Chapter 5

Convergence analysis of new complex Durrmeyer operators

5.1 Introduction

In 2017, Chen et al. [44] have introduced a new modification of Bernstein operators depending on shape parameter $0 \leq \alpha \leq 1$ that provides more modeling flexibility in Bernstein operators, as follows:

$$P_m^{\alpha}(\Psi; x) = \sum_{j=0}^{m} b_{m,j}^{\alpha}(x) \Psi\left(\frac{j}{m}\right), \tag{5.1}$$

where Ψ is a continuous function defined on $[0, 1]$ and the mth degree of α-Bernstein basis function $b_{m,j}^{\alpha}(x)$ are defined by $b_{1,0}^{\alpha}(x) = 1 - x,\ b_{1,1}^{\alpha}(x) = x$ and

$$\begin{aligned} b_{m,j}^{\alpha}(x) = & \left[\binom{m-2}{j}(1-\alpha)x + \binom{m-2}{j-2}(1-\alpha)(1-x) \right. \\ & \left. + \binom{m}{j}\alpha x(1-x)\right] x^{j-1}(1-x)^{m-j-1}, \quad m \geq 2,\ x \in [0,1]. \end{aligned}$$

Clearly, $P_m^{\alpha}(\Psi; x)$ are positive linear operators for $\alpha \in [0, 1]$. In particular, at $\alpha = 1$, the α-Bernstein operators (5.1) yields Bernstein operators (1.1). Further, Mohiuddine et al. [90] have introduced a new sequence of α-Bernstein-Kantorovich operators (5.1) to approximate the integrable functions. Within this series, Kajla and Acar [68] have explored the Durrmeyer variant of the operators (5.1). These operators are defined as follows:

$$\bar{P}_m^{\alpha}(\Psi;x)=(m+1)\sum_{j=0}^{m}b_{m,j}^{\alpha}(x)\int_0^1 q_{m,j}(t)\Psi(t)dt. \tag{5.2}$$

The authors also delved into the examination of local approximation techniques, error estimation utilizing the Ditzian-Totik modulus of smoothness, and the convergence of these operators towards specific functions, accompanied by illustrative graphics.

Cetin [42] have discussed the geometric properties of the complex variant of the α-Bernstein operators (5.1) defined as follows:

$$\mathcal{B}_m^{\alpha}(\Psi;z)=\sum_{j=0}^{m}b_{m,j}^{\alpha}(z)\Psi\left(\frac{j}{m}\right)$$

where $z\in\mathbb{C}$ and Ψ is a complex-valued analytic function in an open disk centered at the origin with $R>1$. For $j=0,1,\cdots,m\in\mathbb{N}$, the complex Bernstein basis function is defined by $q_{m,j}(z)=\binom{m}{j}z^j(1-z)^{m-j}$ and the complex α-Bernstein basis function $b_{m,j}^{\alpha}(z)$ of mth degree is defined as $b_{1,0}^{\alpha}(z)=1-z,\ b_{1,1}^{\alpha}(z)=z$ and

$$\begin{aligned}b_{m,j}^{\alpha}(z)=&\left[\binom{m-2}{j}(1-\alpha)z+\binom{m-2}{j-2}(1-\alpha)(1-z)\right.\\&\left.+\binom{m}{j}\alpha z(1-z)\right]z^{j-1}(1-z)^{m-j-1},\quad m\geq 2.\end{aligned}$$

Moreover, many researchers have delved into analyzing the quantitative approximation characteristics of various complex versions of positive linear operators, see, [14, 43, 60, 63, 71, 81, 82] and references cited therein. Motivated by the above works, we introduce a complex variant of α-Bernstein Durrmeyer operator (5.2), as follows:

$$\mathfrak{D}_m^{\alpha}(\Psi; z) = (m+1)\sum_{j=0}^{m} b_{m,j}^{\alpha}(z)\int_0^1 q_{m,j}(y)\Psi(y)dy, \tag{5.3}$$

We first establish a recurrence relation for the moments of the operators (5.3) and then establish some quantitative upper estimation on some compact disks. At the end, we establish qualitative Voronovskaja-type results for the newly introduced operators as well as their derivatives and estimate the exact order of approximation using Big-θ notation.

