

Chapter 5

A robust domain decomposition method for singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions

In this chapter, we consider the following singularly perturbed parabolic reaction-diffusion problem

$$Lu := u_t(x, t) - \varepsilon u_{xx}(x, t) + a(x, t)u(x, t) = f(x, t), \quad (x, t) \in \Omega := D \times (0, T], \quad (5.1)$$

with the Robin boundary conditions

$$u(0, t) - \sqrt{\varepsilon}u_x(0, t) = g_\ell(t), \quad u(1, t) + \sqrt{\varepsilon}u_x(1, t) = g_r(t), \quad t \in (0, T], \quad (5.2)$$

and initial condition

$$u(x, 0) = g_b(x), \quad x \in \overline{D}, \quad (5.3)$$

where $D = (0, 1)$, $a(x, t) \geq \alpha > 0$ on $\overline{\Omega}$, and $0 < \varepsilon \leq 1$ is known as the perturbation parameter. It is known that problem (5.1)-(5.3) has a unique solution exhibiting boundary layers near $x = 0$ and $x = 1$ [72, 76, 77]. Such problems arise in the modelling of certain types of bioswitches [121].

There are only a very few studies of such problems with Robin boundary conditions (RBCs) [72, 73, 76, 77] which are based on the fitted mesh approach. The objective of this chapter is two-fold: first, to introduce a domain decomposition method of SWR type to numerically solve problem (5.1)-(5.3), and second, to present a parameter-uniform convergence analysis of the introduced method. We consider a space-time partitioning of the original domain using the Shishkin transition point. The PDE associated with problem (5.1)-(5.3) in each subdomain is discretized by the finite difference scheme, while the Robin boundary conditions of problem (5.1)-(5.3) are approximated by a special finite difference scheme to maintain the accuracy. Then an iterative algorithm is introduced, where transmission of the information to the neighbours is done using the piecewise-linear interpolation. We discuss parameter-uniform convergence of the constructed method using auxiliary problems that separates the discretization and iteration errors. Note that the convergence analysis of domain decomposition methods for problems with Dirichlet boundary conditions cannot be straightforwardly applied to the present method due to the presence of the Robin boundary conditions. Firstly, we require a different definition of auxiliary problems to handle the Robin boundary conditions. The boundary and initial conditions of the auxiliary problems in earlier works are simply defined using the exact solution of the problem, but that is not entirely true for the present problem. Secondly, we require a more complex constant coefficient problem (in Theorem 3.2) and altogether different barrier functions (in Theorems 3.2 and 3.3) for bounding the error between the auxiliary solution and the Schwarz iterates. It is proved that the resulting numerical approximations are parameter-uniform and, more interestingly, that the iteration convergence is optimal for small ε . Finally, numerical results are provided to support the convergence theory.

This chapter is structured as follows: Section 5.1 provides a priori bounds for the

solution derivatives, and Section 5.2 includes the development of the SWR method for problem (5.1)-(5.3). Section 5.3 includes the convergence analysis of the developed method. Finally, in Section 5.4, we present the numerical results for two test examples confirming our convergence theory. Then, we conclude the work in Section 5.5.

5.1 A priori bounds on the solution derivatives

We discuss a priori bounds on the solution derivatives of (5.1)-(5.3). The existence of a unique solution of problem (5.1)-(5.3) is established in the following lemma.

Lemma 5.1. *Suppose $a, f \in C^{(\beta, \beta/2)}(\bar{\Omega})$, $g_\ell, g_r \in C^{\frac{1+\beta}{2}}([0, T])$, $g_b \in C^{2+\beta}(\bar{D})$, $\beta \in (0, 1)$, and the compatibility conditions up to the first order are true. Then, problem (1)-(3) has a unique solution $u \in C^{(2+\beta, 1+\beta/2)}(\bar{\Omega})$.*

Proof. See [122, Chapter IV, Section 5]. □

In the following lemma, crude bounds for the derivatives of u are given.

Lemma 5.2. *Suppose $a, f \in C^{(2+\beta, 1+\beta/2)}(\bar{\Omega})$, $g_\ell, g_r \in C^{\frac{3+\beta}{2}}([0, T])$, $g_b \in C^{4+\beta}(\bar{D})$, $\beta \in (0, 1)$, and the compatibility conditions up to the second order are true. Then, problem (1)-(3) has a unique solution $u \in C^{(4+\beta, 2+\beta/2)}(\bar{\Omega})$. Moreover, it holds*

$$\left\| \frac{\partial^{q_1+q_2} u}{\partial x^{q_1} \partial t^{q_2}} \right\|_{\bar{\Omega}} \leq C \varepsilon^{-q_1/2} \quad \text{for } 0 \leq q_1 + 2q_2 \leq 4, \quad (5.4)$$

Proof. The proof follows from the arguments in [76, Theorem 5]. □

The above bounds are not enough for the convergence analysis in Section 5.3. We require to split u into regular and singular parts, and bounds for their derivatives as given in the following lemma.

Lemma 5.3. *Suppose $a, f \in C^{(4+\beta, 2+\beta/2)}(\bar{\Omega})$, $g_\ell, g_r \in C^{\frac{5+\beta}{2}}([0, T])$, $g_b \in C^{6+\beta}(\bar{D})$, $\beta \in (0, 1)$, and sufficient compatibility conditions hold. Then, one can split u as $u = u_1 + u_2$, where u_1 and u_2 are the regular and singular parts, respectively, satisfying*

$$|\partial_x^{q_1} u_1(x, t)| \leq C(1 + \varepsilon^{(2-q_1)/2}), \quad (5.5)$$

$$|\partial_x^{q_1} u_2(x, t)| \leq C\varepsilon^{-q_1/2} \left(e^{(-x\sqrt{\alpha/\varepsilon})} + e^{-(1-x)\sqrt{\alpha/\varepsilon}} \right), \quad (5.6)$$

for $(x, t) \in \bar{\Omega}$, $0 \leq q_1 \leq 4$.

Proof. The lemma can be proved following the arguments in [76, Theorem 6]. \square

5.2 Construction and stability analysis of domain decomposition method

We partition Ω into $\Omega_p = D_p \times (0, T]$, $p = \ell, m, r$, where $D_\ell = (0, 2\rho)$, $D_m = (\rho, 1 - \rho)$, $D_r = (1 - 2\rho, 1)$ with the parameter ρ defined by

$$\rho = \min \left\{ \frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\alpha}} \ln N \right\}. \quad (5.7)$$

