

# Chapter 1

## Introduction

“Mathematics reveals its secrets only to those  
who approach it with pure love, for its own beauty.”

–Archimedes.

### 1.1 Background

#### 1.1.1 Linear and Nonlinear Waves

Many stimulating phenomena arise from nonlinearity, such as shock formation, wave interaction, and the evolution of nonlinear waves. Research and development activities on nonlinear waves have seen a noticeable increase in recent years. There is a considerable difference between so-called “linear” phenomena such as sound, light, or electromagnetic signals and the violent disturbances caused by the detonation of explosives, the flow through rocket nozzles, or the impact on solids that result

from the supersonic flight of projectiles or the effect on solids. They propagate via nonlinear differential equations, whereas the latter are governed by linear equations, implying a breakdown of the intimate superposition's law, reflection, and refraction. Further, shock fronts are a prominent feature, among many other novel features. A shock wave is a small transition layer of abrupt changes in physical quantities such as temperature, density, and pressure. Nonlinear wave motion is a prominent subject for study and analysis. During the period beginning more than a hundred years ago, many mathematicians and physicists like Stokes, Earnshaw, Riemann, Rankine, Hugoniot, Lord Rayleigh, and later Hadamard, Von Neumann, Courant, Friedrichs, G. B. Whitham, and others developed the fundamental concepts. They wrote research papers and books initiating this field of research. Since the last few decades, nonlinear wave motion, particularly shock waves, and expansion waves, has gained a great deal of attention when the barriers between applied and pure science have been lowered.

Waves are generally defined as disturbances or variations propagating in a medium and transferring energy from point to point. It may be elastic deformation or a variation of pressure, electric or magnetic intensity, electric potential, or temperature. Inherently, this type of wave involves motion involving the space  $R^n$  and the time  $t$ , naturally leading to time-related problem. It should always be treated separately to distinguish the time variable from other independent variables. The motion of linear waves, such as sound waves, is always accompanied by a speed (relative to the medium) which may vary within the medium. In all conceivable linear wave motions within the medium, the "sound speed" remains the same. Nonlinear wave motion is also affected by such a sound speed. Sound speed is defined as little perturbations, or waves, that significantly alter a given primary wave motion and propagate at a specific speed. Hence, the sound speed depends on both the position within the medium and the state of the medium produced by the primary motion.

Waves are prevalent in the majority of scientific and technical fields. Examples include quantum physics, fluid mechanics, optics, electromagnetics, solid mechanics, and structural mechanics. Two types of partial differential equations describe the waves in these applications: linear and nonlinear. The homogeneous linear partial differential equations satisfy the principle of superposition, that is, the linear combination of two solutions will also be the solution. Accordingly, the solution space of the homogeneous linear partial differential equation forms vector space, and one advantage of using that space's linear structure is that it may be utilized to build solutions with desired properties that satisfy a variety of boundaries and initial conditions. In case of nonlinear partial differential equation, the principle of superposition of solutions does not apply, which holds for linear partial differential equations. In this work, we solve problems involving quasilinear hyperbolic partial differential equations in gas dynamics. In this thesis, our main objective is to analyze the problem of propagation of non-linear waves involving quasilinear hyperbolic partial differential equation. Under certain conditions, such waves can be represented by a quasilinear system of first-order equations, which are linear in the first order derivatives of the dependent variables, but the coefficients may be function of dependent variables. When the effects of viscosity and heat conduction are neglected, the fluid dynamic equations reduce to a hyperbolic system of equations (Euler's equations).

### 1.1.2 Hyperbolic system of PDEs

A mathematical model represented by a hyperbolic system of partial differential equations (PDEs) describes the evolution of certain physical phenomena, often involving waves or propagation of disturbances. In a hyperbolic system, the significant features are the presence of characteristic curves along which information travels at

finite speed. These characteristic curves can be thought of as paths along which changes in the solution propagate. Many scientific and technological applications may be formulated as one-dimensional hyperbolic system of first-order PDEs. In particular, it has many applications in gas-dynamics, fluid dynamics, aerodynamics, multi phase flows, astrophysics, and plasma physics etc. The most interesting feature of a quasilinear hyperbolic system of PDEs lies in the fact that a smooth solution breaks down after a finite time. Breaking of these smooth solutions gives rise to one of the most interesting nonlinear phenomena occurring in nature, that is, shock, which contains a sudden jump in velocity, pressure, and density. Another interesting feature of quasilinear hyperbolic system of PDEs is the interaction of nonlinear waves. The detailed study of mathematical properties and applications of nonlinear wave propagation problems in the context of hyperbolic systems of PDEs may be found in the books and monographs by Courant and Friedrichs [1], Lax [2], Jeffrey [3], Zheng [4], Bressan [5], Dafermos [6], Sharma [7] and Smoller [8] etc. In order to present a mathematical description, let us consider first order partial differential equation of the form

