On the Weyl transform

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by

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Chapter 6

Conclusions

The Weyl transform is a non-commutative analog of the Euclidean Fourier transform. Various results about the Euclidean Fourier transform have simple analogs for the Weyl transform. For instance, the analog of the Riemann-Lebesgue lemma is that the Weyl transform of an L^1 function is a compact operator. In this thesis, we proved that under appropriate curvature and dimension assumptions, the Weyl transform of a measure is compact, and belongs to some Schatten class.

Chapter 1 is introductory in nature, and contains the appropriate background and history of the subject.

In Chapter 2, we proved that if μ is a smooth measure supported on a compact connected smooth hypersurface of positive Gaussian curvature in $\mathbb{R}^{2n}$, $n \geq 2$, then the Weyl transform of μ is compact and belongs to the p-Schatten class if $p > n \geq 6$. Since the Weyl transform of the normalized surface measure on a sphere in $\mathbb{R}^{2n}$ belongs to $S^p(\mathcal{H})$ if and only if $p > 4n/(2n-1)$, with no restriction on n, we conjectured in [32] that if μ is a smooth measure supported on a compact connected smooth hypersurface in $\mathbb{R}^{2n}$ with positive Gaussian curvature, then $W(\mu) \in S^p(\mathcal{H})$ if $p > 4n/(2n-1)$, with no restriction on n.

In Chapter 3, we proved that the Weyl transform of a smooth measure supported on a finite type real-analytic submanifold of $\mathbb{R}^{2n}$ is compact. We also proved that when $n = 1$, we do not need the addition assumption of real-analyticity, i.e., we proved that the Weyl transform of a smooth measure supported on a finite type smooth submanifold of $\mathbb{R}^2$ is compact.

In Chapter 4, we proved that the Weyl transform of a compactly supported distribution on $\mathbb{R}^{2n}$ is p-th power traceable if and only if the Fourier transform of the distribution is p-th power integrable. Moreover, we proved that the Weyl transform of a compactly supported distribution on $\mathbb{R}^{2n}$ is compact if and only if the Fourier transform of the distribution vanishes at infinity. As a consequence of this, we conclude that if μ is a smooth measure supported on a compact connected smooth hypersurface in $\mathbb{R}^{2n}$, whose Gaussian curvature is nonzero everywhere, then $W(\mu)$ is a compact operator, and $W(\mu) \in S^p(\mathcal{H})$ if and only if $p > 4n/(2n-1)$. Moreover if $\mu = \psi\sigma$ is a smooth measure supported on a finite type smooth submanifold of $\mathbb{R}^{2n}$, then $W(\mu)$ is a compact operator, and $W(\mu) \in S^p(\mathcal{H})$ if and only if $p > 2nk$, where k is the type of the submanifold inside the support of ψ.

Finally, in Chapter 5, we provide applications of these results. We described the conditions under which the quantum translates of a non-zero Schatten class operator are linearly independent. Moreover, we proved an analog of the Fourier restriction theorem for the Fourier-Wigner transform.