Chapter 2

On the evolution of finite and small amplitude waves in non-ideal gas with dust particles *

"Live as if you were to die tomorrow,
learn as if you were to live forever."

–Mahatma Gandhi

2.1 Introduction

The present chapter concerns with the study of progressive wave solution for the waves of finite and moderately small amplitude in the mixture of the non-ideal gas

* "The contents of this chapter have been published in ***Physica Scripta (IOP Publishing), Volume 95, 2020.***"

and dust particles governed by quasilinear hyperbolic system of PDEs. The study of non-linear waves has great importance due to its interdisciplinary applications in many physical phenomena. A large number of analytical and numerical techniques have been developed to solve the non-linear system of partial differential equations governing the wave motion (see [4, 1]). Several methods have been proposed to study the asymptotic behavior of weakly non-linear waves described by the quasi linear system of hyperbolic PDEs to establish the transport equation which is used to study the wave process asymptotically by the authors Sharma et al. [2], Hunter [75], Kluwick et al. [76] and many more. In general, shock phenomena can occur in various astrophysical situations, such as stellar winds, photo-ionized gas, supernova blasts, rapid crashes between interstellar gas clusters, and so forth. Shock processes, such as a shock arising from stellar pulsation, a supernova explosion going out through a stellar shell, a shock originating from some extent source such as man-made explosion in Earth's atmosphere or Sun's impulsive flare has considerable significance in astrophysics and space sciences. In the past, the theory of progressive waves have more attention due to its wide applications in aerodynamics, space science, engineering science etc. It has been studied that the non-linearity in the near and far field decisively changes the behavior of the compressive and expansive waves, although the disturbance is small, and causes the behavior of forgetting. Due to this forgetfulness behavior, the shockless disturbances only remember the global initial conditions such as total energy input and forget the previous details of the flow. In comparison to the theory of non-linear geometric acoustics dealing with small amplitude waves, the progressive wave approach deals with finite amplitude pulses; it is exact for the acceleration waves, shock formation, and Riemann waves. Courant et al. [4, 77] have contributed significantly in the context of progressive wave to solve the linear systems that are helpful to introduce the nonlinear system. Varley and Cumberbatch [49] have provided the theory of 'Relatively Undistorted

Waves'to address the waves of finite amplitude in a radially symmetric isentropic gas flow. This theory depends on the successive approximation scheme for the system of hyperbolic partial differential equations (PDEs) which makes no assumption about the magnitude of the disturbance. Seymour and Mortell [54] further applied this technique in detail in which they introduced an expansion scheme that generalizes the previous research and it was used in linear geometric acoustics to inspect the amplitude dispersion and shock formation. Similar approach has been developed by Taniuti [48], Asano [78] and their peers [79]. They discussed a class of nonlinear partial differential equations that perceive a decrease in tractable nonlinear equations such as the Kortweg-deVries equation and Burger equations, and also discussed the applications of this method in plasma physics and hydrodynamics. Gupta et al. [80] and Varley et al. [52] have discussed the concept of simple modulated waves to examine high frequency waves in distinct regime. Further, the complete shock wave classification in a van der Waals fluid is studied by Zhao et al. [81]. Singh et al. [82, 83, 84] and Nath et al. [85], Chaturvedi et al.[86] have discussed about the evolution of weak shock waves and its propagation in distinct material media. Ambika et al. [50] have studied the disturbance and flow patterns of amplitude waves by using progressive wave approach.

Dusty gases have received great attention in numerous science and industrial research since the last few decades owing to their applications to lunar ash burst, nozzle flow, underground explosions, bomb blast, celestial explosions, description of star formation, coma collision with a planet, shocks in supernova explosions and many other physical problems. Applications of dusty-gas flow studies have recently drawn attention to industrial and environmental problems. Removing small solid particles from semiconductor wafers by using shock waves and non-stationary gas dynamic wave systems is an example of industrial applications. Here, it is assumed that the gas consists of the dust particles in which the volume of the particles contribute only five percent of the mixture's total volume. In the present problem, we have considered that the particles are spherical, of uniform size, and their specific heat is constant. Also, we assume that the temperature within each particle is uniform, and there are no collisions between particles. The study of fluid flow containing solid particles is of concern to many research fields of science and engineering such as centrifugal separation of specific matter from liquids, many chemical processing, fluidized beds, strong particle movement in rocket exhaust and dust storm in geophysical and astrophysical issues. Miura and Glass [63] investigated the propagation of a shock wave en-routing through a dusty-gas layer. Pai [15] gave some basic idea of a gas flow in the presence of small solid particles. Carrier [66] has explored the characteristic of shock waves in dusty gas where the planar steady retarded flow of a dusty gas is appropriately studied. Nath et al. [51] and Chaturvedi et al. [87] examined the propagation of weak discontinuities in dusty gas.

In this chapter, we analyze the concept of progressive wave solution in a non-ideal dusty gas for planar, cylindrically symmetric and spherically symmetric waves of finite and small amplitude, and examine how the non-ideal parameter and mass fraction of dust particles affect the shock formation and flow pattern. Besides this, we have discussed some specific cases in which it is assumed that the initial disturbances are either pulse or periodic wave and figured out the entire process of shock decay after the shock formation.