$$u_{i,t} + \sum_{j=1}^m a_{ij}(x, t, \mathbb{W})u_{i,x} + b_i(x, t, \mathbb{W}) = 0 \quad (1.1)$$

for  $i = 1, \dots, m$ . Based on the space  $x$  and the variable  $t$  (time), there are  $m$  equations with  $m$  unknowns  $u_i$ . The dependent variables are  $u_i$ , and the independent variables are  $x, t$ ;  $u_i = u_i(x, t)$ ;  $u_{i,t}$  represents the partial derivative of  $u_i$  with respect to  $t$ ; similarly,  $u_{i,x}$  represents the partial derivative of  $u_i$  with respect to  $x$ . Another way to express the system (1.1) in matrix form is as

$$\mathbb{W}_t + \mathbb{A}\mathbb{W}_x + \mathbb{B} = 0, \quad (1.2)$$

where

$$\mathbb{W} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_m \end{bmatrix}, \mathbb{B} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}, \mathbb{A} = \begin{bmatrix} a_{11} & \cdot & \cdot & a_{1m} \\ a_{21} & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & a_{mm} \end{bmatrix}.$$

It follows that system (1.2) is linear with constant coefficients if matrix  $\mathbb{A}$  with all entries  $a_{ij}$  being constant and vector  $\mathbb{B}$  with all components  $b_i$  being constant. The system is linear with variable coefficients, if  $a_{ij} = a_{ij}(x, t)$  and  $b_i = b_i(x, t)$ . If  $\mathbb{B}$  is linearly dependent on  $\mathbb{W}$ , then the system is linear, but if  $\mathbb{A}$  is a function of the vector  $\mathbb{W}$ , *that is*  $\mathbb{A} = \mathbb{A}(\mathbb{W})$ , it is quasilinear. In general, quasilinear systems involve a system of nonlinear equations. In system (1.2),  $\mathbb{B} = 0$  corresponds to the homogeneous system.

*Definition 1.1.1. (Hyperbolic systems)* Hyperbolic systems can describe a wide range of physical phenomena, including fluid dynamics, electromagnetism, and elasticity, where wave-like behavior and the propagation of disturbances are essential aspects of the dynamics. The system (1.2) is called a first-order hyperbolic system of partial differential equations if the matrix  $\mathbb{A}(\mathbb{W})$  admits  $m$  real eigenvalues

$$\lambda_1(\mathbb{W}) \leq \lambda_2(\mathbb{W}) \leq \lambda_3(\mathbb{W}) \leq \dots, \leq \lambda_m(\mathbb{W}).$$

together with set of  $m$  linearly independent eigenvectors  $L_1, L_2, L_3, \dots, L_m$ . The eigenvalues are also called the characteristic speeds or wave speeds associated with (1.2). If the system's eigenvalues are distinct, it is considered strictly hyperbolic:

$$\lambda_1(\mathbb{W}) < \lambda_2(\mathbb{W}) < \lambda_3(\mathbb{W}) < \dots, < \lambda_m(\mathbb{W}).$$

We now provide a few concepts related to linearity and nonlinearity for every  $j$ -wave family.

*Definition 1.1.2.* For every  $i, j = 1, \dots, m$ , we can say that the  $j$ -characteristic field corresponding to  $j$ -characteristic of (1.2) is **genuinely nonlinear** when

$$\nabla \lambda_j(\mathbb{W}) \cdot L_j(\mathbb{W}) \neq 0,$$

and **linearly degenerate** when

$$\nabla \lambda_j(\mathbb{W}) \cdot L_j(\mathbb{W}) = 0,$$

where  $\nabla = \left( \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_m} \right)$ .

### 1.1.3 The Riemann Problem

The Riemann Problem is a fundamental concept in the study of hyperbolic partial differential equations (PDEs), particularly in the context of conservation laws. It is named after the German mathematician Bernhard Riemann [9]. It plays a crucial role in understanding the behavior of solutions to these equations, especially in the context of shock and rarefaction waves. A hyperbolic PDE is a type of partial differential equation that can describe wave-like phenomena where information travels at finite speed. Conservation laws are a specific class of hyperbolic PDEs describing the conservation of certain quantities in a physical system, such as mass, momentum, or energy. The Riemann Problem involves the study of the behavior of solutions to a conservation law equations that is initially discontinuous, that is, it has a jump or discontinuity in its initial conditions. The problem seeks to find

the solution to this PDE under such conditions and to understand how the solution develops over time. The Riemann Problem is typically stated as follows: given two constant states (values) separated by a discontinuity, what is the solution to the conservation law equation as time progresses? This problem often leads to the formation of shock waves and rarefaction waves, which are fundamental concepts in studying hyperbolic PDEs occurring in fluid dynamics. Solving the Riemann Problem involves determining the structure and evolution of these waves as they interact with each other and propagate through the medium. The Riemann Problem serves as a fundamental building block for understanding more complex solutions in hyperbolic PDEs, and its solutions are often used to validate numerical methods and simulations in various scientific and engineering fields. The study of the Riemann Problem has applications in various areas, including fluid dynamics, gas dynamics, acoustics, traffic flow modeling, and more. Researchers and scientists use analytical and numerical techniques to explore and understand the behavior of solutions to hyperbolic conservation laws, particularly in scenarios involving shocks, rarefactions, and other wave phenomena.