Here, ρ is chosen as the Shishkin transition parameter [6, 9], which enables us to have fine meshes (for small ε) on Ω_ℓ and Ω_r (the subdomains corresponding to the boundary layers). Further, we discretize each subdomain Ω_p with a uniform mesh in both the spatial and temporal directions. On each spatial subdomain $\bar{D}_p = [a, b]$, we

define a uniform mesh $\overline{D}_p^N = \{x_{p;i} = a + ih_p, i = 0, \dots, N, h_p = (b - a)/N\}$. When the domain is obvious, for convenience, the term p will be dropped from the subscript in $x_{p;i}$. On the interval $\overline{\omega} = [0, T]$, we define a uniform mesh $\overline{\omega}^M = \{t_j = j\Delta t, j = 0, \dots, M, \text{ with } \Delta t = T/M\}$. Here, M and N are discretization parameters in time and space directions respectively. Defining $D_p^N = \overline{D}_p^N \cap D_p$, and $\omega^M = \overline{\omega}^M \cap (0, T]$, the mesh $\Omega_p^{N,M}$ corresponding to Ω_p is defined as $\Omega_p^{N,M} = D_p^N \times \omega^M$. Further, we define $\overline{\Omega}_p^{N,M} = \overline{D}_p^N \times \overline{\omega}^M$ and $\gamma_{p,\ell}^{N,M} = \{x_{p;0}\} \times \omega^M$, $\gamma_{p,r}^{N,M} = \{x_{p;N}\} \times \omega^M$, $\gamma_{p,b}^{N,M} = \overline{D}_p^N \times \{t_0\}$. Then, on each $\Omega_p^{N,M}$, $p = \ell, m, r$, the discretization of (5.1) is defined as follows

$$L_p^{N,M} U_{p;i,j} := \delta_t U_{p;i,j} - \varepsilon \delta_x^2 U_{p;i,j} + a_{i,j} U_{p;i,j} = f_{i,j}, \quad (x_{p;i}, t_j) \in \Omega_p^{N,M},$$

where

$$\delta_x^2 U_{p;i,j} = \frac{[F_x^+ U_p - F_x^- U_p]_{i,j}}{h_p},$$

$$F_x^+ U_{p;i,j} = (U_{p;i+1,j} - U_{p;i,j})/h_p, \quad F_x^- U_{p;i,j} = (U_{p;i,j} - U_{p;i-1,j})/h_p,$$

$$\text{and } \delta_t U_{p;i,j} = (U_{p;i,j} - U_{p;i,j-1})/\Delta t.$$

We discretize (5.2) as follows

$$\left\{ \begin{array}{l} \Gamma_\ell^{N,M} U_\ell(0, t_j) := U_\ell(0, t_j) - \sqrt{\varepsilon} F_x^+ U_\ell(0, t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}} (a U_\ell + \delta_t U_\ell)(0, t_j) \\ \quad = g_\ell(t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}} f(0, t_j), \quad t_j \in \omega^M, \\ \Gamma_r^{N,M} U_r(1, t_j) := U_r(1, t_j) + \sqrt{\varepsilon} F_x^- U_r(1, t_j) + \frac{h_r}{2\sqrt{\varepsilon}} (a U_r + \delta_t U_r)(1, t_j) \\ \quad = g_r(t_j) + \frac{h_r}{2\sqrt{\varepsilon}} f(1, t_j), \quad t_j \in \omega^M. \end{array} \right. \quad (5.8)$$

Note that we have approximated the boundary conditions using a special discretization scheme. If we had used the standard upwind scheme for the discretization of the boundary conditions we would have obtained only the first order accuracy. Therefore, we use a special discretization scheme for the boundary conditions, which is

based on improving the truncation error to maintain the second order accuracy.

Consider $\bar{\Omega}^{N,M} = (\bar{\Omega}_\ell^{N,M} \setminus \bar{\Omega}_m) \cup \bar{\Omega}_m^{N,M} \cup (\bar{\Omega}_r^{N,M} \setminus \bar{\Omega}_m)$. To compute the approximate solution of problem (5.1)-(5.3) the algorithm is defined as follows. We start the iterative process with $U^{[0]}(x_i, t_j)$ the initial approximation, defined as follows: $U^{[0]}(x_i, 0) = g_b(x_i), 0 \leq x_i \leq 1; U^{[0]}(x_i, t_j) = 0, 0 < x_i < 1, 0 < t_j \leq T$. Suppose $\mathcal{I}_j Z$ is used to denote the piecewise-linear interpolant function of Z at t_j . For each $k \geq 1$, we solve

$$\left\{ \begin{array}{ll} L_\ell^{N,M} U_\ell^{[k]} = f & \text{in } \Omega_\ell^{N,M}, \\ \Gamma_\ell^{N,M} U_\ell^{[k]}(0, t_j) = g_\ell(t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}} f(0, t_j) & \text{if } t_j \in \omega^M, \\ U_\ell^{[k]}(2\rho, t_j) = \mathcal{I}_j U^{[k-1]}(2\rho, t_j) & \text{if } t_j \in \omega^M, \\ U_\ell^{[k]}(x_i, 0) = g_b(x_i) & \text{if } x_i \in \bar{D}_\ell^N, \end{array} \right.$$

$$\left\{ \begin{array}{ll} L_r^{N,M} U_r^{[k]} = f & \text{in } \Omega_r^{N,M}, \\ U_r^{[k]}(1 - 2\rho, t_j) = \mathcal{I}_j U^{[k-1]}(1 - 2\rho, t_j) & \text{if } t_j \in \omega^M, \\ \Gamma_r^{N,M} U_r^{[k]}(1, t_j) = g_r(t_j) + \frac{h_r}{2\sqrt{\varepsilon}} f(1, t_j) & \text{if } t_j \in \omega^M, \\ U_r^{[k]}(x_i, 0) = g_b(x_i) & \text{if } x_i \in \bar{D}_r^N, \end{array} \right.$$

$$\left\{ \begin{array}{ll} L_m^{N,M} U_m^{[k]} = f & \text{in } \Omega_m^{N,M}, \\ U_m^{[k]}(\rho, t_j) = \mathcal{I}_j U_\ell^{[k]}(\rho, t_j), & \text{for } t_j \in \omega^M, \\ U_m^{[k]}(1 - \rho, t_j) = \mathcal{I}_j U_r^{[k]}(1 - \rho, t_j) & \text{for } t_j \in \omega^M, \\ U_m^{[k]}(x_i, 0) = g_b(x_i) & \text{for } x_i \in \bar{D}_m^N. \end{array} \right.$$

We then compute $U^{[k]}$ by

$$U^{[k]} = \begin{cases} U_\ell^{[k]} & \text{in } \bar{\Omega}_\ell^{N,M} \setminus \bar{\Omega}_m, \\ U_m^{[k]} & \text{in } \bar{\Omega}_m^{N,M}, \\ U_r^{[k]} & \text{in } \bar{\Omega}_r^{N,M} \setminus \bar{\Omega}_m. \end{cases} \quad (5.9)$$

The iterations are performed until

$$\|U^{[k+1]} - U^{[k]}\|_{\bar{\Omega}^{N,M}} \leq tol$$

is reached, where tol is a specified user tolerance.

For $p = \ell, r, m$, suppose the operator $\mathcal{T}_p^{N,M}$ is defined as $\mathcal{T}_p^{N,M} Z(y, t_j) = Z(y, t_j)$, $y = \rho, 2\rho, 1 - \rho, 1 - 2\rho, t_j \in \omega^M$ and $\mathcal{T}_\ell^{N,M} Z(0, t_j) = \Gamma_\ell^{N,M} Z(0, t_j)$, $\mathcal{T}_r^{N,M} Z(1, t_j) = \Gamma_r^{N,M} Z(1, t_j)$, $t_j \in \omega^M$. Now using the arguments in [76, Theorem 7] we can prove the following maximum principle for $L_p^{N,M}$.

