|, \tag{4.14}$$

and $$D^\alpha V(t, x(t)) \le -\gamma_3 \|x\|, \tag{4.15}$$

and $\gamma_1, \gamma_2, \gamma_3$ are arbitrary positive constants. Then $x = 0$ is asymptotically stable for any $\|x_0\| \le r$.

Remark 4.1: In view of the Theorem 4.3, the equilibrium point $x = 0$ of the system (4.1) will be exponentially stable if there exist positive constants c, k, λ such that

$$\|x(t)\| \le k \|x(t_0)\| (t - t_0)^{\alpha - 1} e^{-\lambda(t - t_0)}, \ \forall \|x(t_0)\| < c, \ \ 0 < \alpha < 1. \tag{4.16}$$

Remark 4.2: For α=1, the equation (4.16) will be reduced to the following form

$$\|x(t)\| \le K \|x(t_0)\| e^{-\lambda(t - t_0)}, \ \forall \|x(t_0)\| < c, \tag{4.17}$$

which clearly shows the exponential stability theorem for the case of integer order system.

Remark 4.3: The exponential stability has faster convergence speed for fractional order systems compared to integer order systems at the neighbourhood of $t = 0$

i.e., $$\left[\frac{d}{dt}(e^{-at})\right]_{t=0} = \left[-ae^{-at}\right]_{t=0} = -a$$

and $$\left[\frac{d}{dt}(t^{-b} e^{-at})\right]_{t=0} = \left[-bt^{-b-1} e^{-at} - at^{-b} e^{-at}\right]_{t=0} = \infty, \tag{4.18}$$

where a and b are positive constants.

Remark 4.4: Let the system $D^{\alpha}x(t) = f(x(t))$, $x(t) \in R$, has an equilibrium point at $x = 0$. If the condition $x(t).f(x(t)) \le 0,\ \forall x$ is satisfied, then the equilibrium point is stable. If $x(t).f(x(t)) < 0,\ \forall x \ne 0$, then the equilibrium point is asymptotically stable.

Lemma 4.2: (Yang (2013)) For any $\sigma \in R^{+}$ and $X, Y \in R$, the inequality

$2|X||Y| \le \sigma X^2 + \sigma^{-1} Y^2$ holds.

4.3 Systems' descriptions

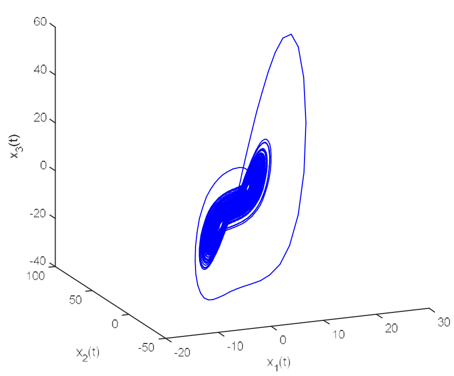
4.3.1 Fractional order complex Lorenz system

The Lorenz system in complex variables was proposed by Fowler et al. (1982) and later it is extended to the complex space system. Luo and Wang (2013) have designed the complex Lorenz system in fractional order system as

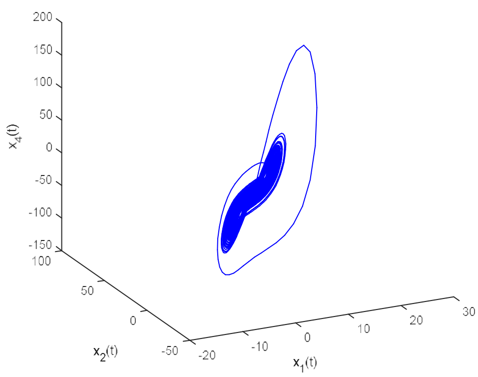
$$\begin{aligned}
\frac{d^{\alpha}x'_1}{dt^{\alpha}} &= a_1(x'_2 - x'_1), \\
\frac{d^{\alpha}x'_2}{dt^{\alpha}} &= a_2 x'_1 - x'_2 - x'_1 x'_3, \\
\frac{d^{\alpha}x'_3}{dt^{\alpha}} &= \frac{1}{2}(\bar{x}'_1 x'_2 + x'_1 \bar{x}'_2) - a_3 x'_3,
\end{aligned} \tag{4.19}$$

where $\alpha \in (0, 1)$ is defined in Caputo sense, $x'_1 = x_1 + i x_2$, $x'_2 = x_3 + i x_4$ are complex state variables with $i = \sqrt{-1}$ and $x'_3 = x_5$ is the real state variable. When the parametric values are taken as $a_1 = 10,\ a_2 = 180,\ a_3 = 1$ and fractional order derivative $\alpha = 0.95$ with initial conditions $(2, 3, -5, -6, 9)$, the system (4.19) shows chaotic behaviour, which is depicted through Fig. 4.1. Since the Caputo derivative also follows the linearity property, therefore separating the real and imaginary parts we can write the equation (4.19) in the following form

$$\begin{aligned}
\frac{d^{\alpha}x_1}{dt^{\alpha}} &= a_1(x_3 - x_1),\\
\frac{d^{\alpha}x_2}{dt^{\alpha}} &= a_1(x_4 - x_2),\\
\frac{d^{\alpha}x_3}{dt^{\alpha}} &= a_2x_1 - x_3 - x_1x_5,\\
\frac{d^{\alpha}x_4}{dt^{\alpha}} &= a_2x_2 - x_4 - x_2x_5,\\
\frac{d^{\alpha}x_5}{dt^{\alpha}} &= x_1x_3 + x_2x_4 - a_3x_5.
\end{aligned} \tag{4.20}$$



(a)



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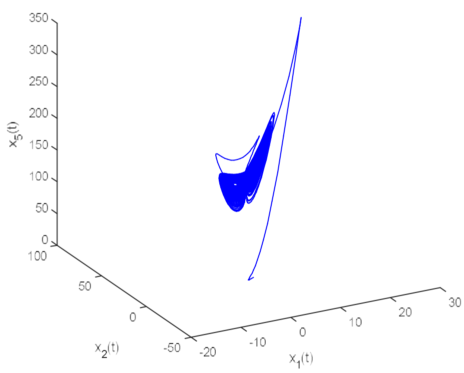

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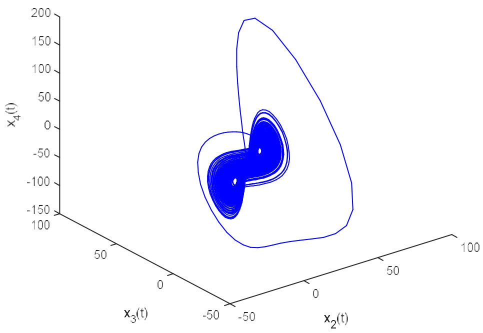

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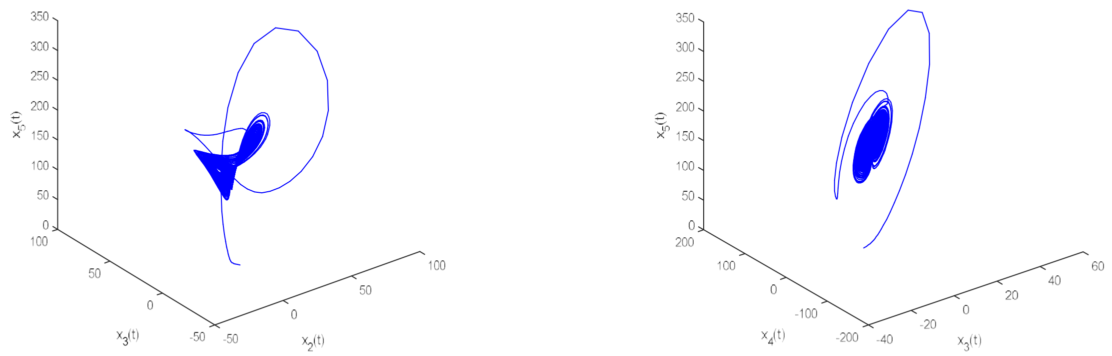


(e) (f)

Fig. 4.1 Chaotic attractors of Lorenz system at $\alpha = 0.95$ in (a) $x_1 - x_2 - x_3$ space; (b) $x_1 - x_2 - x_4$ space; (c) $x_1 - x_2 - x_5$ space; (d) $x_2 - x_3 - x_4$ space; (e) $x_2 - x_3 - x_5$ space; (f) $x_3 - x_4 - x_5$ space.