The Initial Value Problem in conservation laws corresponds to the Riemann problem for one-dimensional Euler equations with time dependence

$$W_t + F(W)_x = 0. \quad (1.3)$$

Here,

$$W = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad F(W) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{bmatrix},$$

where  $(x, t) \in R \times R_+$  and  $x, t$  represents the space and time coordinates, respectively.  $u(x, t)$  and  $\rho(x, t) > 0$  represents the velocity and density, respectively. Since,  $W$  is function of  $(\rho, u, p)$ . In view of this the initial conditions of the Riemann problem for the system (1.3) are given by

$$W(x, 0) = \begin{cases} W_L = (\rho_l, u_l, p_l), & \text{if } x < 0 \\ W_R = (\rho_r, u_r, p_r), & \text{if } x > 0 \end{cases}. \quad (1.4)$$

Here,  $(\rho_l, u_l, p_l)$  and  $(\rho_r, u_r, p_r)$  show the left and right constant states, respectively, separated by the jump discontinuity at  $x = 0$ .

Non-constant initial data determines the Generalized Riemann problem, while a Riemann problem is determined by constant initial data, representing the system's equilibrium state. The Generalized Riemann problem provides a smooth solution for the linear hyperbolic system. In nonlinear hyperbolic systems, Generalized Riemann problem provides bounded discontinuous solutions. The Generalized Riemann problem is usually regarded as a perturbation of the Riemann problem, and it has been demonstrated that the solution obtained in this problem behaves like the classical solution of the Riemann problem.

Applications of the Generalized Riemann Problem can be found in various fields of physics and engineering, including fluid dynamics, gas dynamics, magnetohydrodynamics, traffic flow modeling, and more. Solving the Generalized Riemann Problem can help researchers and practitioners gain insights into the behavior of wave phenomena in these systems, leading to a better understanding of the underlying physics and improved numerical simulations.

The Generalized Riemann problem for governing system (1.3) with initial boundary

condition given as

$$W(x, 0) = \begin{cases} (\rho_l(x), u_l(x)), & \text{if } x < 0, \\ (\rho_r(x), u_r(x)), & \text{if } x > 0. \end{cases} \quad (1.5)$$

Here  $\rho_l(x)$ ,  $\rho_r(x)$ ,  $u_l(x)$  and  $u_r(x)$  are smooth arbitrary functions such that  $\rho_l(0) \neq \rho_r(0)$ ,  $u_l(0) \neq u_r(0)$ .

The Rankine-Hugoniot condition is fundamental in studying shock waves and other discontinuities in fluid dynamics and related fields. It provides a set of jump conditions that describe the conservation of mass, momentum, and energy across a shock wave, which is a sudden and rapid change in the properties of a fluid. The Rankine-Hugoniot conditions are named after two mathematicians and physicists who independently contributed to their development: one is William John Macquorn Rankine (1820–1872): A Scottish engineer and physicist who made significant contributions to thermodynamics and fluid mechanics. The second one is Emil Hugoniot (1827–1883): A French physicist known for his work on the behavior of matter at high pressures.

For a scalar nonlinear case where  $\lambda_1(W) = F'(W)$  is strictly convex, the weak solution to this problem will be either a shock wave or a continuous rarefaction wave. Generally, bounded, piecewise smooth weak solutions to a system of conservation laws satisfy the Rankine-Hugoniot condition at discontinuities. The Rankine-Hugoniot condition is

$$S[W] = [F],$$

where  $[.]$  represents the jump in a quantity across a shock of speed  $S$ , giving a relation between the speed  $S$  of the discontinuity and the constant states  $W_l$  and  $W_r$  on each side of the discontinuity. An additional condition, entropy condition, is required to pick out the physically correct weak solution. One such condition is the Lax entropy condition, which states that if  $F(W)$  is convex, a shock wave would

satisfy the entropy condition

$$\lambda_1(W_l) > S > \lambda_1(W_r).$$

More general weak solutions for (1.3) are the nonlinear wave solutions briefly described below:

- **Shock Wave:** A shock wave is a type of propagating disturbance or wave that carries a sudden change in pressure, temperature, density, and other physical properties through a medium, such as air, water, or solid materials.

If  $\lambda_1(W_l) > \lambda_1(W_r)$ , the entropy satisfying weak solution is a shock wave given by

$$W(x, t) = \begin{cases} W_l & \text{for } \frac{x}{t} < S, \\ W_r & \text{for } \frac{x}{t} \geq S, \end{cases}$$

where  $S$  is the shock speed given by the Rankine-Hugoniot condition.