Lemma 5.4. *Suppose Z_p , $p = m, r, \ell$, satisfies $Z_p(x_i, 0) \geq 0$ for $x_i \in \bar{D}_p^N$ and $\mathcal{T}_p^{N,M} Z_p(x_0, t_j) \geq 0$, $\mathcal{T}_p^{N,M} Z_p(x_N, t_j) \geq 0$, $t_j \in \omega^M$. Then, if $L_p^{N,M} Z_{p;i,j} \geq 0$ in $\Omega_p^{N,M}$, it holds that $Z_{p;i,j} \geq 0$ in $\bar{\Omega}_p^{N,M}$.*

The stability of the numerical scheme is given by the following lemma.

Lemma 5.5. *For any mesh function Z_p , we have*

$$\|Z_p\|_{\bar{\Omega}_p^{N,M}} \leq \max \left\{ \|\mathcal{T}_p^{N,M} Z_p\|_{\gamma_{p,\ell}^{N,M}}, \|\mathcal{T}_p^{N,M} Z_p\|_{\gamma_{p,r}^{N,M}}, \|Z_p\|_{\gamma_{p,b}^{N,M}}, \frac{1}{\alpha} \|L_p^{N,M} Z_p\|_{\Omega_p^{N,M}} \right\}.$$

Proof. Suppose $\Phi^\pm(x_i, t_j) = \Theta \pm Z(x_i, t_j)$ is a mesh function with

$$\Theta = \max \left\{ \|\mathcal{T}_p^{N,M} Z_p\|_{\gamma_{p,\ell}^{N,M}}, \|\mathcal{T}_p^{N,M} Z_p\|_{\gamma_{p,r}^{N,M}}, \|Z_p\|_{\gamma_{p,b}^{N,M}}, \frac{1}{\alpha} \|L_p^{N,M} Z_p\|_{\Omega_p^{N,M}} \right\}.$$

Then, it is easy to verify that $\mathcal{T}_\ell^{N,M} \Phi^\pm(x_0, t_j) \geq 0$, $\mathcal{T}_r^{N,M} \Phi^\pm(x_N, t_j) \geq 0$, $t_j \in \omega^M$, $\Phi^\pm(x_i, 0) \geq 0$ for $x_i \in \bar{D}_p^N$, and $L_p^{N,M} \Phi^\pm(x_i, t_j) \geq 0$, $(x_i, t_j) \in \Omega_p^{N,M}$. Hence, from Lemma 5.4, the proof follows. \square

5.3 Convergence analysis

We now establish that the method gives parameter-uniform approximations to the solution of problem (5.1)-(5.3). Further, we prove that the iterative process converges optimally for small ε . We consider the following auxiliary mesh function

$$\tilde{U} = \begin{cases} \tilde{U}_\ell & \text{in } \bar{\Omega}_\ell^{N,M} \setminus \bar{\Omega}_m, \\ \tilde{U}_m & \text{in } \bar{\Omega}_m^{N,M}, \\ \tilde{U}_r & \text{in } \bar{\Omega}_r^{N,M} \setminus \bar{\Omega}_m, \end{cases} \quad (5.10)$$

where \tilde{U}_p , $p = \ell, m, r$, are such that

$$\begin{cases} L_\ell^{N,M} \tilde{U}_\ell = f & \text{in } \Omega_\ell^{N,M}, \\ \Gamma_\ell^{N,M} \tilde{U}_\ell(0, t_j) = \Gamma_\ell^{N,M} U^{[k]}(0, t_j) & \text{for } t_j \in \omega^M, \\ \tilde{U}_\ell(2\rho, t_j) = u(2\rho, t_j) & \text{for } t_j \in \omega^M, \\ \tilde{U}_\ell(x_i, 0) = u(x_i, 0) & \text{if } x_i \in \bar{D}_\ell^N, \end{cases}$$

$$\begin{cases} L_m^{N,M} \tilde{U}_m = f & \text{in } \Omega_m^{N,M}, \\ \tilde{U}_m(\rho, t_j) = u(\rho, t_j) & \text{for } t_j \in \omega^M, \\ \tilde{U}_m(1 - \rho, t_j) = u(1 - \rho, t_j) & \text{for } t_j \in \omega^M, \\ \tilde{U}_m(x_i, 0) = u(x_i, 0) & \text{if } x_i \in \bar{D}_m^N, \end{cases}$$

$$\begin{cases} L_r^{N,M} \tilde{U}_r = f & \text{in } \Omega_r^{N,M}, \\ \tilde{U}_r(1 - 2\rho, t_j) = u(1 - 2\rho, t_j) & \text{for } t_j \in \omega^M, \\ \Gamma_r^{N,M} \tilde{U}_r(1, t_j) = \Gamma_r^{N,M} U^{[k]}(1, t_j) & \text{for } t_j \in \omega^M, \\ \tilde{U}_r(x_i, 0) = u(x_i, 0) & \text{if } x_i \in \bar{D}_r^N. \end{cases}$$

Here, the discrete operators $L_\ell^{N,M}$, $L_m^{N,M}$, $L_r^{N,M}$, $\Gamma_r^{N,M}$, and $\Gamma_\ell^{N,M}$ are the ones that were defined in the previous section. The main difference between these problems and the problems in the previous section is that we used the solution u of problem (5.1)-(5.3) in the boundary conditions for \tilde{U}_ℓ at $(2\rho, t_j)$, \tilde{U}_r at $(1 - 2\rho, t_j)$, and \tilde{U}_m at (ρ, t_j) and $(1 - \rho, t_j)$. By the triangle inequality, we get

$$\|u - U^{[k]}\|_{\bar{\Omega}^{N,M}} \leq \|u - \tilde{U}\|_{\bar{\Omega}^{N,M}} + \|\tilde{U} - U^{[k]}\|_{\bar{\Omega}^{N,M}}. \quad (5.11)$$

The following lemma gives the bound of the first term on the LHS of the inequality in (5.11).

Lemma 5.6. *Suppose u is the solution of (5.1)-(5.3) and \tilde{U} is the auxiliary mesh function defined in (5.10). Then*

$$\|u - \tilde{U}\|_{\bar{\Omega}^{N,M}} \leq C(\Delta t + N^{-2} \ln^2 N). \quad (5.12)$$

Proof. Note that $(u - \bar{U}_\ell)(2\rho, t_j) = 0$ for $t_j \in \omega^M$. On the other hand, from (5.1)-(5.2) and (5.8) it follows that

$$\begin{aligned} \Gamma_\ell^{N,M}(u - \tilde{U}_\ell)(0, t_j) &= \left(u - \sqrt{\varepsilon} F_x^+ u + \frac{h_\ell}{2\sqrt{\varepsilon}}(au + \delta_t u) - (u - \sqrt{\varepsilon} u_x) - \frac{h_\ell}{2\sqrt{\varepsilon}} f \right) (0, t_j) \\ &= \left(\sqrt{\varepsilon}(u_x - F_x^+ u) + \frac{h_\ell}{2\sqrt{\varepsilon}}(au - f + u_t) \right) (0, t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}(\delta_t u - u_t)(0, t_j) \\ &= \sqrt{\varepsilon} \left(u_x - F_x^+ u + \frac{h_\ell}{2} u_{xx} \right) (0, t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}(\delta_t u - u_t)(0, t_j), \quad t_j \in \omega^M. \end{aligned}$$

