Chapter 4
PROPOSED TECHNIQUES FOR THE REDUCTION OF CONTINUOUS TIME INTERVAL SYSTEMS BASED ON MIXED METHODS

4.1 INTRODUCTION

In the previous chapter we have given an overview of various Routh based techniques for model order reduction of interval systems. Some inherent drawbacks associated with Routh based methods are:

1. They may lead to unstable approximation for a stable system.
2. Poor transient response matching.
3. Poor high frequency response matching.
4. Non- minimum phase problems.

In this chapter some new mixed methods are proposed for reduction of Continuous time interval systems. These methods are conceptually simple and give good approximation in both the transient and steady state periods.

4.2 PROBLEM FORMULATION

Let the transfer function of a higher order continuous interval systems be

$$G_n(s) = \frac{[b_{1,1}^-, b_{1,1}^+] + [b_{1,2}^-, b_{1,2}^+]s + ... + [b_{1,n}^-, b_{1,n}^+]s^{n-1}}{[a_{1,1}^-, a_{1,1}^+] + [a_{1,2}^-, a_{1,2}^+]s + ... + [a_{1,n+1}^-, a_{1,n+1}^+]s^{n}} = \frac{\sum_{i=1}^{n} \left[b_{1i}^-, b_{1,i}^+ \right] s^{i-1}}{\sum_{i=1}^{n+1} \left[a_{1i}^-, a_{1,i}^+ \right] s^{i}} = \frac{N(s)}{D(s)} \qquad (4.1)$$

where $[b_{1,i}^-, b_{1,i}^+]$ for i = 1 to n and $[a_{1,i}^-, a_{1,i}^+]$ for i = 1 to n+1 are the uncertain parameters.

The reduced order model $R_k(s)$ is expressed as

$$R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{[d_{1,1}^-, d_{1,1}^+] + [d_{1,2}^-, d_{1,2}^+]s + ... + [d_{1,k}^-, d_{1,k}^+]s^{k-1}}{[c_{1,1}^-, c_{1,1}^+] + [c_{1,2}^-, c_{1,2}^+]s + ... + [c_{1,k+1}^-, c_{1,k+1}^+]s^{k}} \qquad (4.2)$$

where k = 1 to n-1

The rules of the interval arithmetic have been defined in Chapter 1.

4.2.1 INTEGRAL SQUARE ERROR

The integral square error (ISE) between the transient response of higher order system (HOS) and lower order system (LOS) is determined to compare different approaches of model reduction

$$ISE = \int_0^\infty \left[y(t) - y_r(t) \right]^2 dt \tag{4.3}$$

where, $y(t)$ and $y_r(t)$ are the unit step responses of original system and reduced order system.

4.2.2 INTEGRAL ABSOLUTE ERROR

The accuracy of the proposed method also verified by using absolute magnitude error index in between transient parts of the original and reduced models

$$IAE = \int_0^\infty \left| \left[y(t) - y_r(t) \right] \right| dt \tag{4.4}$$

where, $y(t)$ and $y_r(t)$ are the unit step responses of original system and reduced order system.

4.3 MIHAILOV CRITERION APPROXIMATION

The Mihailov stability criterion an advanced technique for the general case of the Pade approximation method. In this method, several reduced models can be obtained depending upon the different values of the constant in the reduced model and bring the Mihailov frequency characteristic of the reduced model to approximate that of the original system at the low frequency region.

4.3.1 MIHAILOV CRITERION AND ROUTH APPROXIMATION METHOD (MCRAM)

The denominator of the polynomial is reduced by Mihailov criterion and numerator polynomial is reduced by Routh approximation

Step 1: Determination of the denominator polynomial of the k^{th} order reduced model obtain by Mihailov Criterion

Substituting $s = j\omega$ in $D_n\left(s\right)$ and separating the denominator into real and imaginary parts

$$\begin{aligned} D\left(j\omega\right) &= \left[a_{11}^{-}, a_{11}^{+}\right] + \left[a_{12}^{-}, a_{12}^{+}\right]s + \left[a_{13}^{-}, a_{13}^{+}\right]s^2 + \dots\dots + \left[a_{1,n+1}^{-}, a_{1,n+1}^{+}\right]s^n \\ D\left(j\omega\right) &= \left(\left[a_{11}^{-}, a_{11}^{+}\right] - \left[a_{13}^{-}, a_{13}^{+}\right]\omega^2 + \dots\dots\right) + j\omega\left(\left[a_{12}^{-}, a_{12}^{+}\right] - \left[a_{14}^{-}, a_{14}^{+}\right]\omega^2 + \dots\dots\right) \\ D\left(j\omega\right) &= \xi\left(\omega\right) + j\omega\eta\left(\omega\right) \end{aligned} \tag{4.5}$$

where $s = j\omega$, ω is the angular frequency, rad/sec

Step 2: $\xi\left(\omega\right) = 0$ and $\eta(\omega) = 0$ the frequencies which are intersecting a

$\left\|\left[\omega_1^{-}, \omega_1^{+}\right]\right\| \langle \left\|\left[\omega_2^{-}, \omega_2^{+}\right]\right\| \langle \dots\dots \langle \left\|\left[\omega_{n-1}^{-}, \omega_{n-1}^{+}\right]\right\|$ are obtained,

where $\omega_0 = 0, \pm\left[\omega_1^{-}, \omega_1^{+}\right], \dots\dots\dots, \pm\left[\omega_{n-1}^{-}, \omega_{n-1}^{+}\right]$

Step 3: Similarly substituting $s = j\omega$ in $D_k\left(s\right)$, then obtains

$$D_k\left(j\omega\right) = \phi\left(\omega\right) + j\omega\psi\left(\omega\right)$$

where

$$\begin{aligned} \phi\left(\omega\right) &= \left[c_{11}^{-}, c_{11}^{+}\right] - \left[c_{13}^{-}, c_{13}^{+}\right]\omega^2 + \dots\dots \\ \psi\left(\omega\right) &= \left[c_{12}^{-}, c_{12}^{+}\right] - \left[c_{14}^{-}, c_{14}^{+}\right]\omega^2 + \dots\dots \end{aligned} \tag{4.6}$$

Step 4: Putting $\phi\left(\omega\right) = 0$ and $\psi\left(\omega\right) = 0$ then we get k number of roots and it must be positive real and alternately distributed along ω axis. The first k numbers of frequencies are $0,\left[\omega_1^-,\omega_1^+\right],\left[\omega_2^-,\omega_2^+\right],.......,\left[\omega_{r-1}^-,\omega_{r-1}^+\right]$ kept unchanged and the roots of $\phi(\omega)=0$ and $\psi(\omega)=0$.

$$\phi(\omega)=\left[\lambda_1^-,\lambda_1^+\right]\left(\omega^2-\left[\left(\omega_1^-\right)^2,\left(\omega_1^+\right)^2\right]\right)\left(\omega^2-\left[\left(\omega_3^-\right)^2,\left(\omega_3^+\right)^2\right]\right).........$$

$$\psi(\omega)=\left[\lambda_2^-,\lambda_2^+\right]\left(\omega^2-\left[\left(\omega_2^-\right)^2,\left(\omega_2^+\right)^2\right]\right)\left(\omega^2-\left[\left(\omega_4^-\right)^2,\left(\omega_4^+\right)^2\right]\right)........ \tag{4.7}$$

Step 5: For finding the coefficient values of $\left[\lambda_1^-,\lambda_1^+\right]$ and $\left[\lambda_2^-,\lambda_2^+\right]$ are calculated from $\xi(0)=\phi(0)$ and $\eta\left(\left[\omega_1^-,\omega_1^+\right]\right)=\psi\left(\left[\omega_1^-,\omega_1^+\right]\right)$ keeping these values of $\left[\lambda_1^-,\lambda_1^+\right]$ and $\left[\lambda_2^-,\lambda_2^+\right]$ in the above equations respectively $\phi(\omega)$ and $\psi(\omega)$ are obtained as

$$D_k(j\omega)=\phi(\omega)+j\omega\psi(\omega) \tag{4.8}$$

Now replace $j\omega$ by s and then the k^{th} reduced order denominator interval polynomial is

$$D_k(s)=\left[c_{11}^-,c_{11}^+\right]+\left[c_{12}^-,c_{12}^+\right]s+....+\left[c_{1,k+1}^-,c_{1,k+1}^+\right]s^k \tag{4.9}$$

Step 6: Routh approximation is applied to obtain reduced order numerator polynomial Here the numerator does not depend on reduced denominator coefficients

Table 4.1: Routh Table for numerator polynomial for method MCRA

$d_{11}=\left[b_{n-1}^-,b_{n-1}^+\right]$	$d_{12}=\left[b_{n-3}^-,b_{n-3}^+\right]$	$d_{13}=\left[b_{n-5}^-,b_{n-5}^+\right]$	$d_{16}=\left[b_{n-7}^-,b_{n-7}^+\right]$	......
$d_{21}=\left[b_{n-2}^-,b_{n-2}^+\right]$	$d_{22}=\left[b_{n-4}^-,b_{n-4}^+\right]$	$d_{23}=\left[b_{n-6}^-,b_{n-6}^+\right]$	$d_{24}=\left[b_{n-8}^-,b_{n-8}^+\right]$	......
d_{31}	d_{32}	d_{33}	d_{34}	......
d_{41}	d_{42}	d_{43}	d_{44}	......

d_{51}	d_{52}	d_{53}	. . . .	
. . . .	. . . .	. . . .		
. . . .	. . . .	. . . .		
d_{n1}				

$$d_{ij}=\left[d_{ij}^{-},d_{ij}^{+}\right]=\frac{\left(\left[d_{i-2,j+1}^{-},d_{i-2,j+1}^{+}\right].\left[d_{i-1,1}^{-},d_{i-1,1}^{+}\right]\right)-\left(\left[d_{i-2,1}^{-},d_{i-2,1}^{+}\right].\left[d_{i-1,j+1}^{-},d_{i-1,j+1}^{+}\right]\right)}{\left[d_{i-1,1}^{-},d_{i-1,1}^{+}\right]} \tag{4.10}$$

where $i\geq 3;1\leq j\leq\left[\frac{(n-i+3)}{2}\right]$

$$R_k(s)=\frac{N_k(s)}{D_k(s)} \tag{4.11}$$

4.3.2 MIHAILOV CRITERION AND FACTOR DIVISION METHOD (MCFDM)

The denominator of the polynomial is reduced by Mihailov criterion and numerator polynomial is reduced by factor division method

FACTOR DIVISION METHOD

Determination of the numerator coefficients of the reduced order model by using factor division method: Any method of reduction which relies upon calculating the reduced denominator first and then the numerator, where has reduced denominator has already been calculated by Mihailov criterion.

$$G_n(s) = \frac{N(s)D_k(s) \big/ D(s)}{D(s)}$$

$$N(s)D_k(s) = \left[u_{11}^-, u_{11}^+\right] + \left[u_{12}^-, u_{12}^+\right]s + ...$$

$$\frac{N(s)D_k(s)}{D(s)} = \frac{\left[u_{11}^-, u_{11}^+\right] + \left[u_{12}^-, u_{12}^+\right]s + ...}{\left[a_{11}^-, a_{11}^+\right] + \left[a_{12}^-, a_{12}^+\right]s + ...} \tag{4.12}$$

$$\left[\alpha_{11}^-, \alpha_{11}^+\right] = \frac{\left[u_{11}^-, u_{11}^+\right]}{\left[a_{11}^-, a_{11}^+\right]} \left\{ \begin{matrix} \left[u_{11}^-, u_{11}^+\right]\left[u_{12}^-, u_{12}^+\right]..... \\ \left[a_{11}^-, a_{11}^+\right]\left[a_{12}^-, a_{12}^+\right]..... \end{matrix} \right\}$$

$$\left[\alpha_{12}^-, \alpha_{12}^+\right] = \frac{\left[r_{11}^-, r_{11}^+\right]}{\left[a_{11}^-, a_{11}^+\right]} \left\{ \begin{matrix} \left[r_{11}^-, r_{11}^+\right]\left[r_{12}^-, r_{12}^+\right]..... \\ \left[a_{11}^-, a_{11}^+\right]\left[a_{12}^-, a_{12}^+\right]..... \end{matrix} \right\} \tag{4.13}$$

....

$$\left[\alpha_{1,k-2}^-, \alpha_{1,k-2}^+\right] = \frac{\left[x_{11}^-, x_{11}^+\right]}{\left[a_{11}^-, a_{11}^+\right]} \left\{ \begin{matrix} \left[x_{11}^-, x_{11}^+\right]\left[x_{12}^-, x_{12}^+\right] \\ \left[a_{11}^-, a_{11}^+\right]\left[a_{12}^-, a_{12}^+\right] \end{matrix} \right\}$$

$$\left[\alpha_{1,k-1}^-, \alpha_{1,k-1}^+\right] = \frac{\left[y_{11}^-, y_{11}^+\right]}{\left[a_{11}^-, a_{11}^+\right]} \left\{ \begin{matrix} \left[y_{11}^-, y_{11}^+\right] \\ \left[a_{11}^-, a_{11}^+\right] \end{matrix} \right\}$$

$$\left[r_{1,i}^-, r_{1,i}^+\right] = \left[u_{1,i+1}^- u_{1,i+1}^+\right] - \left[\alpha_{11}^-, \alpha_{11}^+\right]\left[a_{1,i+1}^-, a_{1,i+1}^+\right]; \text{where i} = \ 0,\ 1,\ 2,\ ...\ \text{k} - 2$$

$$\left[s_{1,i}^-, s_{1,i}^+\right] = \left[r_{1,i+1}^- r_{1,i+1}^+\right] - \left[\alpha_{12}^-, \alpha_{12}^+\right]\left[a_{1,i+1}^-, a_{1,i+1}^+\right]; \text{where i} = \ 0,\ 1,\ 2,\ ...\ \text{k} - 3$$

....

....

$$\left[y_{1,i}^-, y_{1,i}^+\right] = \left[x_{11}^-, x_{11}^+\right] - \left[\alpha_{1,k-2}^-, \alpha_{1,k-2}^+\right]\left[a_{11}^-, a_{11}^+\right]$$

The reduced transfer function is

$$R_k(s) = \frac{\left[\alpha_{11}^-, \alpha_{11}^+\right] + \left[\alpha_{12}^-, \alpha_{12}^+\right]s + ...}{D_k(s)} \tag{4.14}$$

4.3.3 MIHAILOV CRITERION AND CAUER SECOND FORM (MCCSF)

The denominator of the polynomial is reduced by Mihailov criterion and numerator polynomial is reduced by cauer second form

CAUER SECOND FORM

Determination of numerator coefficients of reduced order model by using Cauer second form

Coefficient values from Cauer second form $\left[h_i^-, h_i^+\right]$, where $i = 1, 2, 3, ..., k$ are evaluated by forming Routh array as

$$\left[h_1^-, h_1^+\right] = \frac{\left[c_{11}^-, c_{11}^+\right]}{\left[c_{21}^-, c_{21}^+\right]} \begin{Bmatrix} \left[c_{11}^-, c_{11}^+\right]\left[c_{12}^-, c_{12}^+\right]..... \\ \left[c_{21}^-, c_{21}^+\right]\left[c_{22}^-, c_{22}^+\right]..... \end{Bmatrix} \tag{4.15}$$

$$\left[h_2^-, h_2^+\right] = \frac{\left[c_{21}^-, c_{21}^+\right]}{\left[c_{31}^-, c_{31}^+\right]} \begin{Bmatrix} \left[c_{21}^-, c_{21}^+\right]\left[c_{22}^-, c_{22}^+\right]..... \\ \left[c_{31}^-, c_{31}^+\right]\left[c_{32}^-, c_{32}^+\right]..... \end{Bmatrix}$$

....

The first two rows are copied from the original system numerator and denominator coefficients and rest of the elements are calculated by using well known Routh approximation

$$\left[c_{i,j}^-, c_{i,j}^+\right] = \left[c_{i-2,j+1}^-, c_{i-2}^+\right] - \left[h_{i-2}^-, h_{i-2}^+\right]\left[c_{i-1,j+1}^-, c_{i-1,j+1}^+\right] \tag{4.16}$$

where $i = 3, 4....$ and $j = 1, 2, ...$

$$\left[h_i^-, h_i^+\right] = \frac{\left[c_{i,1}^-, c_{i,1}^+\right]}{\left[c_{i+1,1}^-, c_{i+1,1}^+\right]} \tag{4.17}$$

The inverse Routh array is constructed as

$$\left[d_{i+1,1}^{-},d_{i+1,1}^{+}\right]=\frac{\left[d_{i,1}^{-},d_{i,1}^{+}\right]}{\left[h_{i}^{-},h_{i}^{+}\right]} \tag{4.18}$$

where $i=1,2,3,.....,k$ and $k\le n.$

also

$$\left[d_{i+1,j+1}^{-},d_{i+1,j+1}^{+}\right]=\frac{\left[d_{i,j+1}^{-},d_{i,j+1}^{+}\right]-\left[d_{i+2,j}^{-},d_{i+2,j}^{+}\right]}{\left[h_{i}^{-},h_{i}^{+}\right]} \tag{4.19}$$

where $i=1,2,3,.....,k$ and $k\le n$ and $j=1,2,3,...,k$

$$N_k\left(s\right)=\left[d_{21}^{-},d_{21}^{+}\right]+\left[d_{22}^{-},d_{22}^{+}\right]s+......+\left[d_{2k}^{-},d_{2k}^{+}\right]s^{k-1} \tag{4.20}$$

Example 4.1: Consider a third order system described by the transfer function [157]

$$G_3(s)=\frac{[2,3]s^2+[17.5,18.5]s+[15,16]}{[2,3]s^3+[17,18]s^2+[35,36]s+[20.5,21.5]} \tag{4.21}$$

Case1: MCRAM

Step 1: Put $s=j\omega$ in the denominator of the higher order polynomial

$$D\left(j\omega\right)=\left(\left[20.5,21.5\right]-\left[17,18\right]\omega^2\right)+j\omega\left(\left[35,36\right]-\left[2,3\right]\omega^2\right) \tag{4.22}$$

Step 2: The intersecting frequencies are

$$\left[\omega_i^{-},\omega_i^{+}\right]=0,\left[1.0929,1.0981\right],\left[3.4641,4.1833\right] \tag{4.23}$$

Step 3: The denominator of the second order model is taken as

$$D_2\left(j\omega\right)=\left[\lambda_1^{-},\lambda_1^{+}\right]\left(\omega^2-\left[1.0929,1.0981\right]^2\right)+j\omega\left[\lambda_2^{-},\lambda_2^{+}\right] \tag{4.24}$$

Step 4: Substitute the values $\left[\lambda_1^-, \lambda_1^+\right] = -\left[17.0011, 18.0007\right]$ and $\left[\lambda_2^-, \lambda_2^+\right] = \left[31.3826, 33.6111\right]$, The reduced denominator is

$$D_2\left(s\right) = \left[17.0011, 18.0007\right]s^2 + \left[31.3826, 33.6111\right]s + \left[20.3061, 21.7052\right] \qquad (4.25)$$

Step 5: Construct Routh table for higher order numerator for obtaining reduced order numerator polynomial

Table 4.2: Routh Table for numerator polynomial of example 4.1

$\left[2,3\right]$	$\left[15,16\right]$
[17.5,18.5]	
[14.1892,16.9143]	

Step 6: Numerator of the reduced order polynomial

$$N_2\left(s\right) = \left[17.5, 18.5\right]s + \left[14.1892, 16.9143\right] \qquad (4.26)$$

Step7: The second order reduced order model

$$R_2\left(s\right) = \frac{\left[17.5, 18.5\right]s + \left[14.1892, 16.9143\right]}{\left[17.0011, 18.0007\right]s^2 + \left[31.3826, 33.6111\right]s + \left[20.3061, 21.7052\right]} \qquad (4.27)$$

The step response and impulse response of second order model and original interval model are compared in the Fig. 4.1, 4.2, 4.3 and 4.4.

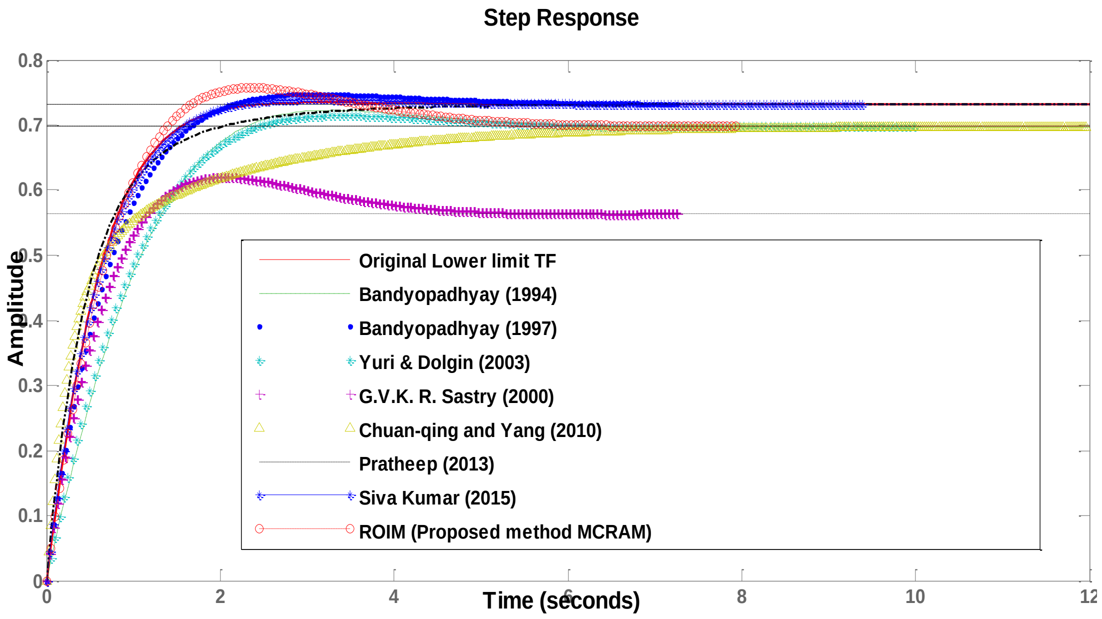

Fig.4.1 Step Response for Mihailov Criterion and Routh Approximation (Lower limit TF)

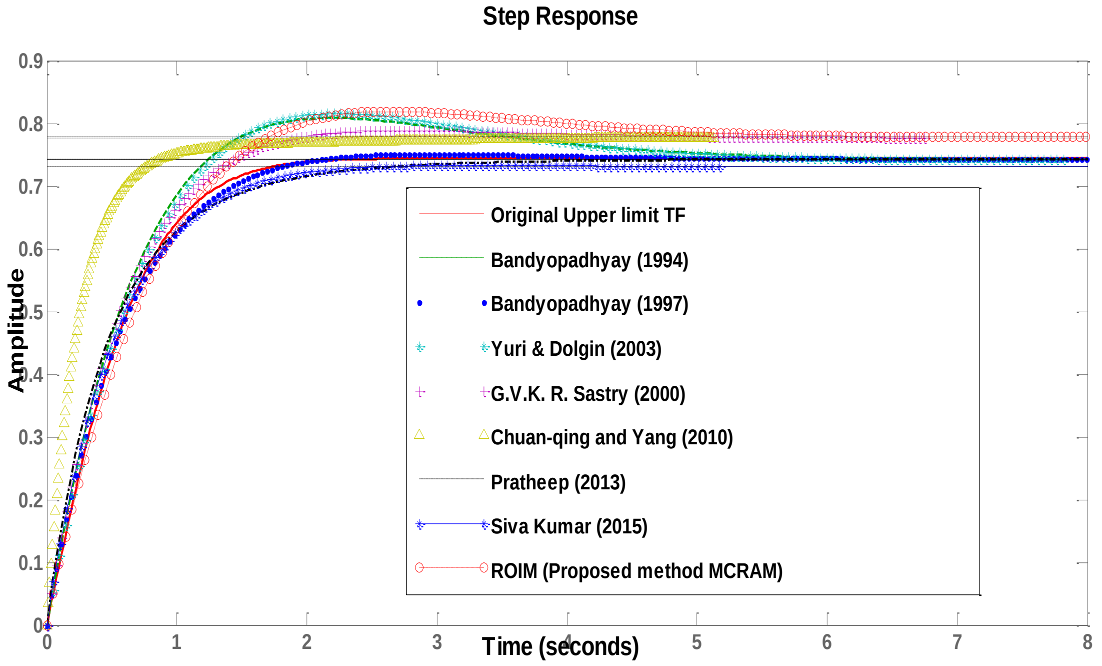

Fig.4.2 Step Response for Mihailov Criterion and Routh Approximation (Upper limit TF)

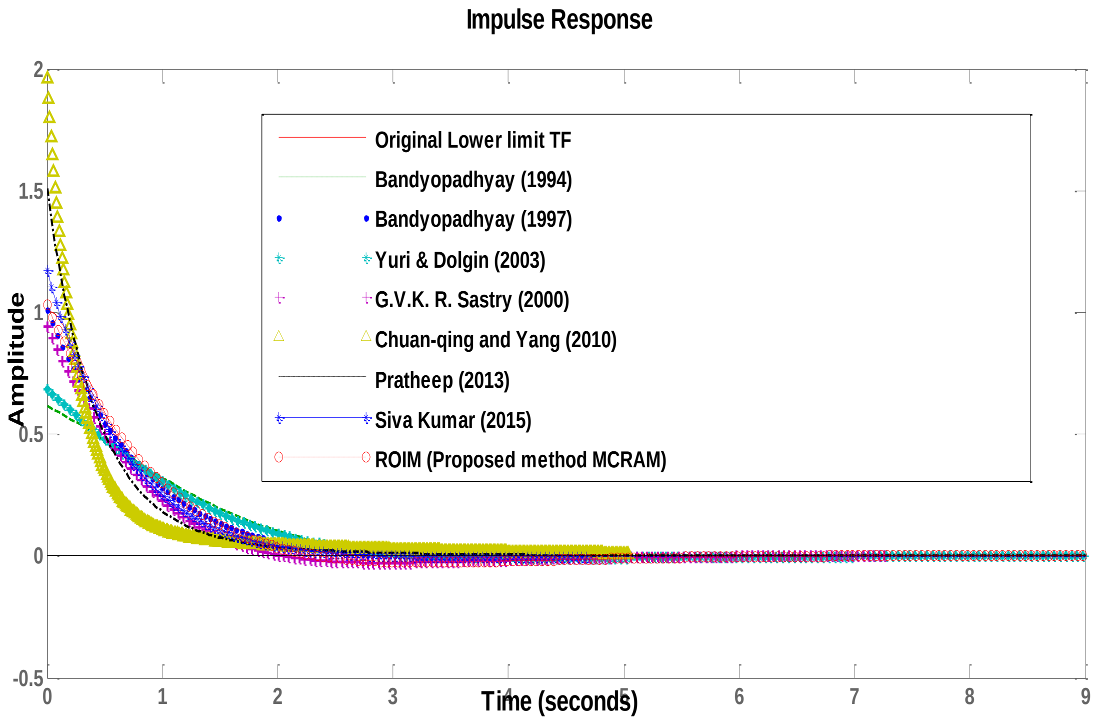

Fig.4.3. Impulse response for Mihailov Criterion and Routh Approximation (Lower limit TF)

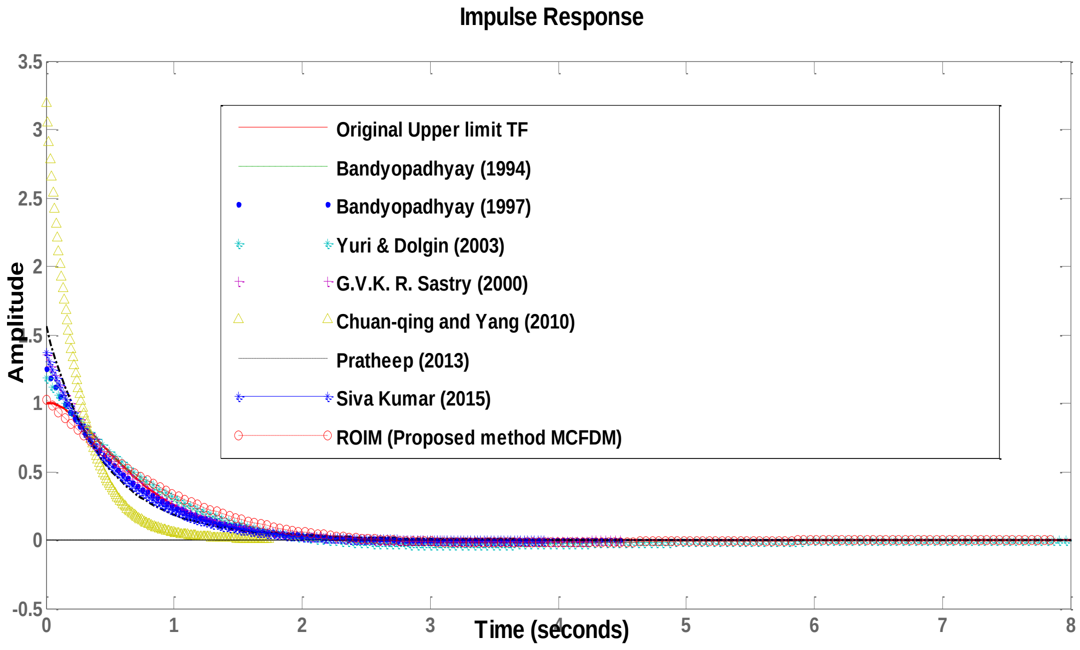

Fig.4.4. Impulse response for Mihailov Criterion and Routh Approximation (Upper limit TF)

The bode plot of the high-order system and reduced order models by proposed method is shown in Fig. 4.5 and Fig. 4.6. A comparison has been made with the existing methods

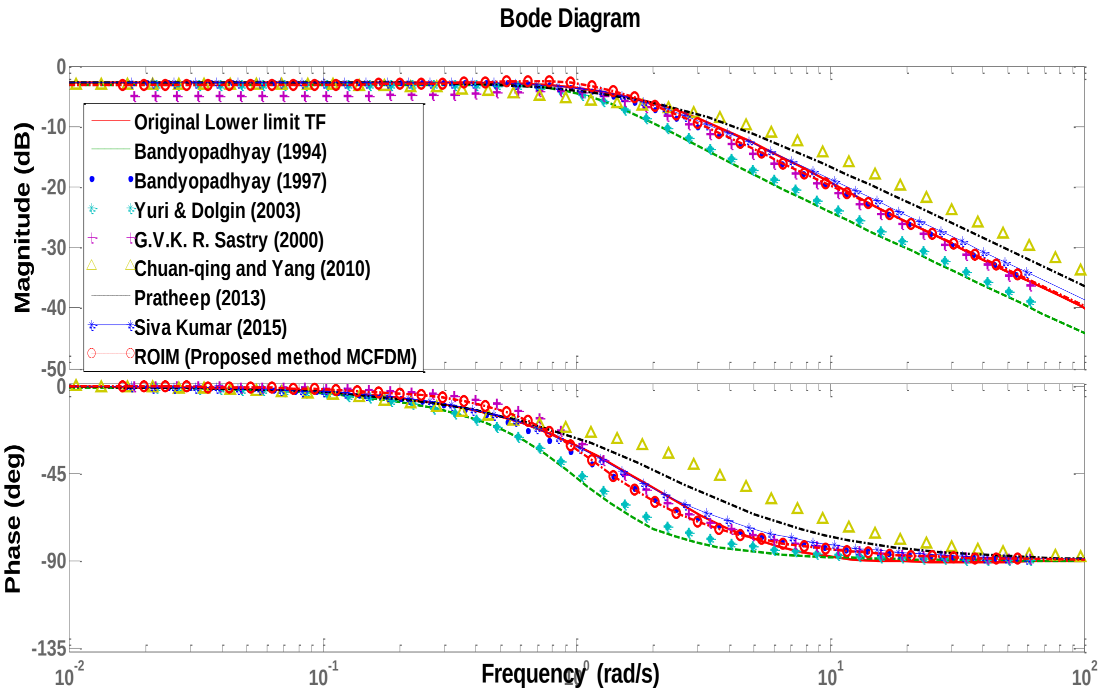

Fig.4.5. Bode plot for Mihailov Criterion and Routh Approximation (Lower limit TF)

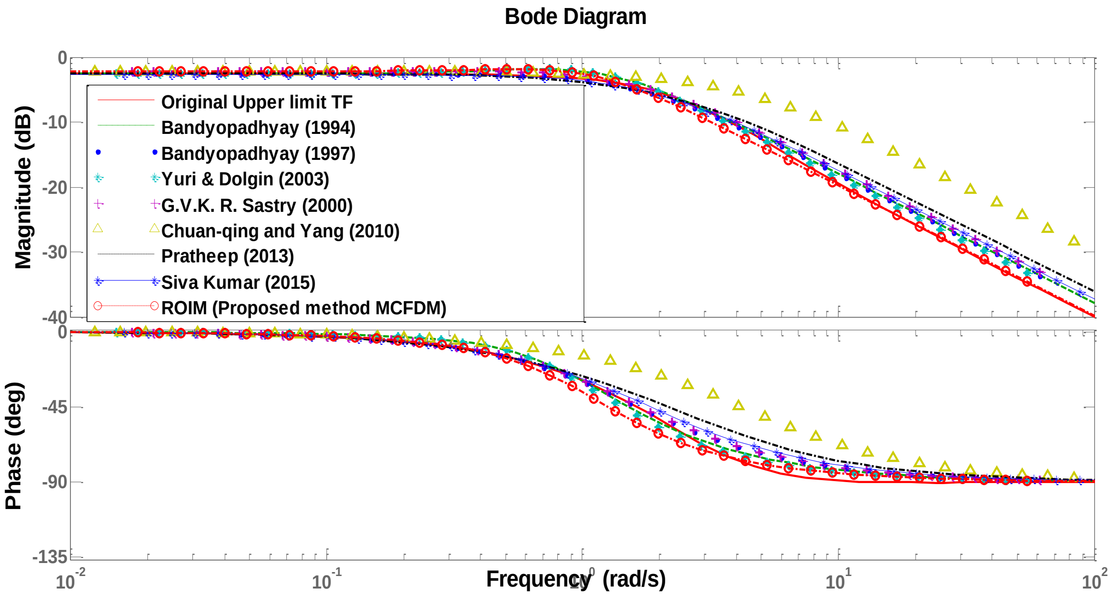

Fig.4.6. Bode plot for Mihailov Criterion and Routh Approximation (Upper limit TF)

The nyquist plot of the high-order system and reduced order models by proposed method is shown in Fig. 4.7 and Fig. 4.8. A comparison has been made with the existing methods.