2.2 Governing equations

The basic equations that govern the one dimensional unsteady inviscid non-ideal dusty gas flow, may be written as (see [88])

$$\begin{cases} \varrho_t + u\varrho_r + \varrho u_r + \frac{m\varrho u}{r} = 0, \\ u_t + uu_r + \frac{p_r}{\varrho} = 0, \\ E_t + uE_r - \frac{p}{\varrho^2}(\varrho_t + u\varrho_r) = 0, \end{cases} \tag{2.1}$$

where u denotes the particle velocity, t is the time, ϱ is the density, p is the pressure and r represents the spatial coordinate. The subscripts have been used for partial derivative of the flow parameters with indicated variables. The internal energy E per unit mass of the mixture is given by

$$E = \frac{(1-Z)(1-\tilde{b}\varrho)p}{(\Gamma - 1)\varrho}. \tag{2.2}$$

Here, $k_p = m_{sp}/m_g$ and $Z = V_{sp}/V_g$ define the mass fraction of the solid dust particles present in the flow and the volume fraction, respectively, where V_{sp} and m_{sp} denote volume of the dust particles and the total mass respectively. Also, V_g represents the total volume with m_g as the total mass of the mixture; $\tilde{b} = b(1 - k_p)$, where b is the van der Waals parameter. The Grüneisen coefficient Γ is given by $\Gamma = \gamma(1 + \lambda\beta)/(1 + \lambda\beta\gamma), \lambda = k_p/(1 - k_p)$, $\beta = c_{sp}/c_p$, $\gamma = c_p/c_v$, where c_{sp} denotes the specific heat of dust particles, c_p and c_v denote the specific heat of the gas at constant pressure and at constant volume, respectively. The volume fraction Z and the mass fraction k_p are related by the relation $Z = \theta\varrho$, $\theta = k_p/\varrho_{sp}$, where ϱ_{sp} is the specific density of the dust particles. The equation of state for the non-ideal dusty gas is given by

$$p = \frac{(1-k_p)}{(1-Z)(1-\tilde{b}\varrho)}\varrho RT, \tag{2.3}$$

where R and T denote the specific gas constant and the temperature of the mixture of gas and solid dust particles respectively.

On the insertion of equation (2.3) and neglecting the terms containing $\mathcal{O}(b^2)$, equations $(2.1)_3$ can be written as

$$p_t + up_r + \varrho A^2\left(u_r + \frac{mu}{r}\right) = 0, \tag{2.4}$$

where

$$A = \left(\frac{(\Gamma - \hat{b}Z)p}{(1-(\hat{b}+Z)+\hat{b}Z)\varrho}\right)^{1/2}, \tag{2.5}$$

is the speed of sound with $\hat{b} = \varrho\tilde{b}$. From equation (2.5), it is noticeable that

$$\frac{(\Gamma - \hat{b}Z)p}{(1-(\hat{b}+Z)+\hat{b}Z)\varrho} > 0, \tag{2.6}$$

which leads that the governing system (2.1) is hyperbolic in nature

2.3 Progressive waves

Progressive waves are the solution of the system (2.1), if a family of propagating wavelets $\omega(r,t) = constant$, such that the magnitude of the derivative of flow variables u, ϱ and p with respect to r along a wavelet is much smaller than the magnitude of the derivative of the flow variables with respect to r for a fixed time t [89]. The aforementioned notion of motion can be considered as a generalization of simple wave theory, where the variables ϱ, u and p can be defined in the form of a single variable $\omega(r,t)$; it means that flow parameters remain unaltered along a wavelet. This exhibits that these waves are similar to slowly modulated simple waves. Now, the transformation is considered from (r,t) to (r,ω) through $t=\tau(r,\omega)$, then the governing equations $(2.1)_1$, $(2.1)_2$ and (2.4) in terms of the $\varrho(r,t)=\bar{\varrho}(r,\omega)$, $u(r,t)=\bar{u}(r,\omega)$ and $p(r,t)=\bar{p}(r,\omega)$ transform into the following form:

$$\begin{cases} (1-u\tau_r)\varrho_t-\varrho u_t\tau_r+\bar{u}\bar{\varrho}_r+\bar{\varrho}\bar{u}_r+\frac{m\bar{\varrho}\bar{u}}{r}=0, \\ (1-u\tau_r)u_t-\frac{1}{\varrho}\tau_r p_t+\bar{u}\bar{u}_r+\frac{1}{\bar{\varrho}}\bar{p}_r=0, \\ (1-u\tau_r)p_t-\varrho A^2\tau_r u_t+\bar{u}\bar{p}_r+\bar{\varrho}\bar{A}^2\left(\bar{u}_r+\frac{m\bar{u}}{r}\right)=0, \end{cases} \tag{2.7}$$

where $\bar{A}=\left(\frac{(\Gamma-\bar{\hat{b}}\bar{Z})\bar{p}}{(1-(\bar{\hat{b}}+\bar{Z})+\bar{\hat{b}}\bar{Z})\bar{\varrho}}\right)^{1/2}$.

Since it was assumed that the system has progressive wave solution, therefore

$$\begin{cases} |\bar{\varrho}_r| << |\varrho_r| \Rightarrow \varrho_r \simeq \varrho_t\tau_r \Rightarrow |\bar{\varrho}_r| << |\varrho_t|, \\ |\bar{u}_r| << |u_r| \Rightarrow u_r \simeq u_t\tau_r \Rightarrow |\bar{u}_r| << |u_t|, \\ |\bar{p}_r| << |p_r| \Rightarrow p_r \simeq p_t\tau_r \Rightarrow |\bar{p}_r| << |p_t|. \end{cases}$$