5.2 Recurrence relation

Let for $z \in \mathbb{C}$ and $s \in \mathbb{N} \cup \{0\}$, denote $e_s(z) = z^s$. It is obvious that our defined operator (5.3) preserves e_0 and e_1. For the other moments, we have the following recurrence relation:

Lemma 5.1. *The following recurrence relation holds for all* $s \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}, z \in \mathbb{C}$ *and* $\alpha \in [0, 1]$*:*

$$\begin{aligned}\mathfrak{D}_m^{\alpha}(e_{s+1}; z) = & \frac{z(1-z)}{m+s+2}(\mathfrak{D}_m^{\alpha}(e_s; z))' + \frac{mz+s+1}{m+s+2}\mathfrak{D}_m^{\alpha}(e_s; z) \\ & - \frac{z}{m+s+2}G(e_s; z) + \frac{1-z}{m+s+2}H(e_s; z),\end{aligned} \tag{5.4}$$

where, $G(e_s; z) = (m+1)\sum_{j=0}^{m}(1-\alpha)q_{m-2,j}(z)(1-z)\int_0^1 q_{m,j}(y)y^s dy$ *and* $H(e_s; z) = (m+1)\sum_{j=0}^{m}(1-\alpha)zq_{m-2,j-2}(z)\int_0^1 q_{m,j}(y)y^s dy$.

Proof. Let us denote $\mathcal{I} = \int_0^1 q_{m,j}(y)y^s dy$. Now, from the definition of our operator, we have

$$\begin{aligned}\mathfrak{D}_m^{\alpha}(e_s;z) =&(m+1)\sum_{j=0}^{m} b_{m,j}^{\alpha}(z) * \mathcal{I} \qquad (5.5)\\ =&(m+1)\sum_{j=0}^{m}(1-\alpha)q_{m-2,j}(z)(1-z) * \mathcal{I}\\ &+(m+1)\sum_{j=0}^{m}(1-\alpha)zq_{m-2,j-2}(z) * \mathcal{I} + (m+1)\sum_{j=0}^{m}\alpha q_{m,j}(z) * \mathcal{I}\\ =&G(e_s;z)+H(e_s;z)+J(e_s;z) \qquad \text{(say)}.\end{aligned}$$

Now,

$$\begin{aligned}G'(e_s;z) =&(m+1)\sum_{j=0}^{m}(1-\alpha)\big(q_{m-2,j}(z)(1-z)\big)' * \mathcal{I}\\ =&\frac{(m+1)(1-\alpha)}{z(1-z)}\sum_{j=0}^{m}(1-z)q_{m-2,j}(z)(j-(m-1)z) * \mathcal{I}\\ =&\frac{1-m}{1-z}G(e_s;z)+\frac{(m+1)(1-\alpha)}{z(1-z)}\sum_{j=0}^{m}j(1-z)q_{m-2,j}(z) * \mathcal{I}.\end{aligned}$$

Note that, $\mathcal{I} = \binom{m}{j}\beta(j+s+1, m-j+1)$, where $\beta(a,b)$ is the beta function defined by

$$\begin{aligned}\beta(a,b) =& \int_0^1 t^{a-1}(1-t)^{b-1}dt, \qquad a,b>0\\ =&\frac{(a-1)!(b-1)!}{(a+b-1)!}, \qquad \text{if } a,b\in\mathbb{N}.\end{aligned}$$

We know that, the relation: $u\beta(u,v) = (u+v)\beta(u+1,v), \forall u,v > 0$. So, we can write $(j+s+1)\beta(j+s+1,m-j+1) = (m+s+2)\beta(j+s+2,m-j+1) \implies j\beta(j+s+1,m-j+1) = (m+s+2)\beta(j+s+2,m-j+1)-(s+1)\beta(j+s+1,m-j+1)$.

Therefore, we obtain

$$G'(e_s;z) = \frac{1-m}{1-z}G(e_s;z) + \frac{(m+1)(1-\alpha)}{z(1-z)}\sum_{j=0}^{m}(1-z)q_{m-2,j}(z)\binom{m}{j}$$
$$\times\{(m+s+2)\beta(j+s+2,m-j+1)-(s+1)\beta(j+s+1,m-j+1)\}$$

and hence

$$z(1-z)G'(e_s;z) = (m+s+2)G(e_{s+1};z) - \{(m-1)z+s+1\}G(e_s;z). \tag{5.6}$$

Similarly, we can obtain

$$z(1-z)H'(e_s;z) = (m+s+2)H(e_{s+1};z) - \{(m-1)z+s+2\}H(e_s;z) \tag{5.7}$$

and

$$z(1-z)J'(e_s;z) = (m+s+2)J(e_{s+1};z) - \{(m-1)z+s\}J(e_s;z). \tag{5.8}$$

As a consequence of (5.6)-(5.8), we get the desired relation. □

Lemma 5.2. *Let $|z| < r$ with $r \geq 1$. Then $\forall s, m \in \mathbb{N}\cup\{0\}$, the relation $|\mathfrak{D}_m^\alpha(e_s;z)| \leq r^s$ holds.*