- **Rarefaction Wave:** In a rarefaction wave, the particles of the medium move apart, leading to a decrease in density and pressure. This occurs when a disturbance or change in a medium causes the particles to spread out, creating lower-density areas.

If  $\lambda_1(W_l) < \lambda_1(W_r)$ , then the correct entropy solution is a rarefaction wave given by

$$W(x, t) = \begin{cases} W_l, & \text{for } \lambda_1(W_l) > \frac{x}{t}, \\ W(\frac{x}{t}), & \text{for } \lambda_1(W_l) \leq \frac{x}{t} \leq \lambda_1(W_r), \\ W_r, & \text{for } \lambda_1(W_r) \leq \frac{x}{t}, \end{cases}$$

where  $W(\frac{x}{t})$  is the solution of  $F'(W(\frac{x}{t})) = \frac{x}{t}$ .

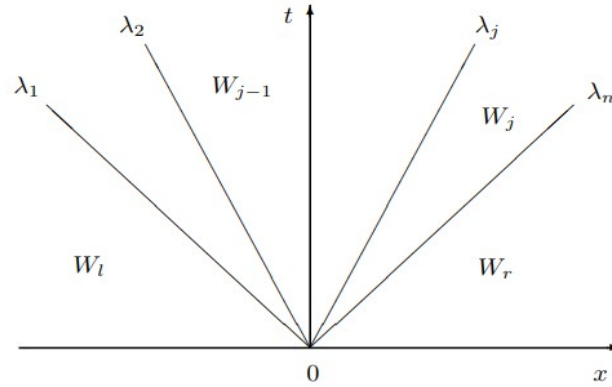


FIGURE 1.1: Structure of the Riemann solution for a system of conservation laws.

These elementary wave solutions also will be the key elements to describe the structure of the solution of the Riemann problem for nonlinear systems. Riemann's solution (1.3) consists of  $n + 1$  constant states, divided by  $n$  waves corresponding to the various characteristic fields. A representation of the solution structure in the  $x - t$  plane is shown in figure (1.1).

#### 1.1.4 Dusty Gas

Dusty gas is a mixture of gas and small solid particles or dust suspended within it. In various natural and artificial scenarios, we can observe this phenomenon. Due to the numerous applications of two-phase flows in different engineering fields, it has been of great interest to analyze a two-phase flow of gas and dust particles. Gas flows, which carry a remarkable amount of solid particles, may exhibit significant relaxation effects due to particles being unable to follow rapid change in the velocity and temperature of the gas. The flow properties of a pure gas significantly differ from those of particles when their mass concentration is comparable to that of the gas. Here, we determine a mixture of a perfect gas and small dust particles of uniform

spherical shape.

A mixture of gas and tiny solid dust particles that comprise less than 5% of the total volume is known as dusty gas [10]. These tiny solid particles behave like a pseudo fluid at a very high speed of fluid [11]. We describe the mixture as the combination of two fluids: gas and the pseudo fluid of solid particles. In this experiment, solid particles consist of identical spheres with mass  $m_{sp}$ , radius  $r_{sp}$  and specific heat  $c_{sp}$ . We examine a dusty gas element, a mixture of gas and solid particles, with a total mass of  $M = M_g + M_{sp}$  and a total volume  $V = V_g + V_{sp}$ . The values of the gas and solid particles are denoted by the subscripts  $g$  and  $sp$ , respectively. In the mixture, solid particles are measured as follows:

$$V_{sp} = n_{sp} \cdot V \cdot \tau_{sp},$$

where  $\tau_{sp}$  and  $n_{sp}$  is the volume of a solid particle and the number of solid dust particles per unit volume of dusty gas respectively. The mass of the solid particles in mixture volume  $V$  is expressed as follows:

$$M_{sp} = n_{sp} \cdot V \cdot m_{sp}.$$

Solid particles have the following species density:

$$\rho_{sp} = \frac{M_{sp}}{V_{sp}} = \frac{m_{sp}}{\tau_{sp}}.$$

Also, the following defines the partial density of the solid particle pseudo-fluid:

$$\bar{\rho}_{sp} = \frac{M_{sp}}{V} = n_{sp} \cdot m_{sp} = Z \rho_{sp} = n_{sp} \cdot \rho_{sp} \cdot \tau_{sp}.$$