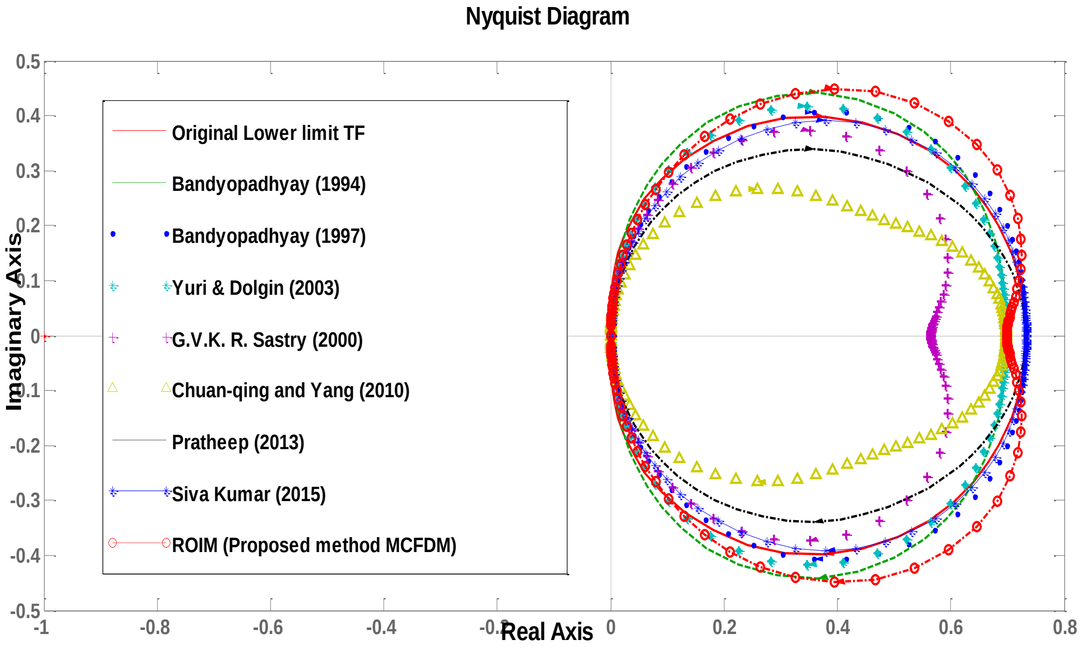

Fig.4.7. Nyquist plot for Mihailov Criterion and Routh Approximation (Lower limit TF)

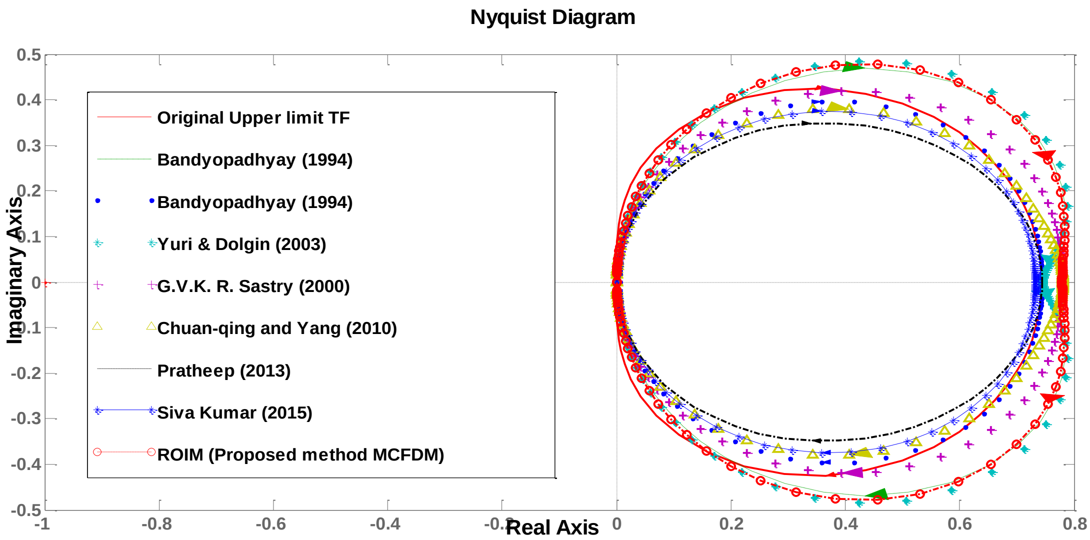

Fig.4.8. Nyquist plot for Mihailov Criterion and Routh Approximation (Upper limit TF)

Case2: MCFDM

Step 1: The denominator of the second order interval polynomial is

$$D_2(s)=[17.0011,18.0007]s^2+[31.3826,33.6111]s+[20.3061,21.7052] \tag{4.28}$$

Step 2: Factor division method is used to obtain reduced numerator polynomial

$$\frac{N(s)D_2(s)}{D(s)}=\frac{[304.5915,347.2832]+[826.0957,939.3238]s+.......}{[20.3061,21.7052]+[31.3826,33.6111]s+[17.0011,18.0007]s^2} \tag{4.29}$$

Step 3: Finding the values of numerator coefficients

$$[\alpha_{11}^-,\alpha_{11}^+]=[14.0331,17.1024];[\alpha_{12}^-,\alpha_{12}^+]=[35.6065,49.4454]$$

$$N_2(s)=[35.6065,49.4454]s+[14.0331,17.1024] \tag{4.30}$$

Step 4: The second order reduced interval model is

$$R_2(s)=\frac{[35.6065,49.4454]s+[14.0331,17.1024]}{[17.0011,18.0007]s^2+[31.3826,33.6111]s+[20.3061,21.7052]} \tag{4.31}$$

The step response of the high-order system and reduced order models by proposed method is shown in Fig. 4.9 and Fig. 4.10. A comparison has been made with the existing methods

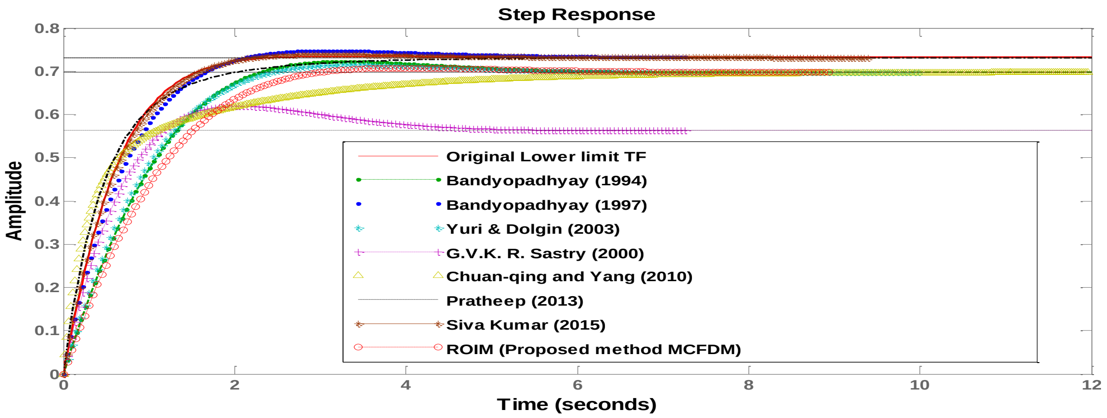

Fig.4.9. Step response for Mihailov Criterion and Factor division method (lower limit TF)

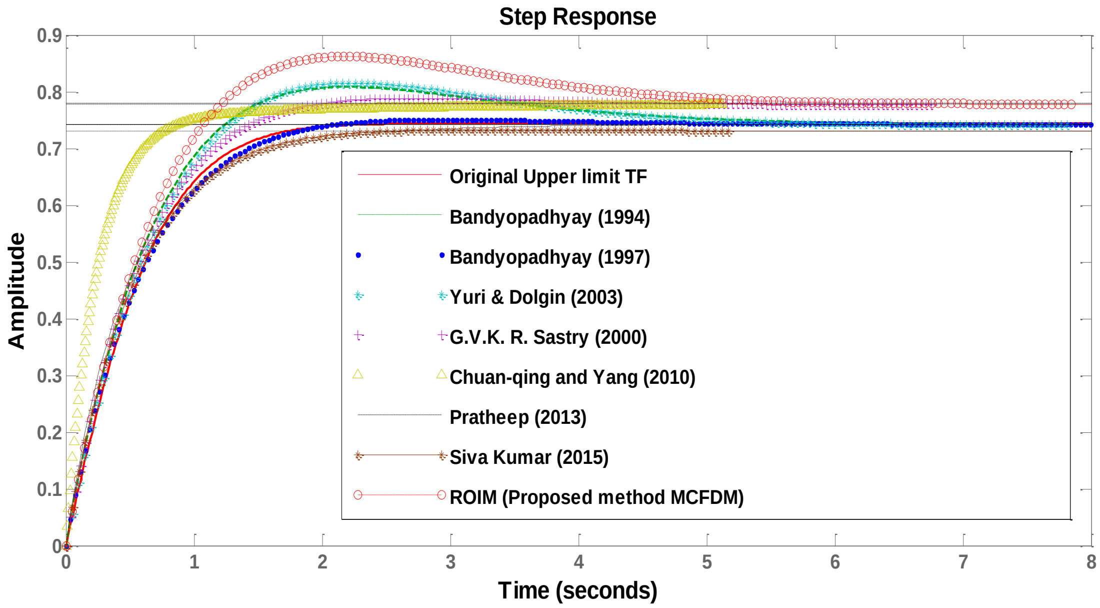

Fig.4.10. Step response for Mihailov Criterion and Factor division method (upper limit TF)

The impulse response of the high-order system and reduced order models by proposed method is shown in Fig. 4.11 and Fig. 4.12. A comparison has been made with the existing methods

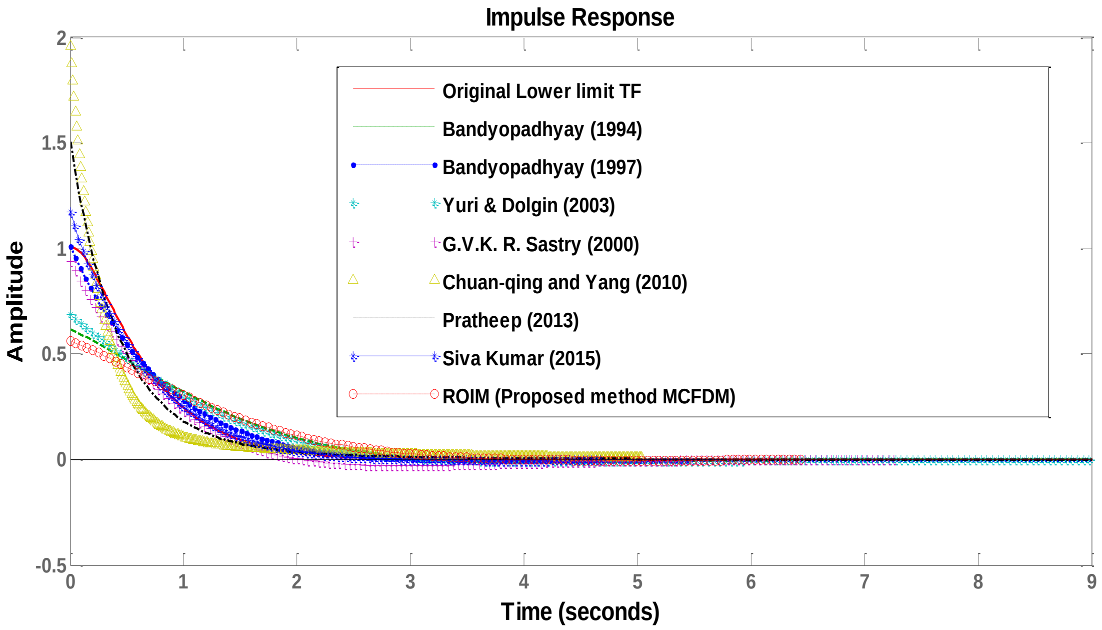

Fig.4.11. Impulse response for Mihailov Criterion and Factor division method (lower limit TF)

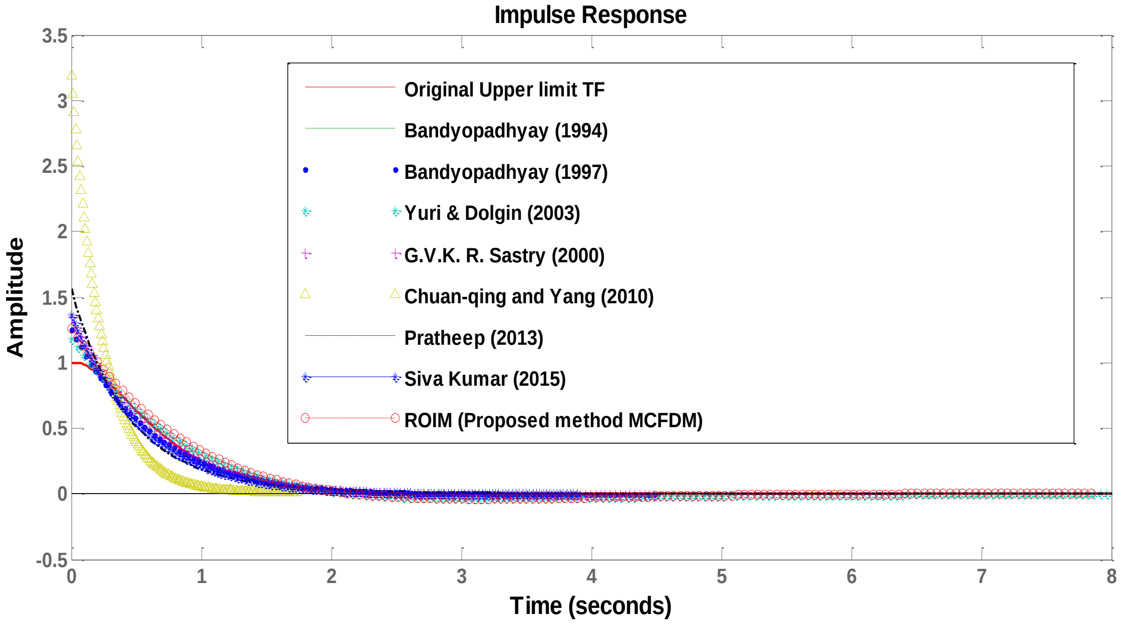

Fig.4.12. Impulse response for Mihailov Criterion and Factor division method (Upper limit TF)

The Bode plot of the high-order system and reduced order models by proposed method is shown in Fig. 4.13 and Fig. 4.14. A comparison has been made with the existing methods.

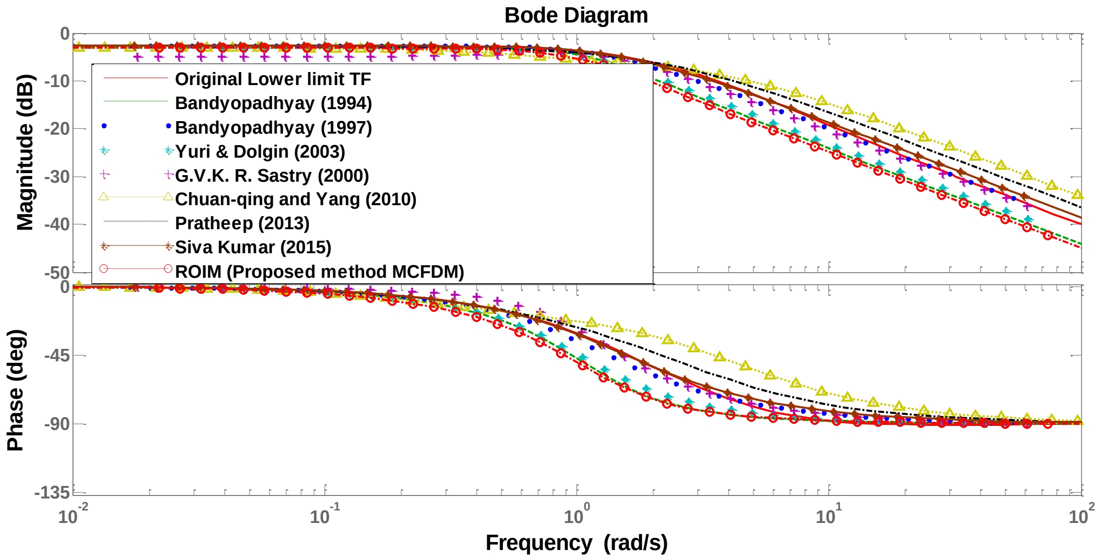

Fig.4.13. Bode plot for Mihailov Criterion and Factor division method (lower limit TF)

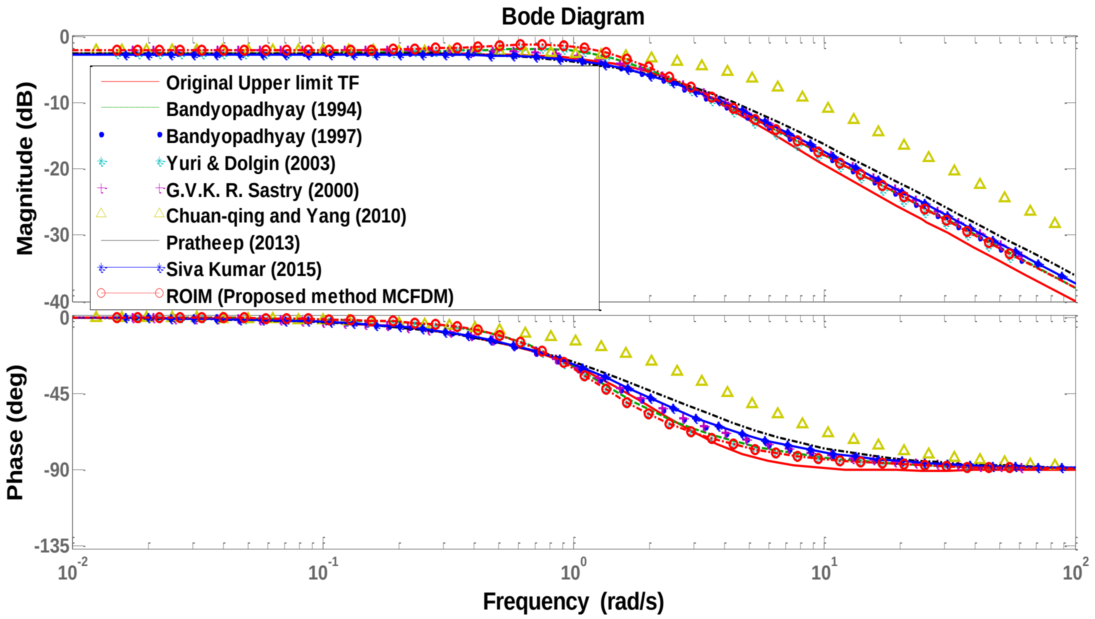

Fig. 4.14. Bode plot for Mihailov Criterion and Factor division method (upper limit TF)

The nyquist plot of the high-order system and reduced order models by proposed method is shown in Fig. 4.15 and Fig. 4.16. A comparison has been made with the existing methods.

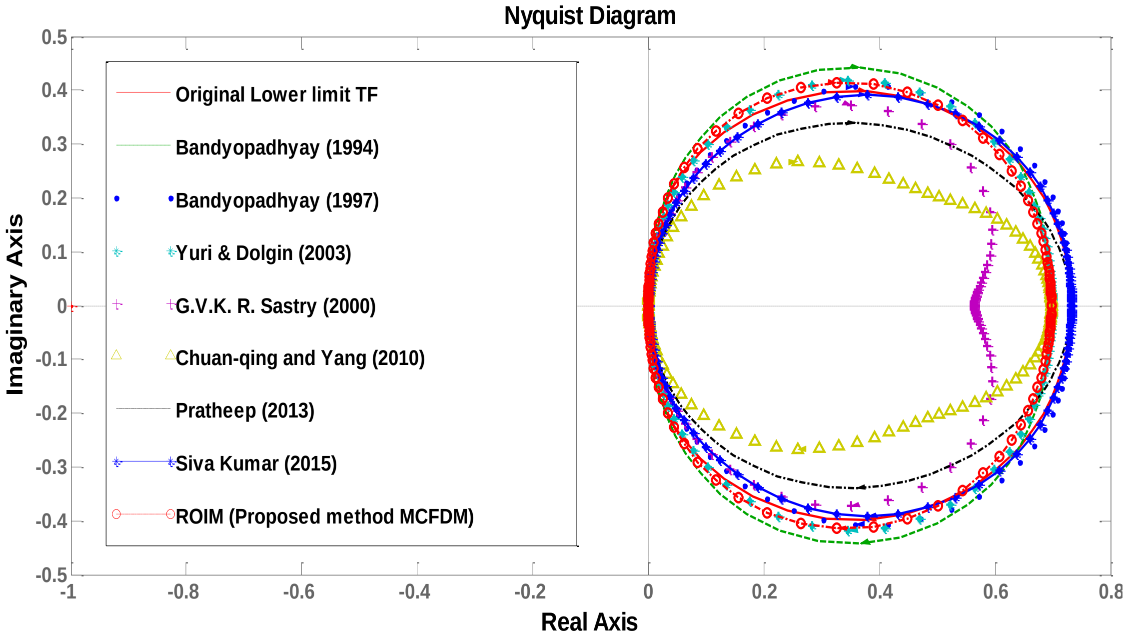

Fig.4.15. Nyquist plot for Mihailov Criterion and Factor division method (lower limit TF)

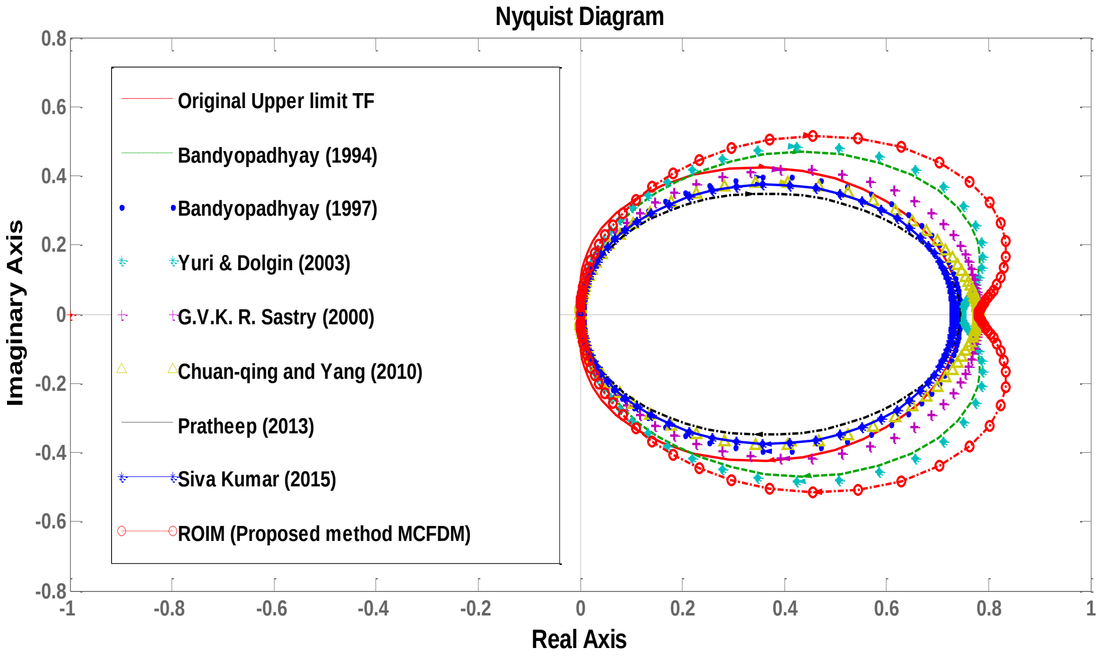

Fig.4.16. Nyquist plot for Mihailov Criterion and Factor division method (upper limit TF)

Case3: MCCSF

Step 1: The denominator of the second order interval polynomial is

$$D_2(s)=[17.0011,18.0007]s^2+[31.3826,33.6111]s+[20.3061,21.7052] \quad (4.32)$$

Step2: Cauer second form is used to obtain reduced numerator polynomial

$$\begin{gathered}[h_1^-,h_1^+]=[1.2812,1.4333];[h_2^-,h_2^+]=[1.1046,1.8859]\\ [d_{21}^-,d_{21}^+]=[14.1674,16.9413];[d_{22}^-,d_{22}^+]=[11.1949,20.3706]\end{gathered} \quad (4.33)$$

Step 3: The reduced numerator interval polynomial is

$$N_2(s)=[11.1949,20.3706]s+[14.1674,16.9413] \quad (4.34)$$

Step4: The second order reduced model is

$$R_2(s)=\frac{[11.1949,20.3706]s+[14.1674,16.9413]}{[17.0011,18.0007]s^2+[31.3826,33.6111]s+[20.3061,21.7052]} \quad (4.35)$$

The step response of the high-order system and reduced order models by proposed method is shown in Fig. 4.17 and Fig. 4.18. A comparison has been made with the existing methods.

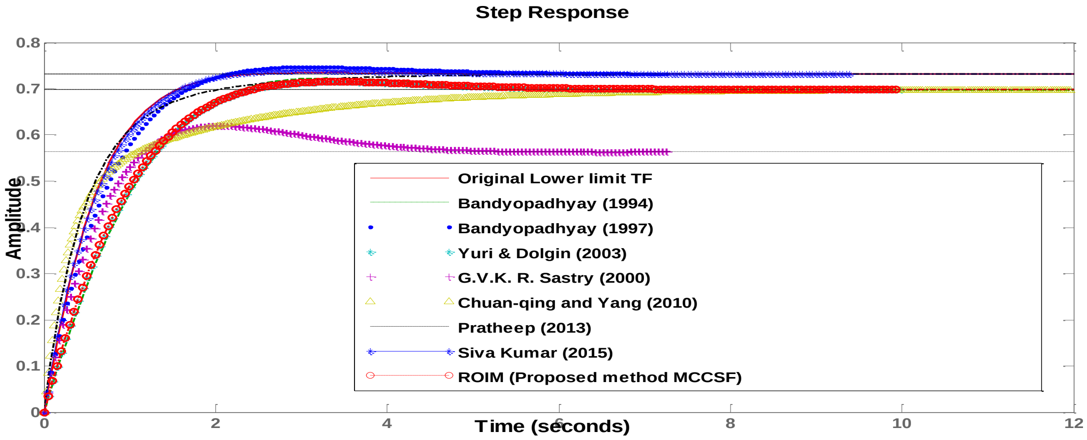

Fig.4.17. Step response for Mihailov Criterion and Cauer Second Form (lower limit TF)

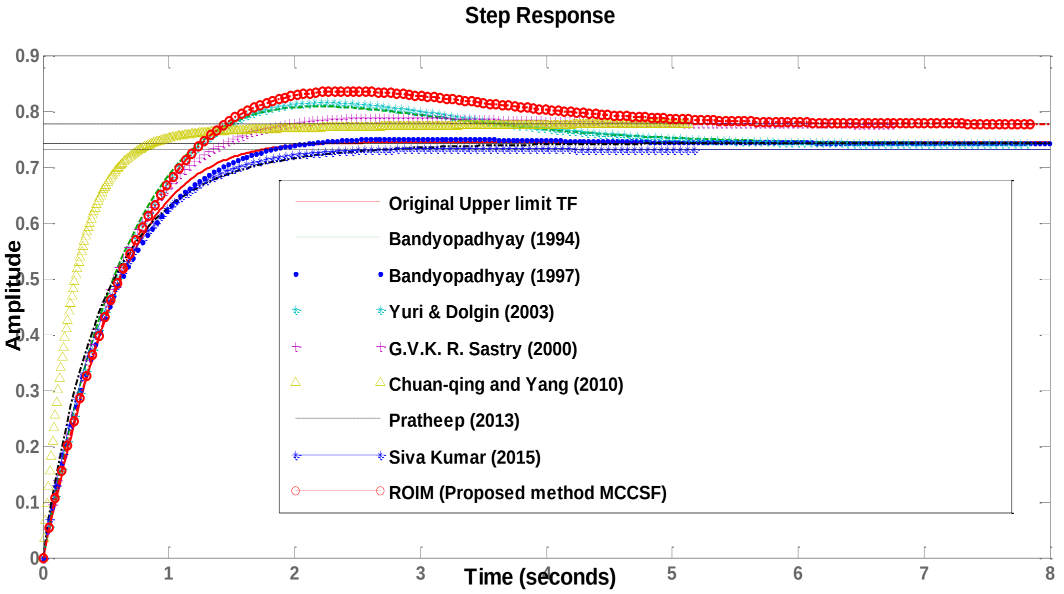

Fig.4.18. Step response for Mihailov Criterion and Cauer Second Form (upper limit TF)

The impulse response of the high-order system and reduced order models by proposed method is shown in Fig. 4.19 and Fig. 4.20. A comparison has been made with the existing methods.