Note that the approximation considered here does not involve the magnitude of the flow variables but only their derivatives. Moreover, if $\bar{\varrho}_r=\mathcal{O}(\bar{\varrho}/r)$, $\bar{u}_r=\mathcal{O}(\bar{u}/r)$ and $\bar{p}_r=\mathcal{O}(\bar{p}/r)$, then equation (2.7) can be rewritten as

$$\begin{cases} (1-u\tau_r)\varrho_t-\varrho u_t\tau_r=0, \\ (1-u\tau_r)u_t-\frac{1}{\varrho}\tau_r p_t=0, \\ (1-u\tau_r)p_t-\varrho A^2\tau_r u_t=0, \end{cases} \tag{2.8}$$

which gives

$$\tau_r=(u+A)^{-1}. \tag{2.9}$$

From equation (2.9), it can be seen that the wavelets, considered here are the characteristic curves of the system $(2.1)_1$, $(2.1)_2$ and (2.4). Using equation (2.9) in (2.8), we obtain

$$\bar{\varrho}\bar{A}\bar{u}_\omega = \bar{A}^2\bar{\varrho}_\omega = \bar{p}_\omega. \tag{2.10}$$

By using equation $(2.7)_2$ and $(2.7)_3$, the compatibility condition involving the flow parameters $\bar{\varrho}, \bar{u}, \bar{p}$ and their derivatives, is given by

$$(\bar{u}+\bar{A})(\bar{\varrho}\bar{A}\bar{u}_r + \bar{p}_r) + \frac{m\bar{\varrho}\bar{u}\bar{A}^2}{r} = 0. \tag{2.11}$$

2.4 Finite amplitude waves

Now, let us assume that the wave propagation into a uniform region is given by $u = 0$, $\varrho = \varrho_0$ and $p = p_0$. Consider the boundary conditions for τ and $\bar{\varrho}$, at $r = r_0$ to be

$$\bar{\varrho} = h(\omega), \tau = \omega. \tag{2.12}$$

Here, h is a smooth and bounded function. Under the progressive wave approximation, in the light of equation (2.10), we can write

$$\bar{u}(r,\omega) = U(\bar{\varrho}(r,\omega)), \bar{p} = P(\varrho(r,\omega)). \tag{2.13}$$

Using equation (2.13), we can solve (2.9) for $t = \tau(r,\omega)$ as

$$\tau = \omega + \int_{r_0}^{r} \frac{1}{U(\bar{\varrho}) + F(\bar{\varrho})} dr. \tag{2.14}$$

The equation (2.11) together with the equation (2.13) can be solved for $\bar{\varrho}$ as a function of r and ω given by

$$\bar{\varrho}U(\bar{\varrho}) = H(\omega)(r/r_0)^{-m/2}, \tag{2.15}$$

where $H(\omega)$=$h(\omega)U(h(\omega))$ and

$$\begin{cases} F(s) = \left(\frac{(\Gamma-\theta\tilde{b}s^2)P(s)}{(1-(\theta+\tilde{b})s+\theta\tilde{b}s^2)s}\right)^{1/2}, \\ U(\bar{\varrho}) = \int_{\varrho_0}^{\bar{\varrho}} \frac{F(s)}{s}ds, \\ P(\bar{\varrho}) = p_0\left(\frac{1-\bar{Z}}{1-\bar{Z}_0}\right)^{\left(\frac{\bar{\hat{b}}-\bar{Z}\Gamma}{\bar{Z}-\bar{\hat{b}}}\right)}\left(\frac{1-\bar{\hat{b}}}{1-\hat{b}_0}\right)^{\left(\frac{\bar{\hat{b}}\Gamma-\bar{Z}}{\bar{Z}-\bar{\hat{b}}}\right)}\left(\frac{\bar{\varrho}}{\varrho_0}\right)^{\Gamma}, \end{cases}$$

where $Z_0 = \theta\varrho_0$ and $\hat{b}_0 = \tilde{b}\varrho_0$. Equation (2.14) shows that the shock first forms at $r = r_s$ on the wavelet ω_s, where r_s can be obtained as the solution of

$$1 - \int_{r_0}^{r_s}\left(\frac{(\Gamma(\Gamma+1) - (\Gamma+3)\bar{Z}\bar{\hat{b}} + 2\bar{Z}\bar{\hat{b}}(\bar{Z}+\bar{\hat{b}}))P(\bar{\varrho})}{2\bar{\varrho}^2(1-(\bar{Z}+\bar{\hat{b}})+\bar{Z}\bar{\hat{b}})^2F(\bar{\varrho})(U(\bar{\varrho})+F(\bar{\varrho}))^2}\right)\left(\frac{\partial\bar{\varrho}}{\partial\omega}\right)_{\omega=\omega_s} dr = 0. \tag{2.16}$$

Equation (2.12) – (2.16) describe the simple wave solution. Also, the disturbance propagating into the region $\varrho = \varrho_0$, $p = p_0$ and $u = 0$ is described by equation (2.12) – (2.16), can be determined from equation (2.14) and (2.15). The density $\bar{\varrho}$ can be obtained from the equation (2.15), thereafter the velocity $\bar{u}$, pressure $\bar{p}$ are determined from equation (2.13). Equation (2.16) implies that at the distance r_s, the solution will be discontinuous and the discontinuity depends on the values of θ and b. Now, we shall envisage the wave propagation into an unperturbed region with $u = 0$ ahead of the shock.