$Z$  represents the volume fraction of solid particles in the mixture. Further, volume fraction of solid particles is written as:

$$Z = \frac{V_{sp}}{V} = n_{sp} \cdot \tau_{sp}.$$

Gas or fluids have the following species density:

$$\rho_g = \frac{M_g}{V_g}.$$

Similarly, a gas's partial density is described as follows:

$$\bar{\rho}_g = \frac{M_g}{V} = (1 - Z)\rho_g.$$

Let's examine the conditions for thermodynamic equilibrium, like:

$$T_{sp} = T_g = T.$$

The mixture's density is determined by

$$\rho = Z\rho_{sp} + (1 - Z)\rho_g = \bar{\rho}_{sp} + \bar{\rho}_g.$$

The following formula yields the mass concentration of the solid particles pseudo fluid:

$$k_p = \frac{\bar{\rho}_{sp}}{\rho} = \frac{Z\rho_{sp}}{\rho}.$$

The pressure of the mixture is written as:

$$p = p_{sp} + p_g.$$

According to the perfect gas law,  $p$  is the total pressure of the mixture:

$$p = R\rho_g T_g.$$

With the help of the above analysis, the pressure of the whole mixture is obtained as follows:

$$p_m = p = R\rho_g T_g = R \left( \frac{\rho_m - Z\rho_{sp}}{1 - Z} \right) T_g = R\rho_m \left( \frac{1 - k_{sp}}{1 - Z} \right) T.$$

Therefore,  $p_m = \frac{\rho_m R_m T}{1 - Z}$ , where,  $R_m = (1 - k_p)R$ . In this case, subscript  $m$  denotes the value of the gas constant throughout the mixture, and  $R$  may be thought of as the mixture's effective gas constant.

### 1.1.5 Ideal and Non-ideal Gas

A theoretical model of an ideal gas simplifies the behavior of gases under certain conditions. In an ideal gas, molecules are considered point masses with no volume and do not interact with each other except through elastic collisions. The following law describes the behavior of an ideal gas:

$$PV = nRT,$$

where gas has a pressure  $P$ , a volume  $V$ , number of molecules  $n$ , and an ideal gas constant  $R$ , as well as an absolute temperature  $T$ . It describes most gases with low density, where molecules are normally spread out a long way from one another.

The expression of real gases is  $\lim_{P \rightarrow 0} PV/RT = 1$  with compressibility factor  $Z$  defined as

$$Z(P, T) = PV/RT.$$

The degree of departure from ideal behaviour can be inferred from the variation from unity. Two presumptions must be made to derive the ideal gas law theoretically: the molecules in the gas must be extremely tiny (having no volume) and they do not interact. However, the assumption that the gas is perfect is no longer true if the temperature of the gas is extremely high and the density is too low. The streamlined van der Waals model is a widely accepted substitute for the ideal gas. Dutch physicist van der Waals developed the van der Waals equation of state without using the ideal gas assumptions, which is expressed as follows

$$\left(P + \frac{na}{V^2}\right)(V - nb) = nRT,$$

where  $a$  and  $b$  are constants and determined experimentally and  $n$  is the number of molecules of the gas. Considering that the specific volume of a gas which is reciprocal to its density, the van der Waals equation of state, in terms of density  $\rho$ , may be expressed as

$$(P + na\rho^2)(1 - nb\rho) = n\rho RT.$$

The van der Waals model closely simulates real gas behavior for a wide range of temperatures and pressures. In the case of compressible flows, gas pressure is very high; consequently, the term  $a\rho^2$  is very small compared to the gas pressure  $P$ .

### 1.1.6 Chaplygin Gas

The Chaplygin gas is a theoretical cosmology and physics model proposed to explain dark energy and dark matter. The Chaplygin gas equation can describe a mathematical problem related to dark energy and dark matter. It is named after the Russian mathematician and physicist Sergey Alexeyevich Chaplygin, who introduced this concept in 1922. N. Ye heavily influenced Chaplygin's theories. Zhukovsky, who

also established the Central Institute of Aerodynamics. In his honour, the town of Chaplygin and the lunar crater Chaplygin are named.

The Chaplygin gas is a type of exotic matter with unusual properties. It is described by an equation of state that connects its pressure ( $p$ ) and energy density ( $\rho$ ) as follows:

$$p(\rho) = \sum_{j=1}^m A_j \rho^j - \frac{B}{\rho^\alpha},$$

where  $A_j$  and  $B$  are a positive constants. This is called the extended Chaplygin gas equation of state. By setting  $m = 2$ , it can recover barotropic fluid with quadratic equation of state. It reduces to modified Chaplygin gas (MCG) equation of state [12] as  $m = 1$ . Furthermore, we propose extended Chaplygin gas because higher  $m$  may recover higher-order barotropic fluid. The generalised Chaplygin gas equation of state is recovered for  $A_1 = 0$ , and the original Chaplygin gas equation of state is recovered in the combination of values  $A_1 = 0$  and  $\alpha = 1$ . Chaplygin gas is a hypothetical substance that fulfils an exotic equation of state in the form [13]

$$p(\rho) = \frac{-B}{\rho^\alpha}.$$

It appears in several cosmological theories. Sergey Chaplygin is honored in the substance's name. In some other models, Generalized Chaplygin gas is taken into account, where  $\alpha$  is a parameter with values ranging from  $0 < \alpha < -1$ .