Proof. We note that

$$\begin{aligned}(m+1)\mathcal{I} =& \frac{1}{\beta(j+1,m-j+1)}\int_0^1 y^{s+j}(1-y)^{m-j}dy\\ =& \frac{\beta(s+j+1,m-j+1)}{\beta(j+1,m-j+1)} = \frac{(s+j)!(m-j)!}{(m+s+1)!}\times\frac{(m+1)!}{j!(m-j)!}\\ =& \frac{(j+1)...(j+s)}{(m+2)(m+s+1)} = \frac{[j+1]_s}{[m+2]_s},\end{aligned}$$

where $[c]_i, c \neq 0$ is the Pochhammer symbol defined by

$$[c]_i = \begin{cases} c(c+1)...(c+i-1), & i \geq 1 \\ 1, & i = 0. \end{cases}$$

So, by (5.5), we can write

$$\mathfrak{D}_m^{\alpha}(e_s; z) = \frac{1}{[m+2]_s} \sum_{j=0}^{m} [j+1]_s b_{m,j}^{\alpha}(z).$$

One can also write $[j+1]_s = \prod_{k=0}^{s-1}(j+1+k) = \sum_{k=0}^{s} \varrho_k(s) j^k$, where $\varrho_k(s), k = 0, ..., s$ are positive constants independent of j. Therefore,

$$\begin{aligned} \mathfrak{D}_m^{\alpha}(e_s; z) =& \frac{1}{[m+2]_s} \sum_{j=0}^{m} \sum_{k=0}^{s} \varrho_k(s) j^k b_{m,j}^{\alpha}(z) = \frac{1}{[m+2]_s} \sum_{k=0}^{s} \varrho_k(s) m^k \sum_{j=0}^{m} b_{m,j}^{\alpha}(z) \left(\frac{j}{m}\right)^k \\ =& \frac{1}{[m+2]_s} \sum_{k=0}^{s} \varrho_k(s) m^k \mathcal{B}_m^{\alpha}(e_k; z). \end{aligned}$$

Thus, we can easily deduce the identity

$$1 = \mathfrak{D}_m^{\alpha}(e_s; 1) = \frac{1}{[m+2]_s} \sum_{k=0}^{s} \varrho_k(s) m^k \mathcal{B}_m^{\alpha}(e_k; 1) = \frac{1}{[m+2]_s} \sum_{k=0}^{s} \varrho_k(s) m^k.$$

Finally, using the triangle inequality, we have

$$\begin{aligned} |\mathfrak{D}_m^{\alpha}(e_s; z)| \leq & \frac{1}{[m+2]_s} \sum_{k=0}^{s} \varrho_k(s) m^k \left|\mathcal{B}_m^{\alpha}(e_k; z)\right| \\ \leq & \frac{1}{[m+2]_s} \sum_{k=0}^{s} \varrho_k(s) m^k r^k \leq r^s. \end{aligned}$$

Here, we used the inequality that $|\mathcal{B}_m^{\alpha}(e_k; z)| \leq r^k$, for $k \in \mathbb{N} \cup \{0\}$, which is shown in [42]. This completes the proof. □

5.3 Quantitative upper estimation

Initially, we establish a crucial result, using which we will proceed to prove the main results.

Proposition 5.3. For $m, s \in \mathbb{N}, z \in \mathbb{C}$ and $\alpha \in [0,1]$, the following relation holds:

$$\mathfrak{D}_m^{\alpha}(e_s; z) = \frac{(m+1)!}{(m+s+1)!}\left[(1-\alpha)\sum_{p=0}^{min\{m-1,s\}}\binom{m-1}{p} z^p \Delta_1^p F_s(0) + \alpha \sum_{p=0}^{min\{m,s\}}\binom{m}{p} z^p \Delta_1^p E_s(0)\right], \tag{5.9}$$

where, $E_s(j) = \prod_{k=1}^{s}(j+k)$,
$F_s(j) = \left(1 - \frac{j}{m-1}\right)E_s(j) + \frac{j}{m-1}E_s(j+1)$, $\Delta_1^p E_s(0) = \sum_{j=0}^{p}(-1)^j\binom{p}{j}E_s(p-j)$,
$\Delta_1^p F_s(0) = \sum_{j=0}^{p}(-1)^j\binom{p}{j}F_s(p-j)$
and $\Delta_1^p E_s(0), \Delta_1^p F_s(0) \geq 0$ for all $p, s \in \mathbb{N}$.