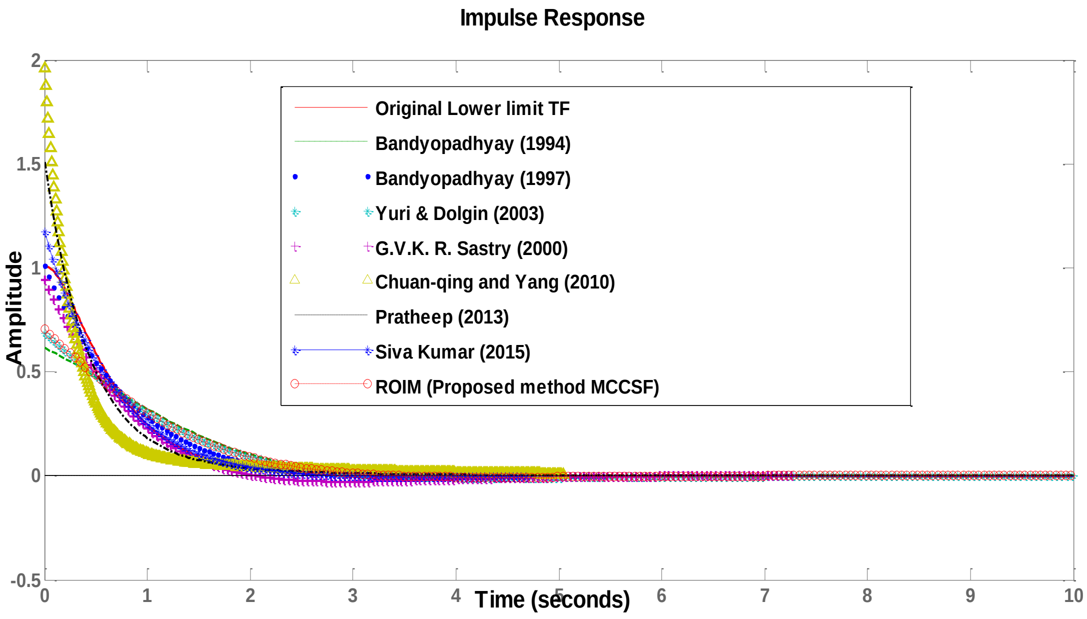

Fig.4.19. Impulse response for Mihailov Criterion and Cauer Second Form (lower limit TF)

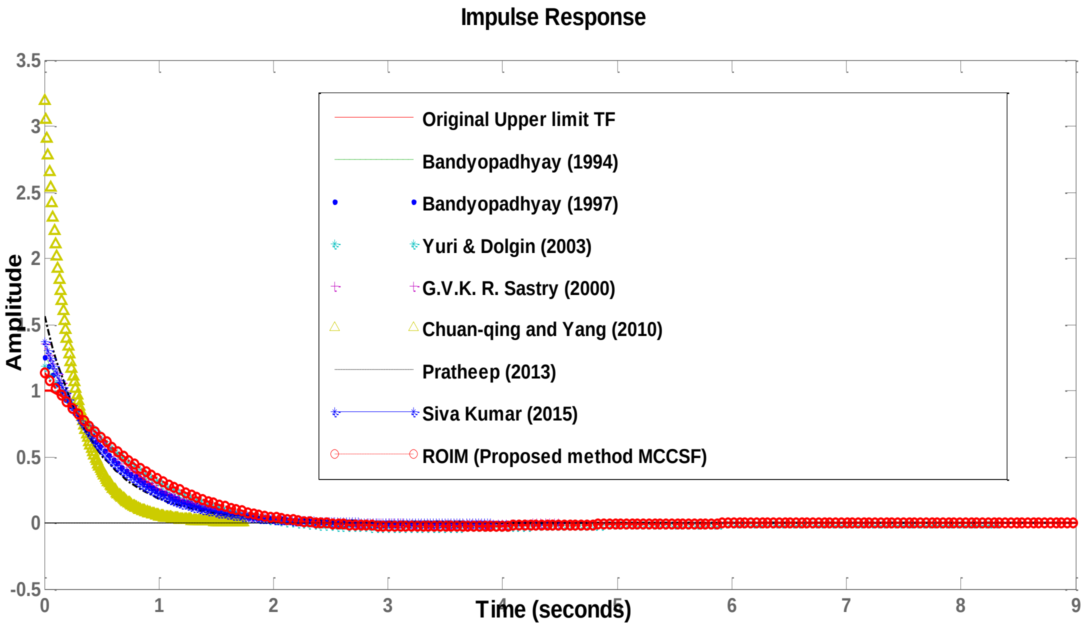

Fig.4.20. Impulse response for Mihailov Criterion and Cauer Second Form (upper limit TF)

The Bode plot of the high-order system and reduced order models by proposed method is shown in Fig. 4.21 and Fig. 4.22. A comparison has been made with the existing methods.

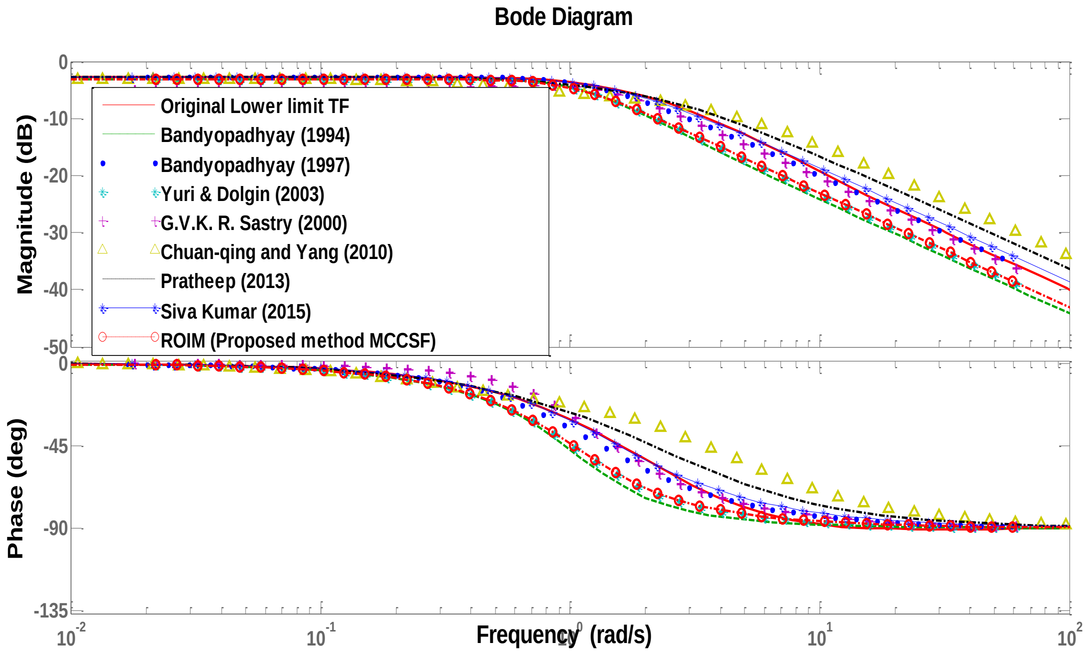

Fig.4.21. Bode plot for Mihailov Criterion and Cauer Second Form (lower limit TF)

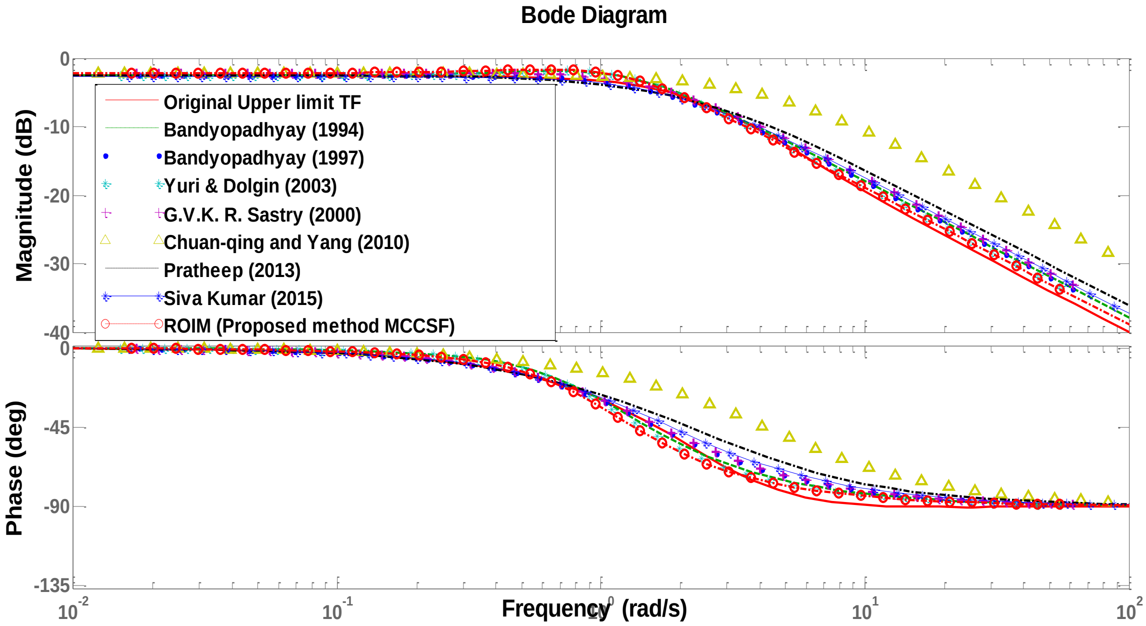

Fig.4.22. Bode plot for Mihailov Criterion and Cauer Second Form (upper limit TF)

The nyquist plot of the high-order system and reduced order models by proposed method is shown in Fig. 4.23 and Fig. 4.24. A comparison has been made with the existing methods.

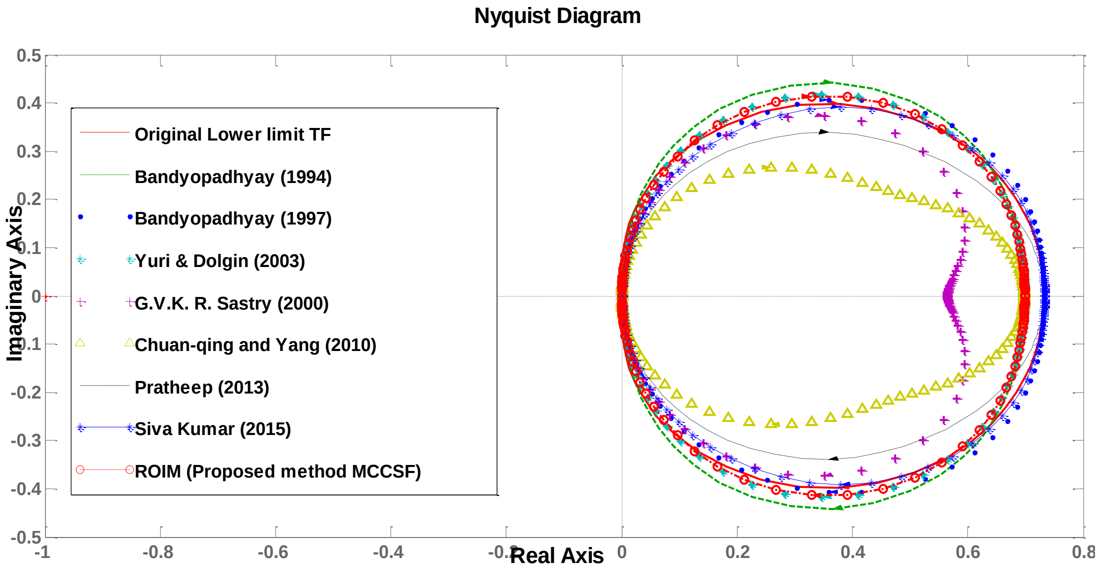

Fig.4.23. Nyquist plot for Mihailov Criterion and Cauer Seond Form (lower limit TF)

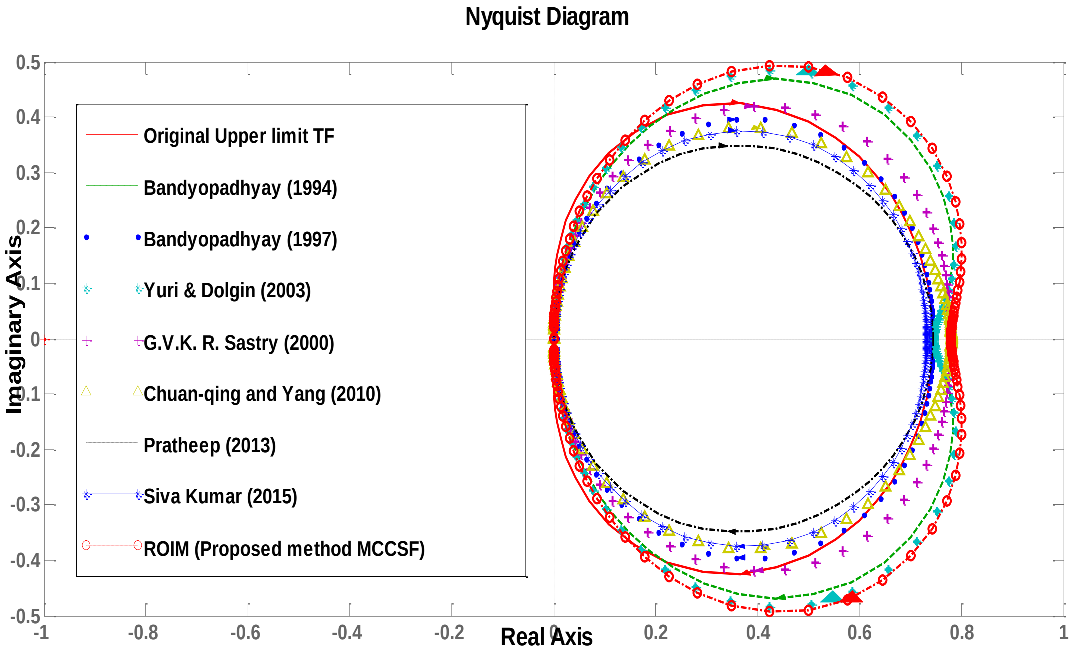

Fig.4.24. Nyquist plot for Mihailov Criterion and Cauer Seond Form (upper limit TF)

The comparison of integral square error and absolute error are verified in Table 4.3

Table 4.3: Comparison of Reduced Order Models of example 4.1

S.No	**Methods**	**ISE**		**IAE**	
		Lower limit	**Upper limit**	**Lower limit**	**Upper limit**
1	Bandypodayay et al. , [156]	0.00878538	8.87354E-05	0.264278635	0.009542405
2	Bandypodayay et al., [157]	2.36319E-05	4.39626E-06	0.005001512	0.002108643
3	Dolgin and Zeheb [162]	0.008876522	8.01481E-05	0.265977631	0.009118821
4	Sastry et al., [161]	0.225673362	0.00949689	1.343641109	0.275550326
5	Chuan-qing and Yang [171]	0.010817047	0.010362791	0.290640962	0.287856756
6	Pratheep et al., [176]	1.19446E-05	9.93222E-07	0.003594415	0.001092456
7	Siva Kumar et al., [177]	2.93246E-06	0.001205443	0.002187121	0.098172255
8	Proposed Method (MCRAM)	0.00826521	0.010482829	0.256432732	0.288782171
9	Proposed Method (MCFDM)	0.008926763	0.011278187	0.266835208	0.299404733

10	Proposed Method (MCCSF)	0.008325873	0.01051877	0.25759018	0.289194223

4.4 ALPHA TRUNCATION METHOD

4.4.1 ALPHA TRUNCATION AND FACTOR DIVISION METHOD

The denominator polynomial is reduced by α -truncation and numerator polynomial is reduced by Factor division method

Factor division method is explained in previous section, so here it is ignored

ALPHA TRUNCATION METHOD (Algorithm)

(1) Determination of the reciprocal transformation of the higher order denominator interval polynomial

(2) Construct the alpha table

(3) Determine the reduced order denominator interval polynomial.

(4) Inverse reciprocal transformation of reduced order denominator polynomial.

Note: Reciprocal transformation places greater emphasis on steady state approximation

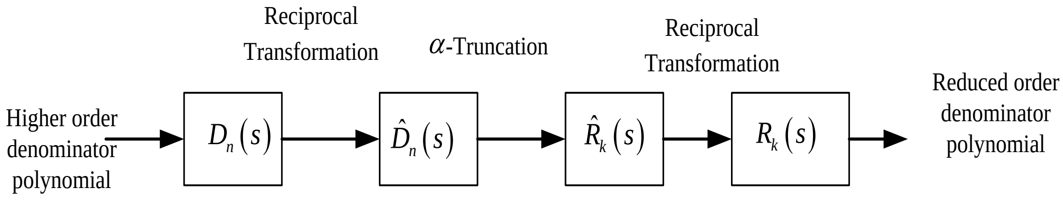

Fig.4.25. Block diagram for alpha truncation method for interval polynomial

Proposed Method: Alpha Truncation Method

Step1: Consider a higher order denominator polynomial is

$$D_n(s) = \left[a_{1,n+1}^{-}, a_{1,n+1}^{+}\right]s^n + \left[a_{1,n}^{-}, a_{1,n}^{+}\right]s^{n-1} + ... + \left[a_{1,2}^{-}, a_{1,2}^{+}\right]s + \left[a_{1,1}^{-}, a_{1,1}^{+}\right] \tag{4.36}$$

Step 2: Reciprocal transformation of higher order polynomial

$$\hat{D}_n(s) = \left[a_{1,1}^{-}, a_{1,1}^{+}\right]s^n + \left[a_{1,2}^{-}, a_{1,2}^{+}\right]s^{n-1} + ... + \left[a_{1,n}^{-}, a_{1,n}^{+}\right]s + \left[a_{1,n+1}^{-}, a_{1,n+1}^{+}\right] \tag{4.37}$$

Step3: Constructing Alpha table

Table 4.4: Alpha Table

	$a_0^0 = \left[a_{11}^{-}, a_{11}^{+}\right]$ $a_0^1 = \left[a_{1,2}^{-}, a_{1,2}^{+}\right]$	$a_2^0 = \left[a_{1,3}^{-}, a_{1,3}^{+}\right]$ $a_2^1 = \left[a_{1,4}^{-}, a_{1,4}^{+}\right]$	....	....
$\left[\alpha_1^{-}, \alpha_1^{+}\right] = \frac{a_0^0}{a_0^1}$	$a_0^2 = a_2^0 - \left[\alpha_1^{-}, \alpha_1^{+}\right]a_2^1$	$a_2^2 = a_4^0 - \left[\alpha_1^{-}, \alpha_1^{+}\right]a_4^1$	....	....
$\left[\alpha_2^{-}, \alpha_2^{+}\right] = \frac{a_0^1}{a_0^2}$	$a_0^3 = a_2^1 - \left[\alpha_2^{-}, \alpha_2^{+}\right]a_2^2$	....		
$\left[\alpha_3^{-}, \alpha_3^{+}\right] = \frac{a_0^2}{a_0^3}$	....			
....	....			

Step 4: The Routh convergent reduced order is an approximation of higher original

system

$$\hat{D}_1(s) = \left[\alpha_1^-, \alpha_1^+\right] s + 1$$

$$\hat{D}_2(s) = \left[\alpha_1^-, \alpha_1^+\right]\left[\alpha_2^-, \alpha_2^+\right] s^2 + \left[\alpha_2^-, \alpha_2^+\right] s + 1$$

....

....

$$\hat{D}_k(s) = s\left[\alpha_k^-, \alpha_k^+\right]\left[\hat{D}_{k-1}^-, \hat{D}_{k-1}^+\right](s) + \left[\hat{D}_{k-2}^-, \hat{D}_{k-2}^+\right](s) \tag{4.38}$$

where $k = 1, 2, 3, ..., n-1$

$D_{-1}(s) = 0; D_0(s) = 1$

Step 5: The inverse reciprocal transformation of the above equation gives reduced order system

4.4.2 ALPHA TRUNCATION AND CAUER SECOND FORM

Alpha truncation and Cauer second form is explained in previous section, so here it is ignored. An example is considered to illustrate this method.

Example 4.1: Consider a third order system described by the transfer function [157]

$$G_3(s) = \frac{[2,3]s^2 + [17.5,18.5]s + [15,16]}{[2,3]s^3 + [17,18]s^2 + [35,36]s + [20.5,21.5]} \tag{4.39}$$

Case1: Alpha Truncation and Factor division method (AFDM)

Denominator is reduced by alpha method

Step 1: Apply reciprocal transform of $\hat{D}_3(s)$ to get

$$\hat{D}_3(s) = \left[20.5, 21.5\right] s^3 + \left[35, 36\right] s^2 + \left[17, 18\right] s + \left[2, 3\right] \tag{4.40}$$

Step 2: Construct Alpha table

Table 4.5: Alpha table for denominator table

	[20.5,21.5]	[17,18]
	[35,36]	[2,3]
$\alpha_1 = [0.5694, 0.6143]$	[15.1571,16.8612]	
$\alpha_2 = [2.0758, 2.3751]$	[2,3]	
$\alpha_3 = [5.0523, 8.4306]$		

Step 3: Reduced order denominator

$$\hat{D}_2(s) = \alpha_1\alpha_2 s^2 + \alpha_2 s + 1$$

$$D_2(s) = [1.1896, 1.459]s^2 + [2.0758, 2.3751]s + 1 \tag{4.41}$$

Step4: Reciprocal of reduced order denominator

$$D_2(s) = s^2 + [2.0758, 2.3751]s + [1.1896, 1.459] \tag{4.42}$$

Step 5: Numerator is reduced by using Factor division method

$$\frac{N(s)D_2(s)}{D(s)} = \frac{[17.844, 23.344] + [531.955, 64.9931]s + ...}{[20.5, 21.5] + [35, 36]s + ...} \tag{4.43}$$

Step 6: Finding the values of $\left[\alpha_{11}^{-}, \alpha_{11}^{+}\right]$ and $\left[\alpha_{12}^{-}, \alpha_{12}^{+}\right]$

$$\left[\alpha_{11}^{-},\alpha_{11}^{+}\right]=\left[0.8299,1.1387\right];\left[\alpha_{12}^{-},\alpha_{12}^{+}\right]=\left[0.4861,1.8389\right];$$

$$N_2\left(s\right)=\left[0.8299,1.1387\right]+\left[0.4861,1.8389\right]s \tag{4.44}$$

Step 7: Using gain correction factor $\left(\eta=0.993\right)$

$$N_2\left(s\right)=\left[0.4828,1.826\right]s+\left[0.8241,1.1309\right] \tag{4.45}$$

Step 8: The second order reduced order model is

$$R_2\left(s\right)=\frac{\left[0.4828,1.826\right]s+\left[0.8241,1.1309\right]}{s^2+\left[2.0758,2.3751\right]s+\left[1.1896,1.459\right]} \tag{4.46}$$

A comparison of the step response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.26 and Fig.4.27.

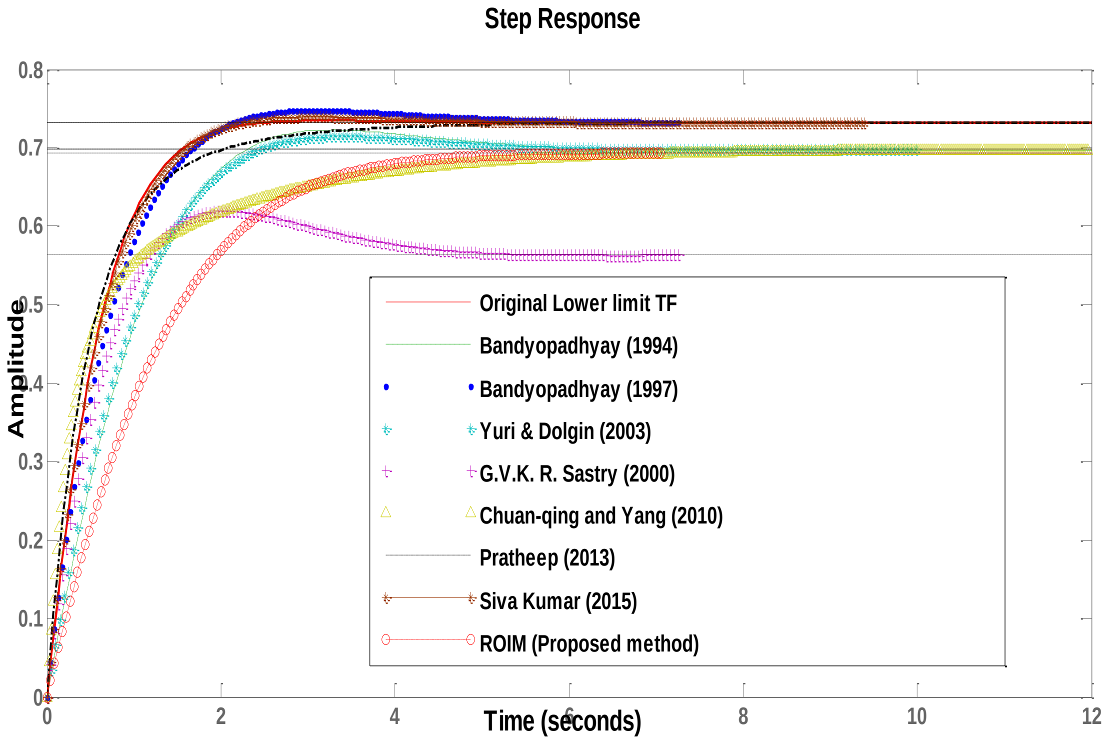

Fig.4.26. Step response for Alpha and Factor division method (lower limit TF)

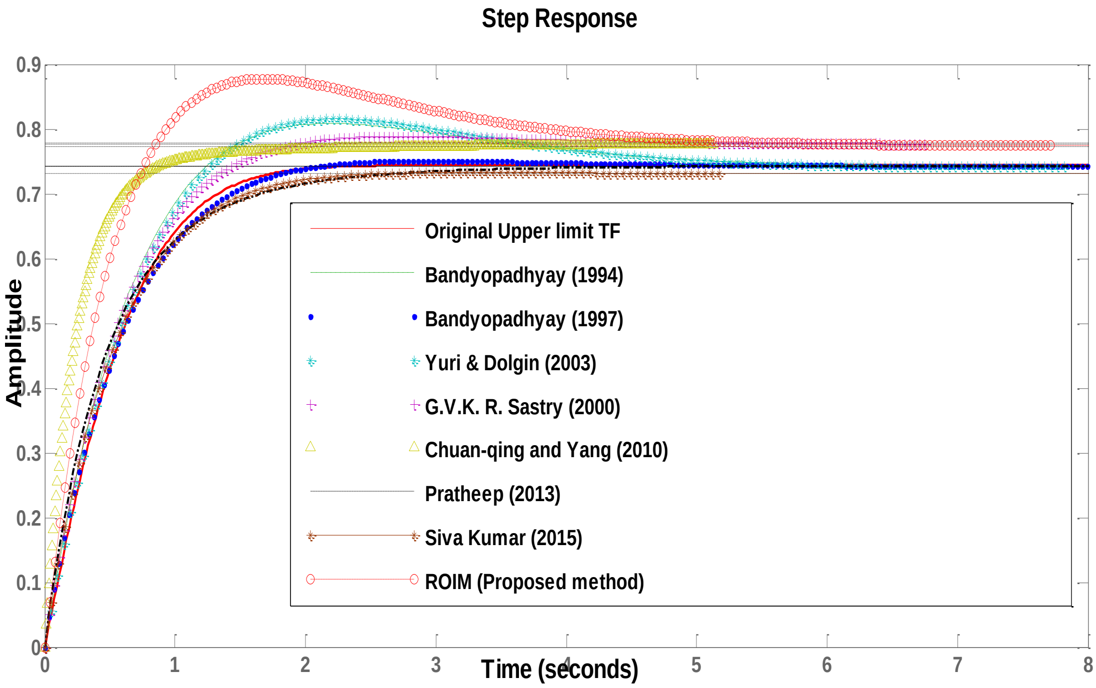

Fig.4.27. Step response for Alpha and Factor division method (upper limit TF)

A comparison of the impulse response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.28 and Fig.4.29.

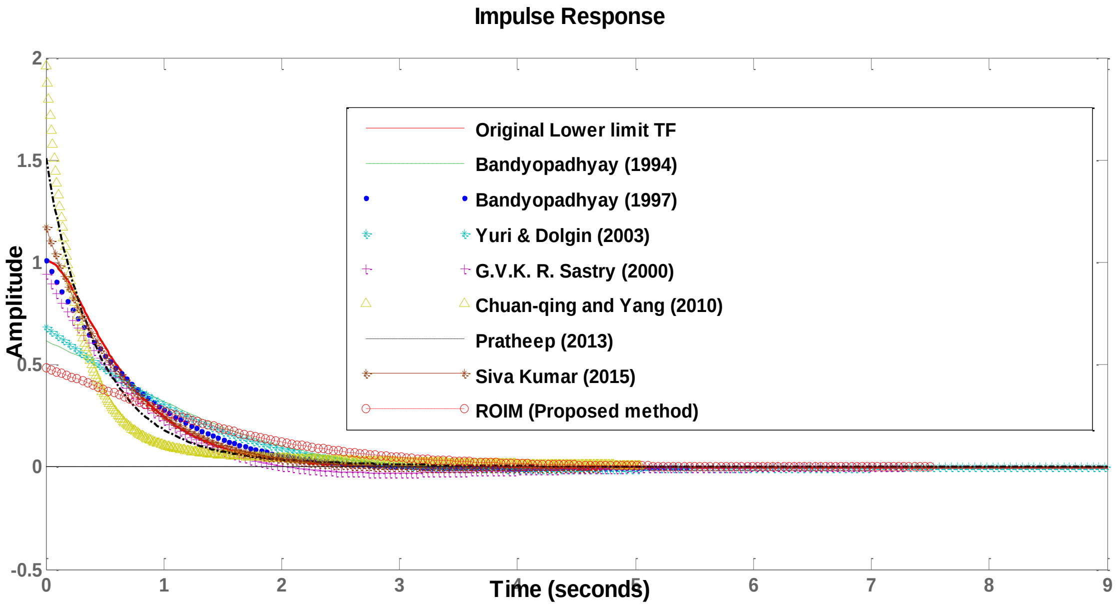

Fig.4.28. Impulse response for Alpha and Factor division method (lower limit TF)

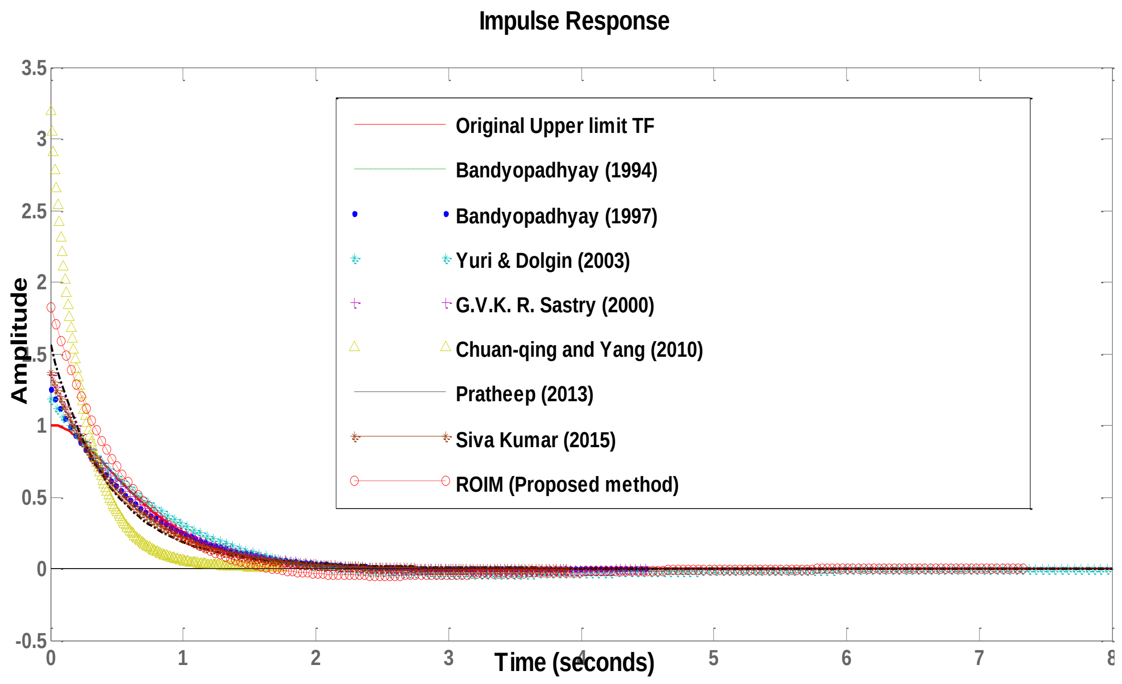

Fig.4.29. Impulse response for Alpha and Factor division method (upper limit TF)

A comparison of the bode plot of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.30 and Fig.4.31.