The Chaplygin gas equation plays an important role in describing the universe's accelerated expansion and evolution of energy density perturbations in exotic background fluids.

## 1.2 Motivation

Partial differential equations (PDEs) describe various natural phenomena and complex physical processes in mathematics, science, and engineering. Hyperbolic PDEs have distinct characteristics that make them particularly well-suited for modeling and understanding such phenomena:

**Wave Propagation:** Hyperbolic PDEs are highly effective in describing the propagation of waves, which are fundamental to a wide range of natural phenomena. Examples include sound waves, light waves, water waves, and electromagnetic waves. By formulating in terms of hyperbolic PDEs, researchers can study the behavior of these waves as they travel through different mediums.

**Shock Waves and Discontinuities:** Hyperbolic PDEs are essential for capturing shock waves and other discontinuities in fluid dynamics, gas dynamics, and other physical systems. Shock waves occur when abrupt changes in properties, such as pressure or density, propagate through a medium. Understanding their formation and propagation is crucial for predicting and controlling dynamic processes.

**Fluid Dynamics and Gas Dynamics:** Hyperbolic PDEs play a significant role in modeling fluid flows and gas dynamics. They provide a framework for studying complex flow patterns, turbulence, and the interaction of different flow regimes, such as subsonic, transonic, and supersonic flows.

**Aerospace and Aerodynamics:** In aerospace engineering, hyperbolic PDEs are used to simulate the behavior of aircraft, rockets, and other vehicles traveling at high speeds. They help researchers analyze lift, drag, and other aerodynamic properties and the impact of shock waves on vehicle performance.

**Nonlinear Phenomena:** Modern mathematical research faces several challenges related to nonlinear hyperbolic conservation laws. When we deal with the hyperbolic system of PDEs, it is challenging to determine the exact solution to the problem.

Discontinuities, such as shocks, slip surfaces, etc., always appear in the solutions despite their smooth initial appearance. There are two ways to determine the solution of the hyperbolic system: analytical and numerical approaches. During the last decades, many analytical and numerical methods have been developed to study the nonlinear waves described by the hyperbolic system of PDEs. Several analytical techniques, such as the method of characteristics, progressive wave approach, wave-front analysis, perturbation method, differential constraint method, and many others, have been developed to understand the physical properties of the waves governed by a quasilinear hyperbolic system of PDEs. Also, various numerical methods such as finite difference, finite element, finite volume, and many other numerical techniques have been made to generalize the study of hyperbolic PDEs. Several books and monographs provide detailed analyses of mathematical properties, analytical and numerical methods, and applications of nonlinear wave propagation problems in hyperbolic systems of PDEs, including those by Courant and Friedrichs [1], Jeffrey [3], Zheng [4], Bressan [5], Dafermos [6], Sharma [7], Smoller [14], Holden and Risebro [15], LeVeque [16], Whitham [17] etc.

Overall, for studying hyperbolic partial differential equations lies in their ability to capture various physical systems' dynamic and wave-like behavior. These equations have applications across different scientific and engineering disciplines, providing valuable insight into fundamental processes and enabling advancements in technology and knowledge.

The detailed study of mathematical properties and applications of nonlinear wave propagation problems in the context of hyperbolic systems of PDEs may be found in the books and monographs by Courant and Friedrichs [18], Jeffrey [3], Whitham [17], Zheng [4], Bressan [5], Dafermos [6], Sharma [7], Smoller [14], Holden and Risebro [15] etc. Two significant obstacles are identified when solving a Cauchy problem

for hyperbolic conservation laws. The first is that after a finite amount of time, a solution might not be continuous, and the second is how to select an acceptable solution among several weak alternatives. Terms like proper wave, stable solution, physically relevant solution, and so on are presented to derive a unique solution to the Cauchy problem. According to Lax [2], the weak solution is admissible, and the conservation laws are completely explained.

Significant attention has been given to the Riemann Problem in recent decades due to its applications in various phenomena such as granular flow, shallow water flow, traffic flow, etc. The solution of hyperbolic systems of conservation laws always involves discontinuities, such as shocks, slip surfaces, etc. Consequently, discontinuous initial data should naturally be considered to smooth ones. The simplest discontinuous initial data problem is the Riemann Problem, which is an initial value problem. In 1860, Riemann took the lead and solved the problem of one-dimensional isentropic flow [9]. This invention is the pioneer of the shock wave in mathematical theory. This work is said to be the most important contribution to mathematical physics Riemann [19]. Courant and Friedrichs [1] extended Riemann's result to one-dimensional adiabatic flow. In the past few decades, many pieces of work have been done on the one-dimensional Riemann problem for various conservation laws. Corresponding to various physical processes Godunov [20] introduced a conservative extension of Courant, Isaacson, and Rees's [21] first-order upwind scheme to nonlinear hyperbolic conservation laws in 1959. The solution to the Riemann problem is the crucial content of the text. A theory has been established for the appropriate amplitude of Riemann data [2, 22] for strictly hyperbolic system and general Riemann data [14, 23, 24, 25, 26] for compressible Euler equations. The Riemann problem is widely acknowledged as the most fundamental in nonlinear hyperbolic conservation laws. Thus, it serves as a touchstone for numerical schemes due to