Using Taylor expansions and (5.4) we get

$$\begin{aligned} \left| \Gamma_\ell^{N,M}(u - \tilde{U}_\ell)(0, t_j) \right| &\leq \frac{\sqrt{\varepsilon}}{6} h_\ell^2 \|u_{xxx}(\cdot, t_j)\|_{[x_0, x_1]} + \frac{h_\ell}{4\sqrt{\varepsilon}}(t_j - t_{j-1}) \|u_{tt}(x_i, \cdot)\|_{[t_{j-1}, t_j]} \\ &\leq C(\Delta t + (N^{-1} \ln N)^2). \end{aligned}$$

Now,

$$L_p^{N,M}(u - \tilde{U}_p) = (L_p^{N,M} - L)u = (\delta_t u - u_t) - \varepsilon (\delta_x^2 u - u_{xx}) \quad \text{in } \Omega_p^{N,M}, \quad p = \ell, m, r, \quad (5.13)$$

and

$$\left| L_p^{N,M}(u - \tilde{U}_p)_{i,j} \right| \leq \frac{(t_j - t_{j-1})}{2} \|u_{tt}(x_i, \cdot)\|_{[t_{j-1}, t_j]} + \frac{\varepsilon}{12} h_p^2 \|u_{xxxx}(\cdot, t_j)\|_{[x_{i-1}, x_{i+1}]}, \quad (5.14)$$

where we have used Taylor expansions to get (5.14). Since $h_\ell \leq C\sqrt{\varepsilon}N^{-1} \ln N$, using (5.4) it follows from (5.14) that

$$\left| L_\ell^{N,M}(u - \tilde{U}_\ell)_{i,j} \right| \leq C(\Delta t + N^{-2} \ln^2 N) \quad \text{in } \Omega_\ell^{N,M}.$$

Hence, by Lemma 5.4 with $C(\Delta t + N^{-2} \ln^2 N) \pm (u - \tilde{U}_\ell)_{i,j}$, one gets

$$\|u - \tilde{U}_\ell\|_{\bar{\Omega}_\ell^{N,M}} \leq C(\Delta t + N^{-2} \ln^2 N).$$

A similar analysis gives

$$\|u - \tilde{U}_r\|_{\bar{\Omega}_r^{N,M}} \leq C(\Delta t + N^{-2} \ln^2 N).$$

Suppose $\rho = 1/4$. Then, we have $h_m = 0.5N^{-1}$ and $\varepsilon^{-1} \leq C \ln^2 N$. Again, using (5.4) it follows from (5.14) that

$$\left| L_m^{N,M}(u - \tilde{U}_m)_{i,j} \right| \leq C(\Delta t + N^{-2} \ln^2 N) \quad \text{in } \Omega_m^{N,M}.$$

If $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}} \ln N$, then $h_m \leq CN^{-1}$. For $(\delta_t u - u_t)$ use Taylor expansion and (5.4) to get

$$\left| (\delta_t u - u_t)_{i,j} \right| \leq C\Delta t \quad \text{in } \Omega_m^{N,M}.$$

For $\varepsilon(\delta_x^2 u - u_{xx})$ use the decomposition $u = u_1 + u_2$, Taylor expansions, and (5.5)-(5.6) to get

$$\begin{aligned} \varepsilon \left| (\delta_x^2 u - u_{xx})_{i,j} \right| &\leq \varepsilon \left(\left| \left(\delta_x^2 u_1 - \frac{\partial^2 u_1}{\partial x^2} \right)_{i,j} \right| + \left| \left(\delta_x^2 u_2 - \frac{\partial^2 u_2}{\partial x^2} \right)_{i,j} \right| \right) \\ &\leq C\varepsilon h_m^2 \left\| \frac{\partial^4 u_1(\cdot, t_j)}{\partial x^4} \right\|_{[x_{i-1}, x_{i+1}]} + C\varepsilon \left\| \frac{\partial^2 u_2(\cdot, t_j)}{\partial x^2} \right\|_{[x_{i-1}, x_{i+1}]} \\ &\leq CN^{-2} + 2Ce^{-\rho\sqrt{\alpha/\varepsilon}} \leq CN^{-2}. \end{aligned}$$

Thus, $\left| L_m^{N,M}(u - \tilde{U}_m)_{i,j} \right| \leq C(\Delta t + N^{-2} \ln^2 N)$ in $\Omega_m^{N,M}$. So, applying Lemma 5.4 to $C(\Delta t + (N^{-1} \ln N)^2) \pm (u - \tilde{U}_m)$, one gets

$$\|u - \tilde{U}_m\|_{\bar{\Omega}_m^{N,M}} \leq C(\Delta t + N^{-2} \ln^2 N).$$

This completes the proof. □

The following notation will be used in the next two theorems.

$$\begin{aligned} \mu^{[k]} &= \max \left\{ \max_{t_j \in \omega^M} |(\tilde{U}_\ell - \mathcal{I}_j U^{[k-1]})(2\rho, t_j)|, \max_{t_j \in \omega^M} |(\tilde{U}_r - \mathcal{I}_j U^{[k-1]})(1 - 2\rho, t_j)| \right\}, \\ \mu_\rho &= \max \left\{ \max_{t_j \in \omega^M} |(\tilde{U}_m - \tilde{U}_\ell)(\rho, t_j)|, \max_{t_j \in \omega^M} |(\tilde{U}_m - \tilde{U}_r)(1 - \rho, t_j)| \right\}. \end{aligned}$$

We will show the parameter-uniform convergence of the method in two cases: when $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}} \ln N$ and $\rho = 1/4$. In the next theorem, for $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}} \ln N$, we first obtain a bound of $\|\tilde{U} - U^{[1]}\|_{\bar{\Omega}^{N,M}}$ and then combine it with Lemma 5.6 to get a bound

of $\|u - U^{[1]}\|_{\bar{\Omega}^{N,M}}$. So, this result demonstrates that only one iteration is enough to attain the required accuracy.