2.5 Small amplitude waves

In this section, we consider the disturbed flow as a perturbation of the uniform state given as $\bar{\varrho} = \varrho_0 + \varrho_1$, where the disturbance ϱ_1 is supposed to be very small. Now from equation (2.13) − (2.16), we have

$$\bar{p}(r,\omega) = p_0 + \varrho_1(r,\omega)F_0^2, \bar{u}(r,\omega) = \varrho_1(r,\omega)F_0/\varrho_0. \tag{2.17}$$

Under the hypothesis $|h(\omega) - \varrho_0| \leq 1$, the perturbed density ϱ_1 is given by

$$\varrho_1(r,\omega) = h(\omega)\,(r/r_0)^{-m/2}\,, \tag{2.18}$$

and the perturbed wavelets are given as

$$\tau(r,\omega) = \omega + \frac{1}{F_0}(r - r_0) - \alpha_1 h(\omega)J(r), \tag{2.19}$$

where

$$\alpha_1 = \frac{(\Gamma(\Gamma+1) - (\Gamma+3)Z_0\hat{b}_0 + 2Z_0\hat{b}_0(Z_0 + \hat{b}_0))p_0}{2\varrho_0^2 F_0^3(1 - (Z_0 + \hat{b}_0) + Z_0\hat{b}_0)^2}$$

and

$$J(r) = \begin{cases} r - r_0 & \text{if } m = 0, \\ 2r_0\left((\frac{r}{r_0})^{1/2} - 1\right) & \text{if } m = 1, \\ r_0 \log\left(\frac{r}{r_0}\right) & \text{if } m = 2. \end{cases} \tag{2.20}$$

Equation (2.19) shows that for $\alpha_1 > 0$, a shock forms on a compression wavelet $(dh(\omega)/d\omega > 0)$ at a distance r_s, given by

$$\alpha_1 \left(\frac{dh(\omega)}{d\omega}\right)_{\omega=\omega_s} = 1. \tag{2.21}$$

In view of equation (2.18), we see that along a wavelet, the density ϱ_1 remains same for $m = 0$ and it decays for $m = 1$ and $m = 2$.

We can obtain the position of weak shock wave by using the following equal area rule (See [1]).

$$2\int_{\omega_1}^{\omega_2} h(\xi)d\xi = (\omega_2 - \omega_1)(h(\omega_1) + h(\omega_2)), \tag{2.22}$$

where ω_1 is the wavelet just ahead of the shock and ω_2 is the wavelet behind the shock.

2.6 Steepening of shock

In this section, we describe the process of shock formation and decay on the leading wave front $\omega = 0$. We assume the two cases; the first one is the case when the disturbance propagating at the boundary (pulse) and second one is the case when the disturbance propagating at the boundary (periodic wave).

Case 1: In this case, we assume that the disturbance propagating at the boundary $r = r_0$, is a pulse which can be written as

$$h(\omega) = \begin{cases} 0 & \text{if } \omega < 0, \\ \varrho_0 \delta \sin\left(\frac{\omega F_0}{r_0}\right) & \text{if } 0 < \omega < \frac{\pi r_0}{F_0}, \\ 0 & \text{if } \omega > \frac{\pi r_0}{F_0}. \end{cases} \tag{2.23}$$

In view of equation (2.23), the progressive wave solution for a small amplitude disturbance is given by (2.17) – (2.19) as

$$\begin{cases} \bar{\varrho}(r,\omega) = \varrho_0 + \varrho_0\delta\sin\left(\frac{\omega F_0}{r_0}\right)\left(\frac{r}{r_0}\right)^{-m/2}, \\ \bar{p}(r,\omega) = p_0 + F_0^2\varrho_0\delta\sin\left(\frac{\omega F_0}{r_0}\right)\left(\frac{r}{r_0}\right)^{-m/2}, \\ \bar{u}(r,\omega) = F_0\varrho_0\delta\sin\left(\frac{\omega F_0}{r_0}\right)\left(\frac{r}{r_0}\right)^{-m/2}, \end{cases} \tag{2.24}$$

and $t = \tau(r,\omega)$ can be written as

$$\tau(r,\omega) = \omega + \frac{1}{F_0}(r - r_0) - \frac{1}{F_0\alpha_{11}}\sin\left(\frac{\omega F_0}{r_0}\right)J(r). \tag{2.25}$$

The shock formation distance r_s/r_0 is given by (2.21) as

$$\frac{r_s}{r_0} = \begin{cases} 1 + \frac{\alpha_{11}}{\cos\left(\frac{\omega F_0}{r_0}\right)} & \text{if } m = 0, \\ \left(1 + \frac{\alpha_{11}}{2\cos\left(\frac{\omega F_0}{r_0}\right)}\right)^2 & \text{if } m = 1, \\ \exp\left(\frac{\alpha_{11}}{\cos\left(\frac{\omega F_0}{r_0}\right)}\right), & \text{if } m = 2, \end{cases} \tag{2.26}$$

where $\alpha_{11} = \frac{2(1-Z_0)(1-\hat{b}_0)(\Gamma - Z_0\hat{b}_0)}{\delta\left[\Gamma(\Gamma+1)-(\Gamma+3)Z_0\hat{b}_0+2(Z_0+\hat{b}_0)Z_0\hat{b}_0\right]}$, is a constant quantity. From equation (2.26), it may be noted that for a given value of k_p and b, $\alpha_{11} > 0$ ($\alpha_{11} < 0$) according as $\Gamma(\Gamma+1) - (\Gamma+3)Z_0\hat{b}_0 + 2(Z_0+\hat{b}_0)Z_0\hat{b}_0$ is positive (negative), hence a shock forms (since $r_s > r_0$) on the leading wave front $\omega = 0$ (respectively on the trailing wave front $\omega = \pi$). Partial derivatives of α_{11} with respect to Z_0 and $\hat{b}_0$ is given as