4.3.2 Fractional order complex Lu system

The complex Lu chaotic system in integer order was proposed by Mahmoud et al. (2007). Jiang et al. (2014) were proposed fractional order Lu chaotic system in complex variables as

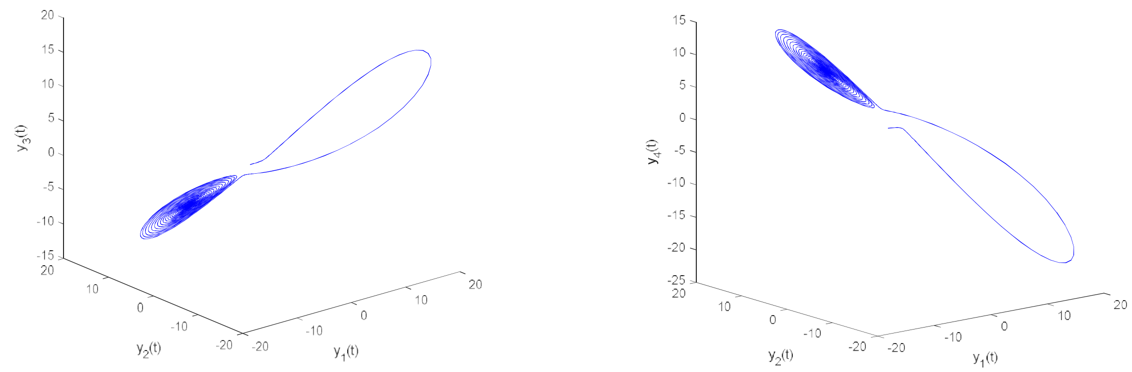
$$\frac{d^\alpha y'_1}{dt^\alpha} = b_1(y'_2 - y'_1),$$

$$\frac{d^\alpha y'_2}{dt^\alpha} = b_2 y'_2 - y'_1 y'_3, \tag{4.21}$$

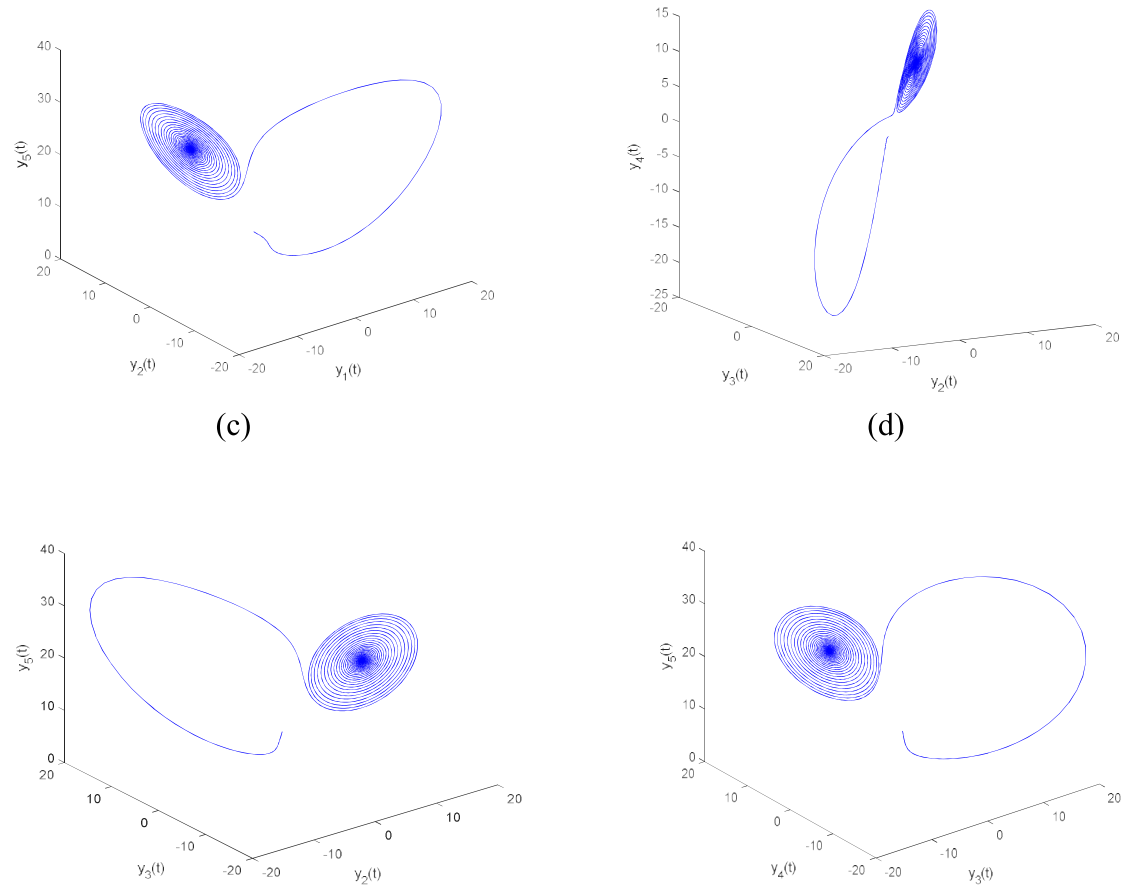
$$\frac{d^\alpha y'_3}{dt^\alpha} = \frac{1}{2}(\bar{y}'_1 y'_2 + y'_1 \bar{y}'_2) - b_3 y'_3,$$

where y'_1, y'_2, y'_3 are state variable, $y'_1 = y_1 + i\,y_2$, $y'_2 = y_3 + i\,y_4$ are complex state variables and $y'_1 = y_5$ is the real variable and $i = \sqrt{-1}$. b_1, b_2 and b_3 are system's parameters. When the parameters' values are taken as $b_1 = 42, b_2 = 22$, $b_3 = 5$ and fractional order derivative $\alpha = 0.95$ with initial condition $(-1, 9, 8, 5, 1)$, the Lu system shows the chaotic behaviour, which is shown through Fig. 4.2. Equation (4.21) can be written as

$$
\begin{aligned}
\frac{d^\alpha y_1}{dt^\alpha} &= b_1(y_5 - y_1), \\
\frac{d^\alpha y_2}{dt^\alpha} &= ab_1(y_4 - y_2), \\
\frac{d^\alpha y_3}{dt^\alpha} &= b_2 y_3 - y_1 y_5, \\
\frac{d^\alpha y_4}{dt^\alpha} &= b_2 y_4 - y_2 y_5, \\
\frac{d^\alpha y_5}{dt^\alpha} &= y_1 y_3 + y_2 y_4 - b_3 y_5.
\end{aligned}
\tag{4.22}
$$



(a) (b)



(c) (d)

(e) (f)

Fig. 4.2 Chaotic attractors of Lu system at $\alpha = 0.95$ in (a) $y_1 - y_2 - y_3$ space; (b) $y_1 - y_2 - y_4$ space; (c) $y_1 - y_2 - y_5$ space; (d) $y_2 - y_3 - y_4$ space; (e) $y_2 - y_3 - y_5$ space; (f) $y_3 - y_4 - y_5$ space.

4.4 Exponential synchronization

4.4.1 Exponential synchronization of identical fractional order complex chaotic systems

In this section, we will study the exponential synchronization of fractional order complex Lorenz chaotic systems in an identical case. The complex Lorenz system is considered as a master system which is defined in equation (4.20) and the response system with control functions is defined by the following system of equations as

$$\frac{d^\alpha z_1}{dt^\alpha} = a_1(z_3 - z_1) + u_1'(t),$$

$$\frac{d^\alpha z_2}{dt^\alpha} = a_1(z_4 - z_2) + u_2'(t),$$

$$\frac{d^\alpha z_3}{dt^\alpha} = a_2 z_1 - z_3 - z_1 z_5 + u_3'(t), \tag{4.23}$$

$$\frac{d^\alpha z_4}{dt^\alpha} = a_2 z_2 - z_4 - z_2 z_5 + u_4'(t),$$

$$\frac{d^\alpha z_5}{dt^\alpha} = z_1 z_3 + z_2 z_4 - a_3 z_5 + u_5'(t),$$

where $u_i'(t),\ i = 1, 2, ..., 5$ are the control functions which will be designed later using exponential stability theory. Defining the error functions as $e_i = z_i - x_i$, $i = 1, 2, ..., 5$, we obtain the error systems as

$$\frac{d^\alpha e_1}{dt^\alpha} = a_1(e_3 - e_1) + u_1'(t),$$

$$\frac{d^\alpha e_2}{dt^\alpha} = a_1(e_4 - e_2) + u_2'(t),$$

$$\frac{d^\alpha e_3}{dt^\alpha} = a_2 e_1 - e_3 - e_1 e_5 - e_1 x_5 - e_5 x_1 + u_3'(t), \tag{4.24}$$

$$\frac{d^\alpha e_4}{dt^\alpha} = a_2 e_2 - e_4 - e_2 e_5 - e_2 x_5 - e_5 x_2 + u_4'(t),$$

$$\frac{d^\alpha e_5}{dt^\alpha} = e_1 e_3 + e_2 e_4 - a_3 e_5 + e_1 x_3 + e_2 x_4 + e_3 x_1 + e_4 x_2 + u_5'(t).$$

Now we will design control functions $u_i(t)$, $i = 1, 2, ..., 5$ using control technique. It is observed that the error states of the error systems will be exponentially converge to zero after a few time, and hence the considered systems are exponentially stabilized for all initial conditions in a set $D_5 \subseteq R^5$.