Proof. We know that

$$\mathcal{I} = \binom{m}{j}\beta(j+s+1, m-j+1) = \frac{m!}{(m+s+1)!}(j+1)...(j+s) = \frac{m!}{(m+s+1)!}\prod_{k=1}^{s}(j+k).$$

Let us denote $\prod_{k=1}^{s}(j+k) = E_s(j)$. Then, we have $\mathcal{I} = \frac{m!}{(m+s+1)!}E_s(j)$.
Clearly, $E_s(j)$ and its derivatives of any order are ≥ 0, i.e., $\Delta_1^j E_s(0) \geq 0, \forall j, s \in \mathbb{N}$. Hence, one can have

$$\begin{aligned}
&\mathfrak{D}_m^{\alpha}(e_s; z)\\
=&\frac{(m+1)!}{(m+s+1)!}\sum_{j=0}^{m} b_{m,j}^{\alpha}(z) * E_s(j)\\
=&\frac{(m+1)!}{(m+s+1)!}\left[(1-\alpha)\sum_{j=0}^{m-1} q_{m-2,j}(z)(1-z) * E_s(j)\right.\\
&\left. + (1-\alpha)\sum_{j=1}^{m-1} q_{m-2,j-1}(z)z * E_s(j+1) + \alpha\sum_{j=0}^{m} q_{m,j}(z) * E_s(j)\right]\\
=&\frac{(m+1)!}{(m+s+1)!}\left[(1-\alpha)\sum_{j=0}^{m-1}\binom{m-1}{k}\left\{\left(1 - \frac{j}{m-1}\right)E_s(j) + \frac{j}{m-1}E_s(j+1)\right\} z^j(1-z)^{m-1-j}\right.
\end{aligned}$$

$$+\alpha\sum_{j=0}^{m} q_{m,j}(z) * E_s(j)\Bigg]$$

$$=\frac{(m+1)!}{(m+s+1)!}\Bigg[(1-\alpha)\sum_{j=0}^{m-1} q_{m-1,j}(z) * F_s(j) + \alpha\sum_{j=0}^{m} q_{m,j}(z) * E_s(j)\Bigg],$$

where, $F_s(j) = \left(1 - \frac{j}{m-1}\right)E_s(j) + \frac{j}{m-1}E_s(j+1)$. By using the Binomial theorem, we obtain

$$\begin{aligned}\mathfrak{D}_m^{\alpha}(e_s; z) =& \frac{(m+1)!}{(m+s+1)!}\Bigg[(1-\alpha)\sum_{j=0}^{m-1}\binom{m-1}{j} z^j * F_s(j)\sum_{k=0}^{m-j-1}(-1)^k\binom{m-1-j}{k}z^k \\ &+ \alpha\sum_{j=0}^{m}\binom{m}{j}z^j * E_s(j)\sum_{k=0}^{m-j}(-1)^k\binom{m-j}{k}z^k\Bigg].\end{aligned}$$

Now, we let $k = p - j$ in the above relation to get

$$\begin{aligned}\mathfrak{D}_m^{\alpha}(e_s; z) =& \frac{(m+1)!}{(m+s+1)!}\Bigg[(1-\alpha)\sum_{p=0}^{m-1}\binom{m-1}{p}z^p\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}F_s(j) \\ &+ \alpha\sum_{p=0}^{m}\binom{m}{p}z^p\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}E_s(j)\Bigg] \\ =& \frac{(m+1)!}{(m+s+1)!}\Bigg[(1-\alpha)\sum_{p=0}^{m-1}\binom{m-1}{p}z^p\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}F_s(p-j) \\ &+ \alpha\sum_{p=0}^{m}\binom{m}{p}z^p\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}E_s(p-j)\Bigg] \\ =& \frac{(m+1)!}{(m+s+1)!}\Bigg[(1-\alpha)\sum_{p=0}^{min\{m-1,s\}}\binom{m-1}{p}z^p\Delta_1^p F_s(0) \\ &+ \alpha\sum_{p=0}^{min\{m,s\}}\binom{m}{p}z^p\Delta_1^p E_s(0)\Bigg].\end{aligned}$$

Thus our claim is established. □

Now, we establish our first main result, which gives a quantitative upper estimation of the error in approximation by the complex α-Durrmeyer operators (5.3).