$$\frac{\partial\alpha_{11}}{\partial Z_0} = \frac{-2(1-\hat{b}_0)\left\{(1-Z_0)A_1 + (\Gamma - Z_0\hat{b}_0)A_2\right\}}{\delta\left[(\Gamma(\Gamma+1) - (\Gamma+3)Z_0\hat{b}_0) + 2Z_0\hat{b}_0(Z_0+\hat{b}_0)\right]^2}, \tag{2.27}$$

where

$$A_1 = \hat{b}_0(\Gamma(\Gamma+1) - (\Gamma+3)Z_0\hat{b}_0) + 2Z_0\hat{b}_0^2(Z_0 + \hat{b}_0), \tag{2.28a}$$

$$A_2 = 2\hat{b}_0(Z_0 + \hat{b}_0) + 2Z_0\hat{b}_0(1 - Z_0) + (\Gamma(\Gamma+1) - (\Gamma+3)\hat{b}_0), \tag{2.28b}$$

and

$$\frac{\partial \alpha_{11}}{\partial \hat{b}_0} = \frac{-2(1-Z_0)\left\{(1-\hat{b}_0)B_1 + (\Gamma - Z_0\hat{b}_0)B_2\right\}}{\delta\left[(\Gamma(\Gamma+1) - (\Gamma+3)Z_0\hat{b}_0) + 2Z_0\hat{b}_0(Z_0+\hat{b}_0)\right]^2}, \tag{2.29}$$

where

$$B_1 = Z_0(\Gamma(\Gamma+1) - (\Gamma+3)Z_0\hat{b}_0) + 2Z_0^2\hat{b}_0(Z_0 + \hat{b}_0), \tag{2.30a}$$

$$B_2 = 2Z_0(Z_0 + \hat{b}_0) + 2Z_0\hat{b}_0(1 - \hat{b}_0) + (\Gamma(\Gamma+1) - (\Gamma+3)Z_0). \tag{2.30b}$$

If $\Gamma(\Gamma+1) - (\Gamma+3)\hat{b}_0 > 0$ and $\Gamma(\Gamma+1) - (\Gamma+3)Z_0 > 0$, then $\frac{\partial \alpha_{11}}{\partial Z_0} < 0$ and $\frac{\partial \alpha_{11}}{\partial \hat{b}_0} < 0$. Consider a weak shock wave propagating into an undisturbed region, where $h(\omega_1) = 0$ for $\omega_1 \leq 0$. On substituting equations (2.23) and (2.25) in (2.22), we have

$$\sin\left(\frac{\omega_2 F_0}{2r_0}\right) = \left(1 - \frac{\alpha_{11} r_0}{J(r)}\right)^{1/2}. \tag{2.31}$$

With the help of equation (2.31) and $(2.24)_1$, the shock strength that is the jump in density $[\varrho]$ is given by the following equation

$$[\varrho] = 2\varrho_0\delta\left(\frac{r}{r_0}\right)^{-m/2}\left(\frac{\alpha_{11} r_0}{J(r)}\left(1 - \frac{\alpha_{11} r_0}{J(r)}\right)\right)^{1/2}. \tag{2.32}$$

Equation (2.32) illustrates that the shock formation on $\omega = 0$ at $r = r_s > r_0$ achieves a maximum strength at $r = r_1 > r_s$, where r_1 is obtained as a solution of the following algebraic equation:

$$\frac{J(r_1)}{r_0} + m\left(\frac{r}{r_0}\right)^{-1+m/2}\left(\frac{J(r_1)}{r_0} - \alpha_{11}\right)\frac{J(r_1)}{r_0} - 2\alpha_{11} = 0, \tag{2.33}$$

and then it decays eventually in proportion to $r^{-m/2}$.

Case 2: Let the small disturbance at the boundary $r = r_0$, having a form of periodic wave defined as

$$h(\omega) = \varrho_0 \delta \sin(\hat{\omega}), \tag{2.34}$$

where $\delta < 0$ and $\hat{\omega} = F_0\omega/r_0$ and let the growth over the period $0 \leq \hat{\omega} \leq 2\pi$ so that the shock first forms on the wavelet $\hat{\omega} = \pi$ at a distance $r = r_s$ nearest to r_0 which can be obtained from the solution of equation (2.21). Equation (2.19) and (2.22) are satisfied on the shock if $\hat{\omega}_1 + \hat{\omega}_2 = 2\pi$ and $\hat{\omega}_1 - \hat{\omega}_2 = 2\mu$, where μ is the solution of

$$\frac{\mu}{\sin\mu} = \frac{J(r)}{r_0\alpha_{11}}. \tag{2.35}$$

Therefore, the jump in density $[\varrho]$ at the shock is given by

$$[\varrho] = 2\varrho_0\delta\sin\mu\left(\frac{r}{r_0}\right)^{-m/2}, \tag{2.36}$$

where r and μ are satisfied by equation (2.35). Equation (2.36) demonstrates that the shock starts with zero strength corresponding to $\mu \to 0$ at $r = r_s$, and goes to a maximum strength for some $\mu = \mu_m$ at $r = r_m$, satisfying the following relation:

$$\sin(2\mu_m) - m\alpha_{11}(\sin(\mu_m)) - \mu_m\cos(\mu_m)(r_m/r_0)^{-1+m/2} = 0. \tag{2.37}$$