Lemma 4.3: (Yang (2013)) For the error system (4.24) if there exits a positive quadratic polynomial $V(E(t)) = V(t)$ such that

$$\psi_1 \, E(t)E^T(t) \le V(t) \le \psi_2 \, E(t)E^T(t), \tag{4.25}$$

$$\frac{d^\alpha V(t)}{dt^\alpha} \le -\psi_3 \, E(t)E^T(t), \tag{4.26}$$

where ψ_1, ψ_2, ψ_3 are positive constants and $\psi_1 \le \psi_2$, then the zero solution of the system (4.24) is stable, and systems (4.20) and (4.23) are said to be exponentialy synchronized.

To show the exponential synchronization between the fractional order complex chaotic systems (4.20) and (4.23), let us define the positive quadratic function candidate as

$$V(t) = \frac{1}{2}(e_1^{\,2} + e_2^{\,2} + e_3^{\,2} + e_4^{\,2} + e_5^{\,2}) = E(t)ME^T(t), \tag{4.27}$$

where the matrix $M = diag(1, 1, 1, 1, 1) \in R^{5\times5}$ is the diagonal and positive definite.

Now taking the fractional derivative of the equation (4.27) w.r.t t, we get

$$\frac{d^\alpha V(t)}{dt^\alpha} \le e_1 \frac{d^\alpha e_1}{dt^\alpha} + e_2 \frac{d^\alpha e_2}{dt^\alpha} + e_3 \frac{d^\alpha e_3}{dt^\alpha} + e_4 \frac{d^\alpha e_4}{dt^\alpha} + e_5 \frac{d^\alpha e_5}{dt^\alpha}. \text{ (Using Lemma 1.7)} \tag{4.28}$$

Putting the values of $\frac{d^\alpha e_i}{dt^\alpha}$, $i = 1, 2, ..., 5$ from equation (4.24) in equation (4.29), we obtain

$$\begin{aligned}\frac{d^\alpha V(t)}{dt^\alpha} \le\ & e_1[a_1(e_3 - e_1) + u_1'(t)] + e_2[a_1(e_4 - e_2) + u_2'(t)] + e_3[a_2 e_1 - e_3 - e_1 e_5 - e_1 x_5 - e_5 x_1 + u_3'(t)] \\ & + e_4[a_2 e_2 - e_4 - e_2 e_5 - e_2 x_5 - e_5 x_2 + u_4'(t)] \\ & + e_5[e_1 e_3 + e_2 e_4 - a_3 e_5 + e_1 x_3 + e_2 x_4 + e_3 x_1 + e_4 x_2 + u_5'(t)].\end{aligned}$$

Using the relations $u_i'(t) = -k_i' e_i(t)$, $i = 1,2,...,5$, we get

$$\frac{d^\alpha V(t)}{dt^\alpha} \le -(a_1 + k_1')e_1^2 - (a_1 + k_2')e_2^2 - (1 + k_3')e_3^2 - (1 + k_4')e_4^2 - (a_3 + k_5')e_5^2 + (a_1 + a_2)e_1 e_3$$

$$+ (a_1 + a_2)e_2 e_4 - e_1 e_3 x_5 - e_2 e_4 x_5 + e_1 e_5 x_3 + e_3 e_5 x_4 . \tag{4.29}$$

Now, taking $|x_i| < M_i$, $i = 1, 2,..., 5$ and using Lemma 4.3, we have

$$\frac{d^\alpha V(t)}{dt^\alpha} < -(a_1 + k_1')e_1^2 - (a_1 + k_2')e_2^2 - (1 + k_3')e_3^2 - (1 + k_4')e_4^2 - (a_3 + k_5')e_5^2 + (a_1 + a_2)|e_1||e_3|$$

$$+ (a_1 + a_2)|e_2||e_4| + |e_1||e_3|M_5 + |e_2||e_4|M_5 + |e_1||e_5|M_3 + |e_3||e_5|M_4 .$$

$$= -(a_1 + k_1')e_1^2 - (a_1 + k_2')e_2^2 - (1 + k_3')e_3^2 - (1 + k_4')e_4^2 - (a_3 + k_5')e_5^2 + (a_1 + a_2)\left[\frac{\sigma_1 e_1^2}{2} + \frac{e_3^2}{2\sigma_1}\right]$$

$$+ (a_1 + a_2)\left[\frac{\sigma_2 e_2^2}{2} + \frac{e_4^2}{2\sigma_2}\right] + \left[\frac{\sigma_1 e_1^2}{2} + \frac{e_3^2}{2\sigma_1}\right]M_5$$

$$+ \left[\frac{\sigma_2 e_2^2}{2} + \frac{e_4^2}{2\sigma_2}\right]M_5 + \left[\frac{\sigma_3 e_1^2}{2} + \frac{e_5^2}{2\sigma_3}\right]M_3 + \left[\frac{\sigma_5 e_3^2}{2} + \frac{e_5^2}{2\sigma_5}\right]M_4 . \tag{4.30}$$

$$\frac{d^\alpha V(t)}{dt^\alpha} < -\left[a_1 + k_1' - \frac{(a_1 + a_2)\sigma_1}{2} - \frac{\sigma_1 M_5}{2} - \frac{\sigma_3}{2}M_3\right]e_1^2 - \left[a_1 + k_2' - \frac{(a_1 + a_2)\sigma_2}{2} - \frac{\sigma_2 M_5}{2}\right]e_2^2$$

$$-\left[1 + k_3' - \frac{(a_1 + a_2)}{2\sigma_1} - \frac{M_5}{2\sigma_1} - \frac{\sigma_4 M_4}{2}\right]e_3^2 - \left[1 + k_4' - \frac{(a_1 + a_2)}{2\sigma_2} - \frac{M_5}{2\sigma_2}\right]e_4^2$$

$$-\left[a_3 + k_5' - \frac{M_3}{2\sigma_3} - \frac{M_4}{2\sigma_4}\right]e_5^2 . \tag{4.31}$$

By choosing the feedback gains k_i', $i = 1, 2,..., 5$ as $k_1' > \frac{(a_1 + a_2)\sigma_1}{2} + \frac{\sigma_1 M_5}{2} + \frac{\sigma_3}{2}M_3 - a_1$,

$k_2' > \frac{(a_1 + a_2)\sigma_2}{2} + \frac{\sigma_2 M_5}{2} - a_1$, $k_3' > \frac{(a_1 + a_2)}{2\sigma_1} + \frac{M_5}{2\sigma_1} + \frac{\sigma_4 M_4}{2} - 1$, $k_4' > \frac{(a_1 + a_2)}{2\sigma_2} + \frac{M_5}{2\sigma_2} - 1$,

$k_5' > \frac{M_3}{2\sigma_3} + \frac{M_4}{2\sigma_4} - a_3$, where σ_1, σ_2, σ_3, σ_4 are positive constants, we finally obtain

$$\frac{d^{\alpha}V(t)}{dt^{\alpha}} < -q' E(t)E^{T}(t), \tag{4.32}$$

where

$$q' = \min\left\{ a_1 + k_1' - \frac{(a_1+a_2)\sigma_1}{2} - \frac{\sigma_1 M_5}{2} - \frac{\sigma_3}{2}M_3,\ a_1 + k_2' - \frac{(a_1+a_2)\sigma_2}{2} - \frac{\sigma_2 M_5}{2}, \right.$$
$$\left. 1 + k_3' - \frac{(a_1+a_2)}{2\sigma_1} - \frac{M_5}{2\sigma_1} - \frac{\sigma_4 M_4}{2},\ 1 + k_4' - \frac{(a_1+a_2)}{2\sigma_2} - \frac{M_5}{2\sigma_2},\ \ a_3 + k_5' - \frac{M_3}{2\sigma_3} - \frac{M_4}{2\sigma_4} \right\}.$$

In the view of Lemmas 4.2 and 4.3, we can conclude that the considered complex chaotic systems (4.20) and (4.23) in fractional order will be exponentially synchronized.