Theorem 5.7. *Suppose u is the exact solution of (5.1)-(5.3) and $U^{[1]}$ is its approximation obtained after the first iterate of the proposed method. Then, for $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}} \ln N$, it holds*

$$\|u - U^{[1]}\|_{\bar{\Omega}^{N,M}} \leq C(\Delta t + N^{-2} \ln^2 N).$$

Proof. Observe that

$$\begin{cases} L_\ell^{N,M}(\tilde{U}_\ell - U_\ell^{[1]}) = 0 & \text{in } \Omega_\ell^{N,M}, \\ (\tilde{U}_\ell - U_\ell^{[1]})(x_i, 0) = 0 & \text{for } x_i \in \bar{D}_\ell^N, \\ \Gamma_\ell^{N,M}(\tilde{U}_\ell - U_\ell^{[1]})(0, t_j) = 0 & \text{for } t_j \in \omega^M, \\ |(\tilde{U}_\ell - U_\ell^{[1]})(2\rho, t_j)| \leq \mu^{[1]} & \text{for } t_j \in \omega^M. \end{cases}$$

For an arbitrary mesh function U , let us define

$$\tilde{\Gamma}_\ell^{N,M}U(0, t_j) := U(0, t_j) - \sqrt{\varepsilon}F_x^+U(0, t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}(\alpha U + \delta_t U)(0, t_j).$$

For $(x_i, t_j) \in \bar{\Omega}_\ell^{N,M}$, consider

$$\chi^\pm(x_i, t_j) = \left[\chi_\ell \pm (\tilde{U}_\ell - U_\ell^{[1]}) \right] (x_i, t_j),$$

where χ_ℓ solves

$$\begin{cases} \delta_t \chi_\ell - \varepsilon \delta_x^2 \chi_\ell + \alpha \chi_\ell = 0 & \text{in } \Omega_\ell^{N,M}, \\ \chi_\ell(x_i, 0) = \mu^{[1]} \frac{B\xi_1^i - A\xi_2^i}{B\xi_1^N - A\xi_2^N} & \text{for } x_i \in \bar{D}_\ell^N, \\ \tilde{\Gamma}_\ell^{N,M} \chi_\ell(0, t_j) = 0 & \text{for } t_j \in \omega^M, \\ \chi_\ell(2\rho, t_j) = \mu^{[1]} & \text{for } t_j \in \omega^M, \end{cases} \quad (5.15)$$

with

$$A = 2\sqrt{\varepsilon}h_\ell - 2\varepsilon(\xi_1 - 1) + h_\ell^2 a, \quad B = 2\sqrt{\varepsilon}h_\ell - 2\varepsilon(\xi_2 - 1) + h_\ell^2 a,$$

and $\xi_i, i = 1, 2$, are as follows

$$\xi_1 = \lambda_1 + \lambda_2 \quad \text{and} \quad \xi_2 = \lambda_1 - \lambda_2$$

with

$$\lambda_1 = 1 + \left(\frac{\rho}{N} \sqrt{\frac{\alpha}{\varepsilon}} \right), \quad \lambda_2 = 2 \left(\frac{\rho}{N} \sqrt{\frac{\alpha}{\varepsilon}} \right) \sqrt{1 + \left(\frac{\rho}{N} \sqrt{\frac{\alpha}{\varepsilon}} \right)^2}.$$

The solution of (5.15) is given by

$$\chi_\ell(x_i, t_j) = \mu^{[1]} \frac{\varphi \xi_1^i - \xi_2^i}{\varphi \xi_1^N - \xi_2^N}, \quad \varphi = B/A, \quad (5.16)$$

which is monotonically increasing and satisfies $\chi_\ell \geq 0$ in $\overline{\Omega}_\ell^{N,M}$. Thus, one gets

$$\chi^\pm(x_i, 0) \geq 0 \quad \text{if} \quad x_i \in \overline{D}_\ell^N, \quad \Gamma_\ell^{N,M} \chi^\pm(0, t_j) \geq 0, \quad \chi^\pm(2\rho, t_j) \geq 0 \quad \text{if} \quad t_j \in \omega^M,$$

and $L_\ell^{N,M} \chi^\pm(x_i, t_j) \geq 0$ if $(x_i, t_j) \in \Omega_\ell^{N,M}$. Hence, applying Lemma 5.4 to χ^\pm , we obtain

$$|(\tilde{U}_\ell - U_\ell^{[1]})_{i,j}| \leq \chi_\ell(x_i, t_j), \quad \text{if} \quad (x_i, t_j) \in \overline{\Omega}_\ell^{N,M}.$$

Further, since $x_i \leq \rho$ for $(x_i, t_j) \in \overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m$, it follows from (5.16) that

$$\chi_\ell(x_i, t_j) \leq \mu^{[1]} \frac{\varphi \xi_1^{N/2} - \xi_2^{N/2}}{\varphi \xi_1^N - \xi_2^N} = \mu^{[1]} \frac{\varphi^2 \xi_1^N - \xi_2^N}{(\varphi \xi_1^N - \xi_2^N)(\varphi \xi_1^{N/2} + \xi_2^{N/2})}.$$

This implies that

$$\chi_\ell(x_i, t_j) \leq \frac{\mu^{[1]}}{\varphi \xi_1^{N/2} + \xi_2^{N/2}} \left[\frac{\varphi^2 - \xi_2^N / \xi_1^N}{\varphi - \xi_2^N / \xi_1^N} \right].$$

Now, since $\varphi > 1$ and $\xi_2/\xi_1 < 1$, we have $(\varphi^2 - \xi_2^N/\xi_1^N)/(\varphi - \xi_2^N/\xi_1^N) \leq C\varphi$. Thus, one gets

$$\chi_\ell(x_i, t_j) \leq \mu^{[1]} \frac{C\varphi}{\varphi\xi_1^{N/2} + \xi_2^{N/2}} \leq \frac{C\mu^{[1]}}{\xi_1^{N/2}}.$$

For $\rho = 2\varepsilon^{1/2}\alpha^{-1/2} \ln N$, it holds

$$\xi_1^{-N/2} \leq \left(1 + \frac{\rho}{N} \sqrt{\frac{\alpha}{\varepsilon}}\right)^{-N} \leq 4N^{-2}, \quad \text{if } N \geq 1,$$

having used $\lambda_2 \geq 2\left(\frac{\rho}{N} \sqrt{\frac{\alpha}{\varepsilon}}\right)$ and the arguments in [9, Lemma 5.1]. Further, since $\mu^{[1]} \leq C$, we have $\chi_\ell(x_i, t_j) \leq CN^{-2}$ in $\bar{\Omega}_\ell^{N,M} \setminus \bar{\Omega}_m$. Thus, we have

$$\|\tilde{U}_\ell - U_\ell^{[1]}\|_{\bar{\Omega}_\ell^{N,M} \setminus \bar{\Omega}_m} \leq CN^{-2}. \quad (5.17)$$

Similarly, it is

$$\|\tilde{U}_r - U_r^{[1]}\|_{\bar{\Omega}_r^{N,M} \setminus \bar{\Omega}_m} \leq CN^{-2}. \quad (5.18)$$

Next, consider $\tilde{U}_m - U_m^{[1]}$ which satisfies

$$\begin{aligned} L_m^{N,M}(\tilde{U}_m - U_m^{[1]}) &= 0 \quad \text{in } \Omega_m^{N,M}, \\ (\tilde{U}_m - U_m^{[1]})(x_i, 0) &= 0, \quad \text{if } x_i \in \bar{D}_m^N, \\ |(\tilde{U}_m - U_m^{[1]})(\rho, t_j)| &= |(\tilde{U}_m - \mathcal{I}_j U_\ell^{[1]})(\rho, t_j)| \\ &\leq |(\tilde{U}_m - \tilde{U}_\ell)(\rho, t_j)| + |(\tilde{U}_\ell - U_\ell^{[1]})(\rho, t_j)| \\ &\leq \mu_\rho + CN^{-2}, \quad t_j \in \omega^M, \end{aligned}$$

since the mesh points (ρ, t_j) and $(1 - \rho, t_j)$ belong to $\Omega_\ell^{N,M}$ and $\Omega_r^{N,M}$, respectively.