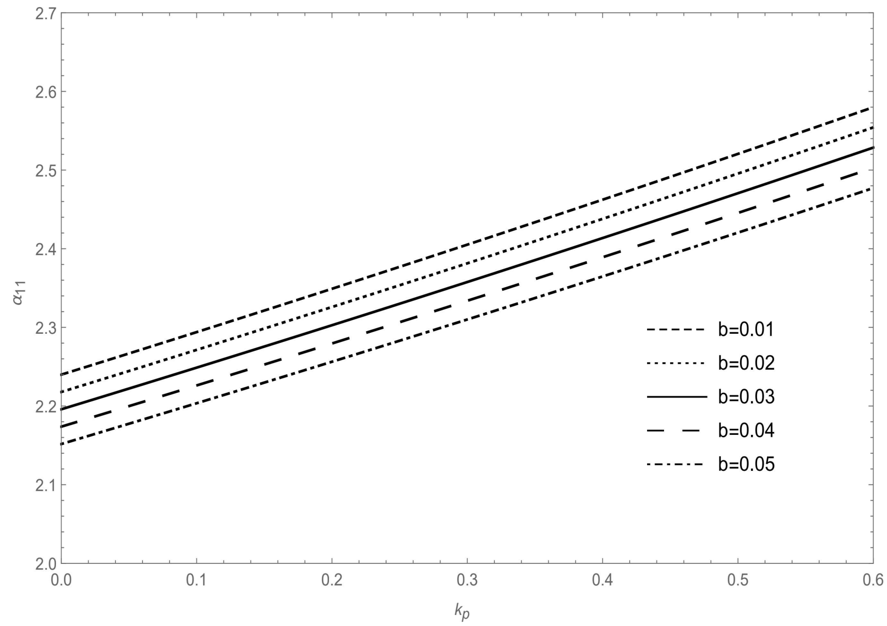
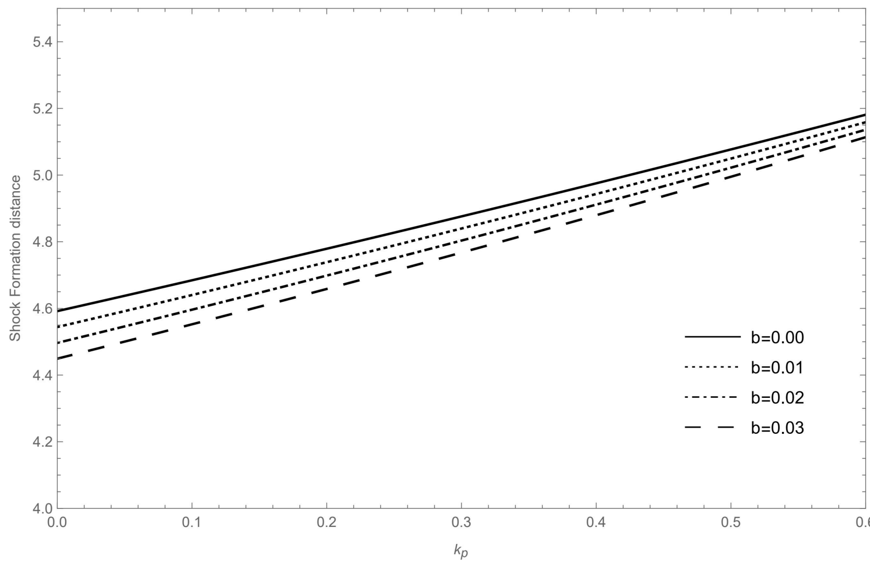


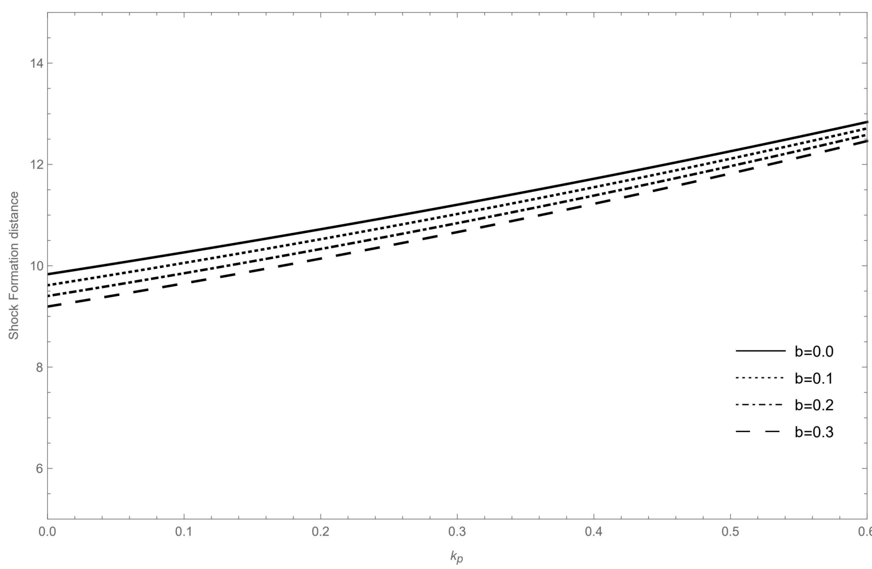
FIGURE 2.1: Variation of α_{11} versus k_p for different values of b with $\gamma = 1.4$ and $\beta = 0.8$.