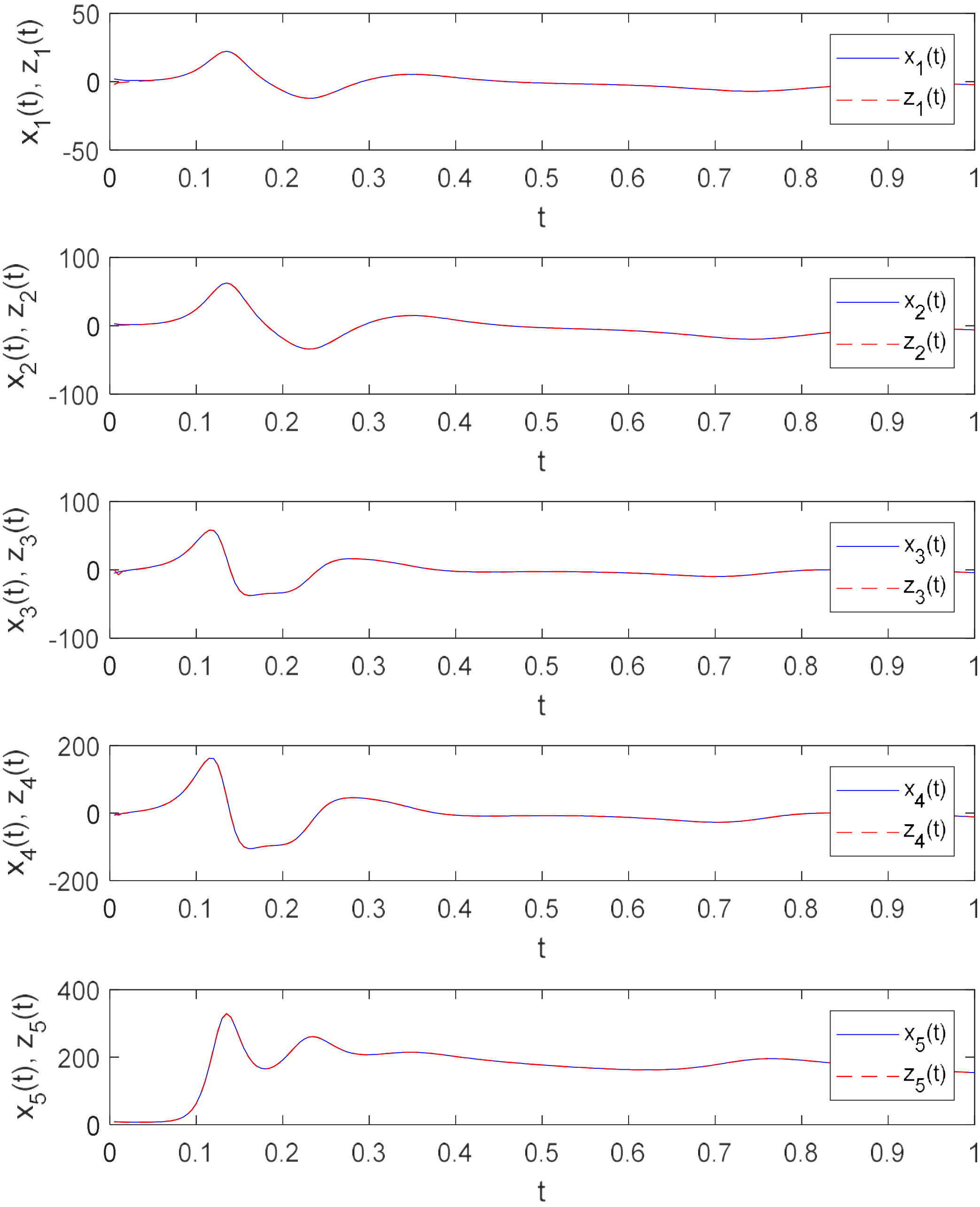
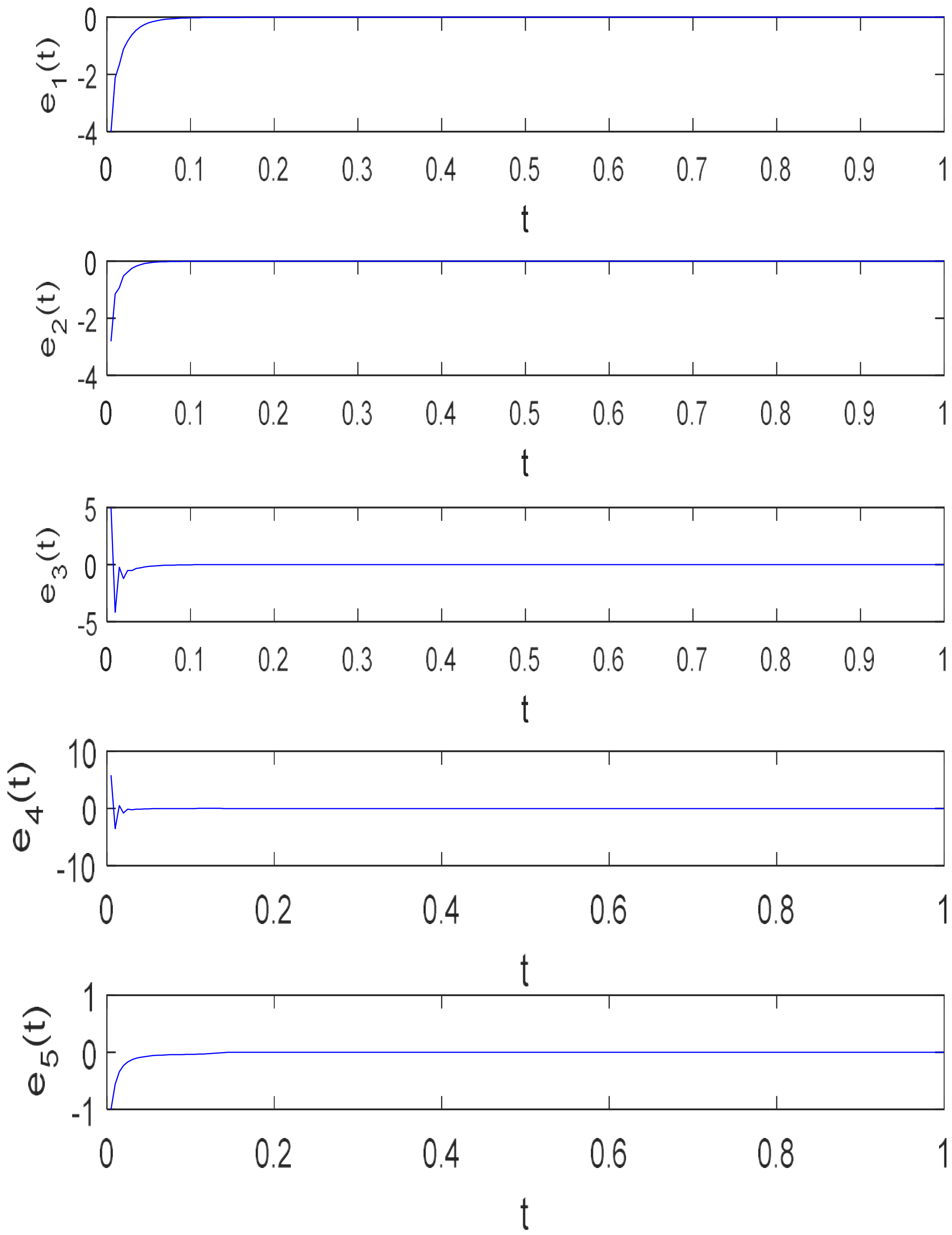
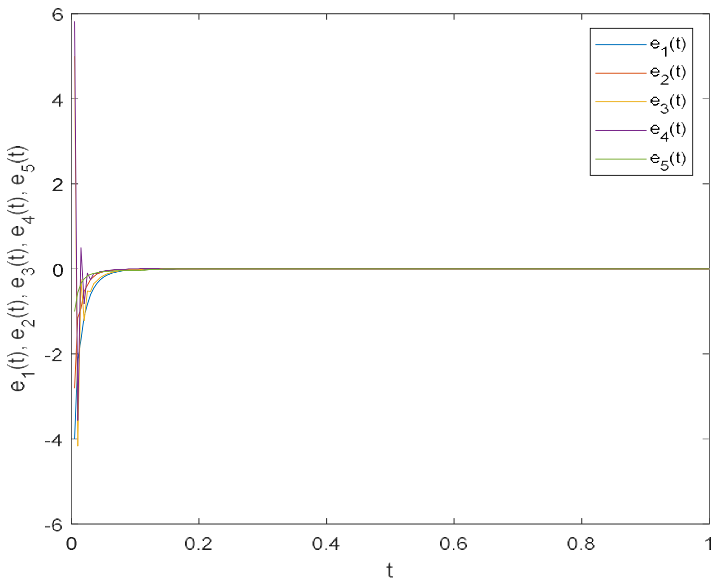


Fig. 4.3 State trajectories for exponential synchronization of systems (4.20) and (4.23) at $\alpha = 0.95$.


e1(t)
0
-2
-4
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
t
e2(t)
0
-2
-4
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
t
e3(t)
5
0
-5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
t
e4(t)
10
0
-10
0 0.2 0.4 0.6 0.8 1
t
e5(t)
1
0
-1
0 0.2 0.4 0.6 0.8 1
t

(a)



(b)

Fig. 4.4 Plots of error functions of exponential synchronization at $\alpha = 0.95$.