Therefore, Lemma 5.4 gives

$$\|\tilde{U}_m - U_m^{[1]}\|_{\bar{\Omega}_m^{N,M}} \leq \mu_\rho + CN^{-2}. \quad (5.19)$$

Combining (5.17), (5.18), and (5.19), we obtain

$$\|\tilde{U} - U^{[1]}\|_{\overline{\Omega}^{N,M}} \leq \mu_\rho + CN^{-2}. \quad (5.20)$$

Since $(\rho, t_j) \in \overline{\Omega}_\ell^{N,M}$ and $(1 - \rho, t_j) \in \overline{\Omega}_r^{N,M}$, by Lemma 5.6 one gets $\mu_\rho \leq C(\Delta t + N^{-2} \ln^2 N)$. Hence, using (5.20) and Lemma 5.6 in (5.11) the proof is complete. \square

In the next theorem, for $\rho = 1/4$, we first obtain a bound on $\|\tilde{U} - U^{[k]}\|_{\overline{\Omega}^{N,M}}$ and then combine it with Lemma 5.6 to get a bound on $\|u - U^{[k]}\|_{\overline{\Omega}^{N,M}}$.

Theorem 5.8. *Suppose $U^{[k]}$ is the approximation of the exact solution u of (5.1)-(5.3), generated by the k -th iterate of the method. Then, for $\rho = 1/4$, it holds*

$$\|u - U^{[k]}\|_{\overline{\Omega}^{N,M}} \leq C \left((5/6)^k + (\Delta t + N^{-2} \ln^2 N) \right). \quad (5.21)$$

Proof. The following notation is used

$$\begin{aligned} \nu^{[k]} &= \max \left\{ \|\tilde{U}_r - U^{[k]}\|_{\overline{\Omega}_r^{N,M} \setminus \overline{\Omega}_m}, \|\tilde{U}_\ell - U^{[k]}\|_{\overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m}, \|\tilde{U}_m - U^{[k]}\|_{\overline{\Omega}_m^{N,M}} \right\}, \\ \mu_{2\rho} &= \max \left\{ \max_{t_j \in \omega^M} |(\tilde{U}_\ell - \tilde{U}_m)(2\rho, t_j)|, \max_{t_j \in \omega^M} |(\tilde{U}_r - \tilde{U}_m)(1 - 2\rho, t_j)| \right\}. \end{aligned}$$

We have

$$\begin{cases} L_\ell^{N,M}(\tilde{U}_\ell - U_\ell^{[1]}) = 0 & \text{in } \Omega_\ell^{N,M}, \\ (\tilde{U}_\ell - U_\ell^{[1]})(x_i, 0) = 0, & \text{if } x_i \in \overline{D}_\ell^N, \\ \Gamma_\ell^{N,M}(\tilde{U}_\ell - U_\ell^{[1]})(0, t_j) = 0 & \text{if } t_j \in \omega^M, \\ |(\tilde{U}_\ell - U_\ell^{[1]})(2\rho, t_j)| \leq \mu^{[1]} & \text{if } t_j \in \omega^M. \end{cases}$$

We define

$$\Psi^\pm(x_i, t_j) = \mu^{[1]} \frac{x_i + \sqrt{\varepsilon}}{2\rho + \sqrt{\varepsilon}} \pm (\tilde{U}_\ell - U_\ell^{[1]})(x_i, t_j),$$

which verify that $\Psi^\pm(x_i, 0) \geq 0$, $x_i \in \overline{D}_\ell^N$; $\Gamma_\ell^{N,M} \Psi^\pm(0, t_j) \geq 0$, $\Psi^\pm(2\rho, t_j) \geq 0$, $t_j \in \omega^M$;

$$L_\ell^{N,M} \Psi^\pm(x_i, t_j) = \left(\frac{x_i + \sqrt{\varepsilon}}{2\rho + \sqrt{\varepsilon}} \right) \mu^{[1]} a(x_i, t_j) \pm 0 \geq 0, (x_i, t_j) \in \Omega_\ell^{N,M}.$$

So, Lemma 5.4 gives

$$|(\tilde{U}_\ell - U_\ell^{[1]})(x_i, t_j)| \leq \mu^{[1]} \left(\frac{x_i + \sqrt{\varepsilon}}{2\rho + \sqrt{\varepsilon}} \right). \quad (5.22)$$

Since $x_i \leq \rho$ for $(x_i, t_j) \in \overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m$, it follows from (5.22) that

$$\|\tilde{U}_\ell - U_\ell^{[1]}\|_{\overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m} \leq \frac{5}{6} \mu^{[1]}. \quad (5.23)$$

Similarly, we have

$$\|\tilde{U}_r - U_r^{[1]}\|_{\overline{\Omega}_r^{N,M} \setminus \overline{\Omega}_m} \leq \frac{5}{6} \mu^{[1]}. \quad (5.24)$$

For $\tilde{U}_m - U_m^{[1]}$, we have

$$L_m^{N,M}(\tilde{U}_m - U_m^{[1]}) = 0 \quad \text{in } \Omega_m^{N,M}, \quad (\tilde{U}_m - U_m^{[1]})(x_i, 0) = 0, \quad \text{if } x_i \in \overline{D}_m^N;$$

and for $t_j \in \omega^M$,

$$|(\tilde{U}_m - U_m^{[1]})(\rho, t_j)| = |(\tilde{U}_m - \mathcal{I}_j U_\ell^{[1]})(\rho, t_j)| \leq \mu_\rho + \frac{5}{6} \mu^{[1]} \quad (\text{since } (\rho, t_j) \in \overline{\Omega}_\ell^{N,M})$$

$$|(\tilde{U}_m - U_m^{[1]})(1-\rho, t_j)| = |(\tilde{U}_m - \mathcal{I}_j U_r^{[1]})(1-\rho, t_j)| \leq \mu_\rho + \frac{5}{6} \mu^{[1]} \quad (\text{since } (1-\rho, t_j) \in \overline{\Omega}_r^{N,M}).$$

Therefore, Lemma 5.4 gives

$$\|\tilde{U}_m - U_m^{[1]}\|_{\overline{\Omega}_m^{N,M}} \leq \mu_\rho + \frac{5}{6} \mu^{[1]}. \quad (5.25)$$

We next obtain an estimate for $\mu^{[2]}$. Since $(2\rho, t_j)$ and $(1 - 2\rho, t_j)$ belong to $\overline{\Omega}_m^{N,M}$, it follows that

$$\begin{aligned} |(\tilde{U}_\ell - \mathcal{I}_j U^{[1]})(2\rho, t_j)| &\leq \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]} \\ \text{and } |(\tilde{U}_r - \mathcal{I}_j U^{[1]})(1 - 2\rho, t_j)| &\leq \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]}. \end{aligned}$$

Thus, $\mu^{[2]} \leq \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]}$. Therefore, it holds that

$$\max\{\nu^{[1]}, \mu^{[2]}\} \leq \lambda + \frac{5}{6}\mu^{[1]}, \quad \lambda = \mu_{2\rho} + \mu_\rho.$$

After repeating the above arguments one gets

$$\max\{\nu^{[k]}, \mu^{[k+1]}\} \leq \lambda + \frac{5}{6}\mu^{[k]}.$$

We simplify this to get

$$\mu^{[k]} \leq 6\lambda + \left(\frac{5}{6}\right)^{k-1} \mu^{[1]}.$$

Thus,

$$\nu^{[k]} \leq 6\lambda + \left(\frac{5}{6}\right)^k \mu^{[1]}. \quad (5.26)$$

Further, since $(2\rho, t_j), (1 - 2\rho, t_j) \in \overline{\Omega}_m^{N,M}$, $(\rho, t_j) \in \overline{\Omega}_\ell^{N,M}$, and $(1 - \rho, t_j) \in \overline{\Omega}_r^{N,M}$, by Lemma 5.6 $\lambda \leq C(\Delta t + N^{-2} \ln^2 N)$. Since $\mu^{[1]} \leq C$, using (5.26) and Lemma 5.6 in (5.11) the proof is complete. \square

5.4 Numerical results

Numerical results considering a couple of examples are given that verify the convergence theory.