Theorem 5.4. *Let $m \in \mathbb{N}$, $\alpha \in [0,1]$ and $|z| < r$. Suppose that $\Psi(z) = \sum_{s=0}^{\infty} w_s z^s$, $\forall\ |z| < R, 1 \leq r < R$. Then*

$$|\mathfrak{D}_m^{\alpha}(\Psi; z) - \Psi(z)| \leq \frac{A_r(\Psi)}{m+2}, \text{ where } A_r(\Psi) = 2\sum_{s=0}^{\infty} |w_s| s(s+3) r^s < \infty.$$

Proof. Suppose $\Psi_i(z) = \sum\limits_{s=0}^{i} w_s z^s = \sum\limits_{s=0}^{i} w_s e_s(z), |z| \leq r, i \in \mathbb{N}$. Now, we apply the operator (5.3) on both sides and use its linear property, we get $\mathfrak{D}_m^{\alpha}(\Psi_i; z) = \sum\limits_{s=0}^{i} w_s \mathfrak{D}_m^{\alpha}(e_s; z), \forall |z| \leq r$. Hence for $|z| \leq r, r \geq 1$, we have

$$\begin{aligned}
&|\mathfrak{D}_m^{\alpha}(\Psi_i; z) - \mathfrak{D}_m^{\alpha}(\Psi; z)| \\
\leq &(m+1)\sum_{j=0}^{m} b_{m,j}^{\alpha}(z) \int_0^1 q_{m,j}(y)|\Psi_i(y) - \Psi(y)|dy \\
\leq &(m+1)\sum_{j=0}^{m} \left[(1-\alpha)\binom{m-2}{j} r + (1-\alpha)\binom{m-2}{j-2}(1+r)\right. \\
&\left. + \alpha\binom{m}{j} r(r+1)\right] \times r^{j-1}(1+r)^{m-j-1} \int_0^1 q_{m,j}(y)|\Psi_i(y) - \Psi(y)|dy \\
\leq &B_{r,m}^{\alpha} \|\Psi_i - \Psi\|_r.
\end{aligned}$$

Here

$$\begin{aligned}
B_{r,m}^{\alpha} =&(m+1)\sum_{j=0}^{m}\left[(1-\alpha)\binom{m-2}{j} r + (1-\alpha)\binom{m-2}{j-2}(1+r) + \alpha\binom{n}{j} r(r+1)\right] \\
&\times r^{j-1}(1+r)^{m-j-1} \int_0^1 q_{m,j}(y)dy,
\end{aligned}$$

which is independent of i. Hence taking $i \to \infty$, we obtain that $\lim\limits_{i\to\infty} \mathfrak{D}_m^{\alpha}(\Psi_i; z) = \mathfrak{D}_m^{\alpha}(\Psi; z)$, which is same as $\mathfrak{D}_m^{\alpha}(\Psi; z) = \sum\limits_{s=0}^{\infty} w_s \mathfrak{D}_m^{\alpha}(e_s; z)$.

This yields $|\mathfrak{D}_m^{\alpha}(\Psi; z) - \Psi(z)| \leq \sum\limits_{s=0}^{\infty} |w_s||\mathfrak{D}_m^{\alpha}(e_s; z) - e_s(z)|$.

By observing (5.9), we split our calculations into two cases: (1) $2 \leq s < m$, (2) $s \geq$

m.

Case (1): In view of the Proposition 5.3, we have

$$\begin{aligned}
&\mathfrak{D}_m^{\alpha}(e_s;z)-e_s(z)\\
=&\frac{(m+1)!}{(m+s+1)!}\left[(1-\alpha)\sum_{p=0}^{s}\binom{m-1}{p}z^p\Delta_1^pF_s(0)+\alpha\sum_{p=0}^{s}\binom{m}{p}z^p\Delta_1^pE_s(0)\right]-z^s\\
=&\frac{(m+1)!}{(m+s+1)!}\left[(1-\alpha)\sum_{p=0}^{s-1}\binom{m-1}{p}z^p\Delta_1^pF_s(0)+\alpha\sum_{p=0}^{s-1}\binom{m}{p}z^p\Delta_1^pE_s(0)\right]\\
&+\left[\frac{(m+1)!}{(m+s+1)!}\left\{(1-\alpha)\binom{m-1}{s}\Delta_1^sF_s(0)+\alpha\binom{m}{s}\Delta_1^sE_s(0)\right\}-1\right]z^s.
\end{aligned}$$