4.4.2 Exponential synchronization of non-identical fractional order complex chaotic systems

In this section, we will discuss the exponential synchronization between non-identical complex chaotic systems in fractional order. Consider the fractional order complex Lorenz system as a master system defined by equation (4.20). Let Lu system as a response system which is defined through control functions as

$$\frac{d^\alpha y'_1}{dt^\alpha} = b_1(y'_2 - y'_1) + u'_1(t),$$
$$\frac{d^\alpha y'_2}{dt^\alpha} = b_2 y'_2 - y'_1 y'_3 + u'_2(t), \tag{4.33}$$
$$\frac{d^\alpha y'_3}{dt^\alpha} = \frac{1}{2}(\bar{y}'_1 y'_2 + y'_1 \bar{y}'_2) - b_3 y'_3 + u'_3(t),$$

where $u'_1(t) = u_1(t) + u_2(t)$, $u'_2(t) = u_3(t) + u_4(t)$ and $u'_3(t) = u_5(t)$ are the control functions, which will be designed using control technique to achieve exponential synchronization between considered fractional order master and response systems, Now separating the real and imaginary parts of the system (4.33), we get

$$
\begin{aligned}
\frac{d^\alpha y_1}{dt^\alpha} &= b_1(y_5 - y_1) + u_1(t), \\
\frac{d^\alpha y_2}{dt^\alpha} &= ab_1(y_4 - y_2) + u_2(t), \\
\frac{d^\alpha y_3}{dt^\alpha} &= b_2 y_3 - y_1 y_5 + u_3(t), \\
\frac{d^\alpha y_4}{dt^\alpha} &= b_2 y_4 - y_2 y_5 + u_4(t), \\
\frac{d^\alpha y_5}{dt^\alpha} &= y_1 y_3 + y_2 y_4 - b_3 y_5 + u_5(t).
\end{aligned}
\tag{4.34}
$$

The error functions between systems (4.25) and (4.34) are defined as $e_i = y_i - x_i$, $i = 1, 2, \ldots, 5$.

Using the definition of error functions, we can obtain the following error systems as

$$
\begin{aligned}
\frac{d^\alpha e_1}{dt^\alpha} &= b_1(e_3 - e_1) + (b_1 - a_1)(x_4 - x_1) + u_1(t)\ , \\
\frac{d^\alpha e_2}{dt^\alpha} &= b_1(e_4 - e_2) + (b_1 - a_1)(x_4 - x_2) + u_2(t), \\
\frac{d^\alpha e_3}{dt^\alpha} &= b_2 e_3 - e_1 e_5 - e_1 x_5 - e_5 x_1 + b_2 x_3 - a_2 x_1 + x_3 + u_3(t), \\
\frac{d^\alpha e_4}{dt^\alpha} &= b_2 e_4 - e_2 e_5 - e_2 x_5 - e_5 x_2 + b_2 x_4 - a_2 x_2 + x_4 + u_4(t), \\
\frac{d^\alpha e_5}{dt^\alpha} &= e_1 e_3 + e_1 x_3 + e_3 x_1 + e_2 e_4 + e_2 x_4 + e_4 x_2 - b_3 x_5 + (a_3 - b_3) x_5 + u_5(t).
\end{aligned}
\tag{4.35}
$$

Now we will design control functions $u_i(t)$, $i=1, 2,...,5$ for exponential synchronization by using control technique. After that the error states of error systems will be exponentially converge to zero after a few time and hence the considered systems are exponentially stabilized for all initial conditions in a set $D_5 \subseteq R^5$.

Theorem 4.4: The master system (4.20) and response system (4.34) will be exponentially synchronized if the feedback control functions are taken as

$$\begin{aligned} u_1(t) &= -(b_1 - a_1)(x_3 - x_1), \\ u_2(t) &= -(b_1 - a_1)(x_4 - x_2), \\ u_3(t) &= -b_2 x_3 + a_2 x_1 - x_3 - k_3 e_3, \\ u_4(t) &= -b_2 x_4 + a_2 x_2 - x_4 - k_4 e_4, \\ u_5(t) &= -(a_3 - b_3) x_5, \end{aligned} \tag{4.36}$$

where the feedback gains $k_3 > 0$, $k_4 > 0$ satisfy

$$k_3 > b_2 + \frac{b_1}{2\sigma_1} + \frac{M_5}{2\sigma_1} + \sigma_3 M_1, k_4 > b_2 + \frac{b_1}{2\sigma_2} + \frac{M_5}{2\sigma_2}, \tag{4.37}$$

with the conditions

$$b_1 - \left[\frac{b_1\sigma_1}{2} + \frac{\sigma_1 M_5}{2} + \frac{\sigma_5 M_5}{2}\right] > 0, b_1 - \left[\frac{b_1\sigma_2}{2} + \frac{\sigma_2 M_2}{2} + \frac{\sigma_6 M_4}{2}\right] > 0, \tag{4.38}$$

$$b_3 - \left[\frac{M_1}{2\sigma_3} + \frac{M_3}{2\sigma_5} + \frac{M_1}{2\sigma_3} + \frac{M_4}{2\sigma_6}\right] > 0,$$

where $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$ are positive constant and $|x_i| < M_i$, $i = 1, 2,..., 5$.

Proof: Now defining positive quadratic function candidate as defined in Theorem 4.4 and substituting the values of $\frac{d^\alpha e_i}{dt^\alpha}$, $i = 1, 2, ..., 5$ from equation (4.35) in equation (4.28), we can obtain

$$\frac{d^\alpha V(t)}{dt^\alpha} \le e_1[b_1(e_3 - e_1) + (b_1 - a_1)(x_3 - x_1) + u_1(t)] + e_2[b_1(e_4 - e_2) + (b_1 - a_1)(x_4 - x_2) + u_2(t)]$$

$$+ e_3[b_2e_3 - e_1e_5 - e_1x_5 - e_5x_1 + b_2x_3 - a_2x_1 + x_3 + u_3(t)]$$

$$+ e_4[b_2e_4 - e_2e_5 - e_2x_5 - e_5x_2 + b_2x_4 - a_2x_2 + x_4 + u_4(t)]$$

$$+ e_5[e_1e_3 + e_1x_3 + e_3x_1 + e_2e_4 + e_2x_4 + e_4x_2 - b_3x_5 + (a_3 - b_3)x_5 + u_5(t)]. \tag{4.39}$$