the explicit structure of the Riemann solution. For a detailed discussion on nonlinear hyperbolic system of conservation laws, we refer to Lax [2]; Jeffrey [3]; Ruggeri and Simic [27]; Liu [28]; Godlewski [29], and references cited therein. The theory of nonlinear hyperbolic conservation laws in one space dimension usually assumes that the system is strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields. Furthermore, general results on the existence of entropy-weak solutions to these systems are established only for initial values with small total variation (see Lax [2] and Glimm [30]). However, the standard theory of hyperbolic system of conservation laws must apply to most physical systems. If initial data has large total variation, it is natural to ask whether these results remain true for a non-strictly hyperbolic system, as well as for a strictly hyperbolic system. Keyfitz and Kranzer [31] considered the Riemann problem for one strictly hyperbolic system with nonlinear characteristic fields. They found that for extensive initial data, the Riemann problem may admit non classical wave solution (which consists of contact discontinuities, shock, and/or rarefaction waves). Both the Lax-Friedrichs and Glimm methods have been influential in advancing the field of numerical analysis and computational fluid dynamics. They provide ways to approximate solutions to complex system of equations involving hyperbolic partial differential equations, which often have no analytical solution or can be difficult to solve directly.

### 1.3 Literature Review

A review of literature on the study of the propagation of shock waves in gas dynamics is given in this section.

The phenomenon of shock waves is mainly associated with aerospace engineering and in particular with the supersonic flight. The development of this particular

branch of physics began in 1746 when a mathematician Robins determined velocity of the bullet by ballistic pendulum and noticed a growth in aerodynamic drag as velocity tends to the sound speed. However, in the 19th century, the phenomenon of the shock wave was still a mystery to many researchers. In 1759, without mentioning the word shock waves, Euler talked about the “size of disturbance” of a sound wave meaning its amplitude. Nevertheless, his assumption that velocity would diminish with increasing amplitude was incorrect. In 1808, Poisson [32] was the first researcher to solve the Euler equation for the one-dimensional unsteady fluid-flow and got the exact solution. In 1823, Poisson created a milestone in nonlinear wave theory by constructing isentropic gas law for the sound wave with infinitesimal amplitude. In 1848, Stokes [33] used the term “surface of discontinuity”. He further extended the theory of acoustic wave, having a finite amplitude, by considering the problem of wave steepening. Stokes was not sure about the possibility of discontinuous motion, in this confusion he used isentropic relation which decides the role of dissipation and energy conservation in shock formation, instead of the correct energy equation. He derived the conservation laws for mass and momentum which are used very frequently in modern days. In 1889, Hugoniot [34] independently derived the correct jump conditions for the shock waves. The formulated theory of Rankine and Hugoniot is, even at present also, the basic model for the propagation of shock waves.

The renowned mathematician G. F. B. Riemann’s (1859) ”theory of waves of finite amplitude,” which applied to the computation of the propagation of planar waves of finite amplitude flowing in both directions and was not restricted to a single progressive wave, served as the foundation for the investigation of the Riemann problem. In 1860, Riemann [9] introduced the Riemann problem for a system of conservation laws in gas dynamics. For the condition in which the initial data of the problem consists of two constant states  $U_*^1$  and  $U_*^2$ , Lax [2] solved the Riemann problem for

the condition with two constant states  $U_*^1$  and  $U_*^2$  being, respectively, the vectors of conserved variables to either side of  $x = 0$  such that  $\|U_*^1 - U_*^2\|$  is approximately small, where the left and right constant states are divided at  $x = 0$  by a jump discontinuity. The Riemann problem, when considering Euler equations, is the shock tube problem. Courant and Friedrichs' [18] book should be studied to thoroughly understand shock tube problem and other physical problems represented by gas-dynamics conservation laws.

Riemann's problem has great significance in terms of its exact solution. In other words, it serves as the fundamental building block of various solutions to general initial value problems using Glimm's [30] random choice method. The exact solution of the Riemann problem for hyperbolic system of conservation laws are proposed by many authors, for detailed methodologies, the readers are referred to the book by [8, 35, 36, 37]. Also, Chorin in [38, 39] proposed the new approach to obtain the exact solution to the Riemann problem. Another improvement to the Godunov's first Riemann solver was presented in [40].