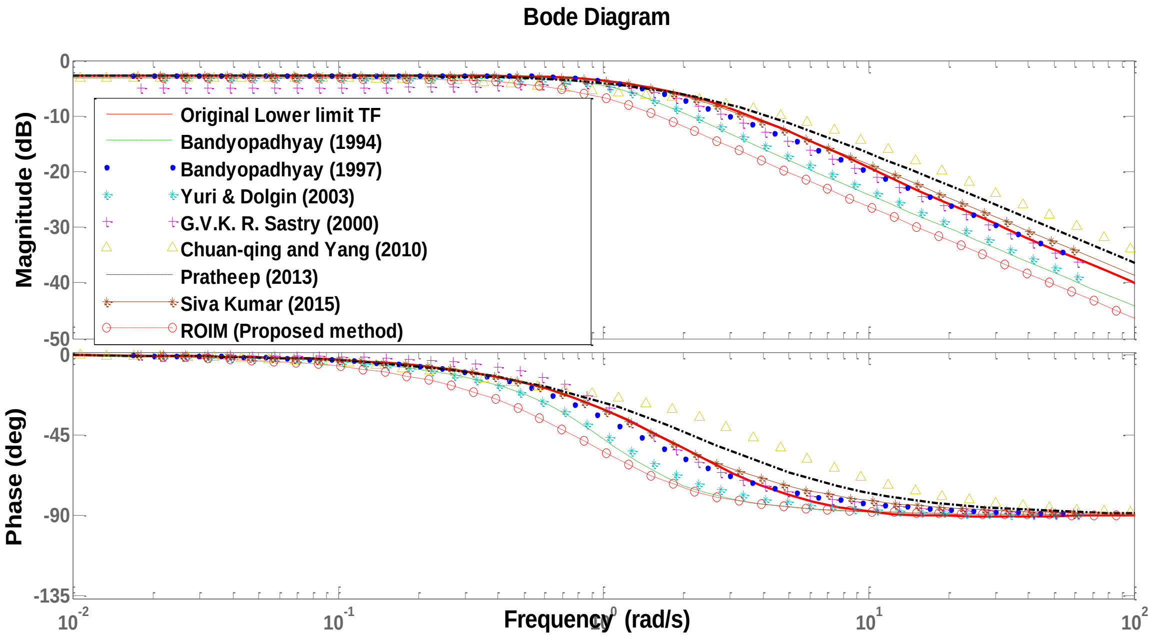

Fig.4.30. Bode plot for Alpha and Factor division method (lower limit TF)

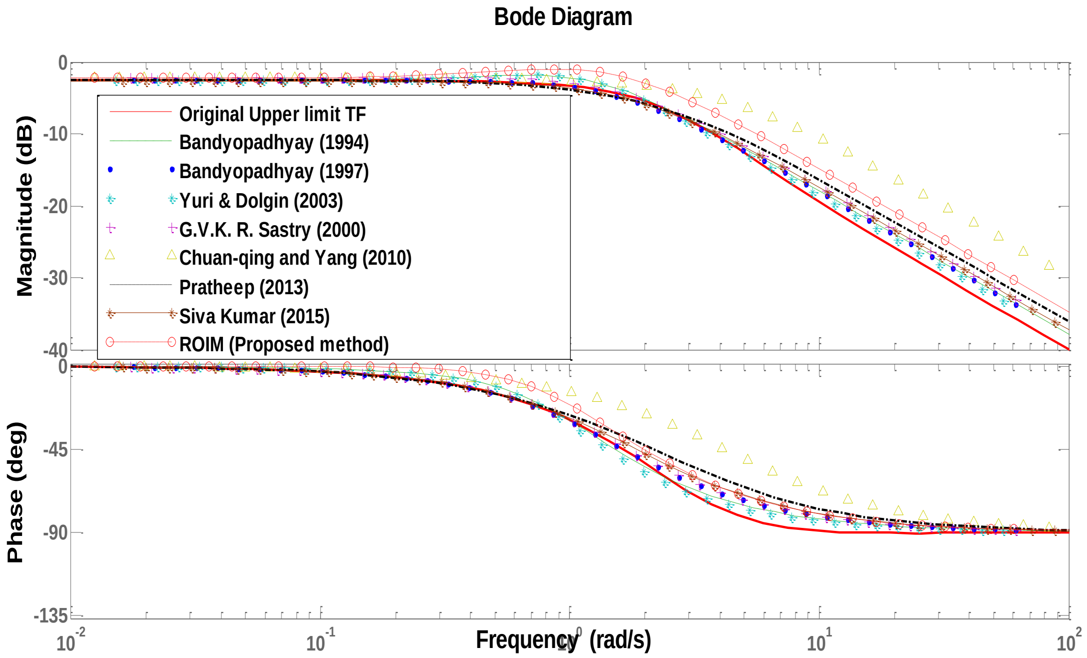

Fig.4.31. Bode plot for Alpha and Factor division method (upper limit TF)

A comparison of the nyquist plot of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.32 and Fig.4.33.

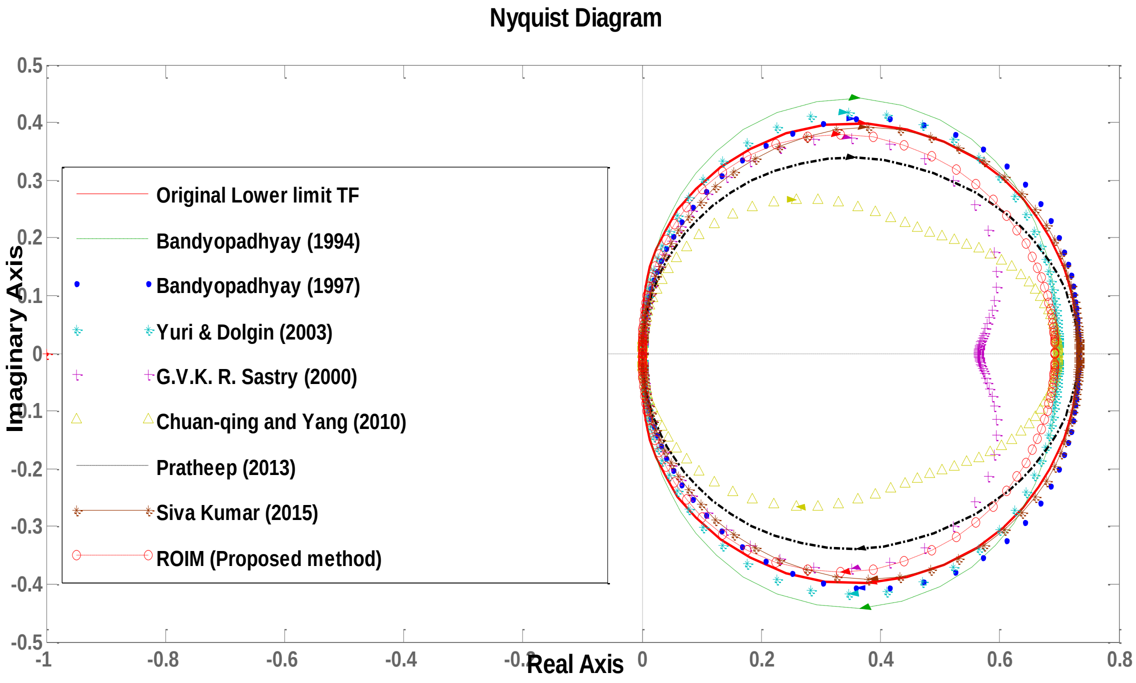

Fig.4.32. Nyquist plot for Alpha and Factor division method (lower limit TF)

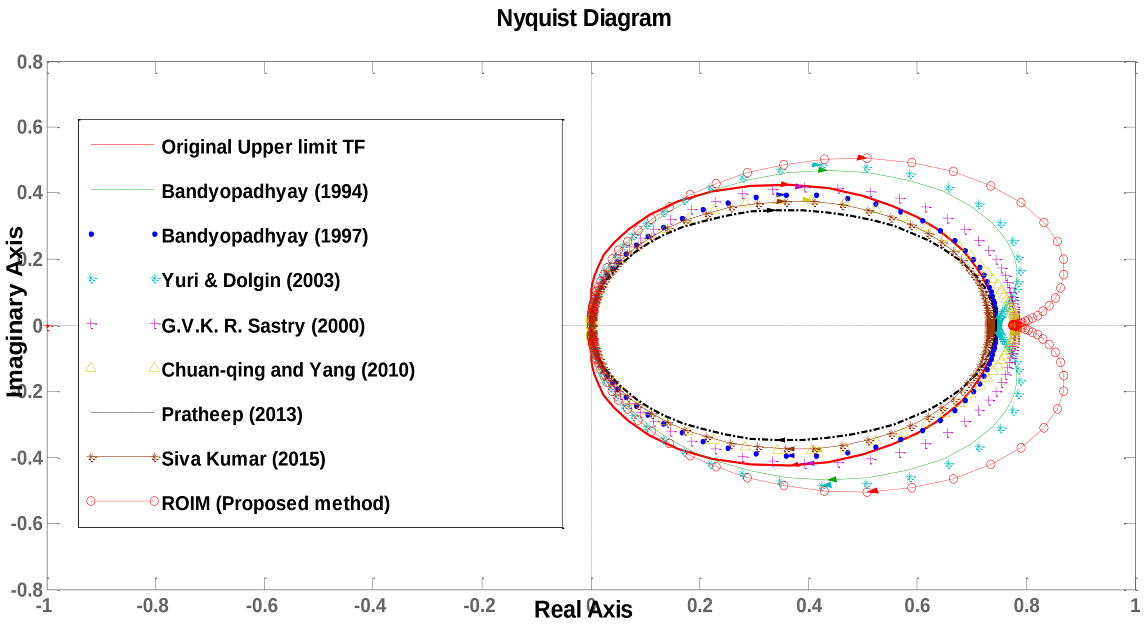

Fig.4.33. Nyquist plot for Alpha and Factor division method (upper limit TF)

Case 2: Alpha Truncation and Cauer second form (ACSF)

Step 1: The denominator of the second order interval polynomial is

$$D_2(s) = s^2 + [2.0758, 2.3751]s + [1.1896, 1.459] \tag{4.47}$$

Step 2: Using second Cauer Form

$$\begin{aligned} &\left[h_1^-, h_1^+\right] = [1.2812, 1.434]; \left[h_2^-, h_2^+\right] = [1.1046, 1.8859] \\ &\left[d_{21}^-, d_{21}^+\right] = [0.8295, 1.1387]; \left[d_{22}^-, d_{22}^+\right] = [0.7286, 1.1505] \end{aligned} \tag{4.48}$$

Step 3: Numerator of the second order polynomial is

$$N_2(s) = [0.7286, 1.5105]s + [0.8295, 1.1387] \tag{4.49}$$

Step 4: Using gain correction factor $(\eta = 0.993)$

$$N_2(s) = [0.7236, 1.5]s + [0.8238, 1.1309] \tag{4.50}$$

Step5: Second order model is

$$R_2(s) = \frac{[0.7236, 1.5]s + [0.8238, 1.1309]}{s^2 + [2.0758, 2.3751]s + [1.1896, 1.459]} \tag{4.51}$$

A comparison of the step response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.34 and Fig. 4.35.

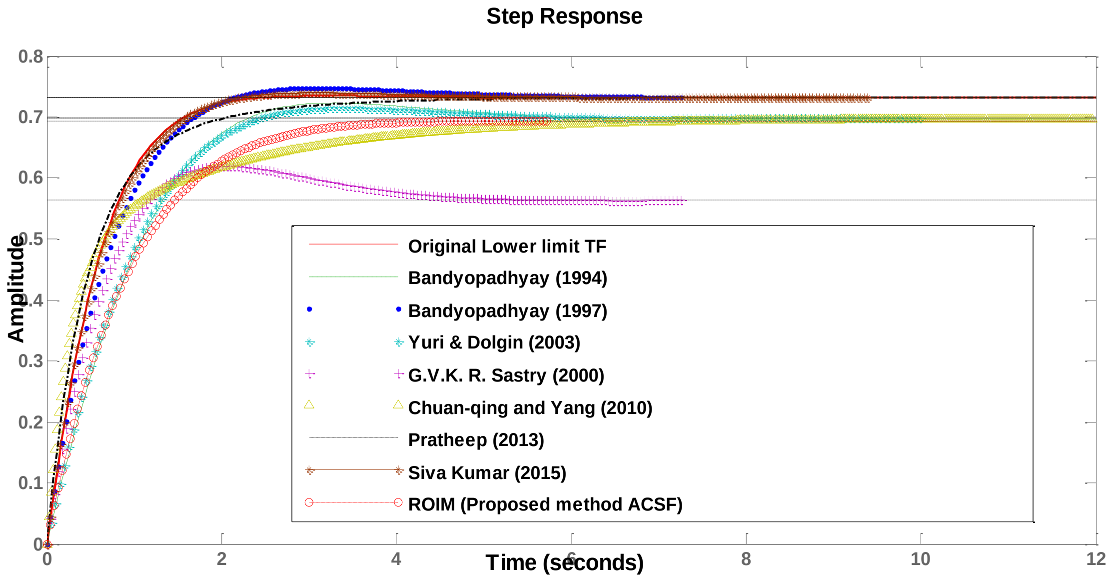

Fig.4.34. Step response for Alpha and Cauer Second form (lower limit TF)

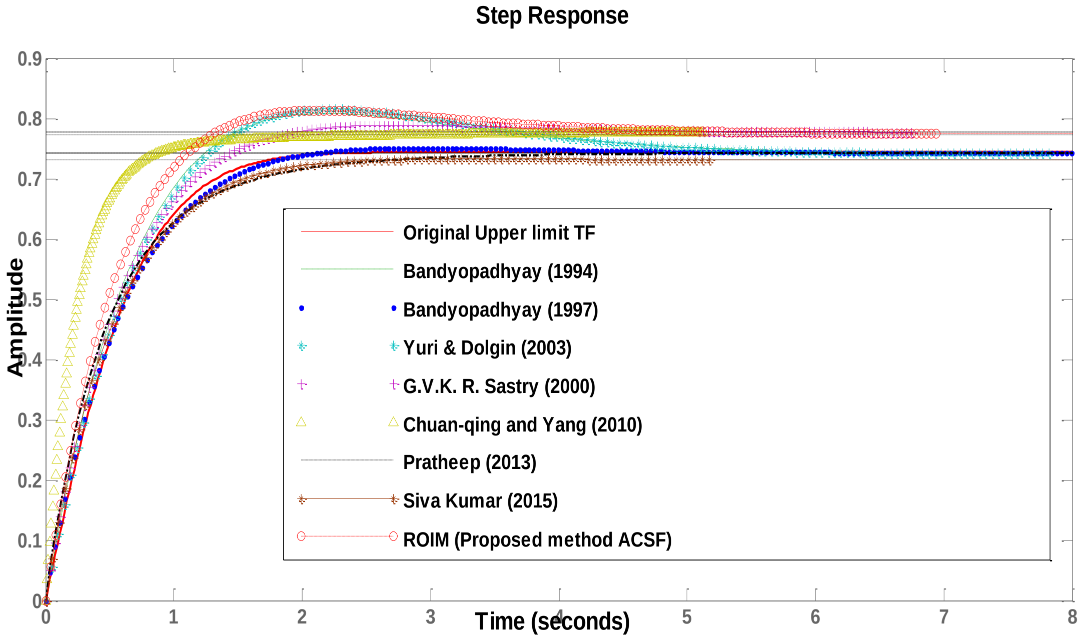

Fig.4.35. Step response for Alpha truncation and Cauer Second form (upper limit TF)

A comparison of the impulse response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.36 and Fig. 4.37.

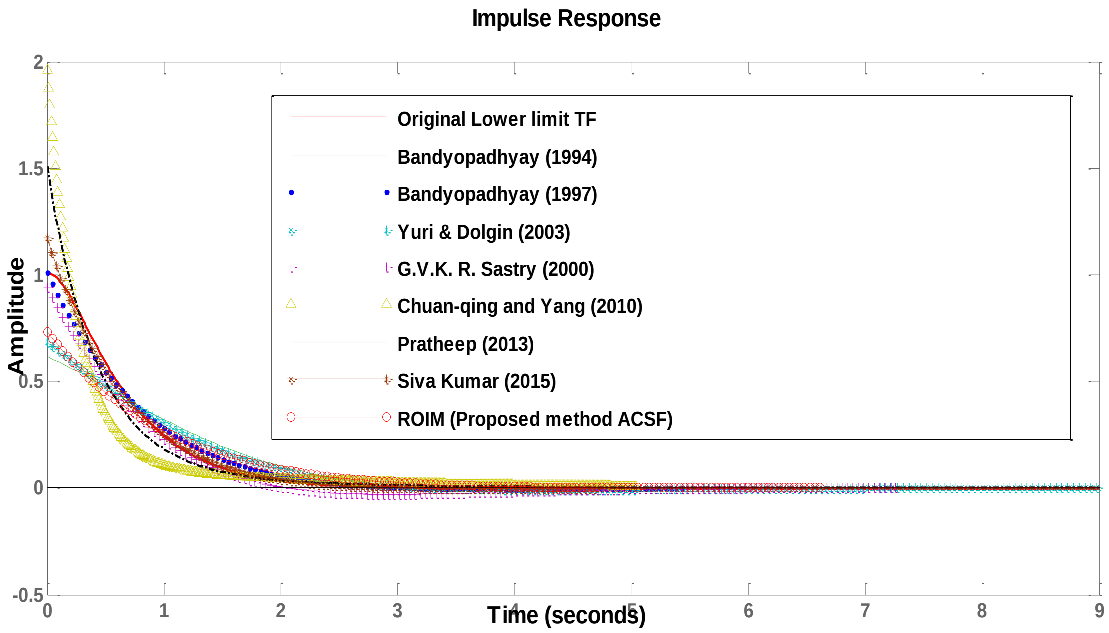

Fig.4.36. Impulse response for Alpha and Cauer Second form (lower limit TF)

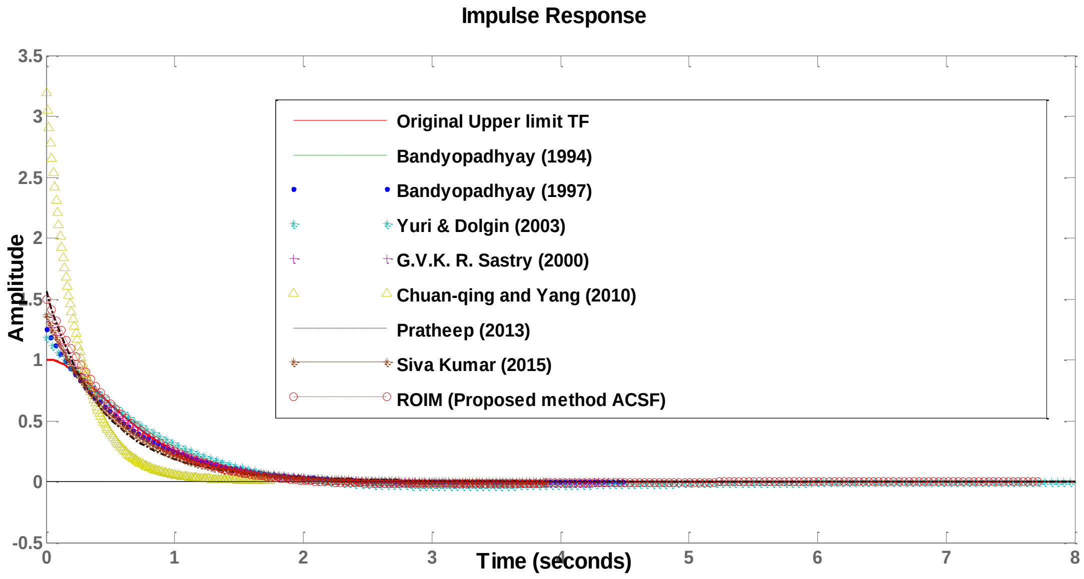

Fig.4.37. Impulse response for Alpha and Cauer Second form (upper limit TF)

A comparison of the bode plot of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.38 and Fig.4.39.

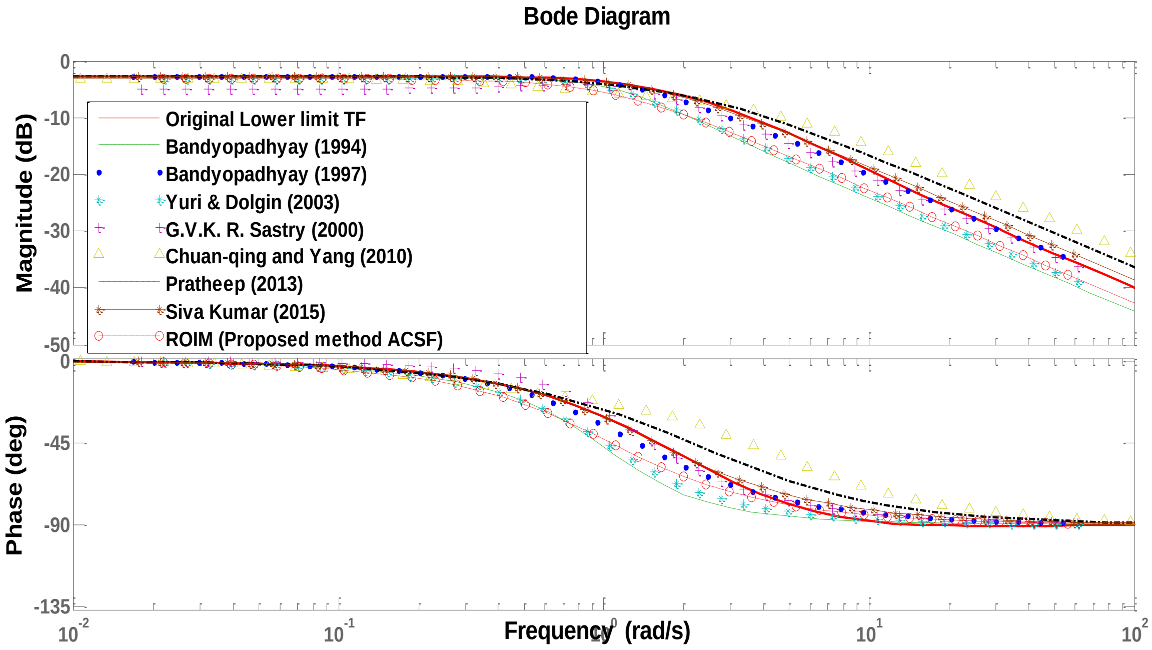

Fig.4.38. Bode plot for Alpha and Cauer second form (lower limit TF)

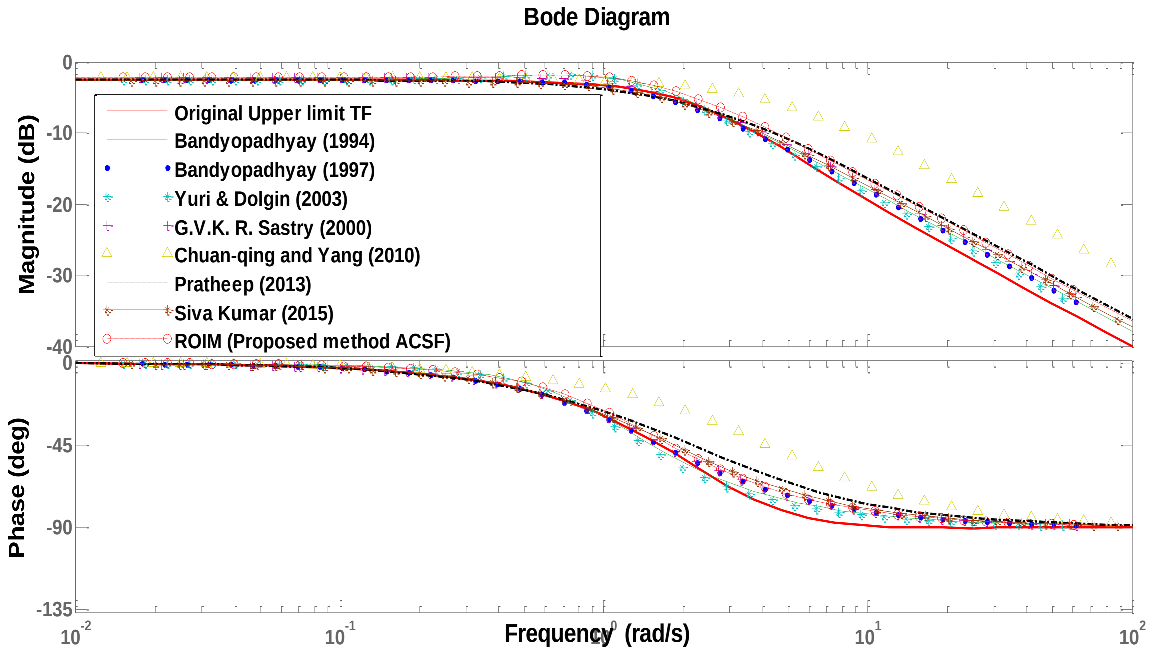

Fig.4.39. Bode plot for Alpha and Cauer second form (upper limit TF)

A comparison of the nyquist plot of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.40 and Fig.4.41.

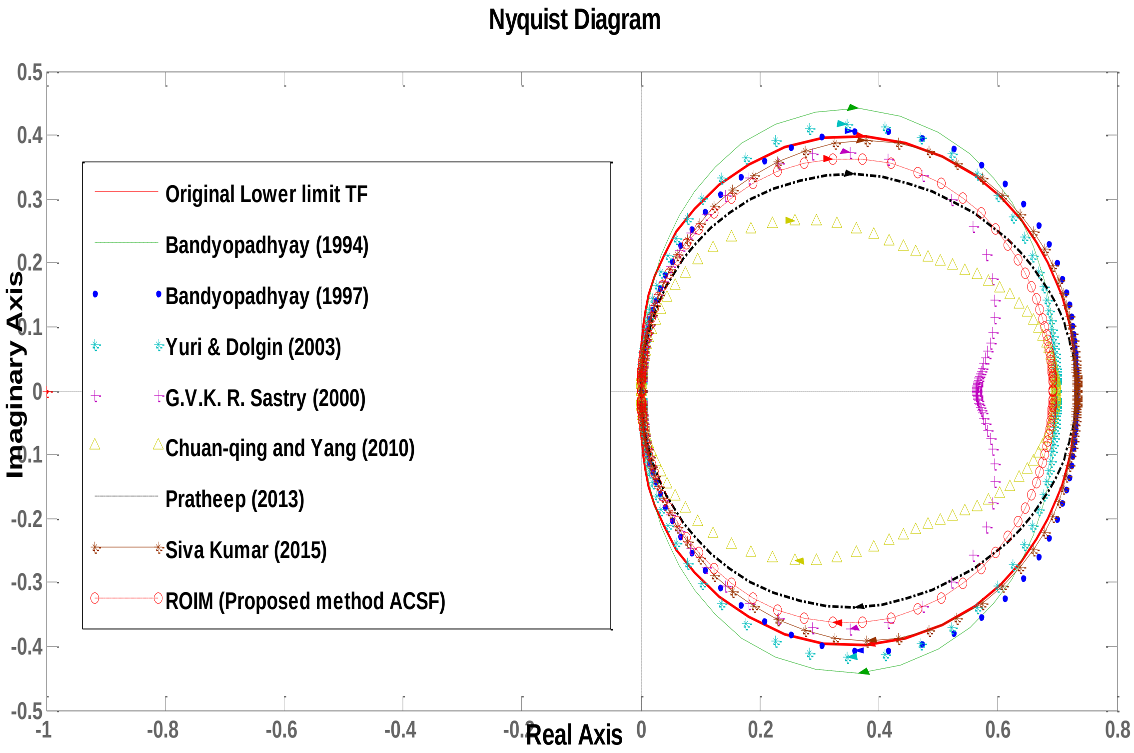

Fig.4.40. Nyquist plot for Alpha and Cauer second form (lower limit TF)

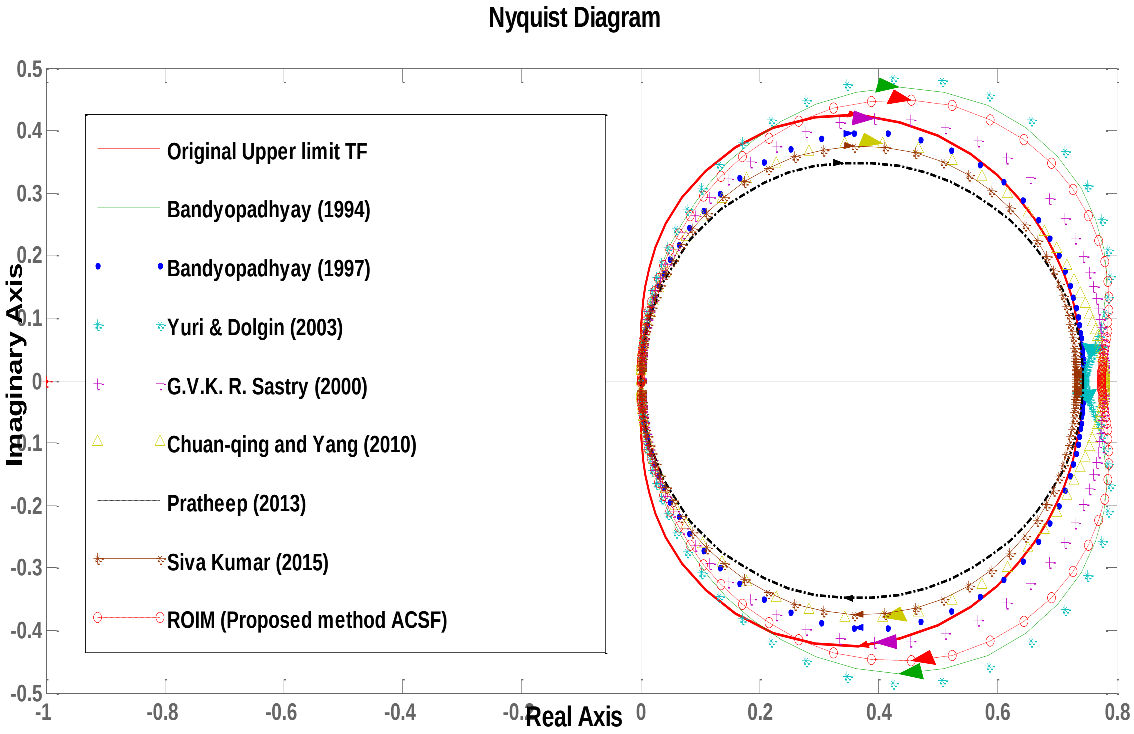

Fig.4.41. Nyquist plot for Alpha and Cauer second form (upper limit TF)

The comparison of integral square error and absolute error are verified in Table 4.6

Table 4.6: Comparison of DTFDM and Existing Reduced Order Models of example 4.1

S.No	**Methods**	**ISE**		**IAE**	
		Lower limit	**Upper limit**	**Lower limit**	**Upper limit**
1	Bandypodayay et al. , [156]	0.00878538	8.87354E-05	0.264278635	0.009542405
2	Bandypodayay et al., [157]	2.36319E-05	4.39626E-06	0.005001512	0.002108643
3	Dolgin and Zeheb [162]	0.008876522	8.01481E-05	0.265977631	0.009118821
4	Sastry et al., [161]	0.225673362	0.00949689	1.343641109	0.275550326
5	Chuan-qing and Yang [171]	0.010817047	0.010362791	0.290640962	0.287856756
6	Pratheep et al., [176]	1.19446E-05	9.93222E-07	0.003594415	0.001092456
7	Siva Kumar et al., [177]	2.93246E-06	0.001205443	0.002187121	0.098172255
8	Proposed Method (AFDM)	0.012504309	0.001205443	0.002187121	0.098172255
9	Proposed Method (ACSM)	0.012320599	0.007999464	0.31394915	0.252610798

4.5 DIRECT ROUTH APPROXIMATION METHOD

Direct Routh Approximation Method (DRAM) for model order reduction of interval systems is proposed. The numerator reduced order polynomial is obtained by β-truncation method and denominator reduced order polynomial is obtained by α-truncation method. This technique has been applied both SISO and MIMO systems. This method is an alternative technique to the previous α-truncation method.

Here simple and direct method applied to obtain reduced order model without using any reciprocal transformation presented in Fig. 4.42.