Using equation (4.36), we get

$$\frac{d^\alpha V(t)}{dt^\alpha} \le -b_1e_1^2 - b_1e_2^2 - k_3e_3^2 - k_4e_4^2 - b_3e_5^2 + b_2e_3^2 + b_2e_4^2 + b_1e_1e_3 + b_1e_2e_4 - e_1e_3x_5 - e_3e_5x_1 - e_2e_4x_5$$

$$+ e_1e_5x_3 + e_3e_5x_1 + + e_2e_5x_4. \tag{4.40}$$

Now, taking $|x_i| < M_i$, $i = 1, 2, ..., 5$ and using Lemma 1.7, we have

$$\frac{d^\alpha V(t)}{dt^\alpha} < -b_1e_1^2 - b_1e_2^2 - k_3e_3^2 - k_4e_4^2 - b_3e_5^2 + b_2e_3^2 + b_2e_4^2 + b_1|e_1||e_3| + b_1|e_2||e_4|$$

$$+ |e_1||e_3|M_5 + |e_3||e_5|M_1 + |e_2||e_4|M_5 + |e_1||e_5|M_3 + |e_3||e_5|M_1 + |e_2||e_5|M_4 .$$

$$= -b_1e_1^2 - b_1e_2^2 - k_3e_3^2 - k_4e_4^2 - b_3e_5^2 + b_2e_3^2 + b_2e_4^2$$

$$+ b_1\left[\frac{\sigma_1}{2}e_1^2 + \frac{e_3^2}{2\sigma_1}\right] + b_1\left[\frac{\sigma_2}{2}e_2^2 + \frac{e_4^2}{2\sigma_2}\right] + \left[\frac{\sigma_1}{2}e_1^2 + \frac{e_3^2}{2\sigma_1}\right]M_5 + 2\left[\frac{\sigma_3}{2}e_3^2 + \frac{e_5^2}{2\sigma_3}\right]M_1$$

$$+ \left[\frac{\sigma_2}{2}e_2^2 + \frac{e_4^2}{2\sigma_2}\right]M_5 + \left[\frac{\sigma_5}{2}e_1^2 + \frac{e_5^2}{2\sigma_5}\right]M_3 + \left[\frac{\sigma_6}{2}e_2^2 + \frac{e_5^2}{2\sigma_6}\right]M_4 . \tag{4.41}$$

i.e., $$\frac{d^{\alpha}V(t)}{dt^{\alpha}} < -\left[b_1 - \frac{b_1\sigma_1}{2} - \frac{\sigma_1 M_5}{2} - \frac{\sigma_1 M_3}{2}\right]e_1^{\,2} - \left[b_1 - \frac{b_1\sigma_2}{2} - \frac{\sigma_2 M_5}{2} - \frac{\sigma_6 M_4}{2}\right]e_2^{\,2}$$

$$-\left[k_3 - b_2 - \sigma_3 M_1 - \frac{b_1}{2\sigma_1} - \frac{M_5}{2\sigma_1}\right]e_3^{\,2} - \left[k_4 - b_2 - \frac{b_1}{2\sigma_2} - \frac{M_5}{2\sigma_2}\right]e_4^{\,2}$$

$$-\left[b_3 - \frac{M_1}{\sigma_3} - \frac{M_3}{2\sigma_5} - \frac{M_4}{2\sigma_6}\right]e_5^{\,2}. \tag{4.42}$$

By choosing $k_3 > 0$, $k_4 > 0$ as $k_3 > b_2 + \frac{b_1}{2\sigma_1} + \frac{M_5}{2\sigma_1} + \sigma_3 M_1$, $k_4 > b_2 + \frac{b_1}{2\sigma_2} + \frac{M_5}{2\sigma_2}$ with the conditions $b_1 - \left[\frac{b_1\sigma_1}{2} + \frac{\sigma_1 M_5}{2} + \frac{\sigma_5 M_5}{2}\right] > 0$, $b_1 - \left[\frac{b_1\sigma_2}{2} + \frac{\sigma_2 M_2}{2} + \frac{\sigma_6 M_4}{2}\right] > 0$ and $b_3 - \left[\frac{M_1}{2\sigma_3} + \frac{M_3}{2\sigma_5} + \frac{M_1}{2\sigma_3} + \frac{M_4}{2\sigma_6}\right] > 0$, we have

$$\frac{d^{\alpha}V(t)}{dt^{\alpha}} < -qE(t)E^T(t), \tag{4.43}$$

where

$$q = \min\left\{b_1 - \frac{b_1\sigma_1}{2} - \frac{\sigma_1 M_5}{2} - \frac{\sigma_1 M_3}{2}, b_1 - \frac{b_1\sigma_2}{2} - \frac{\sigma_2 M_5}{2} - \frac{\sigma_6 M_4}{2}, k_3 - b_2 - \sigma_3 M_1 - \frac{b_1}{2\sigma_1} - \frac{M_5}{2\sigma_1},\right.$$
$$\left. k_4 - b_2 - \frac{b_1}{2\sigma_2} - \frac{M_5}{2\sigma_2}, b_3 - \frac{M_1}{\sigma_3} - \frac{M_3}{2\sigma_5} - \frac{M_4}{2\sigma_6}\right\}.$$

In view of Lemmas 4.3 and 4.4, the considered fractional order complex chaotic systems (4.20) and (4.34) will be exponentially synchronized.

4.4.3 Numerical simulations

During exponential synchronization, the parametric values of the fractional order complex Lorenz and Lu systems are taken as $a_1 = 10, a_2 = 180, a_3 = 1$ and $b_1 = 42,\ b_2 = 22,\ b_3 = 5$ respectively. For Theorem 4.4, the initial values for master and response systems are considered as $(2, 3, -5, -6, 9)$ and $(-2,\ 0.19,\ 0,\ -0.18,\ 8)$

respectively, and hence the initial condition of the error system will be $(-4,\ -2.8,\ 5,\ 5.82,\ -1)$. We have chosen the values of $\sigma_i = \frac{1}{2},\ i = 1, 2, ..., 4$ and upper bounds absolute value $M_i,\ i = 1, 2, ..., 5$ of the state variables $x_i,\ i = 1, 2, ..., 5$ of the system (4.20) can be estimated through numerical simulations. In Theorem 4.5, the initial conditions of master and response systems are considered as $(2, 3, -5, -6, 9)$ and $(-1,\ 9,\ 8,\ 5,\ 1)$, and we have also chosen the values of $\sigma_i, i = 1, 2, ..., 6$ as $\frac{1}{2},\ \frac{1}{3},\ 2,\ 1,\ 3,\ 4$ respectively. During exponential synchronization time step size is taken as 0.005. The state trajectories of the considered master and response systems are presented through Figs. 4.3 and 4.5 for identical and non-identical cases respectively at $\alpha = 0.95$, the considered systems are exponentially synchronized after a very small time. It is also seen from Figs. 4.4 and 4.6, the error functions are exponentially converged to zero after a small time duration. It can be seen from the literature survey that the exponential synchronization takes less time compared to other synchronization schemes such as adaptive synchronization, modified projective synchronization etc. (Nourian and Balochian (2016), Singh et al. (2017), Tian et al. (2016)). In this chapter, the exponential synchronization is achieved after time $t = 0.2$ (Fig. 4.4(b) and 4.6(b)). In the next section, the application of exponential synchronization of fractional order systems via digital cryptography with a secret key system to improve the security of secure communication have been discussed.