Example 5.1. Consider

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + (1 + xe^{-t})u(x,t) = f(x,t) & (x,t) \in \Omega = D \times (0,1], \\ u(0,t) - \sqrt{\varepsilon} \frac{\partial u}{\partial x}(0,t) = g_\ell(t) & t \in (0,1], \\ u(1,t) + \sqrt{\varepsilon} \frac{\partial u}{\partial x}(1,t) = g_r(t) & t \in (0,1], \\ u(x,0) = 0 & x \in [0,1]. \end{cases}$$

where f , g_ℓ , and g_r are such that

$$u(x,t) = t \left[\frac{e^{-x/\sqrt{\varepsilon}} + e^{(x-1)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x) \right].$$

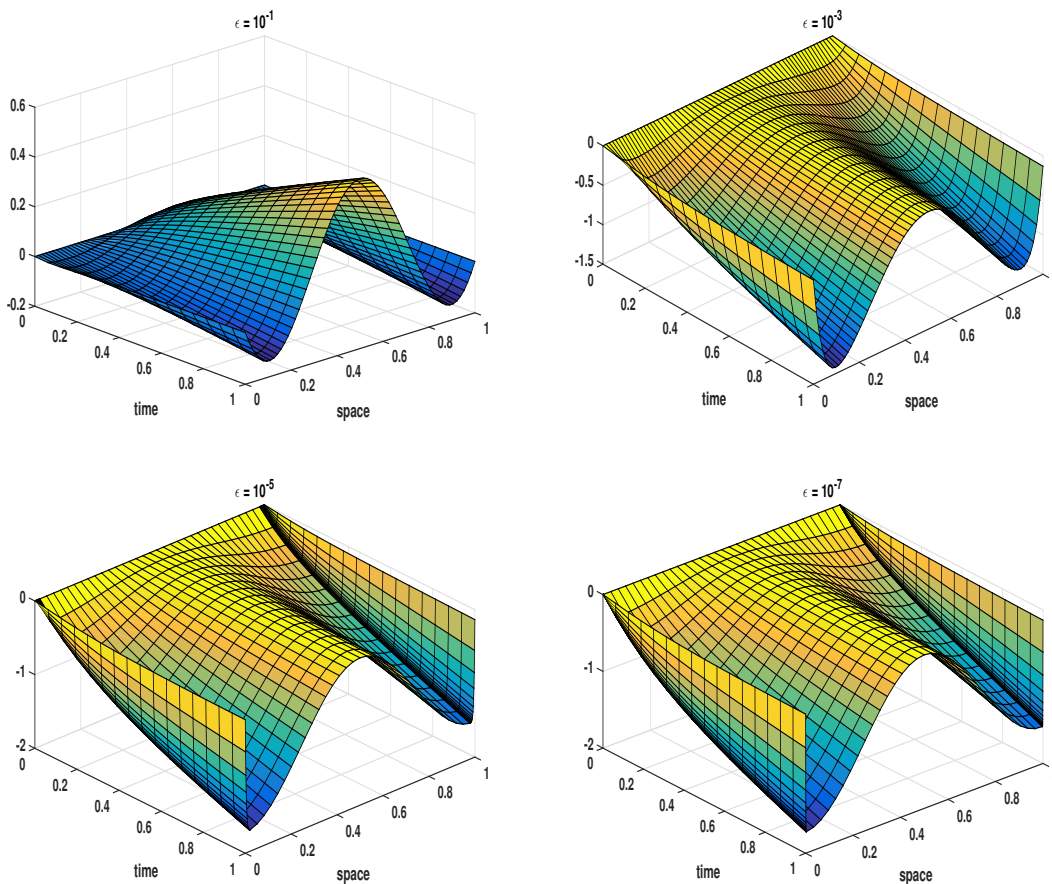


FIGURE 5.1: Solution plots for Example 5.1 taking $N = 64$, $M = 16$, and various values of ε .

TABLE 5.1: Errors and convergence orders for Example 5.1.

$\varepsilon = 10^{-p}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
	$\Delta t = 0.25/4$	$\Delta t = 0.25/4^2$	$\Delta t = 0.25/4^3$	$\Delta t = 0.25/4^4$	$\Delta t = 0.25/4^5$
$p = 1$	6.855E-05	1.712E-05	4.285E-06	1.069E-06	2.674E-07
2	3.193E-04	8.292E-05	2.072E-05	5.181E-06	1.295E-06
3	3.195E-03	8.063E-04	2.020E-04	5.054E-05	1.263E-05
4	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
5	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
6	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
7	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
8	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
$E^{N,\Delta t}$	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
$R^{N,\Delta t}$	1.542	1.609	1.658	1.695	

TABLE 5.2: Iterations for Example 5.1.

$\varepsilon = 10^{-p}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
	$\Delta t = 0.25/4$	$\Delta t = 0.25/4^2$	$\Delta t = 0.25/4^3$	$\Delta t = 0.25/4^4$	$\Delta t = 0.25/4^5$
$p = 1$	4	4	4	5	5
2	1	2	2	2	2
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

The solution plots for various values of ε are given in Figure 5.1. Note that the boundary layers are near to $x = 0, 1$. Taking tolerance error $tol = N^{-2}$, we compute the approximate solution and denote it by $U^{N,\Delta t}$. We then evaluate parameter uniform errors and convergence orders as follows

$$E^{N,\Delta t} = \max_{\forall \varepsilon} E_{\varepsilon}^{N,\Delta t} \quad \text{and} \quad R^{N,\Delta t} = \log_2 \left(\frac{E^{N,\Delta t}}{E^{2N,\Delta t/4}} \right),$$

where $E_{\varepsilon}^{N,\Delta t} = \|u - U^{N,\Delta t}\|_{\overline{\Omega}^{N,M}}$ are the maximum pointwise errors. Table 5.1 displays the computed errors $E_{\varepsilon}^{N,\Delta t}$, $E^{N,\Delta t}$, and parameter uniform orders $R^{N,\Delta t}$ for Example 5.1. Clearly, we can confirm parameter uniform convergence from this

table. In addition, Table 5.2 gives the iterations needed for convergence; from this, we observe that one iteration is enough to get the desired results for very small values of ε .