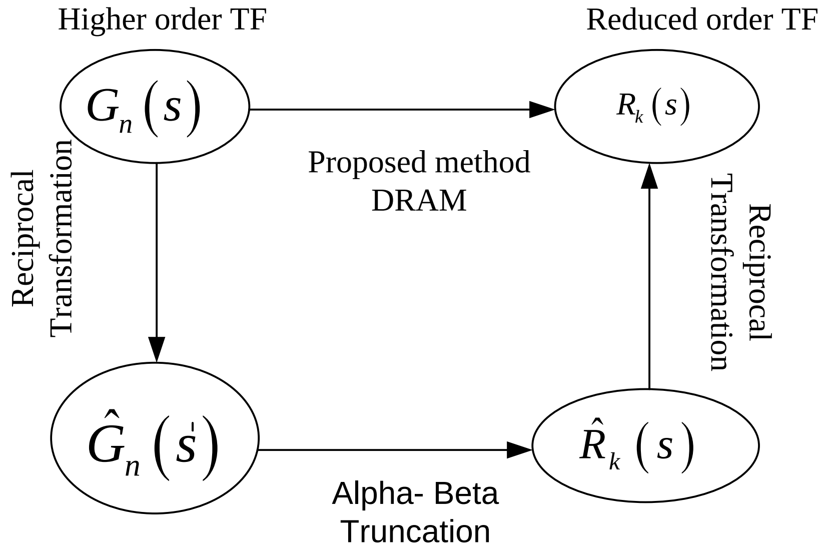

Fig. 4.42: Comparison between proposed method and existing method (alpha- beta truncation method)

Proposed Method:

Step1: Construct Alpha table for the denominator polynomial

Table 4.7: α (Routh) table

$\left[q_{10}^{-},q_{10}^{+}\right]=\left[a_{11}^{-},a_{11}^{+}\right]$	$\left[q_{11}^{-},q_{11}^{+}\right]=\left[a_{12}^{-},a_{12}^{+}\right]$	$\left[q_{12}^{-},q_{12}^{+}\right]=\left[a_{13}^{-},a_{13}^{+}\right]$	....
$\left[q_{11}^{-},q_{11}^{+}\right]$	0	$\left[q_{13}^{-},q_{13}^{+}\right]$	....
$\left[q_{20}^{-},q_{20}^{+}\right]$	$\left[q_{21}^{-},q_{21}^{+}\right]$	$\left[q_{22}^{-},q_{22}^{+}\right]$	....
$\left[q_{21}^{-},q_{21}^{+}\right]$	0	$\left[q_{23}^{-},q_{23}^{+}\right]$	....
$\left[q_{30}^{-},q_{30}^{+}\right]$	$\left[q_{31}^{-},q_{31}^{+}\right]$	....	
....	....	....	
....	....	....	
$\left[q_{n,0}^{-},q_{n,0}^{+}\right]$	$\left[q_{n,1}^{-},q_{n,1}^{+}\right]$		
$\left[q_{n,1}^{-},q_{n,1}^{+}\right]$			

To calculate the values of $\left[\alpha_i^{-},\alpha_i^{+}\right]$

$$\left[\alpha_i^{-},\alpha_i^{+}\right]=\frac{\left[q_{i0}^{-},q_{i0}^{+}\right]}{\left[q_{i1}^{-},q_{i1}^{+}\right]}\text{, where } i=1,2,3,......,n \tag{4.52}$$

For finding the values of $\left[q_{ij}^{-},q_{ij}^{+}\right]$;where $i=2,3,4,.....\&\ j=0,1,2,3,.....$

$$\begin{gathered}\left[q_{ij}^{-},q_{ij}^{+}\right]=\left[q_{i-1,j+1}^{-},q_{i-1,j+1}^{+}\right](j=even);\\ \left[q_{ij}^{-},q_{ij}^{+}\right]=\left[q_{i-1,j+1}^{-},q_{i-1,j+1}^{+}\right]\left(\left[\alpha_{i-1}^{-},\alpha_{i-1}^{+}\right]\times\left[q_{i-1,j+2}^{-},q_{i-1,j+2}^{+}\right]\right)\left(j=odd\right)\end{gathered} \tag{4.53}$$

Step2: The reduced denominator polynomial is

$$\left[D_1^-, D_1^+\right](s) = s + \left[\alpha_1^-, \alpha_1^+\right]; \left[D_{-1}^-, D_{-1}^+\right](s) = \left[\frac{1}{s}, \frac{1}{s}\right]; \left[D_0^-, D_0^+\right](s) = [1,1]$$

$$\left[D_2^-, D_2^+\right](s) = s^2 + \left[\alpha_2^-, \alpha_2^+\right]s + \left[\alpha_1^-, \alpha_1^+\right]\left[\alpha_2^-, \alpha_2^+\right];$$

....

$$\left[D_k^-, D_k^+\right](s) = s^2\left[D_{k-2}^-, D_{k-2}^+\right](s) + \left[\alpha_k^-, \alpha_k^+\right]\left[D_{k-1}^-, D_{k-1}^+\right](s); \tag{4.54}$$

Step3: Construct Beat table for the numerator polynomial

Table 4.8: β (Routh) table

$\left[p_{10}^-, p_{10}^+\right] = \left[b_{11}^-, b_{11}^+\right]$	$\left[p_{11}^-, p_{11}^+\right] = \left[b_{12}^-, b_{12}^+\right]$	$\left[p_{12}^-, p_{12}^+\right] = \left[b_{13}^-, b_{13}^+\right]$	. . . .
$\left[q_{11}^-, q_{11}^+\right]$	0	$\left[q_{13}^-, q_{13}^+\right]$	. . . .
$\left[p_{20}^-, p_{20}^+\right]$	$\left[p_{21}^-, p_{21}^+\right]$	$\left[p_{22}^-, p_{22}^+\right]$	. . . .
$\left[q_{21}^-, q_{21}^+\right]$	0	$\left[q_{23}^-, q_{23}^+\right]$	. . . .
$\left[p_{30}^-, p_{30}^+\right]$	$\left[q_{31}^-, q_{31}^+\right]$	. . . .	
. . . .	. . . .	. . . .	
. . . .	. . . .	. . . .	
$\left[p_{n,0}^-, p_{n,0}^+\right]$			
$\left[q_{n,1}^-, q_{n,1}^+\right]$			

To calculate the values of $\left[\beta_i^-, \beta_i^+\right]$

$$\left[\beta_i^-, \beta_i^+\right] = \frac{\left[p_{i0}^-, p_{i0}^+\right]}{\left[q_{i1}^-, q_{i1}^+\right]}, \text{ where } i = 1,2,3,...... \tag{4.55}$$

For finding the values of $\left[p_{ij}^-, p_{ij}^+\right]$; where $i = 2,3,4,.....\ \&\ j = 0,1,2,3,.....$

$$\left[p_{ij}^-, p_{ij}^+\right] = \left[p_{i-1,j+1}^-, p_{i-1,j+1}^+\right](j = even);$$
$$\left[p_{ij}^-, p_{ij}^+\right] = \left[p_{i-1,j+1}^-, p_{i-1,j+1}^+\right]\left(\left[\beta_{i-1}^-, \beta_{i-1}^+\right] \times \left[p_{i-1,j+2}^-, p_{i-1,j+2}^+\right]\right)(j = odd) \tag{4.56}$$

Step4: The reduced numerator polynomial is

$$\left[N_1^-,N_1^+\right](s)=s+\left[\beta_1^-,\beta_1^+\right];\left[N_{-1}^-,N_{-1}^+\right](s)=0;\left[N_0^-,N_0^+\right](s)=0;$$
$$\left[N_2^-,N_2^+\right](s)=\left[\beta_2^-,\beta_2^+\right]s+\left[\alpha_2^-,\alpha_2^+\right]\left[\beta_1^-,\beta_1^+\right];$$
....
$$\left[N_k^-,N_k^+\right](s)=\left[\beta_k^-,\beta_k^+\right]s^{k-1}+s^2\left[N_{k-2}^-,N_{k-2}^+\right](s)+\left[\alpha_k^-,\alpha_k^+\right]\left[N_{k-1}^-,N_{k-1}^+\right](s); \tag{4.57}$$

Example 4.2: Consider a third order SISO system described by the transfer function [168]

$$G_3(s)=\frac{[1,1]s^2+[3.3,6.5]s+[2.7,10]}{[1,1]s^3+[8.5,8.6]s^2+[18,18.2]s+[10.25,10.76]} \tag{4.58}$$

$$\left[\alpha_1^-,\alpha_1^+\right]=\frac{\left[q_{10}^-,q_{10}^+\right]}{\left[q_{11}^-,q_{11}^+\right]}=[0.5632,0.5978]$$
$$\left[\alpha_2^-,\alpha_2^+\right]=\frac{\left[q_{20}^-,q_{20}^+\right]}{\left[q_{21}^-,q_{21}^+\right]}=[2.2397,2.3031]$$
$$\left[D_2^-,D_2^+\right](s)=s^2+[2.2397,2.3031]s+[1.2614,1.3768] \tag{4.59}$$

$$\left[\beta_1^-,\beta_1^+\right]=\frac{\left[p_{10}^-,p_{10}^+\right]}{\left[q_{11}^-,q_{11}^+\right]}=[0.1483,0.5555]$$
$$\left[\beta_2^-,\beta_2^+\right]=\frac{\left[p_{20}^-,p_{20}^+\right]}{\left[q_{21}^-,q_{21}^+\right]}=[0.4106,0.8225]$$
$$\left[N_2^-,N_2^+\right](s)=[0.4106,0.8225]s+[0.1483,0.5555] \tag{4.60}$$

Therefore the reduced order transfer function is

$$R_2(s)=\frac{[0.4106,0.8225]s+[0.1483,0.5555]}{s^2+[2.2397,2.3031]s+[1.2614,1.3768]} \tag{4.61}$$

A comparison of the step response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.43

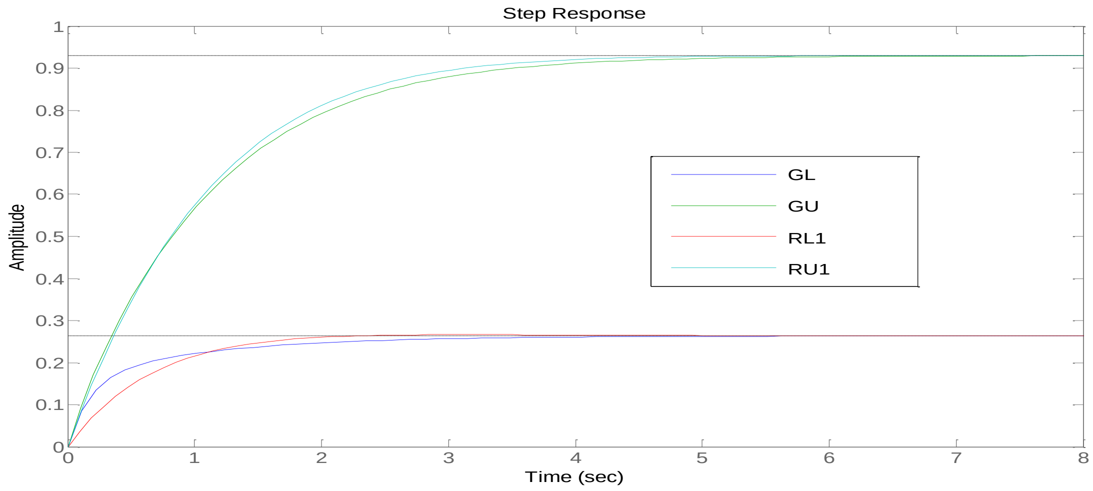

Fig. 4.43: Step response of original model and reduced model

Table 4.9: Comparison of Reduced Order Models of example 4.2

Method of Order reduction	**ISE for lower limit** **errL**	**ISE for upper limit** **errU**
Proposed method	0.000326	0.000774
Saraswathi [168]	0.0034	0.00368

Example 4.3: Consider a fifth order (MIMO) system described by the transfer function

$$G_5(s) = \frac{\begin{bmatrix} 3.83, 4.06 \\ 3.78, 4.00 \end{bmatrix} s^4 + \begin{bmatrix} 118.0, 125.0 \\ 95.8, 101.6 \end{bmatrix} s^3 + \begin{bmatrix} 339.6, 360.1 \\ 267.86, 283.9 \end{bmatrix} s^2 + \begin{bmatrix} 275.50, 280.10 \\ 233.53, 238.54 \end{bmatrix} s + \begin{bmatrix} 66.34, 70.32 \\ 66.34, 70.32 \end{bmatrix}}{\left[1, 1.03\right] s^5 + \left[24.6, 25.34\right] s^4 + \left[136.14, 140.23\right] s^3 + \left[282.72, 291.2\right] s^2 + \left[236.51, 243.61\right] s + \left[66.34, 70.32\right]} \quad (4.62)$$

$$\left[\alpha_1^-, \alpha_1^+\right] = \frac{\left[q_{10}^-, q_{10}^+\right]}{\left[q_{11}^-, q_{11}^+\right]} = \left[0.2723, 0.2973\right]$$

$$\left[\alpha_2^-, \alpha_2^+\right] = \frac{\left[q_{20}^-, q_{20}^+\right]}{\left[q_{21}^-, q_{21}^+\right]} = \left[1.0428, 1.1584\right]$$

$$\left[D_2^-, D_2^+\right](s) = s^2 + \left[1.0428, 1.1584\right] s + \left[0.2723, 0.2973\right] \quad (4.63)$$

$From\beta 1table$

$$\left[\beta_1^-, \beta_1^+\right] = \frac{\left[p_{10}^-, p_{10}^+\right]}{\left[q_{11}^-, q_{11}^+\right]} = \left[0.2723, 0.2973\right]$$

$$\left[\beta_2^-, \beta_2^+\right] = \frac{\left[p_{20}^-, p_{20}^+\right]}{\left[q_{21}^-, q_{21}^+\right]} = \left[1.2147, 1.3319\right]$$

$$\left[B_2^-, B_2^+\right](s) = \left[1.2147, 1.3319\right] s + \left[0.2723, 0.2973\right] \quad (4.64)$$

$From\beta 2table$

$$\left[\beta_1^-, \beta_1^+\right] = \frac{\left[p_{10}^-, p_{10}^+\right]}{\left[q_{11}^-, q_{11}^+\right]} = \left[0.2723, 0.2973\right]$$

$$\left[\beta_2^-, \beta_2^+\right] = \frac{\left[p_{20}^-, p_{20}^+\right]}{\left[q_{21}^-, q_{21}^+\right]} = \left[1.03, 1.1343\right]$$

$$\left[B_2^-, B_2^+\right](s) = \left[1.03, 1.1343\right] s + \left[0.2723, 0.2973\right] \quad (4.65)$$

The reduced order model is

$$R_2(s)=\frac{\begin{bmatrix}[1.2147,1.3319]\\ [1.03,1.1343]\end{bmatrix}s+\begin{bmatrix}[0.2839,0.3444]\\ [0.2839,0.3444]\end{bmatrix}}{s^2+[1.0428,1.1584]s+[0.2839,0.3444]} \tag{4.66}$$

A comparison of the step response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.44 and Fig. 4.45.

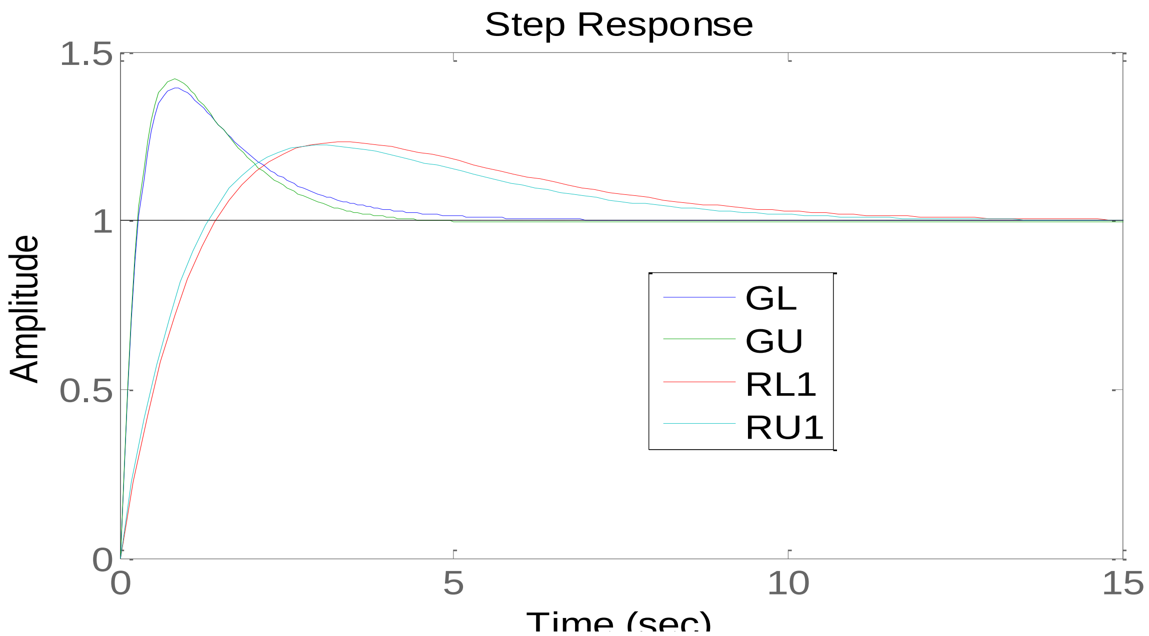

Fig. 4.44: Step Response of original model and reduced model of 1st output and input

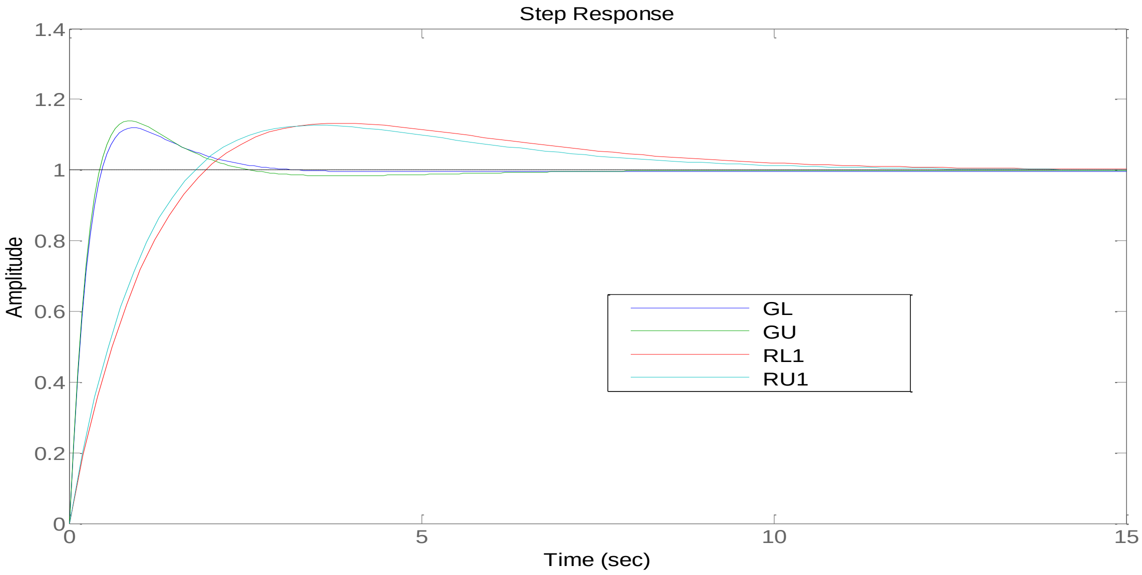

Fig. 4.45: Step Response of original model and reduced model of 2^{nd} output and input

4.6 DIRECT TRUNCATION METHOD FOR INTERVAL SYSTEMS

The truncation method is extended to interval systems, where successively low order models are produced by neglecting progressively higher order terms from numerator and denominator of higher order terms from numerator and denominator of higher order system until the minimum order model with satisfactory performance is achieved. The method compared with other existing Routh based methods of and concluded that the truncation method to interval systems requires no computation is as good as these methods.

4.6.1 Advantages of Direct Truncation Method

- It is easy to prove that reduced order model fits the first k time moments of the original system transfer function.
- The main advantage of the truncation method over the proposed and existing techniques is its simplicity and the fact that it requires no computational effort.

Example 4.1: Consider a third order system described by the transfer function [157]

$$G_3(s)=\frac{[2,3]s^2+[17.5,18.5]s+[15,16]}{[2,3]s^3+[17,18]s^2+[35,36]s+[20.5,21.5]} \tag{4.67}$$

Case1: Direct truncation Method

For getting second order model, eliminating higher order denominator and numerator coefficients directly for obtain reduced order system

$$\begin{aligned}D_2(s)&=[17,18]s^2+[35,36]s+[20.5,21.5]\\N_2(s)&=[17.5,18.5]s+[15,16]\end{aligned} \tag{4.68}$$

The reduced order polynomial is

$$R_2(s)=\frac{[17.5,18.5]s+[15,16]}{[17,18]s^2+[35,36]s+[20.5,21.5]} \tag{4.69}$$

Table 4.10: Comparison of DTM and Existing Reduced Order Models of example 4.1

S.No	**Methods**	**ISE**		**IAE**	
		Lower limit	**Upper limit**	**Lower limit**	**Upper limit**
1	Bandypodayay et al. , [156]	0.00878538	8.87354E-05	0.264278635	0.009542405
2	Bandypodayay et al., [157]	2.36319E-05	4.39626E-06	0.005001512	0.002108643
3	Dolgin and Zeheb [162]	0.008876522	8.01481E-05	0.265977631	0.009118821
4	Sastry et al., [161]	0.225673362	0.00949689	1.343641109	0.275550326

5	Chuan-qing and Yang [171]	0.010817047	0.010362791	0.290640962	0.287856756
6	Pratheep et al., [176]	1.19446E-05	9.93222E-07	0.003594415	0.001092456
7	Siva Kumar et. al [177]	2.93246E-06	0.001205443	0.002187121	0.098172255
8	Proposed Method (Direct Truncation)	1.96148E-05	3.99567E-05	0.00443081	0.006339729

Comparison of the step response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.46 and Fig. 4.47.

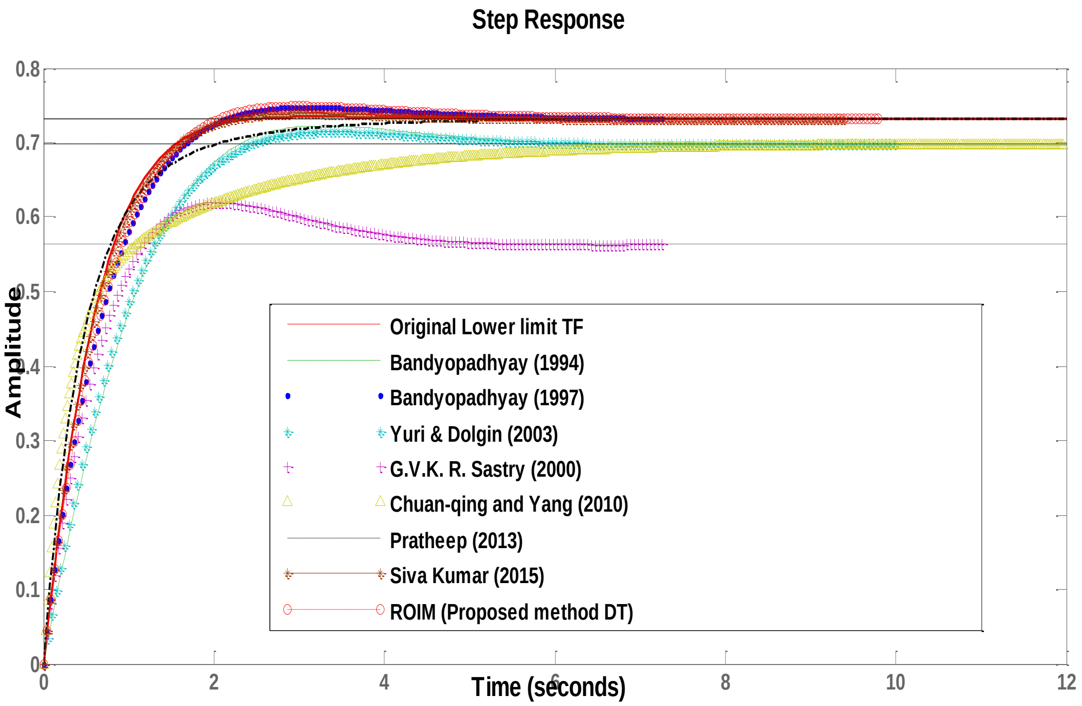

Fig.4.46.Step response for Direct truncation method (lower limit TF)

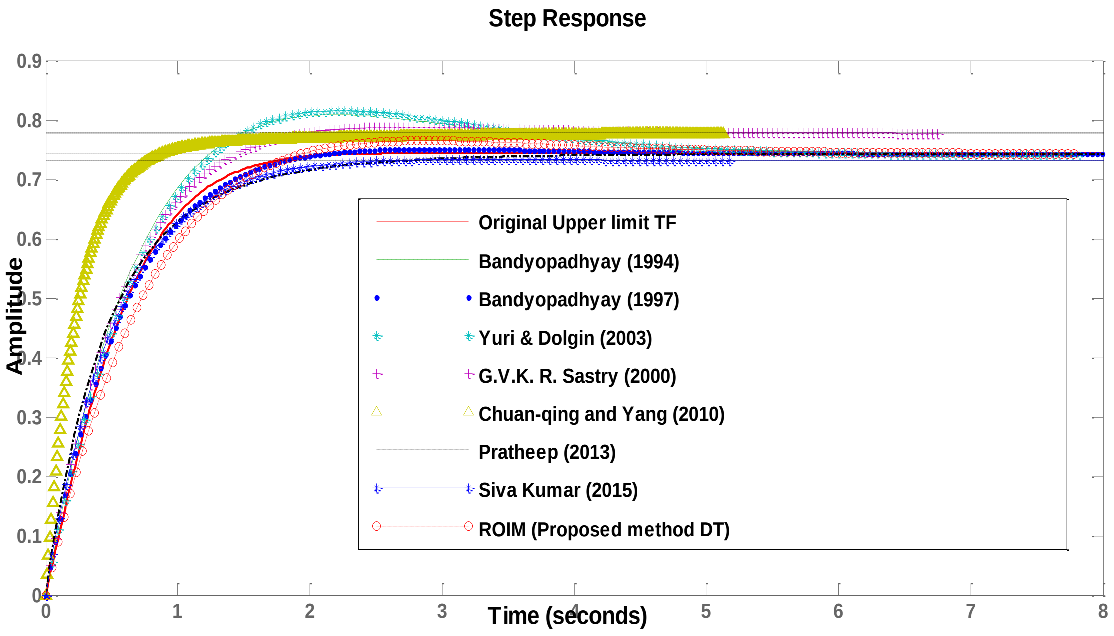

Fig.4.47. Step response for Direct truncation method (upper limit TF)

Comparison of the impulse response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.48 and Fig. 4.49.

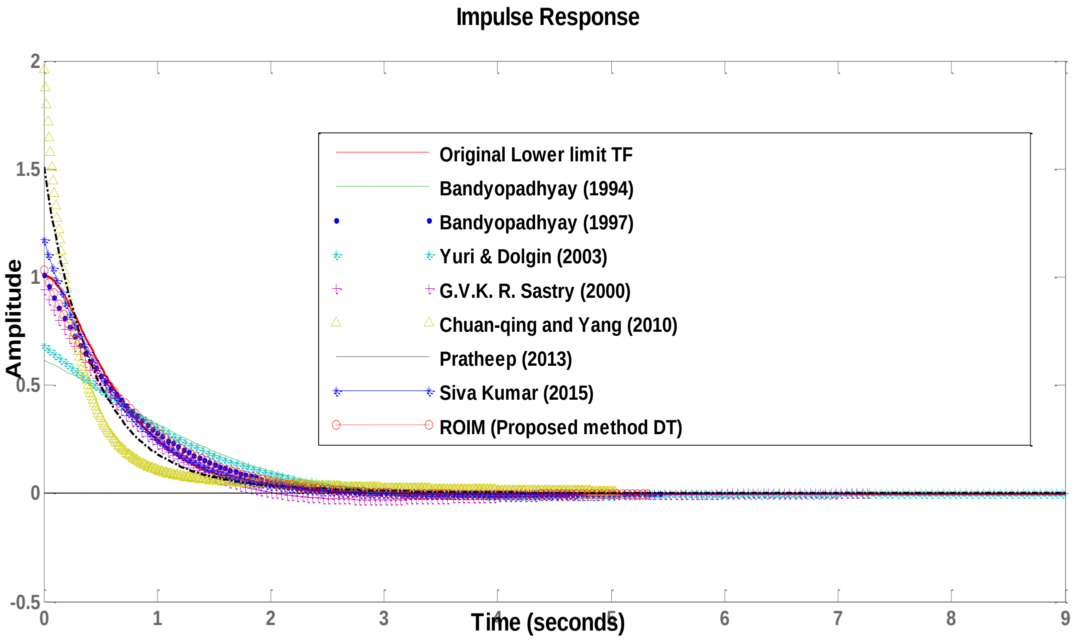

Fig.4.48. Impulse response for Direct truncation method (lower limit TF)

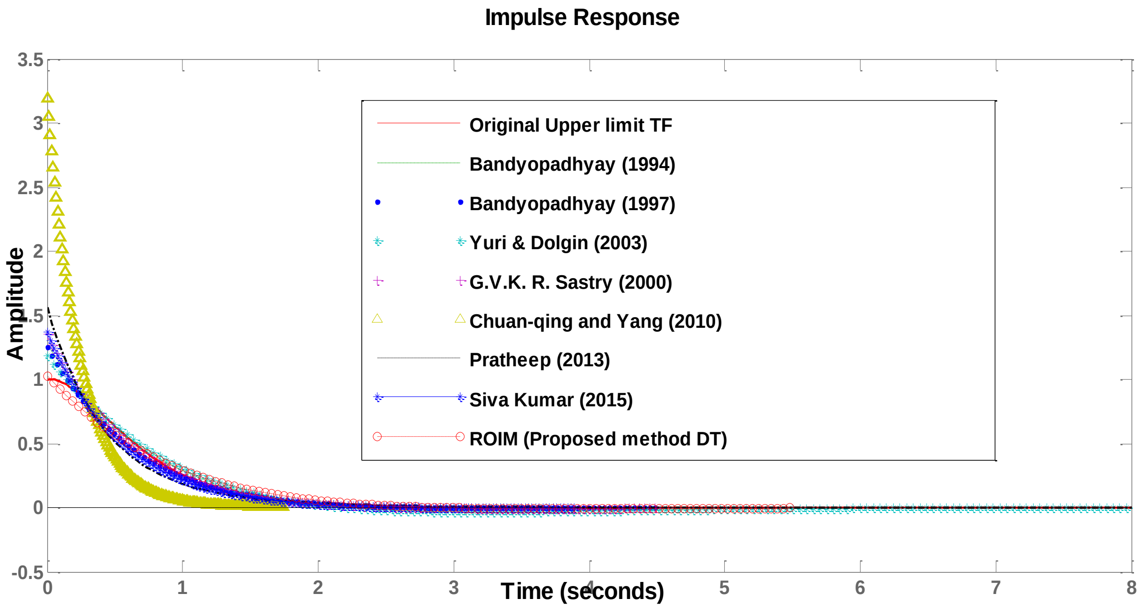

Fig.4.49. Impulse response for Direct truncation method (upper limit TF)

Comparison of the impulse response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.50 and Fig. 4.51.

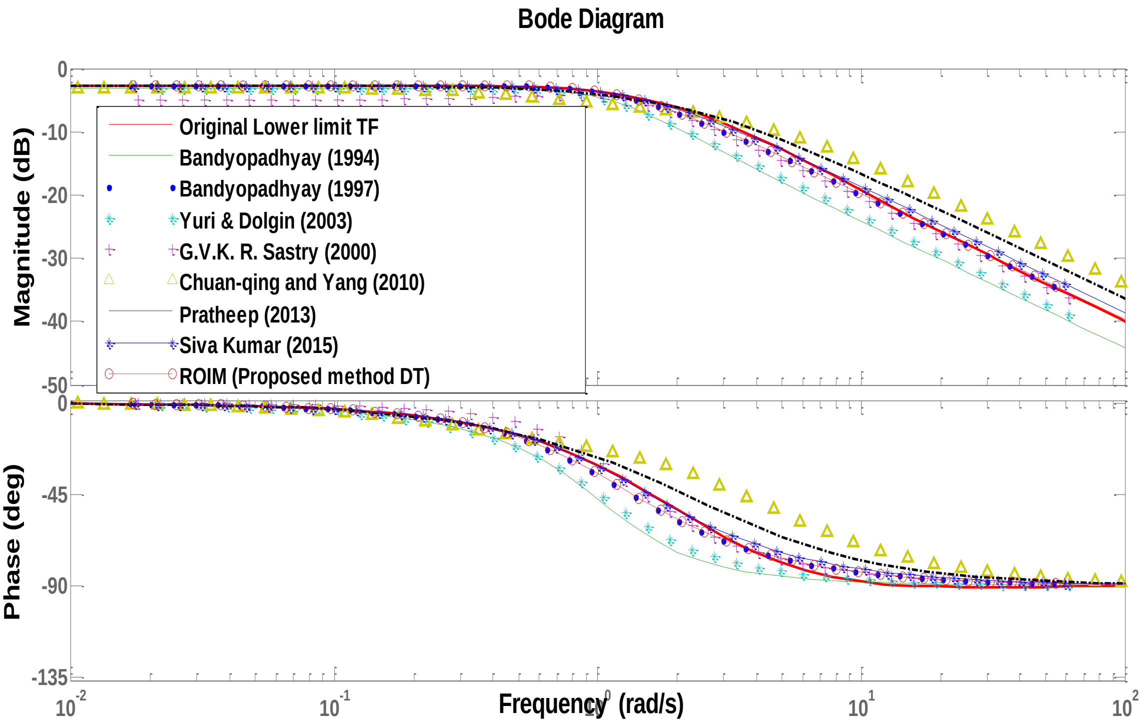

Fig.4.50. Bode plot for Direct truncation method (lower limit TF)

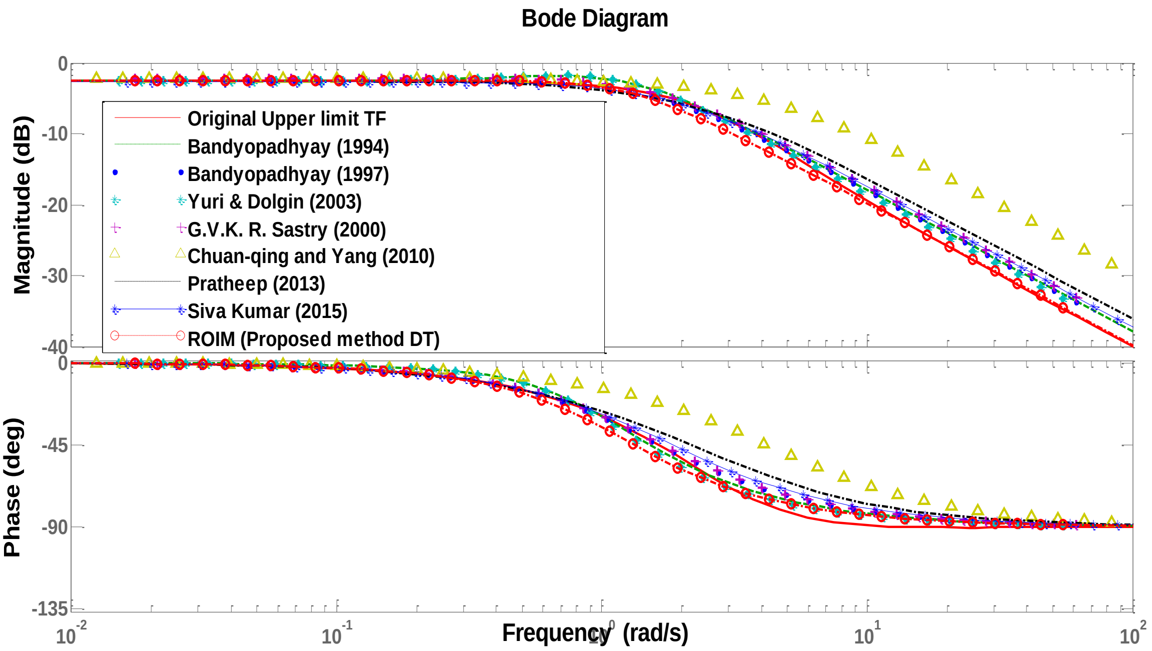

Fig.4.51. Bode plot for Direct truncation method (upper limit TF)

Comparison of the nyquist response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.52 and Fig. 4.53.