2.7 Results and Discussion

In this section, we discuss the solution obtained under the given approximation in the physical plane, and also we examine the effect of mass fraction of dust particles and non-ideal parameter on the propagation of the progressive wave, and the conditions of the shock formation. To clarify more we consider the result in two cases; first one is spherically symmetric case and other one is cylindrically symmetric case. Also, we consider two cases that the disturbance is either a periodic wave or a pulse and we observed that in both the cases the shock formation and decay completely depends on a constant α_{11}. We can obtain the shock formation distance from equation (2.26) for both planar and non-planar case. Also, it can be noticed that in the case of planar flow, the shock formation distance is a linear function of α_{11} while in case of cylindrically symmetric flow, it is quadratic, and for spherically symmetric flow, it is an exponential function of α_{11}. The shock formation rate increases with the increasing value of m. From the curves of the equations (2.27) and (2.29), it is clear that α_{11} is a decreasing function of Z and b. Fig.2.1 shows that an increase in the value of non-ideal parameter b causes to decrease the value of α_{11} while α_{11} is the increasing function of the mass fraction k_p of the dust particles. From the relation (2.26), we observe that the increase in m $(m = 0, 1, 2)$ leads to an increase in the shock formation distance which implies that the shock formation is earlier in case of planar flow than the non-planar flow. Also, equation (2.26) demonstrates that the increasing values of b causes an early shock formation in both cases cylindrically as well as spherically symmetric flows. Fig.2.2 shows that as we increase the value of k_p, the shock formation distance increases. This is due to the fact when a wave is transmitted into dusty gas, the wave velocity decreases due to the hindrance of particles of large inertia and results in an increase in the shock formation distance. The early shock formation is due to non-idealness of the gas whose effect is to destabilize the gas. From Fig.2.2, it is clear that the presence of the non-ideal parameter affects the shock formation distance. The effect of k_p along with b on the shock strength in cylindrically symmetric and spherically symmetric ideal and non-ideal gas flow is shown by the curves in Fig.2.3. We can see that the maximum value of shock strength decreases with an increase in the value of k_p and b. A similar kind of behavior can be observed between shock curvature and the maximum shock strength for different values of the mass fraction k_p and b. The distortion of pulse described by (2.23) for cylindrically symmetric and spherically symmetric flows are shown in Fig.2.4, respectively. Also, Fig.2.4 describes the effect of non-ideal parameter b and mass fraction of the solid dust particles k_p on the distortion of the pulses. On increasing (decreasing) the value of b and k_p causes to slowdown (enhance) the flattening of the wave profiles. Also, the distance $r = r_m$ at which the shock achieves its maximum strength can be determined from equation (2.37).

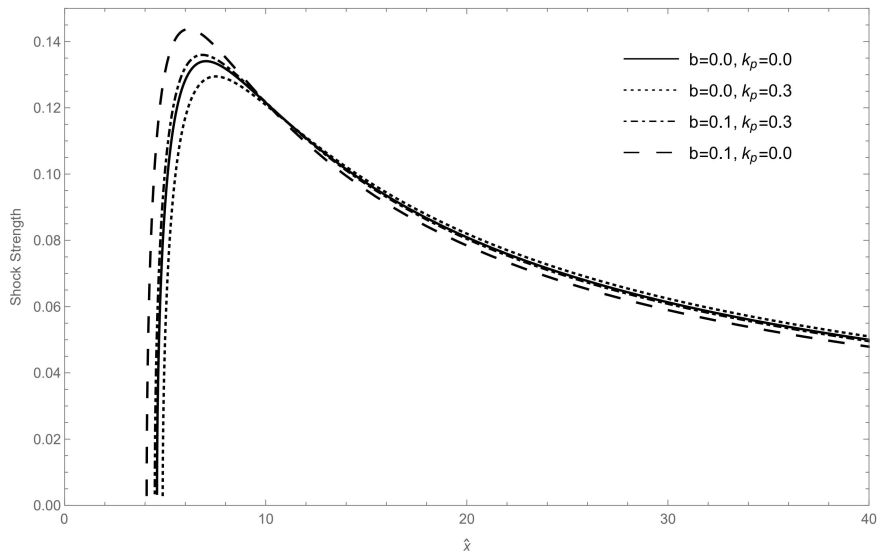


(a)

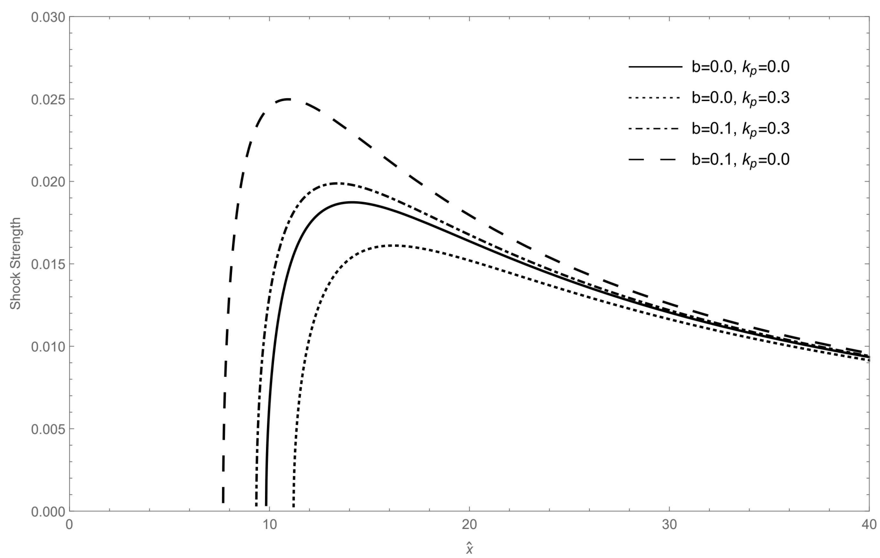


(b)

FIGURE 2.2: Variation of shock formation distance in (a) cylindrically symmetric flow and (b) spherically symmetric flow for different values of b, respectively.