Example 5.2. Consider

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1+x^2}{2} u(x,t) = t^3 & (x,t) \in (0,1) \times (0,1], \\ u(0,t) - \sqrt{\varepsilon} \frac{\partial u}{\partial x}(0,t) = -(128/35)\pi^{-1/2} t^{7/2} & t \in (0,1], \\ u(1,t) + \sqrt{\varepsilon} \frac{\partial u}{\partial x}(1,t) = -(128/35)\pi^{-1/2} t^{7/2} & t \in (0,1], \\ u(x,0) = 0 & x \in [0,1]. \end{cases}$$

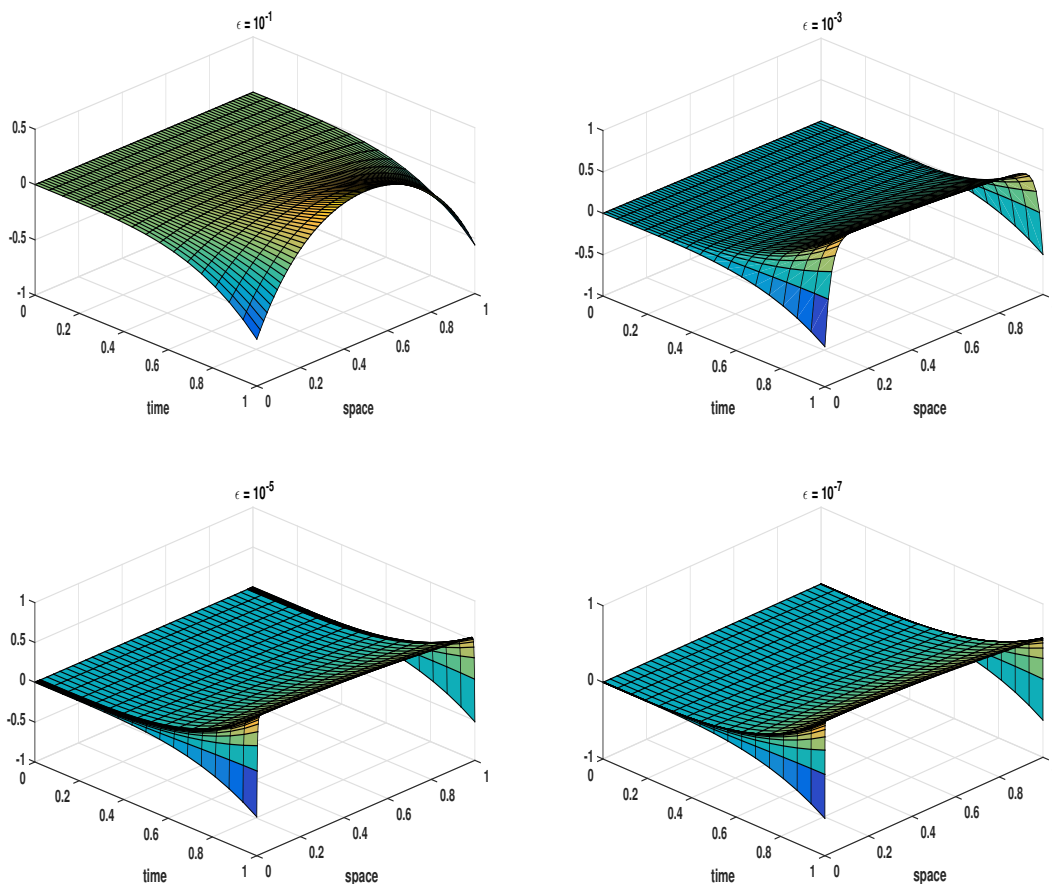


FIGURE 5.2: Solution plots for Example 5.2 taking $N = 64$, $M = 16$, and various values of ε .

The solution plots for various values of ε are given in Figure 5.2. Note that the boundary layers are near to $x = 0, 1$. Since the solution is unknown, we compute another approximate solution to calculate the uniform errors and uniform convergence orders as follows

$$E_\varepsilon^{N,\Delta t} = \max_{(x_i,t_j) \in \bar{\Omega}^{N,M}} |U^{2N,\Delta t/4}(x_i,t_j) - U^{N,\Delta t}(x_i,t_j)|, \quad E^{N,\Delta t} = \max_\varepsilon E_\varepsilon^{N,\Delta t},$$

$$R^{N,\Delta t} = \log_2 \left(\frac{E^{N,\Delta t}}{E^{2N,\Delta t/4}} \right),$$

where $U^{2N,\Delta t/4}$ is obtained taking in each subdomain $2N + 1$ points in space and $\Delta t/4$ mesh size in time, but using the same subdomain parameter ρ as is considered for $U^{N,\Delta t}$. The error plots for various values of ε are given in Figures ?? and ?? for Examples 5.1 and 5.2 respectively.

TABLE 5.3: Errors and convergence orders for Example 5.2.

$\varepsilon = 10^{-p}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
	$\Delta t = 0.25/4$	$\Delta t = 0.25/4^2$	$\Delta t = 0.25/4^3$	$\Delta t = 0.25/4^4$	$\Delta t = 0.25/4^5$
$p = 1$	7.647E-03	1.970E-03	4.965E-04	1.243E-04	3.11E-05
2	1.796E-02	4.449E-03	1.109E-03	2.772E-04	6.929E-05
3	1.858E-02	4.588E-03	1.143E-03	2.856E-04	7.138E-05
4	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
5	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
6	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
7	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
8	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
$E^{N,\Delta t}$	1.888E-02	7.929E-03	2.821E-03	9.318E-03	2.949E-04
$R^{N,\Delta t}$	1.252	1.490	1.598	1.659	

The computed errors $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$, and uniform convergence rates $R^{N,\Delta t}$ for Example 5.2 are tabulated in Table 5.3, showing the uniform convergence of the method. Table 5.4 gives iterations that are performed to achieve the convergence for Example 5.2. One can see that only one iteration is needed to get the solution up to the desired accuracy for very small values ε .

TABLE 5.4: Iterations for Example 5.2.

$\varepsilon = 10^{-p}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
	$\Delta t = 0.25/4$	$\Delta t = 0.25/4^2$	$\Delta t = 0.25/4^3$	$\Delta t = 0.25/4^4$	$\Delta t = 0.25/4^5$
$p = 1$	3	4	4	4	4
2	2	2	2	2	2
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

5.5 Conclusions

An SWR technique to solve singularly perturbed parabolic reaction-diffusion equations with Robin boundary conditions is developed and analyzed in this paper. The original domain is initially divided into three overlapping subdomains. The problem is discretized using the backward difference and central difference schemes for the time and space derivatives, respectively. It is shown that the proposed scheme is unconditionally stable. Error analysis is also discussed in this work, and it is demonstrated that the proposed approach is uniformly convergent with order almost two in space and one in time. Furthermore, it is shown that for small values of ε , one iteration is sufficient to achieve the specified accuracy. The idea discussed in this paper can also be extended to singularly perturbed semilinear differential equations having boundary conditions of Robin type. Further, it is our intention in the future to extend the SWR technique to higher dimensional singularly perturbed problems.