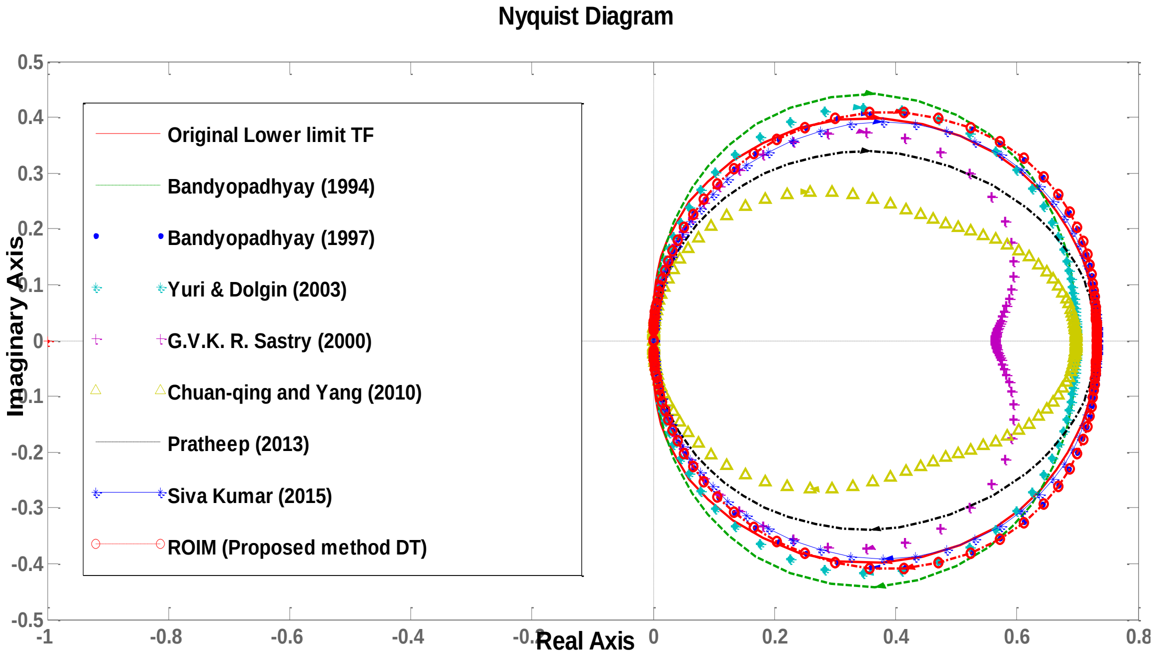

Fig.4.52. Nyquist plot for Direct truncation method (lower limit TF)

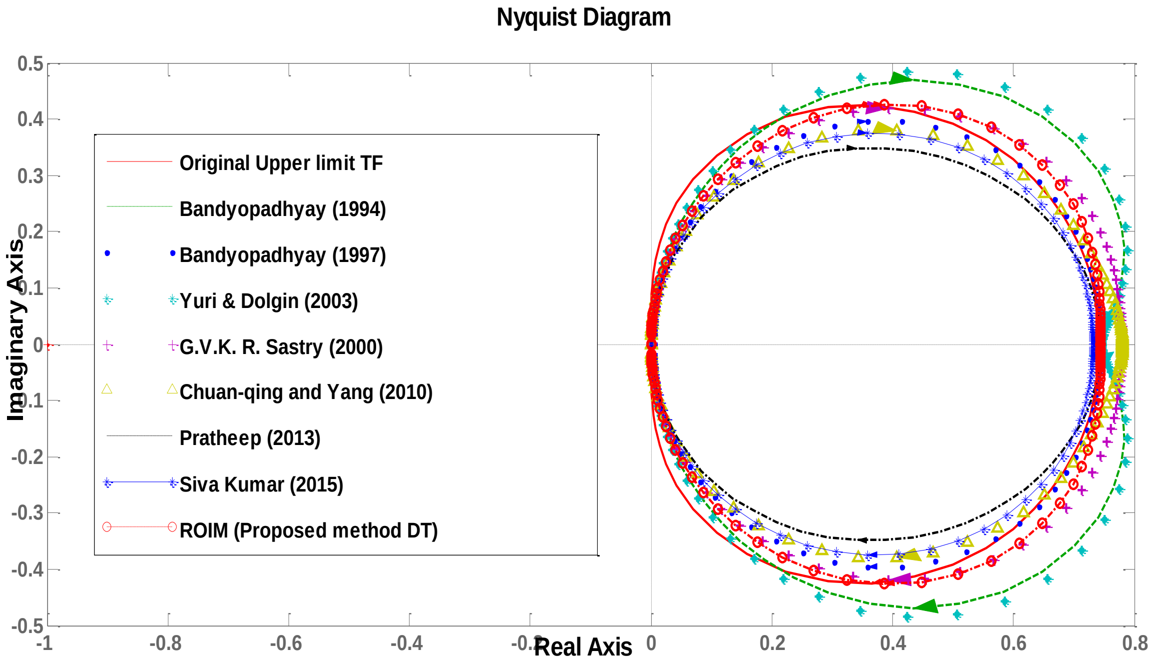

Fig.4.53.Nyquist plot for Direct truncation method (upper limit TF)

Case 2: Reduction by direct truncation method and factor division method

The reduced denominator polynomial is

$$D_2(s)=[17,18]s^2+[35,36]s+[20.5,21.5] \tag{4.70}$$

Reduced order numerator polynomial is obtain by factor division method

Step 1: Using the factor division method

$$\frac{N(s)D_2(s)}{D(s)}=\frac{[307.5,344]+[883.75,973.75]s+.....}{[20.5,21.5]+[35,36]s+......} \tag{4.71}$$

Step 2: Finding the values of $\left[\alpha_{11}^{-},\alpha_{11}^{+}\right]$ and $\left[\alpha_{12}^{-},\alpha_{12}^{+}\right]$

$$\left[\alpha_{11}^{-},\alpha_{11}^{+}\right]=[14.3023,16.7805];\left[\alpha_{12}^{-},\alpha_{12}^{+}\right]=[12.4021,24.2073]$$

$$N_2(s)=[12.4021,24.2073]s+[14.3023,16.7805] \tag{4.72}$$

Step3: The reduced transfer function is

$$R_2(s) = \frac{[12.4021, 24.2073]s + [14.3023, 16.7805]}{[17,18]s^2 + [35,36]s + [20.5, 21.5]} \tag{4.73}$$

Table 4.11: Comparison of DTFDM and Existing Reduced Order Models of example 4.1

S.No	**Methods**	**ISE**		**IAE**	
		Lower limit	**Upper limit**	**Lower limit**	**Upper limit**
1	Bandypodayay et al. , [156]	0.00878538	8.87354E-05	0.264278635	0.009542405
2	Bandypodayay et al., [157]	2.36319E-05	4.39626E-06	0.005001512	0.002108643
3	Dolgin and Zeheb [162]	0.008876522	8.01481E-05	0.265977631	0.009118821
4	Sastry et al., [161]	0.225673362	0.00949689	1.343641109	0.275550326
5	Chuan-qing and Yang [171]	0.010817047	0.010362791	0.290640962	0.287856756
6	Pratheep et al., [176]	1.19446E-05	9.93222E-07	0.003594415	0.001092456
7	Siva Kumar et. al [177]	2.93246E-06	0.001205443	0.002187121	0.098172255
8	Proposed Method (DTFDM)	0.009227579	0.011358432	0.27169542	0.300295425

Comparison of the step response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.54 and Fig. 4.55.

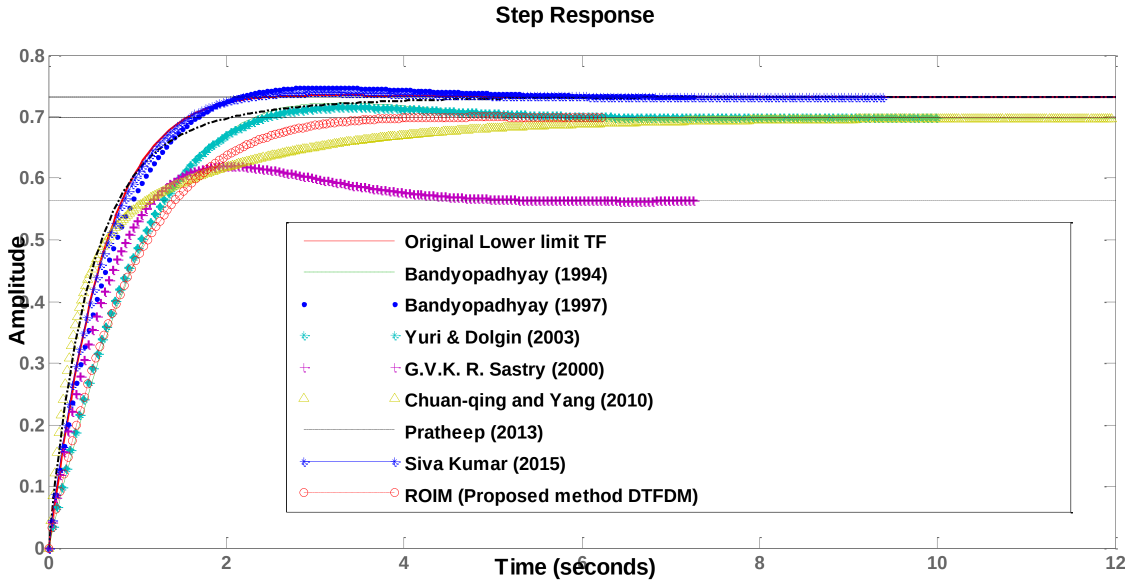

Fig.4.54. Step response for Direct truncation and Factor division method (lower limit TF)

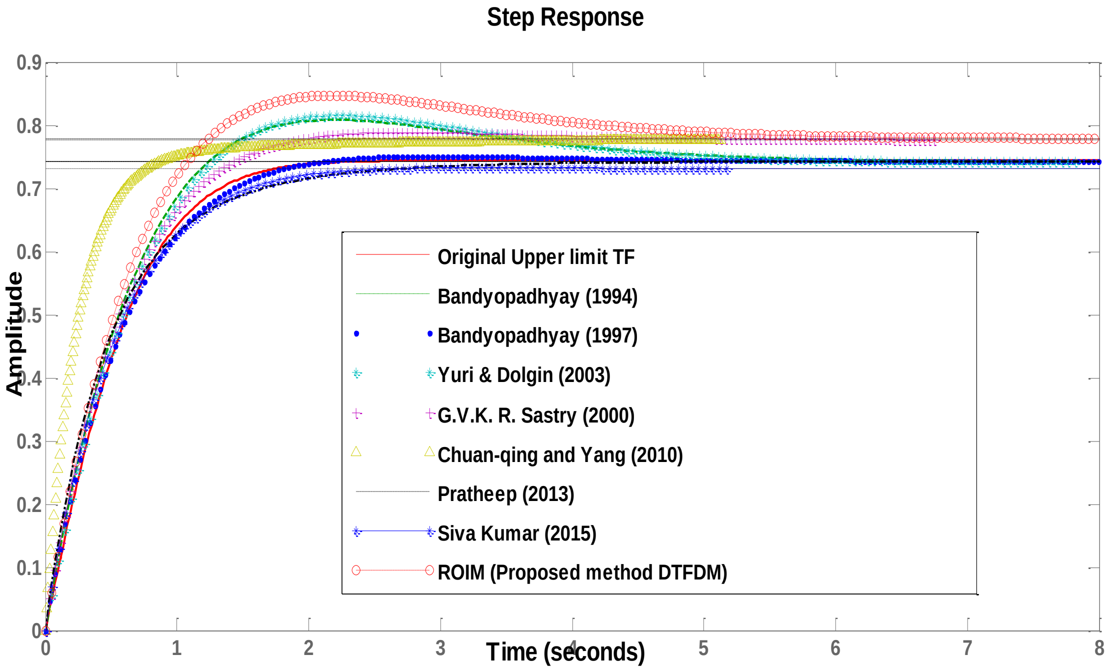

Fig.4.55. Step response for Direct truncation and Factor division method (upper limit TF)

Comparison of the impulse response of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.56 and Fig. 4.57.

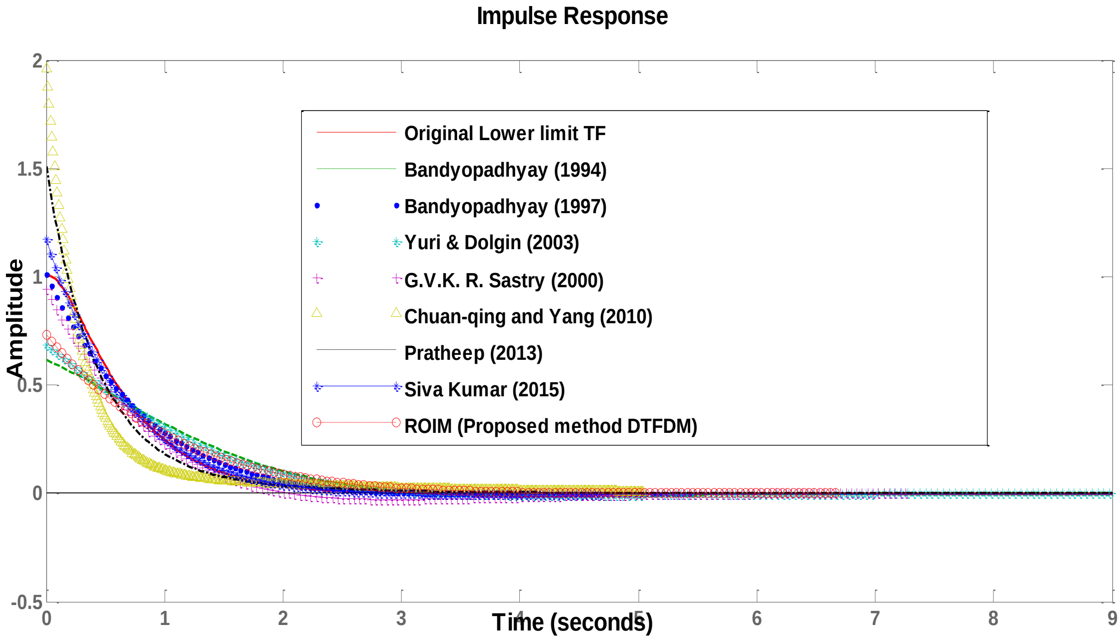

Fig.4.56. Impulse response for Direct truncation and Factor division method (lower limit TF)

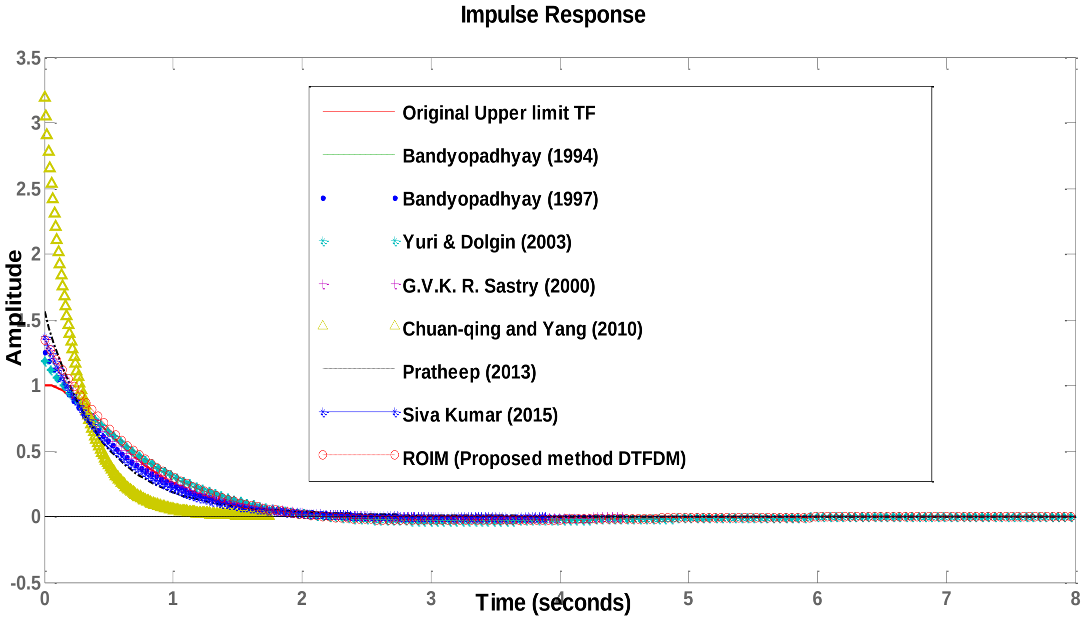

Fig.4.57. Impulse response for Direct truncation and Factor division method (upper limit TF)

Comparison of the bode plot of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.58 and Fig. 4.59.

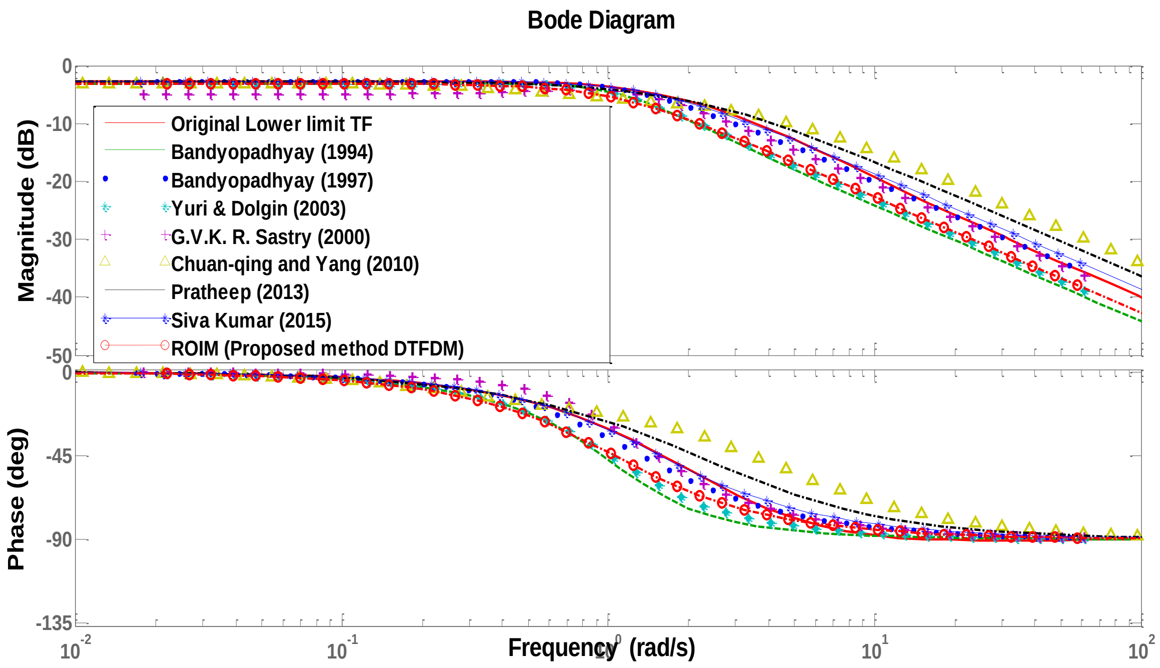

Fig.4.58. Bode plot for Direct truncation and Factor division method (lower limit TF)

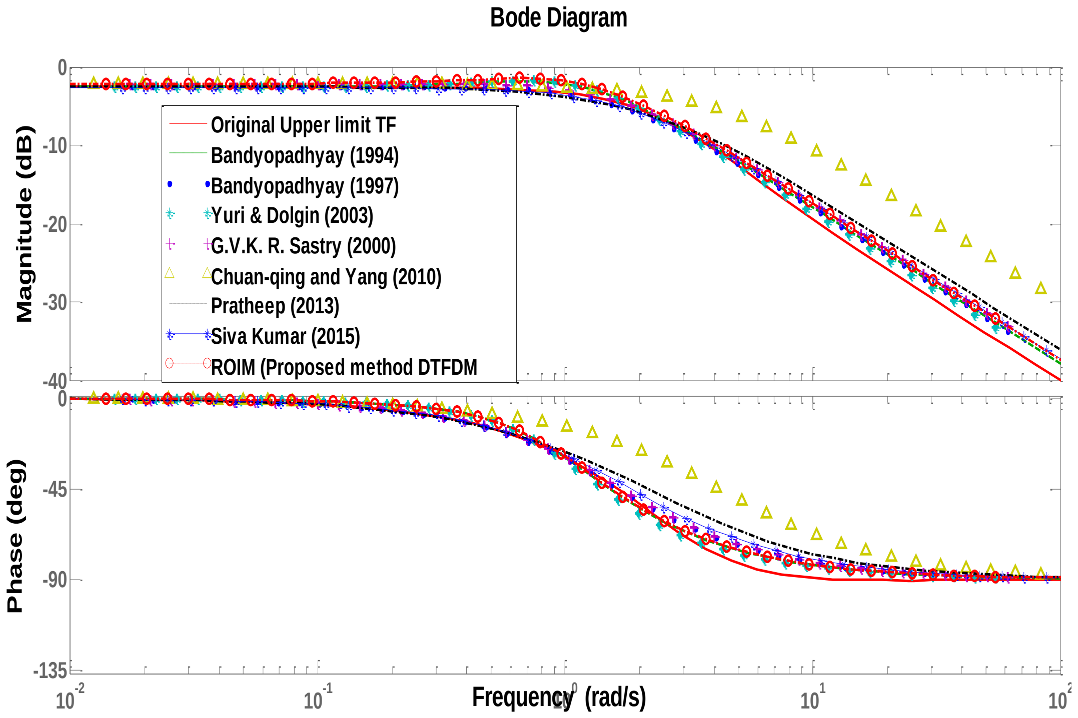

Fig.4.59. Bode plot for Direct truncation and Factor division method (upper limit TF)

Comparison of the nyquist plot of the lower limit and upper limit of the model obtained by proposed method is shown in Fig. 4.60 and Fig. 4.61.

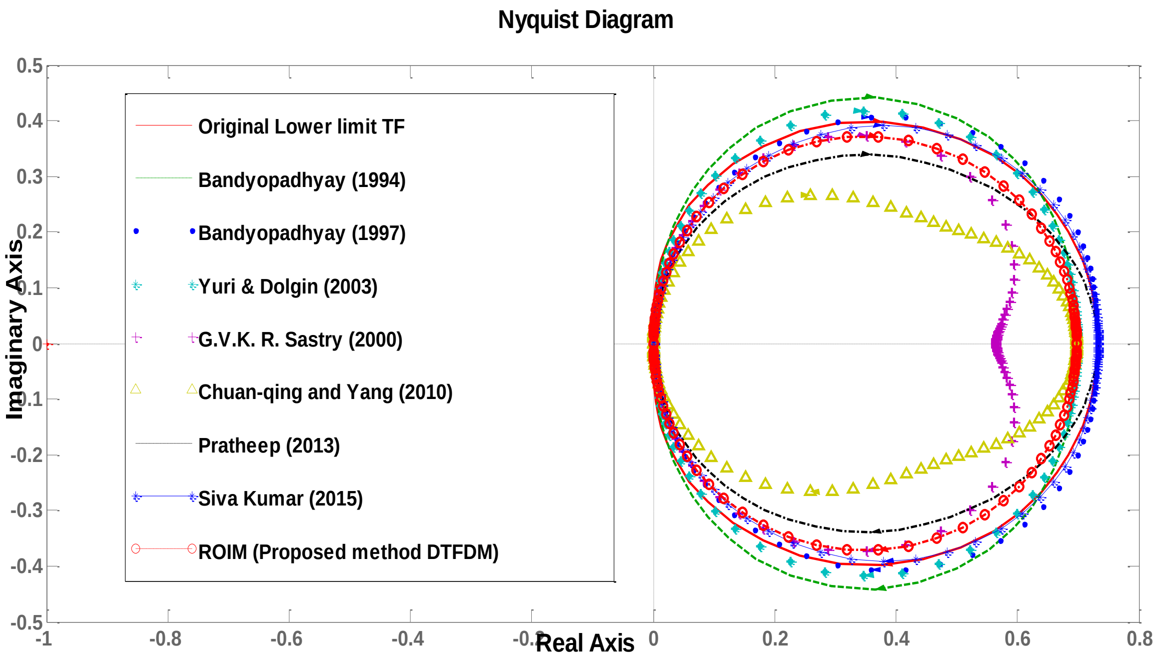

Fig.4.60. Nyquist plot for Direct truncation and Factor division method (lower limit TF)

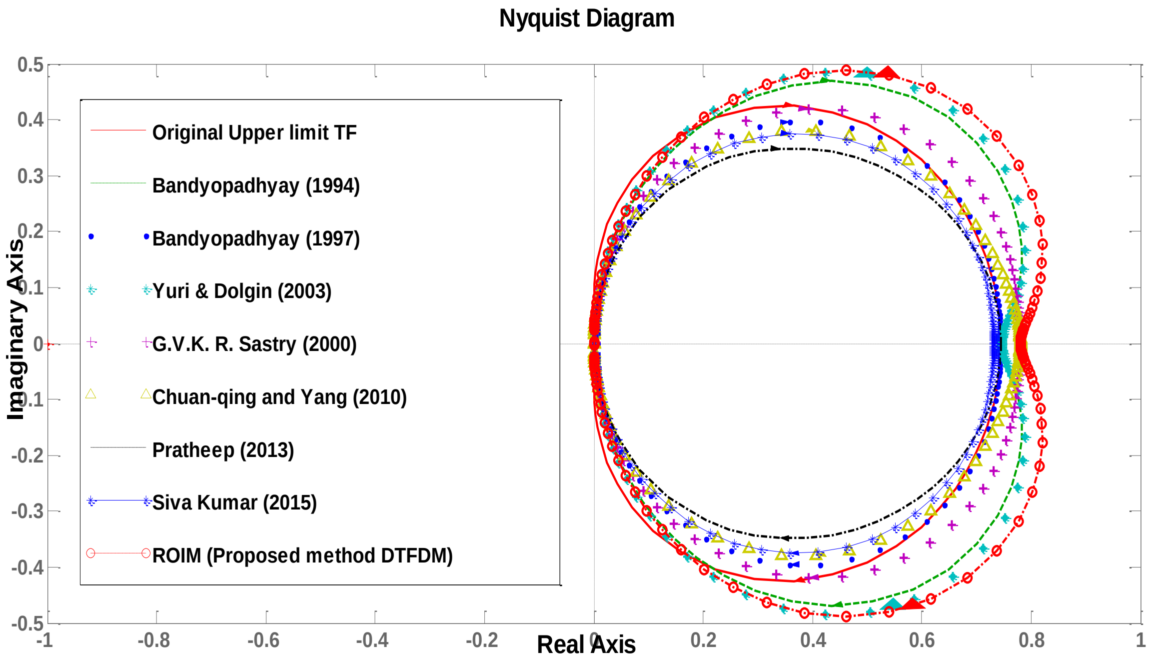

Fig.4.61. Nyquist plot for Direct truncation and Factor division method (upper limit TF)

Example 4.3: Consider a fifth order (MIMO) system described by the transfer function

$$G_5(s)=\frac{\begin{bmatrix}3.83,4.06\\3.78,4.00\end{bmatrix}s^4+\begin{bmatrix}118.0,125.0\\95.8,101.6\end{bmatrix}s^3+\begin{bmatrix}339.6,360.1\\267.86,283.9\end{bmatrix}s^2+\begin{bmatrix}275.50,280.10\\233.53,238.54\end{bmatrix}s+\begin{bmatrix}66.34,70.32\\66.34,70.32\end{bmatrix}}{[1,1.03]s^5+[24.6,25.34]s^4+[136.14,140.23]s^3+[282.72,291.2]s^2+[236.51,243.61]s+[66.34,70.32]} \tag{4.74}$$

The second order reduced order model is

$$G_2(s)=\frac{\begin{bmatrix}275.50,280.10\\233.53,238.54\end{bmatrix}s+\begin{bmatrix}66.34,70.32\\66.34,70.32\end{bmatrix}}{[282.72,291.2]s^2+[236.51,243.61]s+[66.34,70.32]} \tag{4.75}$$

4.7 DIFFERENTIATION METHOD FOR INTERVAL SYSTEMS

4.7.1 Direct Differentiation method for Interval Systems.

The differentiation method for model reduction of fixed coefficients was first proposed by Gutman et al., [115], later this method was extended to interval systems. In this method there is no necessary for the modifications to achieve stable reduced order models. But Routh approximation based methods applying directly to interval systems fails to produced stable reduced order interval systems. This is an advantage of the differentiation method compared with Routh based approximations. The method is based on differentiation of interval polynomials. The differentiation method is one of the stability preservation methods.

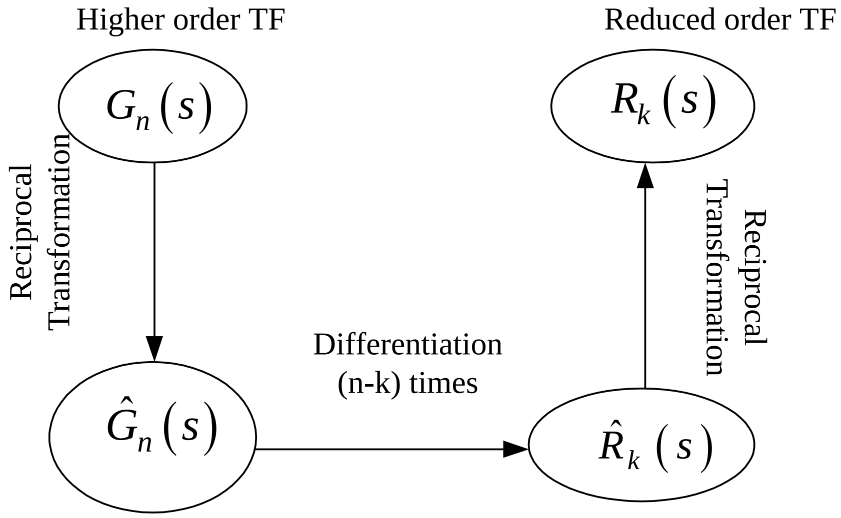

Fig. 4.62. Block diagram of differentiation method.

Algorithm:

Step1: Reciprocal transformation of the higher order interval system.

Step2: Differentiate the denominator and numerator interval polynomial $(n-k)$ times.

Step3: Second reciprocal transformation of the interval system.

