Chapter 2

Approximation by Bernstein type operators

This chapter deals with Bernstein-type generalizations i,e,. approximating continuous functions in finite intervals. Here, we have two sub-chapters, the first one is about a modified Bernstein-Kantorovich operator, and the second one is about a blending type Bernstein-Beta operator.

Throughout this chapter, we consider the set $X = [0, 1]$.

2.1 Modified Bernstein-Kantorovich Operators

2.1.1 Introduction

In 2018, Usta [109] has introduced a new modification of Bernstein operators for a continuous function Ψ on X and $x \in (0, 1)$ as follows:

$$\mathfrak{B}_m^*(\Psi; x) = \frac{1}{m} \sum_{j=0}^{m} \binom{m}{j} (j - mx)^2 x^{j-1} (1-x)^{m-j-1} \Psi\left(\frac{j}{m}\right) \qquad m \in \mathbb{N}. \tag{2.1}$$

In that paper, he has investigated some convergence properties and established a Voronovskaja-type asymptotic result for the operators (2.1) to estimate its convergence speed. Furthermore, he added some numerical examples to illustrate the convergence of such operators.

In recent years, the generalization and modification of the Kantorovich variant has become a popular area of research. Several researchers have contributed some works in this area. For more details, we refer to the readers the papers [6, 7, 28, 51, 65, 75, 76, 79, 80, 90, 103, 115] and the references cited therein.

Motivated by this, we introduce a Kantorovich variant of the operators (2.1) for $\Psi \in \mathfrak{C}(X)$ and $x \in (0,1)$, as follows:

$$\mathcal{K}_m^*(\Psi)(x) = \mathcal{K}_m^*(\Psi; x) = \frac{m+1}{m} \sum_{j=0}^{m} \binom{m}{j} (j-mx)^2 x^{j-1} (1-x)^{m-j-1} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} \Psi(t)dt, \tag{2.2}$$

where $m \in \mathbb{N}$. Note that, for $\Psi, \Phi \in \mathfrak{C}(X)$, and $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
\mathcal{K}_m^*(a\Psi + b\Phi; x) =& \frac{m+1}{m} \sum_{j=0}^{m} \binom{m}{j} (j-mx)^2 x^{j-1} (1-x)^{m-j-1} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} (a\Psi + b\Phi)(t)dt \\
=& \frac{a(m+1)}{m} \sum_{j=0}^{m} \binom{m}{j} (j-mx)^2 x^{j-1} (1-x)^{m-j-1} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} \Psi(t)dt \\
&+ \frac{b(m+1)}{m} \sum_{j=0}^{m} \binom{m}{j} (j-mx)^2 x^{j-1} (1-x)^{m-j-1} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} \Phi(t)dt \\
=& a\mathcal{K}_m^*(\Psi; x) + b\mathcal{K}_m^*(\Phi; x).
\end{aligned}$$

Also, for any $\Psi \geq 0$, we must have $\mathcal{K}_m^*(\Psi; x) \geq 0$. All these clearly imply that the operators (2.2) are linear and positive.

We equip the spaces $\mathfrak{C}(X)$, $\mathfrak{C}_B^1(X)$ and $\mathfrak{C}_B^2(X)$ with the sup norm:

$$\|\Psi\| = \sup\{|\Psi(x)| : x \in [0,1]\}.$$

The structure of this work is organized as follows. In Subsection 2.1.2, we compute some moments and central moments and establish some basic results for the operators (2.2). In Subsection 2.1.3, we discuss the order of convergence in terms of the usual moduli of continuity as well as the Ditzian-Totic modulus of smoothness and then for the function belonging to the Lipschitz-type class. In Subsection 2.1.4, we derive a quantitative Voronovskaya-type asymptotic theorem. In the final subsection, we show the convergence of the considered operators (2.2) with the help of some numerical examples.

2.1.2 Preliminary Results

In this subsection, we discuss some auxiliary results which will be used to prove our main results.

Lemma 2.1. *Let* $e_i(t) := t^i$ *for* $i = 0, 1, 2, 3$*. Then, the moments for the operators* (2.2) *are calculated to be:*

$$\begin{aligned}
\mathcal{K}_m^*(e_0;x) =& 1,\\
\mathcal{K}_m^*(e_1;x) =& \left(\frac{m-2}{m+1}\right)x + \frac{3}{2(m+1)},\\
\mathcal{K}_m^*(e_2;x) =& \left\{\frac{m^2-7m+6}{(m+1)^2}\right\}x^2 + \left\{\frac{6m-8}{(m+1)^2}\right\}x + \frac{7}{3(m+1)^2},\\
\mathcal{K}_m^*(e_3;x) =& \left\{\frac{m^3-15m^2+38m-24}{(m+1)^3}\right\}x^3 + \left\{\frac{\frac{27}{2}m^2-\frac{117}{2}m+45}{(m+1)^3}\right\}x^2\\
&+ \left\{\frac{\frac{43}{2}m-25}{(m+1)^3}\right\}x + \frac{15}{4(m+1)^3}.
\end{aligned}$$

Proof. Using the Lemma 1 of [109] and the operators (2.1), the above moments can be easily derived. □

As a consequence of the above lemma, in the next lemma we calculate the central moments for the operators (2.2), which are defined by $\mathcal{K}_m^*((t-x)^r;x),\ r = 0, 1, 2, \cdots$.