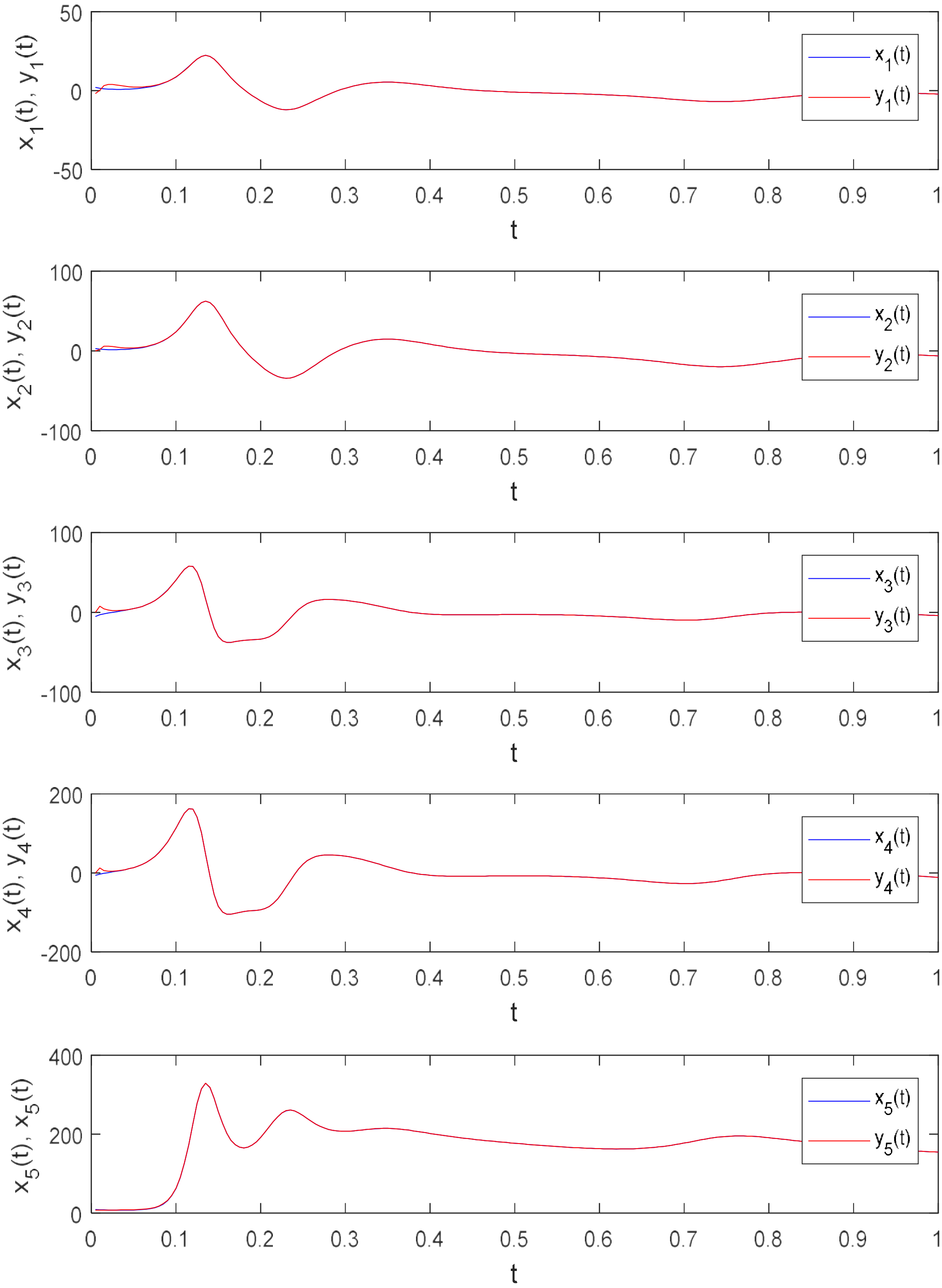
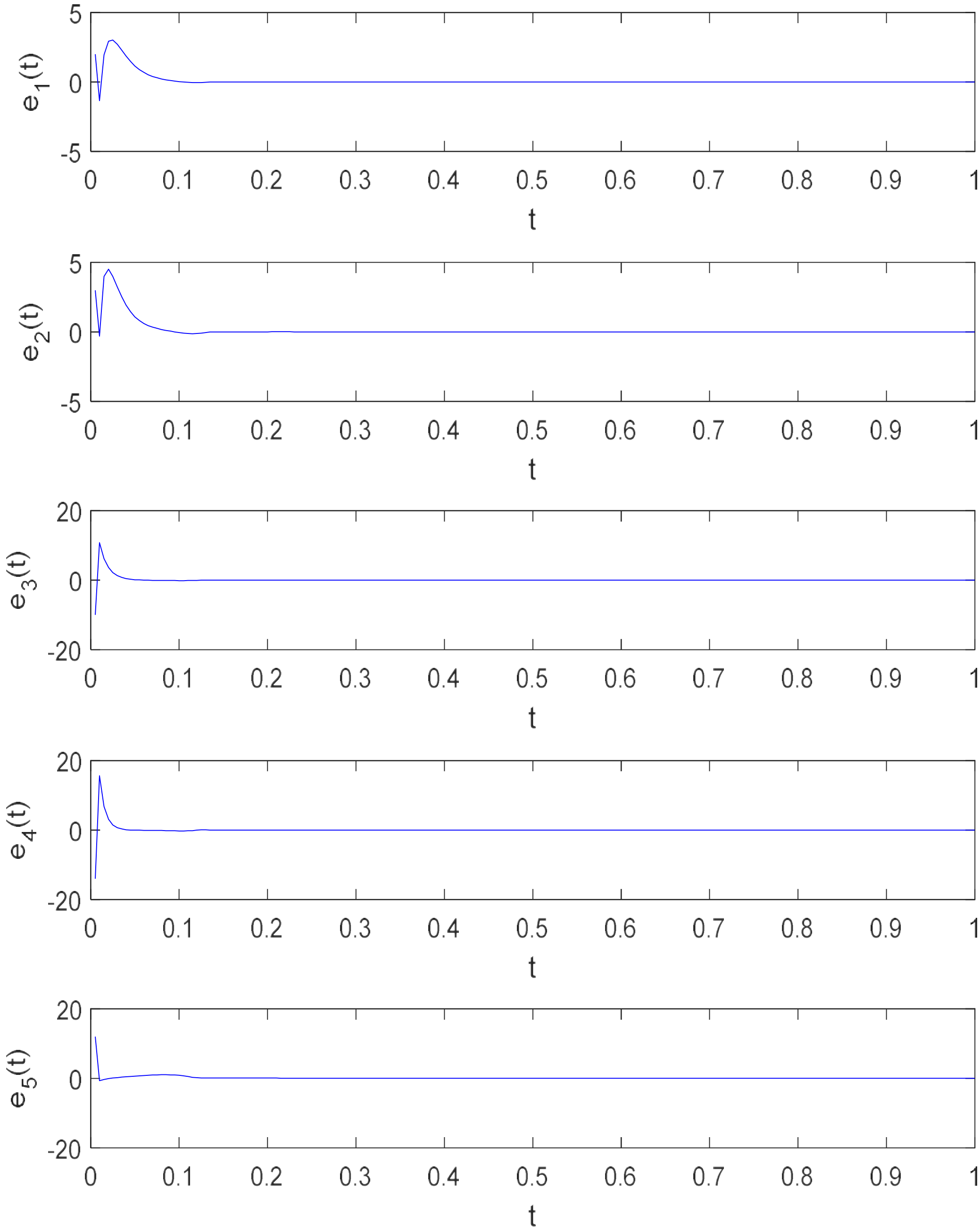
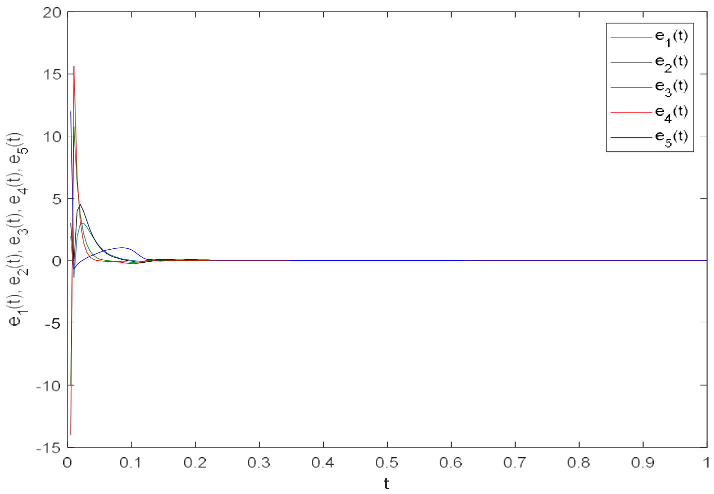


Fig. 4.5 State trajectories for exponential synchronization of systems (4.20) and (4.34) at $\alpha = 0.95$.



(a)



(b)

Fig. 4.6 Plots of error functions of exponential synchronization at $\alpha = 0.95$.

4.5 Application

4.5.1 Synchronized systems and cryptography (Mitra and Banerjee (2011), Muthukumar and Balasubramaniam (2013), Zhen et al. (2013))

We have utilized exponential synchronization scheme of two identical systems in communications with the help of cryptographic encoding. First, we will consider a sender system (driving system (4.20)) and a receiver system (response system (4.23)). For this case the error functions converge to zero after $t > 0.1$ at order of the fractional derivative $\alpha = 0.95$. After $t > 0.1$, the state variables of the systems (4.20) and (4.23) will be $x_i = z_i$, $i = 1, 2, ..., 5$, as the systems are synchronized. Therefore, both the sender and receiver will choose the same values of x_1 and z_1 variables after time $t = 0.1$ to obtain the secret keys. During communication, Fibonacci Q- matrices will be used to create secret keys.

4.5.2 Fibonacci Q- matrix

The Fibonacci sequence is

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

and it is defined by $F_0 = 0$, $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.

The Fibonacci Q- matrix is introduced in the article (Gould (1981)) and it is a (2×2) square matrix defined in Lemma 4.5.

Lemma 4.4: Let $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$.

Property 4.2: $Q^{m+n} = Q^m Q^n = Q^n Q^m$.

4.5.3 The process of generation of the secret keys

To generate secret keys between sender and receiver during communication, both will take the values $[100 \,|\, x_1 \,|\, \alpha]$ and $[100 \,|\, z_1 \,|\, \alpha]$ respectively after time $t > t_0$, where t_0 is initial time. The plaintext (p) and ciphertext (c) are used between sender and receiver, and these can be divided into message units. The message that will be sent to the receiver from the sender is called plaintext (p). We have considered a task for the numbers for a unit message as given in (Mitra and Banerjee (2011)). The unit message will be in alphabet, decimals, numbers, space between two words or sentences or any punctuation marks. Numeric numbers 0 to 9 are assigned by 0, 1, 2, ..., 9. Alphabet (26 letters) from A to Z are assigned by the numbers 12 to 37. The blank space (gap) between two words is assigned by 10, and the decimal (full stop or any punctuations marks) is assigned by 11.