Next, taking the modulus on both sides and applying the triangle inequality, we obtain

$$\begin{aligned}
&|\mathfrak{D}_m^{\alpha}(e_s;z)-e_s(z)|\\
\leq&\left|\frac{(m+1)!}{(m+s+1)!}\left[(1-\alpha)\sum_{p=0}^{s-1}\binom{m-1}{p}z^p\Delta_1^pF_s(0)+\alpha\sum_{p=0}^{s-1}\binom{m}{p}z^p\Delta_1^pE_s(0)\right]\right|\\
&+\left|1-\frac{(m+1)!}{(m+s+1)!}\left\{(1-\alpha)\binom{m-1}{s}\Delta_1^sF_s(0)+\alpha\binom{m}{s}\Delta_1^sE_s(0)\right\}\right|r^s\\
\leq&\frac{(m+1)!}{(m+s+1)!}r^s\left|(1-\alpha)\sum_{p=0}^{s}\binom{m-1}{p}\Delta_1^pF_s(0)+\alpha\sum_{p=0}^{s}\binom{m}{p}\Delta_1^pE_s(0)\right.\\
&\qquad\left.-(1-\alpha)\binom{m-1}{s}\Delta_1^sF_s(0)-\alpha\binom{m}{s}\Delta_1^sE_s(0)\right|\\
&+\left|1-\frac{(m+1)!}{(m+s+1)!}\left\{(1-\alpha)\binom{m-1}{s}\Delta_1^sF_s(0)+\alpha\binom{m}{s}\Delta_1^sE_s(0)\right\}\right|r^s. \qquad (5.10)
\end{aligned}$$

Now, substituting $z=1$ in (5.9), we have

$$\mathfrak{D}_m^{\alpha}(e_s;1)=\frac{(m+1)!}{(m+s+1)!}\left[(1-\alpha)\sum_{p=0}^{min\{m-1,s\}}\binom{m-1}{p}\Delta_1^pF_s(0)\right.$$

$$+\alpha \sum_{p=0}^{min\{m,s\}} \binom{m}{p} \Delta_1^p E_s(0)\Bigg]$$

and then we can use the fact that, for the operator (5.3), $\mathfrak{D}_m^\alpha(e_s;1)=1$, to conclude

$$\frac{(m+1)!}{(m+s+1)!}\left[(1-\alpha)\sum_{p=0}^{s}\binom{m-1}{p}\Delta_1^p F_s(0)+\alpha\sum_{p=0}^{s}\binom{m}{p}\Delta_1^p E_s(0)\right]=1, \text{ for } 0\leq s<m.$$

Hence the relation (5.10) yields

$$\begin{aligned}|\mathfrak{D}_m^\alpha(e_s;z)-e_s(z)| \leq & 2r^s\Bigg[1-\frac{(m+1)!}{(m+s+1)!}\Bigg\{(1-\alpha)\binom{m-1}{s}\Delta_1^s F_s(0)\\ &+\alpha\binom{m}{s}\Delta_1^s E_s(0)\Bigg\}\Bigg]. \qquad (5.11)\end{aligned}$$

Now, we will use the relation derived in Lemma 3.1 of [44] to obtain

$$\begin{aligned}&\frac{(m+1)!}{(m+s+1)!}\left\{(1-\alpha)\binom{m-1}{s}\Delta_1^s F_s(0)+\alpha\binom{m}{s}\Delta_1^s E_s(0)\right\}\\ &=\frac{(m+1)!}{(m+s+1)!}\left\{(1-\alpha)\binom{m-1}{s}\left(1+\frac{s}{m-1}\right)s!+\alpha\binom{m}{s}s!\right\}\\ &=(1-\alpha)\frac{(m-s)(m-s+1)...(m-2)}{(m+2)(m+3)...(m+s-2)(m+s)(m+s+1)}\\ &\quad+\alpha\frac{(m-s+1)(m-s+2)...m}{(m+2)(m+3)...(m+s+1)}.\end{aligned}$$