Example 4.1: Consider a third order system described by the transfer function [157]

$$G_3(s)=\frac{[2,3]s^2+[17.5,18.5]s+[15,16]}{[2,3]s^3+[17,18]s^2+[35,36]s+[20.5,21.5]} \tag{4.76}$$

Step 1: Reciprocal transformation of higher order system

$$\hat{G}_3(s)=\frac{[15,16]s^2+[17.5,18.5]s+[2,3]}{[20.5,21.5]s^3+[35,36]s^2+[17,18]s+[2,3]} \tag{4.77}$$

Step 2: Differentiation of step1

$$\hat{R}_2(s)=\frac{[30,32]s+[17.5,18.5]}{[61.5,64.5]s^2+[70,72]s+[35,36]} \tag{4.78}$$

Step 3: Reciprocal Transformation of step 2

$$\bar{R}_2(s) = \frac{[17.5, 18.5]s + [30, 32]}{[35, 36]s^2 + [70, 72]s + [61.5, 64.5]} \tag{4.79}$$

Step 4: The above method has large steady state error in the step response. This can be removed by comparing dc gain of the higher order system and reduced order model which results in a gain factor $K = \left[1.3409, 1.6781\right]$. Finally the reduced order system is

$$R_2(s) = \frac{[23.4657, 31.0448]s + [40.22, 53.6992]}{[35, 36]s^2 + [70, 72]s + [61.5, 64.5]} \tag{4.80}$$

The model obtained by Bandyopadhyay et al., [157] is as follows

$$R_2(s) = \frac{\left[1.0091, 1.2552\right]s + \left[0.8409, 1.1168\right]}{s^2 + \left[2.0181, 2.4430\right]s + \left[1.1492, 1.5007\right]} \tag{4.81}$$

The model obtained by Sastry et al., [161] is given by

$$R_2(s) = \frac{\left[0.94, 1.35\right]s + \left[0.8409, 1.168\right]}{s^2 + \left[2.0181, 2.4430\right]s + \left[1.1492, 1.5007\right]} \tag{4.82}$$

A comparison of the step response of the lower limit and upper limit of the model obtained by proposed method and original model is shown in Fig. 4.63 and Fig.4.64.

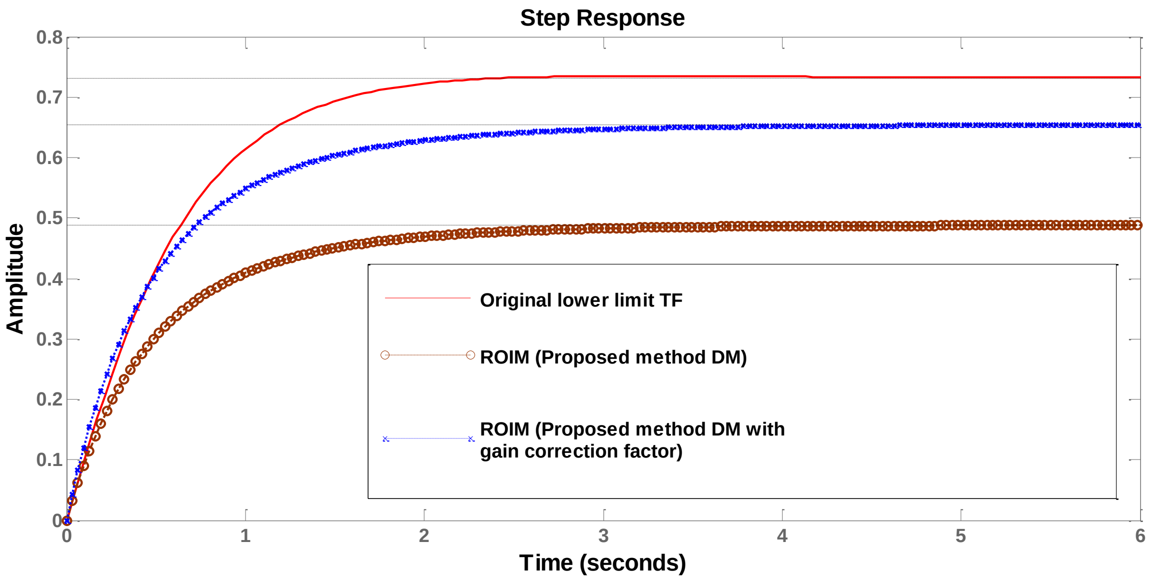

Fig.4.63. Step response for Differentiation method for gain correction factor (lower limit TF)

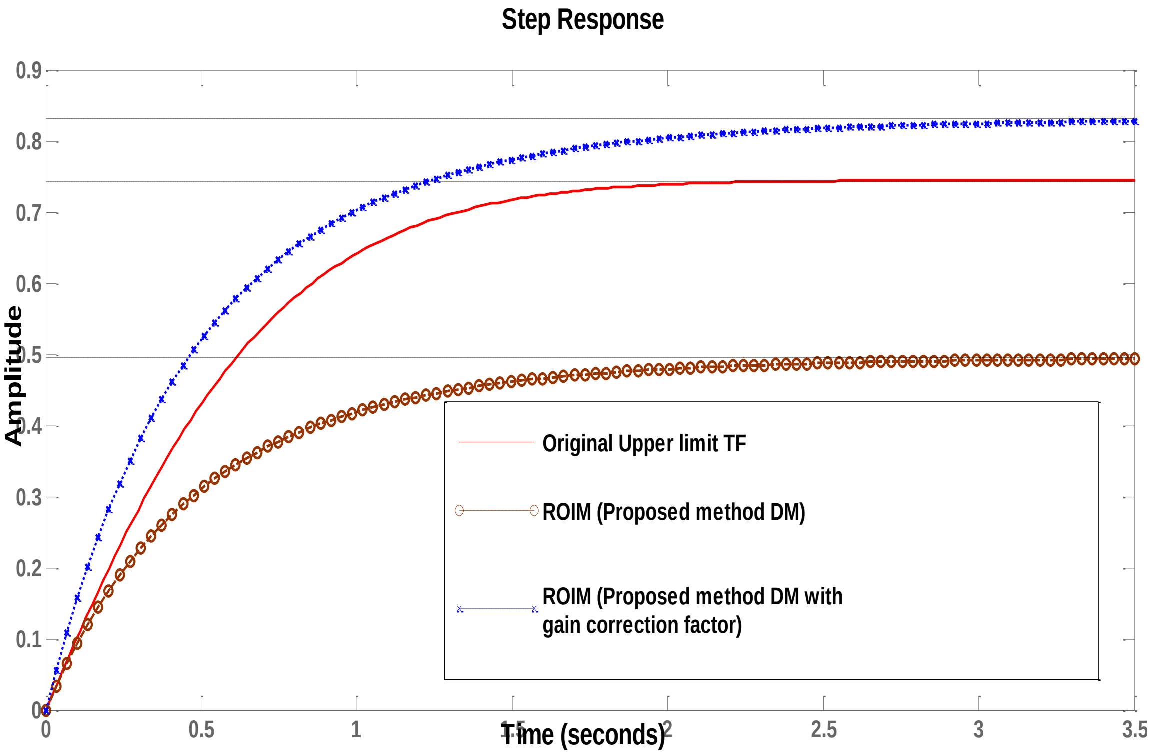

Fig.4.64. Step response for Differentiation method for gain correction factor (upper limit TF)

A comparison of the impulse response of the lower limit and upper limit of the model obtained by proposed method and original model is shown in Fig. 4.65 and Fig.4.66.

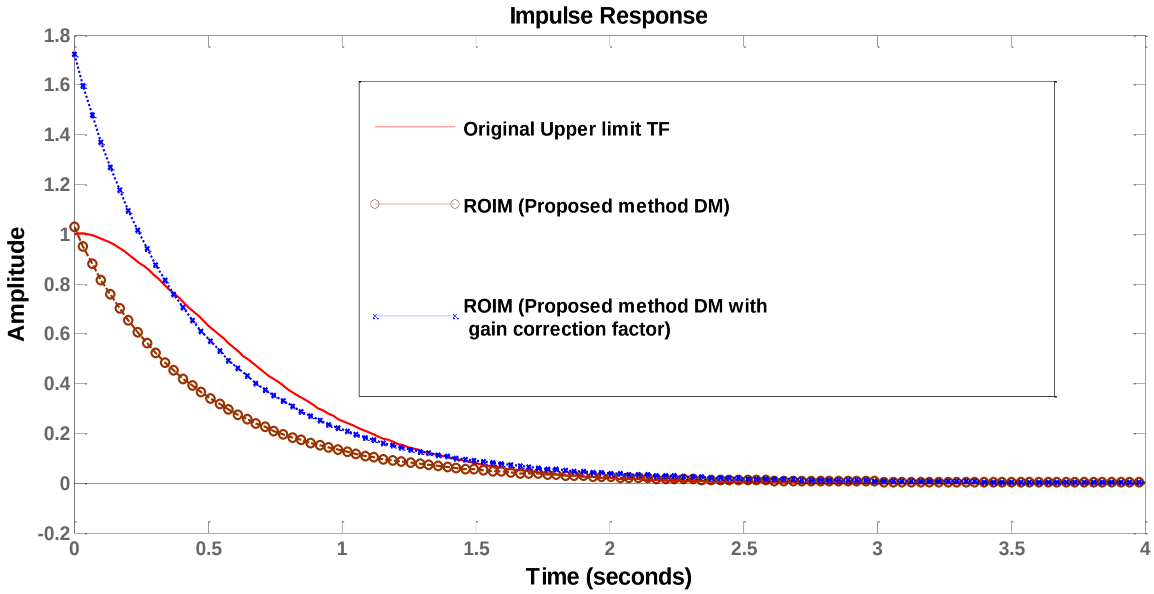

Fig.4.65. Impulse response for Differentiation method for gain correction factor (lower limit TF)

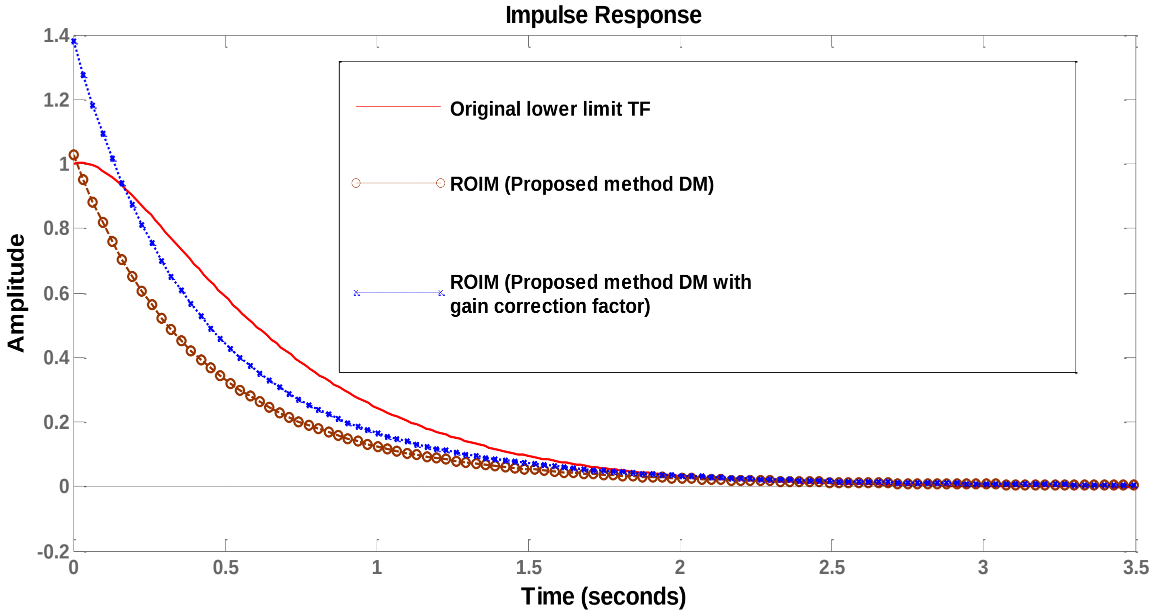

Fig.4.66. Impulse response for Differentiation method for gain correction factor (upper limit TF)

A comparison of the bode plot of the lower limit and upper limit of the model obtained by proposed method and original model is shown in Fig. 4.67 and Fig.4.68.

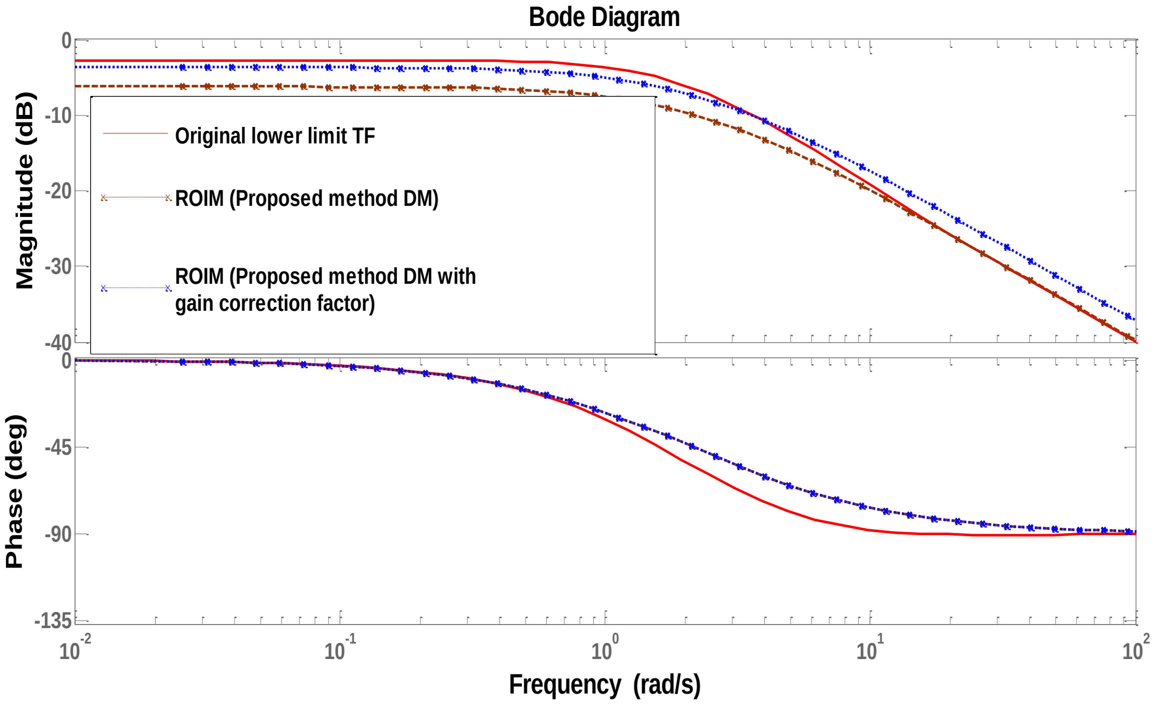

Fig.4.67. Bode plot for Differentiation method for gain correction factor (lower limit TF)

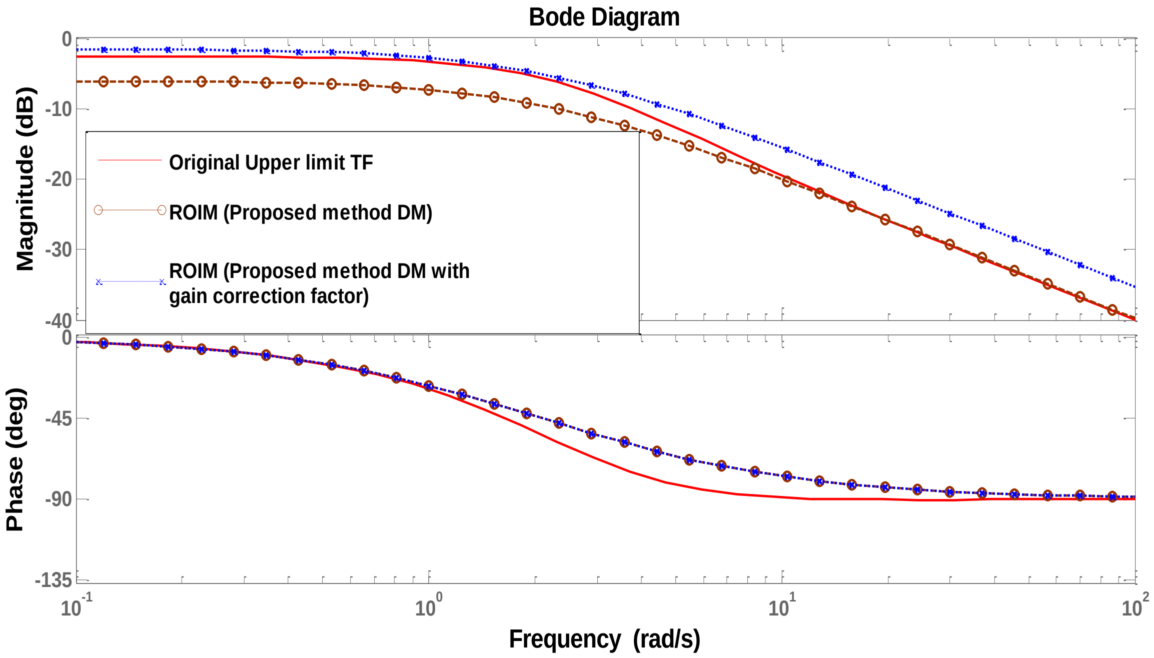

Fig.4.68. Bode plot for Differentiation method for gain correction factor (upper limit TF)

A comparison of the nyquist plot of the lower limit and upper limit of the model obtained by proposed method and original model is shown in Fig. 4.69 and Fig.4.70.

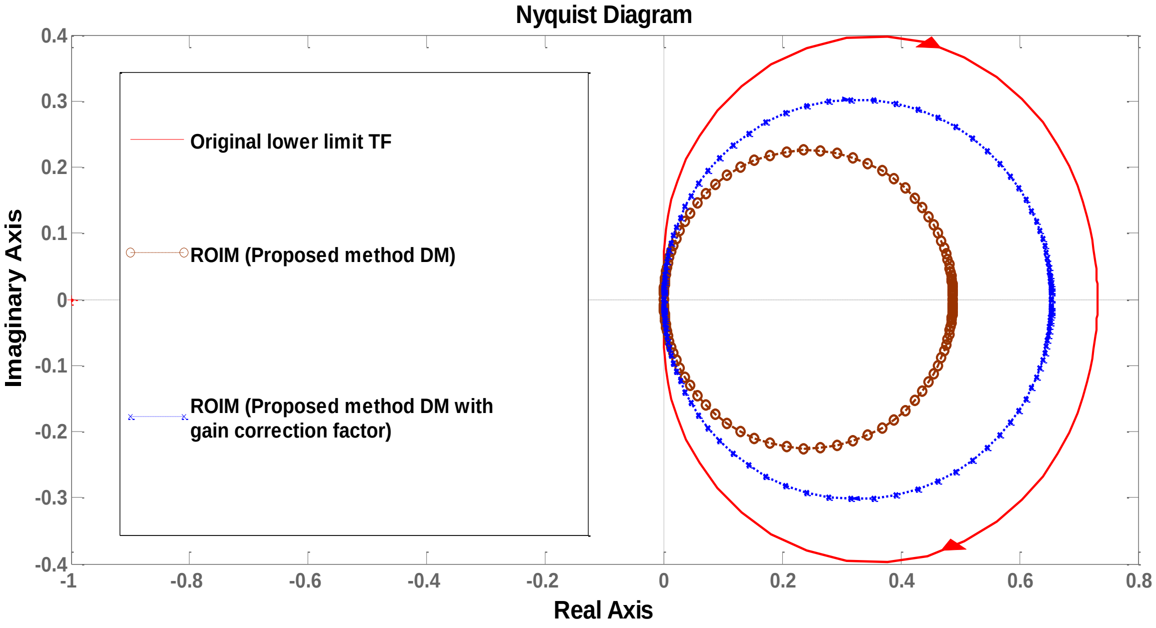

Fig.4.69. Nyquist plot for Differentiation method for gain correction factor (lower limit TF)

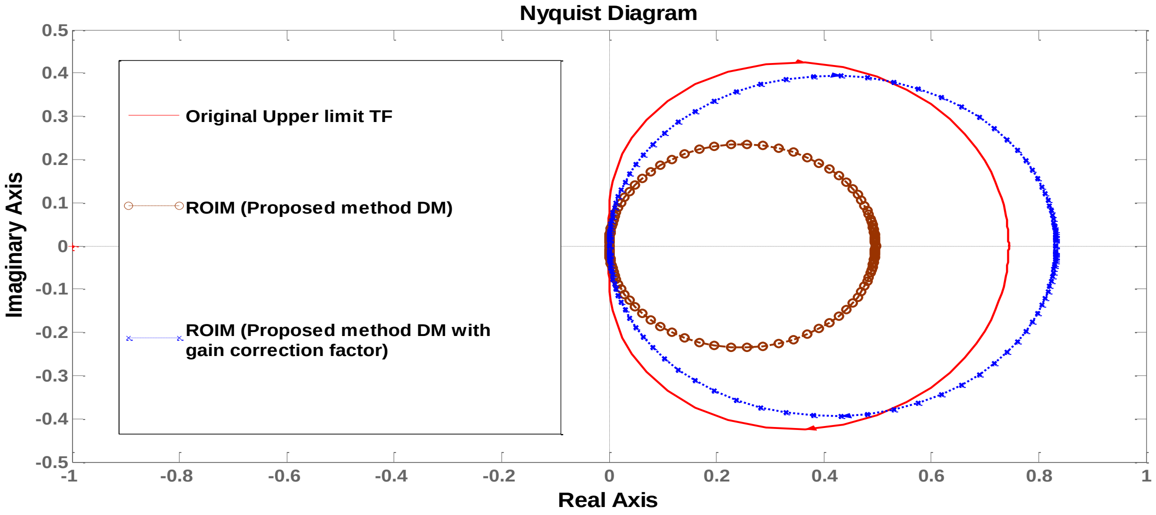

Fig.4.70. Nyquist plot for Differentiation method for gain correction factor (upper limit TF)

Case 2: Differentiation method and factor division method

$$R_2(s)=\frac{[13.923,43.154]s+[42.907,50.34]}{[17,18]s^2+[70,72]s+[61.5,64.5]} \tag{4.83}$$

A comparison of the step response of the lower limit and upper limit of the model obtained by proposed method and existing methods is shown in Fig. 4.71 and Fig.4.72.

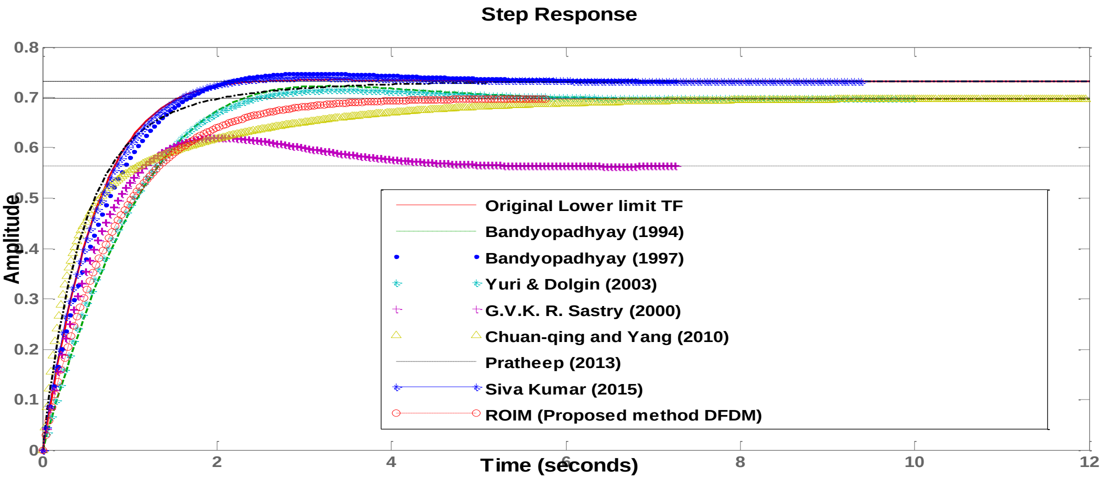

Fig.4.71. Step response for Differentiation method and Factor division method (lower limit TF)

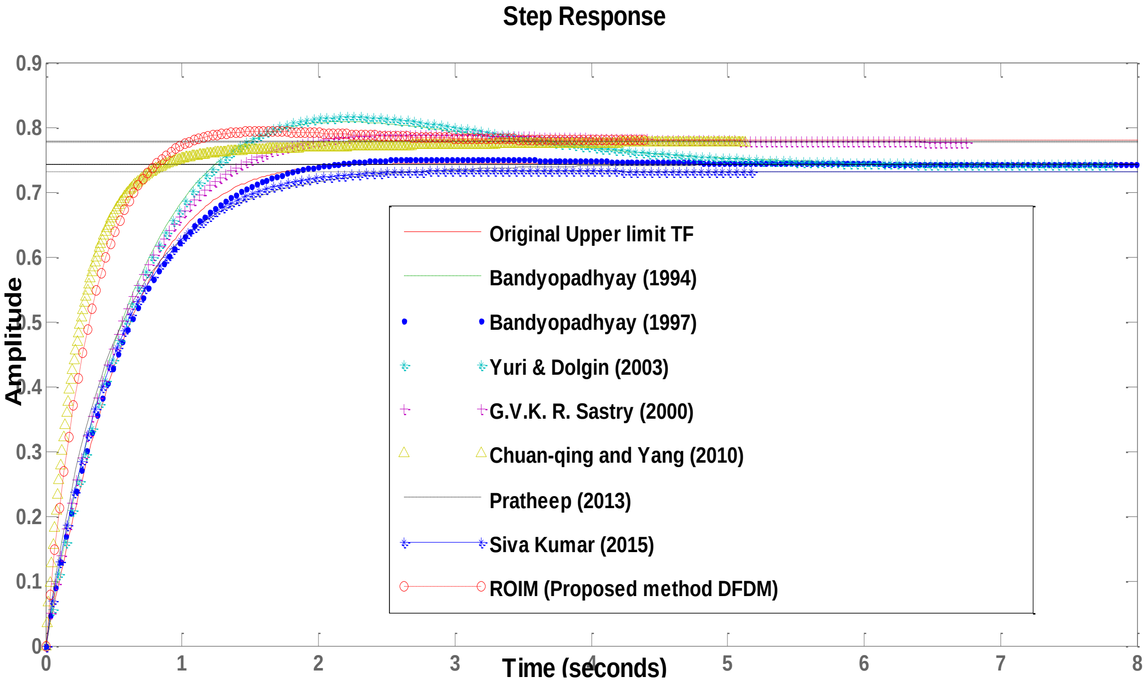

Fig.4.72. Step response for Differentiation method and Factor division method (upper limit TF)

A comparison of the impulse response of the lower limit and upper limit of the model obtained by proposed method and existing methods is shown in Fig. 4.73 and Fig.4.74.

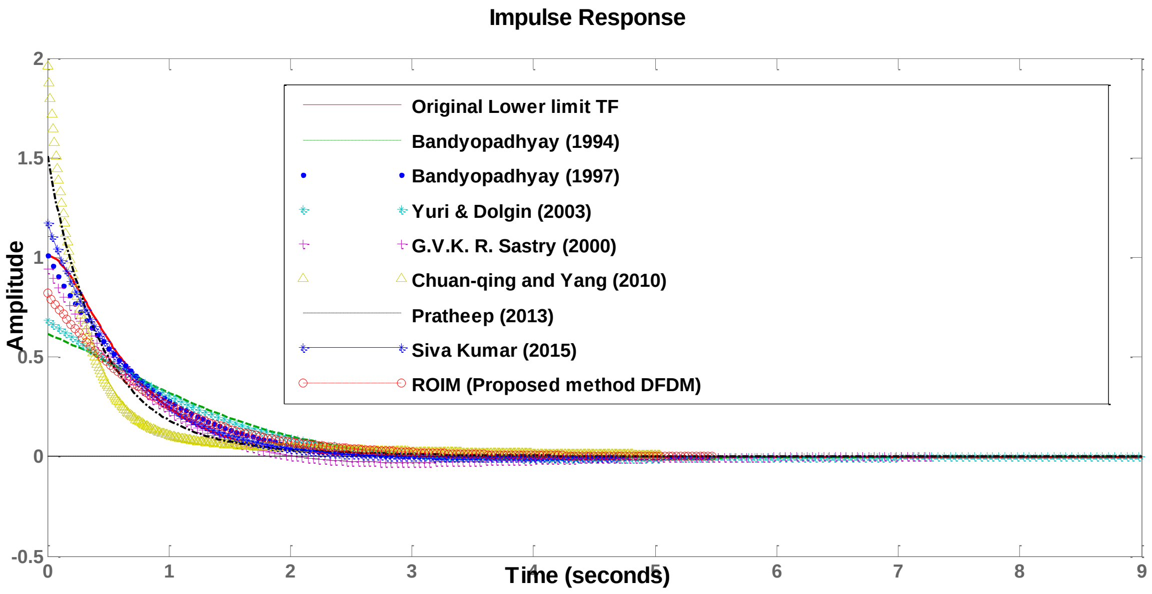

Fig.4.73. Impulse response for Differentiation method and Factor division method (lower limit TF)

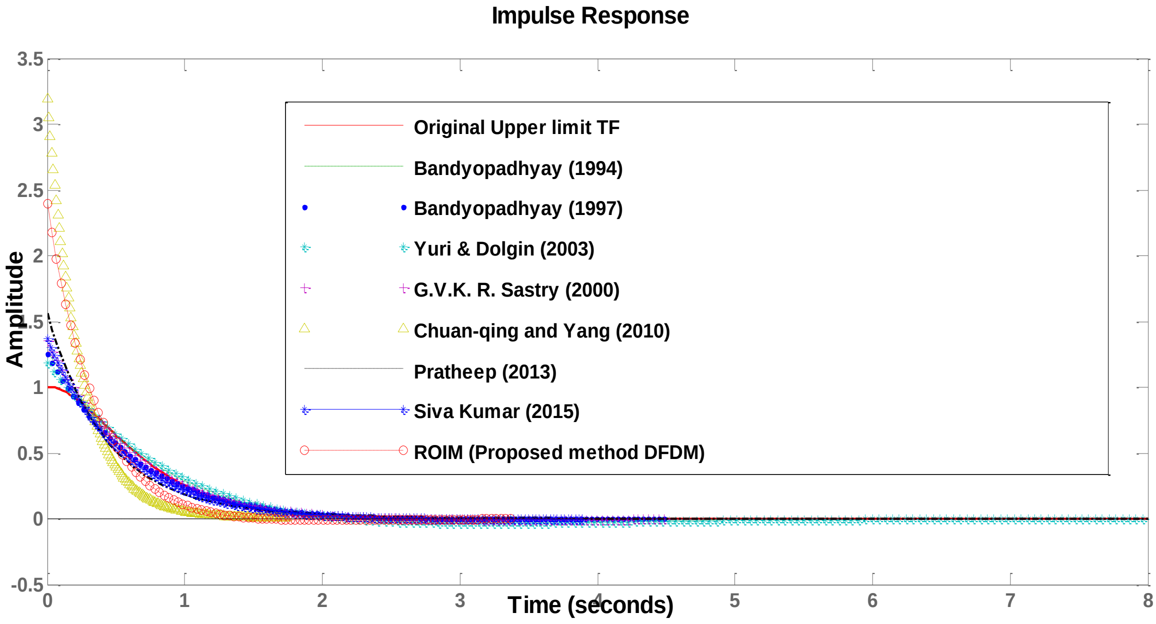

Fig.4.74. Impulse response for Differentiation method and Factor division method (upper limit TF)

A comparison of the bode plot of the lower limit and upper limit of the model obtained by proposed method and existing methods is shown in Fig. 4.75 and Fig.4.76.