We also examine an article by Liu [28] and the books of Dafermos [6], Bressan [5] and LeVeque [16] for a comprehensive study of the Riemann problem. In Euler equations, a simple wave is the special solution with one of the Riemann invariants remaining constant. Introducing shock waves in simple wave solution is necessary since waves break. Entropy and Riemann invariants have been observed to jump remarkably little when shock strength is small or moderate, see Whitham [17]. Friedrichs [1] developed an approximate theory of shocks with weak or moderate strengths based on this, in which the actual shock conditions are replaced with an equivalent simple compression wave.

It is observed that in the compressible fluid flow, if the fluid's speed is greater than the sound speed, then shocks may form when particles collide. For gas dynamics under small pressure, it has been observed numerically (Chang et al. [41]) that,

for instance, in one case, the particles appear sticky and tend to concentrate at the shock location, which moves at the speed of the shock; for example, in the region of rarefaction waves, the particles seem to be far apart and tend to form cavitation. Such phenomena are a tendency towards concentration and cavitation in terms of density. Using the vanishing pressure limit, Chen and Liu [42] rigorously explained concentration and cavitation for isentropic fluids. In recent decades, the problems regarding concentration, cavitation phenomena, the formation of delta shock waves, and vacuum states in solution have become of much more attention ([43, 44, 45, 46, 47, 48]). Using the vanishing pressure technique, the development of the delta shock wave and vacuum state for pressureless Euler systems was examined for the isentropic, isothermal, and non-isentropic fluids of Chen and Liu [42], [49]. See also Li [50] for the study of isothermal Euler equations with zero temperature. The vanishing pressure limit approach has been widely used and the results were extended for the relativistic Euler equations by Yin and Sheng [51], the perturbed Aw-Rascle model by Shen and Sun [52], the modified Chaplygin gas equations by Yang and Wang [53, 54], the Aw-Rascle model of traffic flow by Liu [55] etc. It is clear that the flux approximation method is indeed a natural generalization of the vanishing pressure limit approach.

Moreover, in the literature, a Riemann problem is analyzed by initial constant data, which is the equilibrium state of the governing system. The non-constant initial data characterize a Generalised Riemann problem (GRP). Finding precise solutions to a generalized Riemann problem with non-constant initial data with a discontinuity at  $x = 0$  is a more difficult task. Many mathematical techniques have been presented in recent decades to obtain precise solutions to PDEs. For example, similarity transformation, perturbation, differential constraint, etc., which lead to exact-type or approximate solutions. Curro et al. [56] presented a methodical reduction technique to produce specific classes of solutions for quasilinear hyperbolic problems. Fusco

and Manganaro [57] described the precise solution to the rate-type material model by reduction technique utilizing the differential constraint approach. By applying the Differential Constraints approach, Curro et al. [58, 59] have successfully solved the Generalised Riemann problem for the traffic flow model. This approach offers a systematic method to find the exact solution to the system of quasilinear PDEs. In gasdynamics, this technique was introduced by Janenko [60]. Many contributions on the generalized Riemann problem have been given concerning existence and uniqueness theorems. Specifically, it can be demonstrated that the behavior of the classical solution to the Riemann problem in the neighbourhood of the origin is the same as that of the solution to a generalized Riemann problem. We also refer readers to the articles [61, 62, 63, 64, 65, 66] for further information on the generalized Riemann problem.

## 1.4 Aims and Thesis Objectives

The present thesis deals with nonlinear wave propagation problems governed by the quasilinear hyperbolic system of partial differential equations. The main objective of this work is to determine the analytical solution of the Euler system of PDEs and examine the behavior of propagating waves in certain medium. In this investigation, we have used some analytical techniques, such as the wavefront analysis method and vanishing pressure limit method, to determine the solution of the one-dimensional and two-dimensional system of PDEs. Also, we study wave interactions and stability of the Riemann solution of hyperbolic systems. We have also constructed the solution to the one-dimensional Riemann Problem under the influence of the external force. We are motivated to solve the problem of the non-homogeneous hyperbolic system, which is modified into the homogeneous hyperbolic system of conservation laws, to

study the solution of the Riemann problem with constant initial data by introducing a new variable for the velocity. This thesis aims to generalize the wave propagation process described by hyperbolic system of PDEs by using some analytical approach and examine the solution of the one-dimensional Riemann problem in the presence of external force. In order to address the entire work of the thesis, we have been motivated to work on the following objectives:

1. To study shock waves in a two-dimensional supersonic planar and axisymmetric non-ideal gas flow with magnetic field.
2. To determine the effect of dust particles on the evolution of weak discontinuity in the two-dimensional supersonic flow of van der Waals gas.
3. The analytical Solution of a Hyperbolic system of PDEs with the magnetic field.
4. To Investigate the phenomenon of the concentration and cavitation in the Riemann solutions for a non-homogeneous logarithmic equation of state with the magnetic field.
5. To determine the structure of the solution of the Riemann problem for the one-dimensional compressible hyperbolic system under the influence of external force.

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