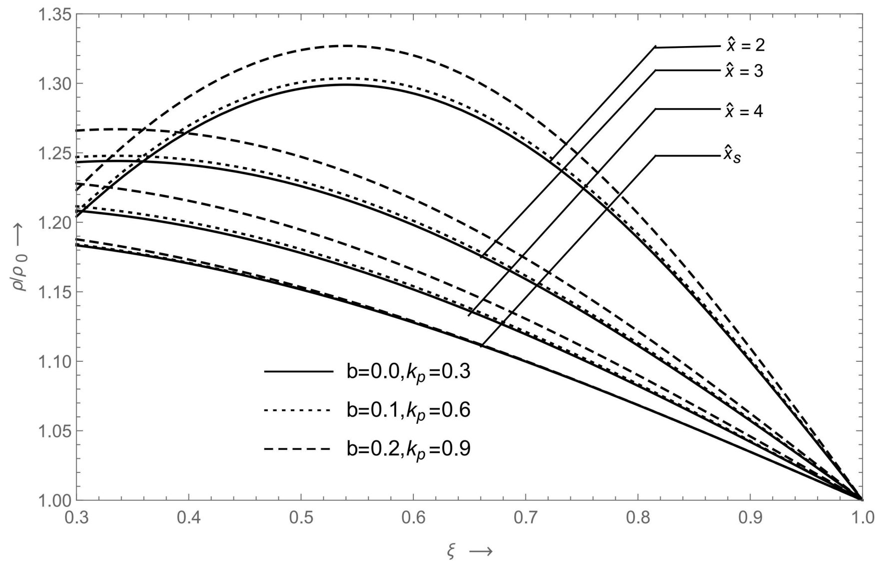


(a)

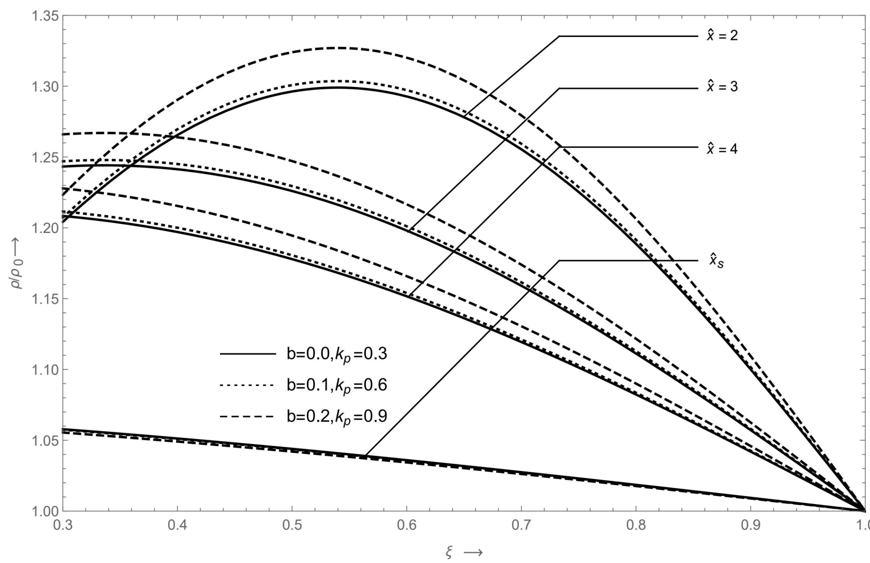


(b)

FIGURE 2.3: Effects of mass fraction k_p and the non-ideal parameter b on the shock strength in (a) cylindrically symmetric flow and (b) spherically symmetric flow for $\gamma = 1.4$ and $\beta = 0.8$, respectively.



(a)



(b)

FIGURE 2.4: Variation of $\hat{\varrho} = \frac{(\varrho_0+\varrho_1)}{\varrho_0}$ with the dimensionless variable $\xi = \frac{(r-F_0 t)}{r_0}$ in (a) cylindrically symmetric flow and (b) spherically symmetric flow of non-ideal gas with dust particles, respectively.

2.8 Conclusion

In this study, we use the progressive wave method to investigate the finite amplitude shock wave, moderately small amplitude shock wave in the mixture of the non-ideal gas and the dust particles, where the motion is perturbed at the boundary. We observed that the van der Waals parameter b, together with mass fraction k_p, influence the shock formation, shock strength, distortion of pulse, and the flow patterns of the solution curves. The amplitude dispersion of the wave depends on the amplitude of the wavelets and the values of b and k_p. Asymptotic behavior for the decay process of planar and non-planar flow configuration is investigated. Also, in this study, we discussed two distinct cases that the disturbance is either a periodic wave or a pulse and found that in both the cases the shock formation and decay relies upon a constant α_{11}. It is also observed that the shock formation distance varies exponentially in case of spherically symmetric flow as compared to cylindrically symmetric flow. The shock strength also varies according to variation in the values of the parameter of non-idealness and the mass fraction of dust particles i.e., an increase in the value of the parameter of non-idealness as well as mass fraction of dust particles cause to decrease (increase) the flattening of the curves for non-planar flow. Also, the condition for obtaining the distance at which the maximum shock strength can be achieved has been determined, and it is found that it gradually decays according to the power law of the distance. It is observed that a rise in the value of b results in a significant increase in the elevation of the expansive part of the profile, and a decrease in the distance of shock formation, thus indicating that the disturbance is being intensified.

****