This gives the following inequality:

$$\begin{aligned}&1-\frac{(m+1)!}{(m+s+1)!}\left\{(1-\alpha)\binom{m-1}{s}\Delta_1^s F_s(0)+\alpha\binom{m}{s}\Delta_1^s E_s(0)\right\}\\ &=[(1-\alpha)+\alpha]-\frac{(m+1)!}{(m+s+1)!}\left\{(1-\alpha)\binom{m-1}{s}\Delta_1^s F_s(0)+\alpha\binom{m}{s}\Delta_1^s E_s(0)\right\}\\ &=(1-\alpha)\left[1-\frac{(m-s)(m-s+1)...(m-2)}{(m+2)(m+3)...(m+s-2)(m+s)(m+s+1)}\right]\\ &\quad+\alpha\left[1-\frac{(m-s+1)(m-s+2)...m}{(m+2)(m+3)...(m+s+1)}\right]\end{aligned}$$

$$\leq(1-\alpha)\Bigg[\Big(1-\frac{m-s}{m+2}\Big)+\Big(1-\frac{m-s+1}{m+3}\Big)+...+\Big(1-\frac{m-4}{m+s-2}\Big)$$
$$+\Big(1-\frac{m-3}{m+s}\Big)+\Big(1-\frac{m-2}{m+s+1}\Big)\Bigg]$$
$$+\alpha\Bigg[\Big(1-\frac{m-s+1}{m+2}\Big)+\Big(1-\frac{m-s+1}{m+3}\Big)+...+\Big(1-\frac{m}{m+s+1}\Big)\Bigg]$$
$$\leq(1-\alpha)\frac{(s-1)(s+3)}{m+2}+\alpha\frac{s(s+3)}{m+2}$$
$$\leq\frac{s(s+3)}{m+2}.$$

Here we have used an important inequality $1-(x_1x_2...x_k)\leq(1-x_1)(1-x_2)(1-x_k)$, where $0\leq x_i\leq 1,\ i=1,...,k$.

Hence from (5.11), we obtain

$$|\mathfrak{D}_m^\alpha(e_s;z)-e_s(z)|\leq 2\frac{s(s+3)}{m+2}r^s.$$

Case (2): In view of Lemma 5.2, for $s>m$, we have

$$|\mathfrak{D}_m^\alpha(e_s;z)-e_s(z)|\leq|\mathfrak{D}_m^\alpha(e_s;z)|+|e_s(z)|\leq 2r^s\leq 2\frac{s(s+3)}{m+2}r^s.$$

Now, considering both the cases as above we deduce that $\forall s,m\in\mathbb{N}$,

$$|\mathfrak{D}_m^\alpha(e_s;z)-e_s(z)|\leq 2\frac{s(s+3)}{m+2}r^s$$
$$\Longrightarrow |\mathfrak{D}_m^\alpha(\Psi;z)-\Psi(z)|\leq\sum_{s=0}^{\infty}|w_s||\mathfrak{D}_m^\alpha(e_s;z)-e_s(z)|\leq\frac{2}{m+2}\sum_{s=0}^{\infty}|w_s|s(s+3)r^s,$$

which is the desired assertion. □

5.4 Voronovskaja type result

This section deals with the exact order of approximation of the defined operators in some open disc. In this sequence, for a being a positive real number, let us denote $\mathcal{B}_a = \{z \in \mathbb{C} : |z| < a\}$.

Theorem 5.5. *Suppose that* $1 < R$ *and* Ψ *be analytic function on* $\{z \in \mathbb{C} : |z| < R\}$ *with* $\Psi(z) = \sum_{s=0}^{\infty} w_s z^s$. *Then*

$$\lim_{m\to\infty} m\left[\mathfrak{D}_m^{\alpha}(\Psi; z) - \Psi(z)\right] = (1-2z)\Psi'(z) + z(1-z)\Psi''(z)$$

uniformly on $\{z \in \mathbb{C} : |z| \le r\}$, *where* $1 \le r < R$.

Proof. We know that for the real case (see [68]),

$$\lim_{m\to\infty} m\left[\mathfrak{D}_m^{\alpha}(\Psi; x) - \Psi(x)\right] = (1-2x)\Psi'(x) + x(1-x)\Psi''(x).$$