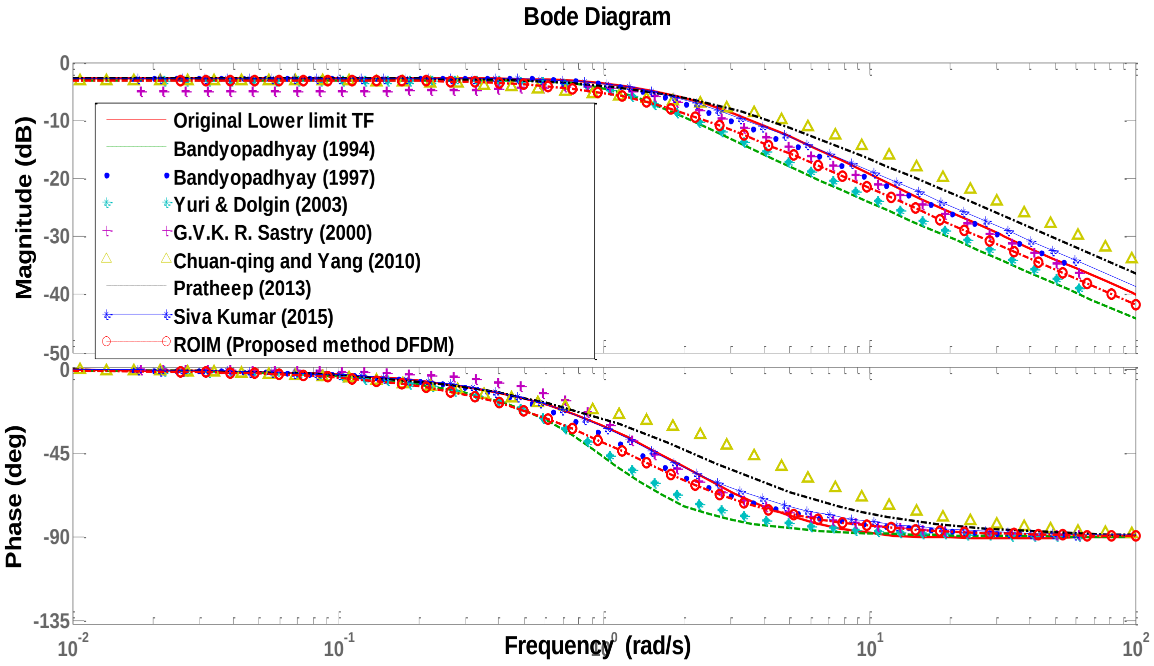

Fig.4.75. Bode plot for Differentiation method and Factor division method (lower limit TF)

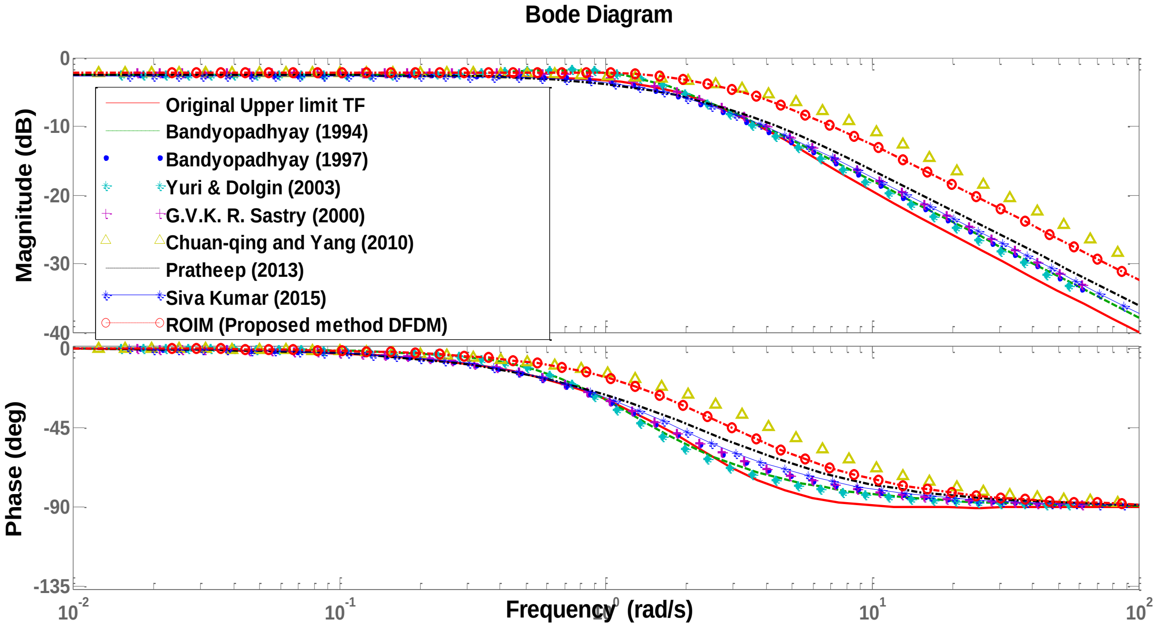

Fig.4.76. Bode plot for Differentiation method and Factor division method (upper limit TF)

A comparison of the nyquist plot of the lower limit and upper limit of the model obtained by proposed method and existing methods is shown in Fig. 4.77 and Fig.4.78.

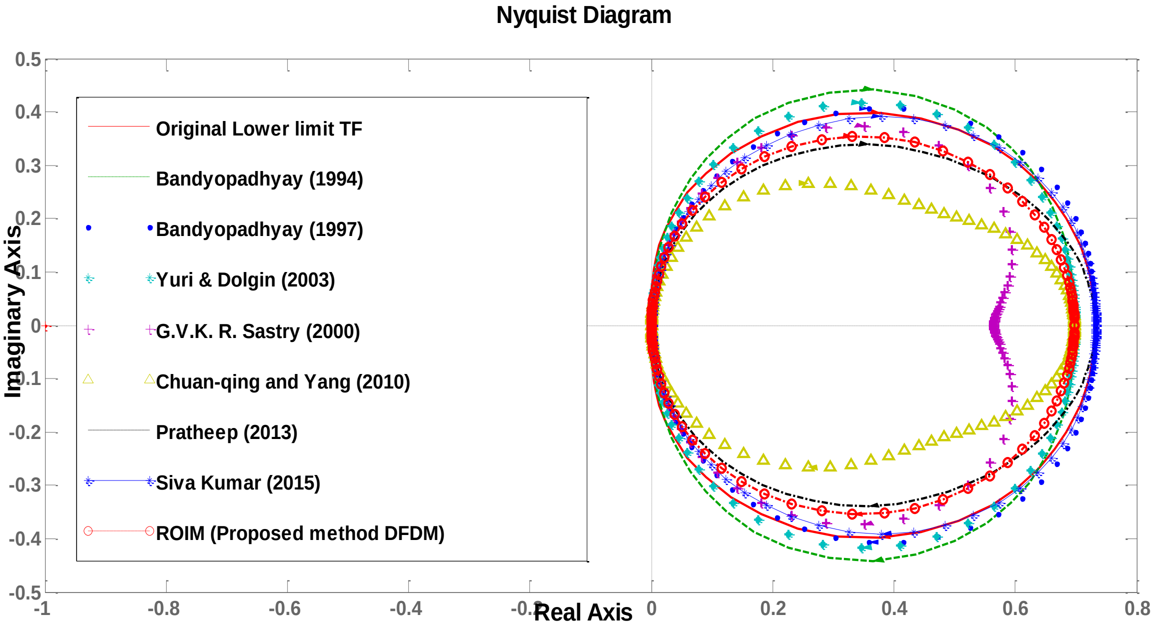

Fig.4.77. Nyquist plot for Differentiation method and Factor division method (lower limit TF)

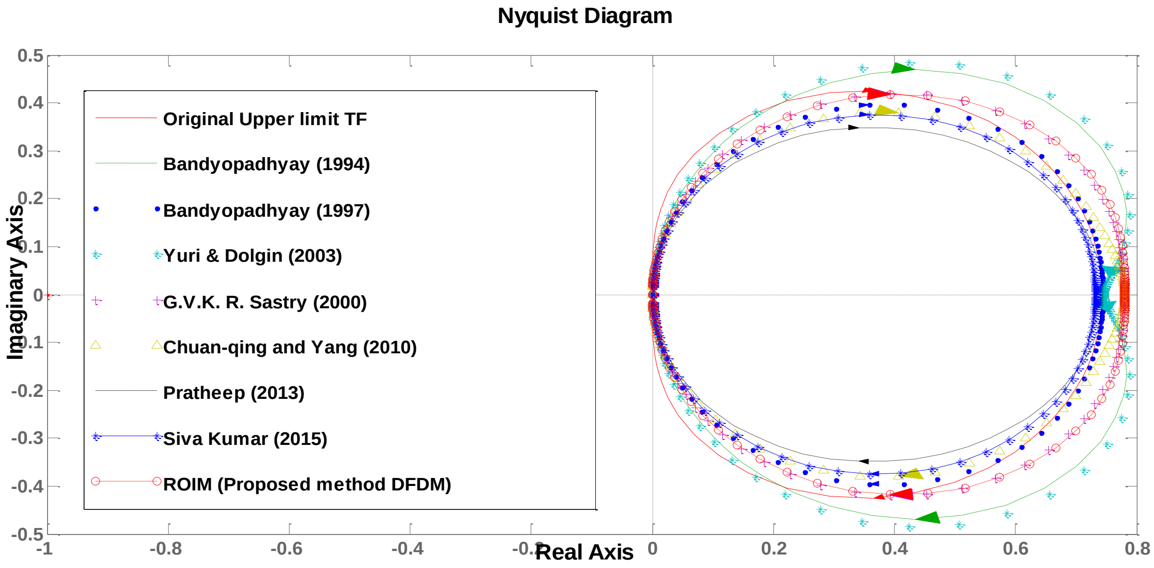

Fig.4.78. Nyquist plot for Differentiation method and Factor division method (upper limit TF)

Case3: Differentiation method and Cauer second form

$$R_2(s) = \frac{[17.04, 38.439]s + [42.908, 50.3434]}{[17,18]s^2 + [70,72]s + [61.5, 64.5]} \tag{4.84}$$

A comparison of the step response of the lower limit and upper limit of the model obtained by proposed method and existing methods is shown in Fig. 4.79 and Fig.4.80.

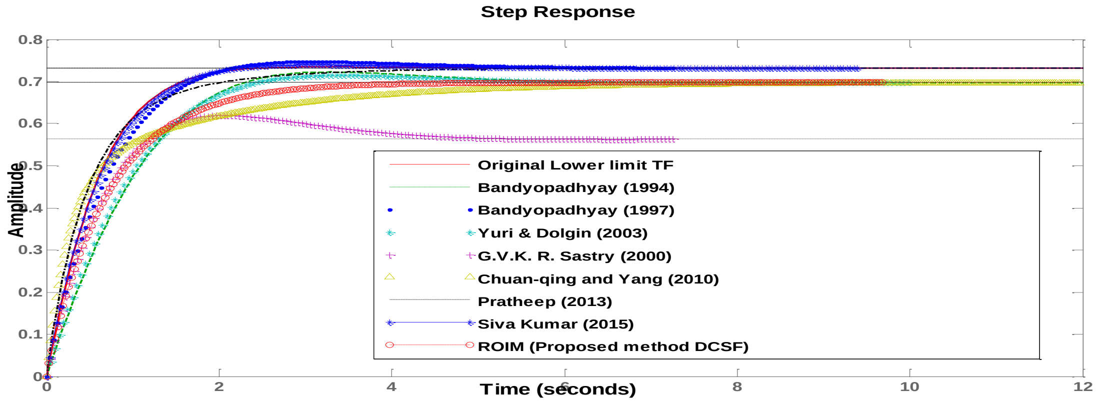

Fig.4.79. Step response for Differentiation method and Cauer second form (lower limit TF)

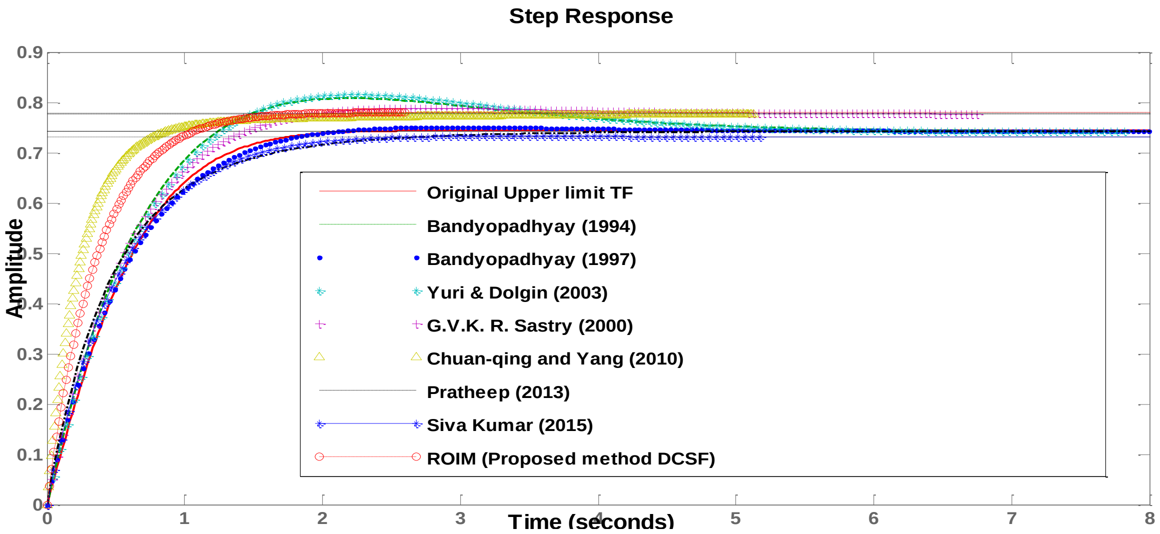

Fig.4.80. Step response for Differentiation method and Cauer second form (upper limit TF)

A comparison of the impulse response of the lower limit and upper limit of the model obtained by proposed method and existing methods is shown in Fig. 4.81 and Fig.4.82.

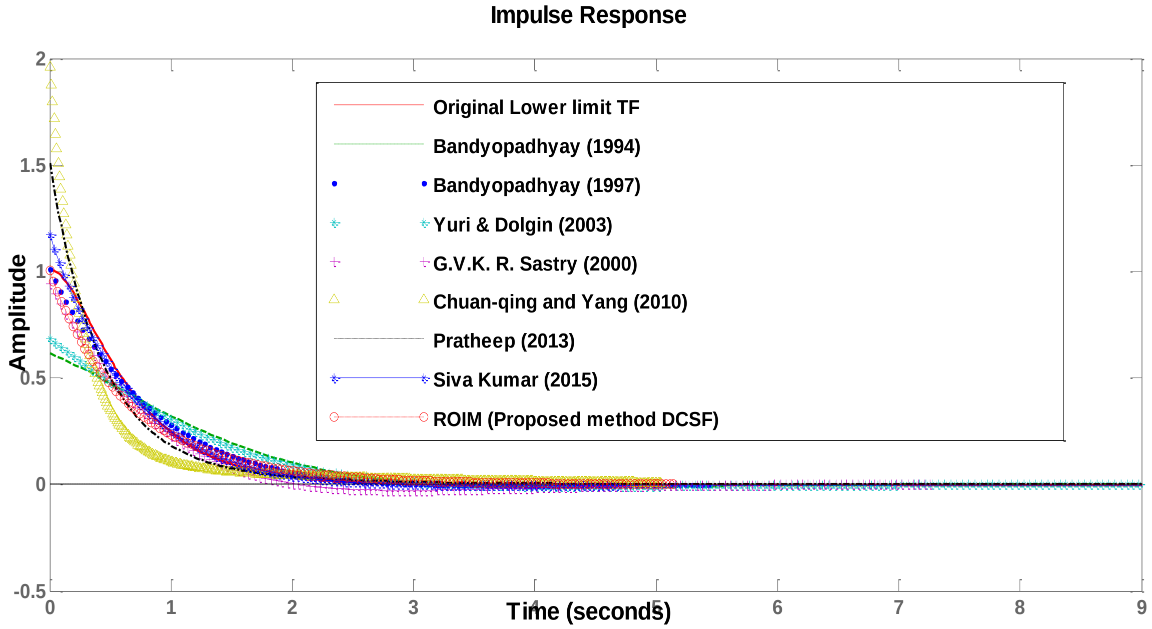

Fig.4.81. Impulse response for Differentiation method and Cauer second form (lower limit TF)

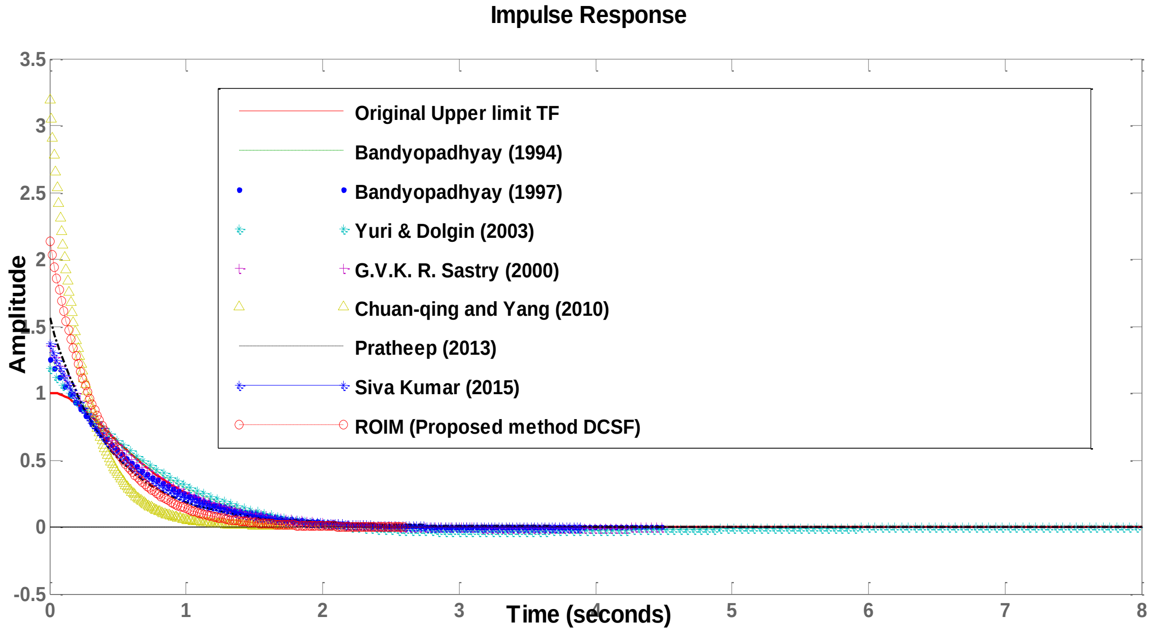

Fig.4.82. Impulse response for Differentiation method and Cauer second form (upper limit TF)

A comparison of the bode plot of the lower limit and upper limit of the model obtained by proposed method and existing methods is shown in Fig. 4.83 and Fig.4.84.

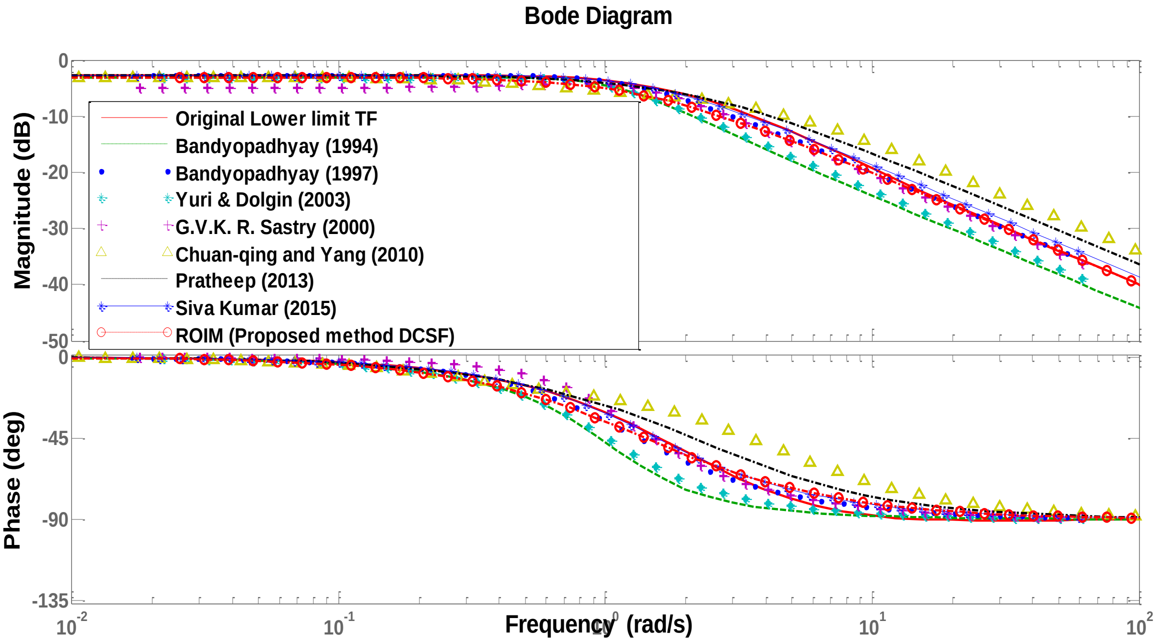

Fig.4.83. Bode plot for Differentiation method and Cauer second form (lower limit TF)

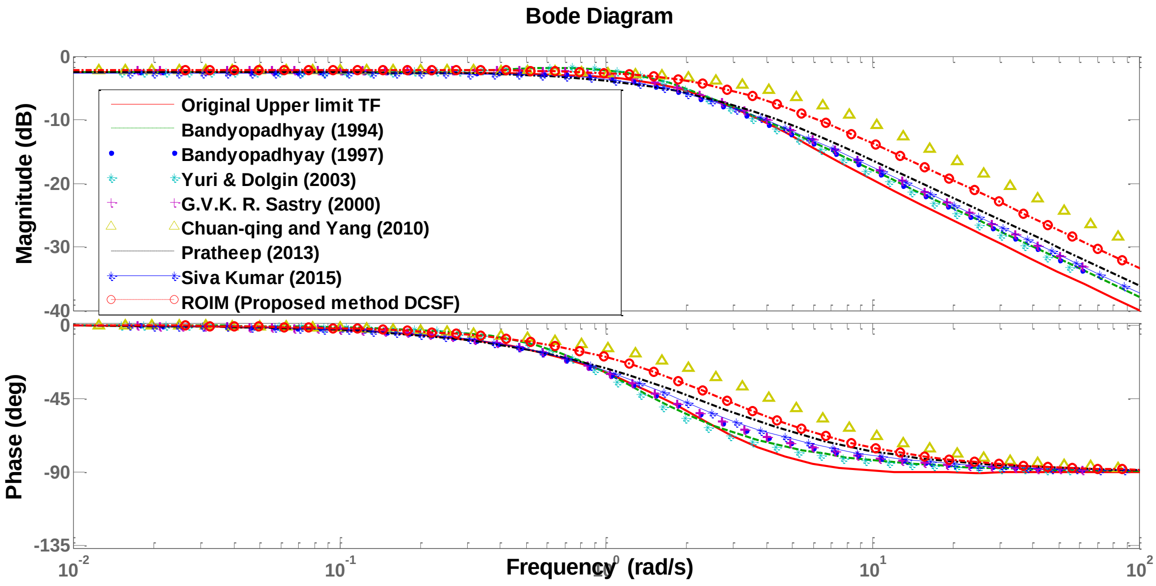

Fig.4.84. Bode plot for Differentiation method and Cauer second form (upper limit TF)

A comparison of the nyquist plot of the lower limit and upper limit of the model obtained by proposed method and existing methods is shown in Fig. 4.85 and Fig.4.86.

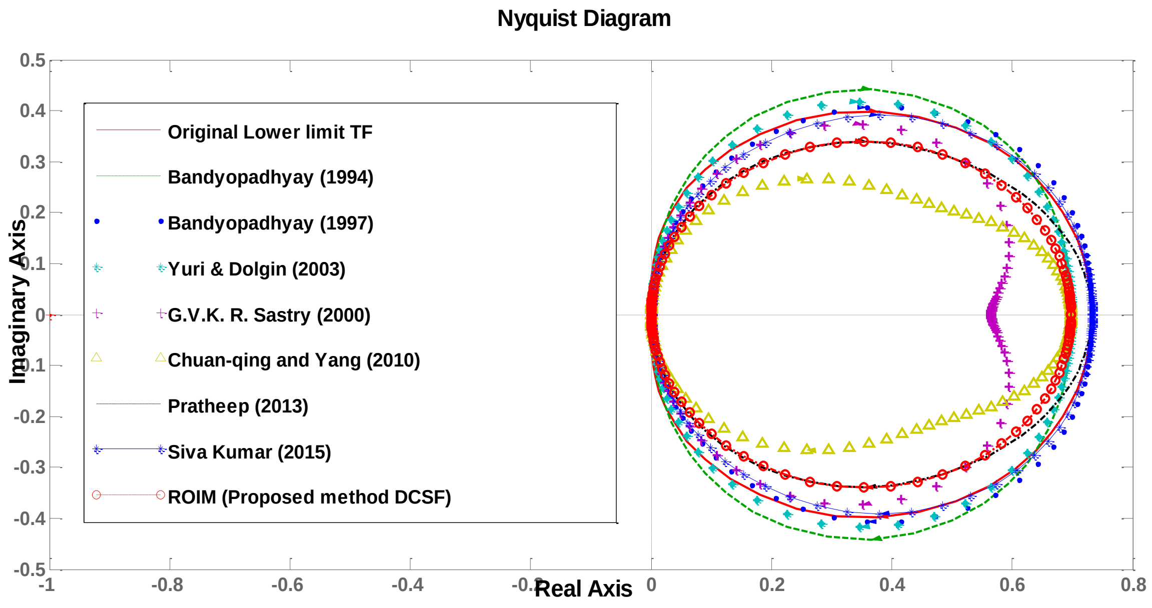

Fig.4.85. Nyquist plot for Differentiation method and Cauer second form (lower limit TF)

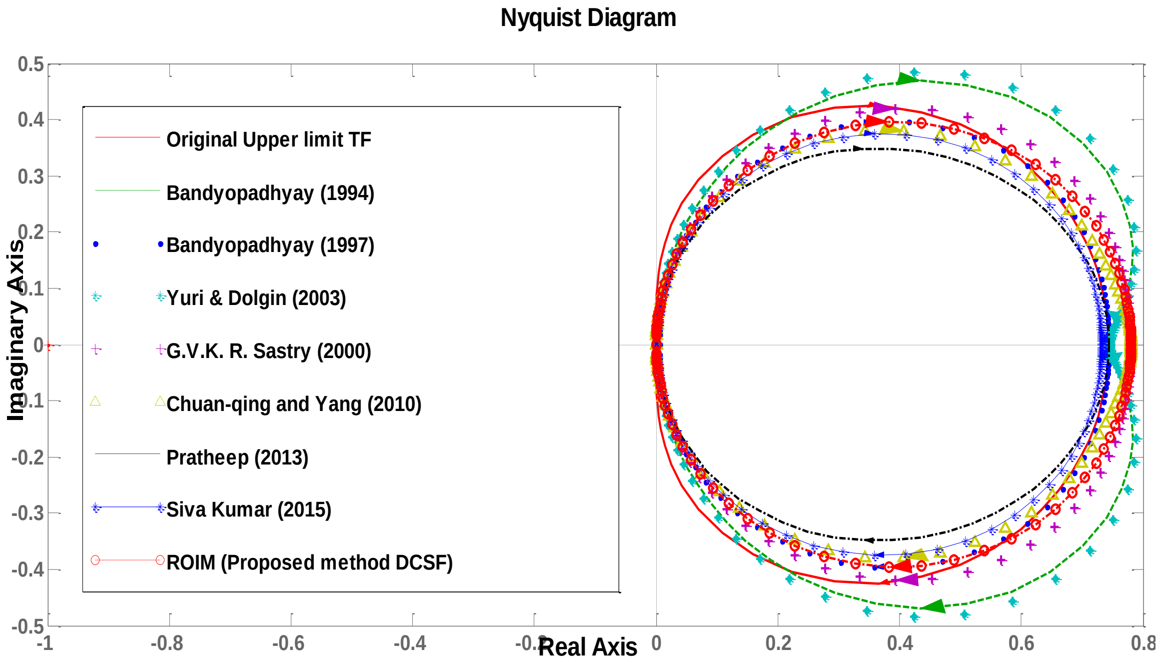

Fig.4.86. Nyquist plot for Differentiation method and Cauer second form (upper limit TF)

Table 4.12: Comparison of Differentiation Method and Existing Reduced Order Models of example 4.1

S.No	Methods	ISE		IAE	
		Lower limit	**Upper limit**	**Lower limit**	**Upper limit**
1	Bandypodayay et al., [156]	0.00878538	8.87354E-05	0.264278635	0.009542405
2	Bandypodayay et al., [157]	2.36319E-05	4.39626E-06	0.005001512	0.002108643
3	Dolgin and Zeheb [162]	0.008876522	8.01481E-05	0.265977631	0.009118821
4	Sastry et al., [161]	0.225673362	0.00949689	1.343641109	0.275550326
5	Chuan-qing and Yang [171]	0.010817047	0.010362791	0.290640962	0.287856756
6	Pratheep et al., [176]	1.19446E-05	9.93222E-07	0.003594415	0.001092456
7	Siva Kumar et. al [177]	2.93246E-06	0.001205443	0.002187121	0.098172255
8	Proposed Method (DMFDM)	0.476571009	0.492473926	1.952576584	1.98490506

9	Proposed Method (DMFDM)	0.009423911	0.010540449	0.274510681	0.290384959
10	Proposed Method (DMCSF)	0.00940033	0.010554188	0.274177914	0.290574356

Example 4.4: Consider a seventh order polynomial and reduced the system

$$P_7(s) = s^7 + [9,10]s^6 + [31,35]s^5 + [71,72]s^4 + [111,112]s^3 + [109,110]s^2 + [76,84]s + [12,13] \tag{4.85}$$

Step1: Reciprocal transformation of seventh order polynomial

$$\hat{P}_7(s) = [12,13]s^7 + [76,84]s^6 + [109,110]s^5 + [111,112]s^4 + [71,72]s^3 + [31,35]s^2 + [9,10] + 1 \tag{4.86}$$

Step 2: Differentiation of the above polynomial

$$\hat{P}_6(s) = [84,91]s^6 + [456,504]s^5 + [545,550]s^4 + [444,448]s^3 + [213,216]s^2 + [62,70]s + [9,10] \tag{4.87}$$

Step 3: Second reciprocal transformation of the above polynomial

$$P_6(s) = [9,10]s^6 + [62,70]s^5 + [213,216]s^4 + [444,448]s^3 + [545,550]s^2 + [456,504]s + [84,91] \tag{4.88}$$

Based on the above steps we get following reduced order polynomials

$$\left.\begin{aligned}
P_5(s) &= [62,70]s^5 + [426,432]s^4 + [1332,1344]s^3 + [2180,2200]s^2 \\
&\qquad + [2280,2520]s + [504,546] \\
P_4(s) &= [426,432]s^4 + [2664,2688]s^3 + [6540,6600]s^2 \\
&\qquad + [9120,10080]s + [2520,2730] \\
P_3(s) &= [2664,2688]s^3 + [13080,13200]s^2 + [27360,30240]s + [10080,10920] \\
P_2(s) &= [13080,13200]s^2 + [54720,60480]s + [30240,32760]
\end{aligned}\right\} \quad (4.89)$$

The number of differentiation depends upon (n-k) times, where n is the order of higher order polynomial and k is the order of reduced order polynomial.

From the literature it is clear that [156], [157], [162], [169] and [172] fails to produces stable reduced order polynomial to the same problem.

Example 4.5: Consider a seventh order polynomial and reduced the system

$$\begin{aligned}
P_6(s) &= [2.1,2.6]s^6 + [76.1,76.7]s^5 + [119.1,119.6]s^4 + [111,111.6]s^3 \\
&\qquad + [71.8,72.3]s^2 + [31,31.7]s + [9,9.9]
\end{aligned} \quad (4.90)$$

Step1: Reciprocal transformation of sixth order polynomial

$$\begin{aligned}
\bar{p}_6(s) &= [9,9.9]s^6 + [31,31.7]s^5 + [71.8,72.3]s^4 + [111,111.6]s^3 + [119.1,119.6]s^2 \\
&\qquad + [76.1,76.7]s + [2.1,2.6]
\end{aligned}$$

(4.91)

Step 2: Differentiation of the sixth order polynomial

$$\begin{aligned}
\bar{p}_5(s) &= [54,59.4]s^5 + [155,158.5]s^4 + [287.2,289.2]s^3 + [333,334.8]s^2 \\
&\qquad + [238.2,239.2]s + [76.1,76.7]
\end{aligned} \quad (4.92)$$

Based on the above steps we get following reduced order polynomials

$$\bar{p}_4(s) = [270,297]s^4 + [620,634]s^3 + [861.2,867.6]s^2 + [666,669.6]s + [238.2,239.2]$$

$$\left.\begin{array}{l}
Diffentiation3 \\
\bar{p}_3(s) = [1080,1188]s^3 + [1860,1902]s^2 + [1722.4,1735.2]s + [666,669.6] \\
Diffentiation2 \\
\bar{p}_2(s) = [3040,3564]s^2 + [3720,3804]s + [1722.4,1735.2] \\
\mathrm{Re}\,ciprocal \\
p_2(s) = [1722.4,1735.2]s^2 + [3720,3804]s + [3040,3564]
\end{array}\right\} \quad (4.93)$$

All the obtain reduced order polynomials are stable.