During communication, we will use one message key corresponding to every message unit, and it will randomly be generated by using Fibonacci Q- matrices. The secret keys will be a series of numbers $\{K_1, K_2, K_3, \ldots, K_n\}$ of the complete message. The message unit p_j is hidden and safed by the key K_j.

The complete method to create secret keys is explained through the following steps:

Step 1. The plaintext is arranged in the form of sequence of integers $p_1, p_2, p_3, p_4, \ldots$ with the help of Table 4.1.

Step 2. Now choose the first four terms p_1, p_2, p_3, p_4 as a first block form a square matrix of dimension (2×2) as $M=\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$, p_5, p_6, p_7, p_8 will be the second block, and so on.

Step 3. If total number of message units (plaintext sequence) is not a multiple of 4, then we will consider the blank spaces at the end of the sequence to complete the sequence as a multiple of 4.

Step 4. The sender and receiver will agree on $\alpha > 0.95$. The secret keys are chosen by the sender and receiver as $K_s = [100\,|\,x_1\,|\,\alpha](\text{mod}\,38)$ and $K_r = [100\,|\,z_1\,|\,\alpha](\text{mod}\,38)$ respectively, at $t = t_1(> t_0)$ for the first four keys, where the $[n]$ is the integer part of n, at $t = t_1$ (during synchronization) the secret key K will be $K = K_s = K_r$.

Step 5. The first four keys K_1, K_2, K_3, K_4 are determined by Fibonacci Q- matrix and Lemma 4.4 as $Q^K = \begin{bmatrix} F_{K+1} & F_K \\ F_K & F_{K-1} \end{bmatrix} = \begin{bmatrix} \overline{K}_1 & \overline{K}_2 \\ \overline{K}_3 & \overline{K}_4 \end{bmatrix}$ and $K_1 = \overline{K}_1(\text{mod}\,38)$, $K_2 = \overline{K}_2(\text{mod}\,38)$, $K_3 = \overline{K}_3(\text{mod}\,38)$, $K_4 = \overline{K}_4(\text{mod}\,38)$.

Step 6. The ciphertext and recovered plaintext are formulated as $c_i = p_i + K_i$ and $p_i = c_i - K_i$ respectively for $i = 1, 2, 3, 4$. Next the value of K from the data series at $t = t_2(> t_1)$ for next four keys is chosen and this way the process continues.

Table 4.1: Assignment of numbers with unit messages

Number assigned	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...	37
Unit message	0	1	2	3	4	5	6	7	8	9	-	.	A	B	C	D	...	Z

Example 4.1: Let us consider a message **"AFTER 2 YEARS"**. This message has 13 message units, which is not multiple of 4. So to complete the sequence we may add the more blank spaces (see Table 4.2). To get the secret keys, the sender (system (4.20)) and receiver (system (4.23)) will agree on $\alpha = 0.95$ and take the values K_s and K_r at the time $t = 1, 2, 3, 4$.

Table 4.2: Indication of the message through keys to complete any message

Keys	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}	K_{13}	K_{14}	K_{15}	K_{16}
Unit Message	A	F	T	E	R	-	2	-	Y	E	A	R	S	-	-	-

At the time $t = 1$, the sender chooses variable x_1 of the system (4.20) and constructs secret keys K_s as $K_s = [100 \,|\, x_1 \,|\, \alpha](\text{mod}\,38) = 206(\text{mod}\,38) = 16$ and the first four message unit can be written as $M = \begin{bmatrix} 12 & 17 \\ 31 & 16 \end{bmatrix}$. The four keys to hide those are to be constructed through the following the step 4.5.

$$Q^{16} = \begin{bmatrix} F_{17} & F_{16} \\ F_{16} & F_{15} \end{bmatrix} = \begin{bmatrix} \overline{K}_1 & \overline{K}_2 \\ \overline{K}_3 & \overline{K}_4 \end{bmatrix} = \begin{bmatrix} 1597 & 987 \\ 987 & 610 \end{bmatrix},$$

i.e., $K_1 = 1$, $K_2 = 37$, $K_3 = 37$, $K_4 = 4$. Hence the cipher text for the first four units will be

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} + \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = \begin{bmatrix} 13 & 54 \\ 68 & 18 \end{bmatrix}.$$

Similarly, the receiver can choose the value z_1 from system (4.23) at the time $t = 1$ and construct the secret key K_r as $K_r = [100 \mid z_1 \mid \alpha](\text{mod}\, 38) = 206(\text{mod}\, 38) = 16$ using step 4.5. The receiver may decode the message by using decryption formula from step 4.6.

The complete structure of message **"AFTER 2 YEARS"** between sender to receiver are given in Table 4.3.

Table 4.3: Structure of the message for sending and receiving **"AFTER 2 YEARS"** during synchronization

Time (t)	$\lvert x_1 \rvert$	**Plaintext (p)**	**Keys** K_s **(mod 38)**	**Ciphertext (c)**	$\lvert z_1 \rvert$	**Keys** K_r **(mod 38)**	**Plaintext recovered (p)**
1	2.175	A(12)	$K_1 = 1$	13	2.175	$K_1 = 1$	12(A)
		F(17)	$K_2 = 37$	54		$K_2 = 37$	17(F)
		T(31)	$K_3 = 37$	68		$K_3 = 37$	31(T)
		E(16)	$K_4 = 2$	18		$K_4 = 2$	16(E)
2	3.723	R(29)	$K_5 = 30$	59	3.723	$K_5 = 30$	29(R)
		-(10)	$K_6 = 13$	23		$K_6 = 13$	10(-)
		2(2)	$K_7 = 13$	15		$K_7 = 13$	2(2)
		-(10)	$K_8 = 17$	27		$K_8 = 17$	10(-)
3	2.879	Y(36)	$K_9 = 21$	57	2.879	$K_9 = 21$	36(Y)
		E(16)	$K_{10} = 13$	29		$K_{10} = 13$	16(E)
		A(12)	$K_{11} = 13$	25		$K_{11} = 13$	12(A)
		R(29)	$K_{12} = 8$	37		$K_{12} = 8$	29(R)

4	4.635	S(30)	$K_{13} = 5$	35	4.635	$K_{13} = 5$	30(S)
		-(10)	$K_{14} = 3$	13		$K_{14} = 3$	10(-)
		-(10)	$K_{15} = 3$	13		$K_{15} = 3$	10(-)
		-(10)	$K_{16} = 2$	12		$K_{16} = 2$	10(-)

4.5.4 Security analysis of the scheme

This scheme is more complicated and contains two parts. The synchronization of identical complex chaotic systems in fractional order in which the sender and receiver take the data after the pre-assigned time interval is considered as the first part, while in the second part the algebraic operations of Fibonacci Q- matrices, which are obtained with the help of Fibonacci sequence, where each unit of a digital message has one key element of synchronized dynamical system.

The sender and receiver keys K_s and K_r, during key construction, will agree with order of the derivative α of the synchronized complex chaotic system. Both will also agree to pick up the data after a pre-assigned time. Here time t is also a secret element for both sender and receiver. The value of the variables x_1 and z_1 are also taken differently for the different time, i.e., at the time $t = 1$ the values of variables are taken as same as 2.175 and at the time $t = 2$, $t = 3$, $t = 4$ are considered as 3.723, 2.879, 4.635 respectively. So it will be very difficult to find the values of x_1 and z_1, time t and fractional derivative to break the security of message from the synchronized fractional order complex chaotic systems. Therefore the data (x_1, t, α) for the proposed key systems are more secure at every step. Further, the transformation of the message is done with the help of Fibonacci

Q- matrices, and it also gives the additional security for the key system. Thus the security of the key system is the more complicated, and it is not easy to break its security.

4.6 Conclusion

In this chapter, the exponential synchronization and its applications in fractional order complex chaotic systems through digital cryptography are discussed. The synchronization is achieved by using exponential stability theorem for fractional order dynamical systems. In exponential synchronization, error functions converge when the time is very small as seen from the Figs. 4.4 and 4.6. The exponential stability has faster convergence speed for fractional order systems compared to integer order systems which is stated in Remark 4.3, which clearly shows that the exponential synchronization scheme is better for fractional order case. The numerical simulation of exponential synchronization between fractional order complex chaotic systems demonstrates the effectiveness and universality of the method. The application in secure communication of exponential synchronization through digital cryptography in fractional order case has been shown. The salient feature of the chapter is the detail description of the proposed scheme through a numerical example to create higher stage security via exponential synchronized systems.

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