Now, with the aid of Vitali's theorem (as outlined in [58]), it is sufficient to prove that the sequence of functions $\left\{m\left(\mathfrak{D}_m^{\alpha}(\Psi; z) - \Psi(z)\right) - (1-2z)\Psi'(z) - z(1-z)\Psi''(z)\right\}_{m\ge 1}$ is bounded in $\{z \in \mathbb{C} : |z| \le r\}$. Using Theorem 5.4, we have

$$\begin{aligned}
&\left|m\left(\mathfrak{D}_m^{\alpha}(\Psi; z) - \Psi(z)\right) - (1-2z)\Psi'(z) - z(1-z)\Psi''(z)\right| \\
&\le \frac{m}{m+2} A_r(\Psi) + (1+2r)\|\Psi'\|_r + r(1+r)\|\Psi''\|_r \\
&\le A_r(\Psi) + (1+2r)\|\Psi'\|_r + r(1+r)\|\Psi''\|_r,
\end{aligned}$$

for $|z| \le r$ with $1 \le r < R$. Also, it can be observed from Theorem 5.4 that $A_r(\Psi)$ is a finite quantity, and hence the right-hand side of the above equation is finite. This shows that the required sequence is bounded, and hence the proof is completed. □

Lastly, we will establish the exact order of approximation for the complex α-Durrmeyer operators as well as for its derivatives in terms of Big-Θ notation.

Theorem 5.6. *Let Ψ be analytic function on $\{z \in \mathbb{C} : |z| < R,\ R > 1\}$ with $\Psi(z) = \sum_{s=0}^{\infty} w_s z^s$, where $w_s, s = 0, 1, 2, \cdots$ are complex constants. then the following holds:*

1. *If there exists $s > 1$ such that $w_s \neq 0$, then for all $1 \leq r < R$, the sequence $\|\mathfrak{D}_m^{\alpha}(\Psi) - \Psi\|_r$ is of $\Theta\left(\frac{1}{m}\right)$ for $|z| \leq r$.*
2. *If there exists $s > max\{1, n-1\}$ such that $w_s \neq 0$, then for all $n \in \mathbb{N}$ and $1 \leq r < r_0 < R$, the sequence $\|\mathfrak{D}_m^{\alpha(n)}(\Psi) - \Psi^{(n)}\|_r$ is of $\Theta\left(\frac{1}{m}\right)$ for $|z| \leq r$.*

Here, $\|\Psi\|_r = \sup\{|\Psi(z)| : |z| \leq r\}$.

Proof. (i). Considering the limiting relation derived in Theorem 5.5, we can get positive constants K_1 and K_2 not depending on m such that for $|z| \leq r$,

$$\begin{aligned} K_1 &\leq m\|\mathfrak{D}_m^{\alpha}(\Psi) - \Psi\|_r \leq K_2 \\ \text{i.e., } \frac{K_1}{m} &\leq \|\mathfrak{D}_m^{\alpha}(\Psi) - \Psi\|_r \leq \frac{K_2}{m}. \end{aligned}$$

Thus, the first part is concluded.

(ii). Let $\mathcal{C}$ be the circle with center at 0 and radius r_0. Clearly then $|u - z| \geq r_0 - r$, $\forall |z| \leq r$ and $u \in \mathcal{C}$.

Then, by Cauchy's integral formula,

$$\mathfrak{D}_m^{\alpha(n)}(\Psi; z) - \Psi^{(n)}(z) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{\mathfrak{D}_m^{\alpha}(\Psi; u) - \Psi(u)}{(u - z)^{n+1}} du.$$

Now, from Theorem 5.5, we can conclude

$$\lim_{m\to\infty} m\big[\mathfrak{D}_m^{\alpha(n)}(\Psi; z) - \Psi^{(n)}(z)\big] = \Big[(1-2z)\Psi'(z) + z(1-z)\Psi''(z)\Big]^{(n)}$$

uniformly in $\{z : |z| \leq r\}$. Hence, there exist positive real numbers P_1 and P_2 not depending on m such that $P_1 \leq m\|\mathfrak{D}_m^{\alpha(n)}(\Psi) - \Psi^{(n)}\|_r \leq P_2$ or $\frac{P_1}{m} \leq \|\mathfrak{D}_m^{\alpha(n)}(\Psi) - \Psi^{(n)}\|_r \leq \frac{P_2}{m}$ in $\{z : |z| \leq r\}$.

Thus, we get the desired assertion. □

Remark 5.7. Theorem 5.6 means that the errors in approximations i.e., $\|\mathfrak{D}_m^{\alpha}(\Psi) - \Psi\|_r$ and $\|\mathfrak{D}_m^{\alpha(n)}(\Psi) - \Psi^{(n)}\|_r$ are equivalent to $\frac{1}{m}$ and the constants in the equivalence depend only on Ψ, r, r_0 and n.

****