Lemma 2.2. *The central moments for the operators* (2.2) *are evaluated to be*

$$\mathcal{K}_m^*((t-x)^0;x) = 1,$$

$$\mathcal{K}_m^*((t-x)^1;x) = \frac{3}{m+1}\left(\frac{1}{2}-x\right) := \zeta_m(x), \tag{2.3}$$

$$\mathcal{K}_m^*((t-x)^2;x) = \left\{\frac{11-3m}{(m+1)^2}\right\}x^2 + \left\{\frac{3m-11}{(m+1)^2}\right\}x + \frac{7}{3(m+1)^2} := \mu_m(x). \tag{2.4}$$

Lemma 2.3. *If* $\Psi \in \mathfrak{C}(X)$, *then* $\|\mathcal{K}_m^*(\Psi)\| \leq \|\Psi\|$.

Proof. Using Lemma 2.1 and the norm defined for $\mathfrak{C}(X)$, the result follows immediately. □

Theorem 2.4. *If* $\Psi \in \mathfrak{C}(X)$, *then* $\lim_{m\to\infty} \mathcal{K}_m^*(\Psi;x) = \Psi(x)$ *uniformly on any compact subset of* $(0,1)$.

Proof. We notice from (2.2) and Lemma 2.1 that

$$\lim_{m\to\infty} \mathcal{K}_m^*(e_i;x) = e_i(x), \text{ for i=0,1,2.}$$

Hence the result follows from Korovkin's theorem (see [74]). □

2.1.3 Local and Global Approximation Results

In this subsection, we estimate the rate of convergence of the newly defined operators (2.2) in terms of the modulus of continuity and with the help of the relation (1.9).

Theorem 2.5. *Let* $\Psi \in \mathfrak{C}(X)$ *and* $x \in (0,1)$. *Then for any* $m \in \mathbb{N}$, *the operators* (2.2) *satisfy*

$$|\mathcal{K}_m^*(\Psi;x) - \Psi(x)| \leq 2\Omega_1(\Psi, \delta_m),$$

where $\delta_m := \sqrt{\mu_m(x)}$ *and* $\mu_m(x)$ *is defined in* (2.4).

Proof. From Lemma 2.1, we know that $\mathcal{K}_m^*(1;x) = 1$, thus

$$|\mathcal{K}_m^*(\Psi;x) - \Psi(x)| \leq |\mathcal{K}_m^*(\Psi;x) - \mathcal{K}_m^*(\Psi(x);x)| \leq \mathcal{K}_m^*(|\Psi(t) - \Psi(x)|;x). \tag{2.5}$$

Remembering the following property of modulus of continuity

$$|\Psi(t) - \Psi(x)| \leq \Omega_1(\Psi, \delta)\left(\frac{|t-x|}{\delta} + 1\right) \tag{2.6}$$

and then applying the operators $\mathcal{K}_m^*$ on both sides, we get,

$$|\mathcal{K}_m^*(\Psi;x) - \Psi(x)| \leq \Omega_1(\Psi, \delta)\left(\frac{\mathcal{K}_m^*(|t-x|;x)}{\delta} + 1\right) \tag{2.7}$$

Next, we apply the Cauchy-Schwarz inequality to obtain

$$|\mathcal{K}_m^*(\Psi;x) - \Psi(x)| \leq \Omega_1(\Psi, \delta)\left(\frac{(\mathcal{K}_m^*((t-x)^2;x))^{1/2}}{\delta} + 1\right)$$

Finally, choosing $\delta = \delta_m := \sqrt{\mu_m(x)}$, we get the desired assertion. ☐

Theorem 2.6. *For* $\Psi \in \mathfrak{C}_B^1(X)$ *and* $x \in (0,1)$, *we have the following inequality:*

$$|\mathcal{K}_m^*(\Psi;x) - \Psi(x)| \leq |\zeta_m(x)||\Psi'(x)| + 2\delta_m\Omega_1(\Psi', \delta_m),$$

where $\delta_m = \sqrt{\mu_m(x)}$ *and* $\zeta_m(x)$ *is defined in* (2.3).

Proof. For any $x, t \in (0,1)$ and $\Psi \in \mathfrak{C}_B^1(X)$, the Taylor series expansion ensures that

$$\Psi(t) - \Psi(x) = \Psi'(x)(t-x) + \int_x^t (\Psi'(u) - \Psi'(x))du.$$

Now, by applying the operators $\mathcal{K}_m^*$ on both sides of the above equations, we obtain

$$\mathcal{K}_m^*((\Psi(t) - \Psi(x); x) = \Psi'(x)\mathcal{K}_m^*((t-x); x) + \mathcal{K}_m^* \left(\int_x^t (\Psi'(u) - \Psi'(x))du; x \right). \tag{2.8}$$

Next, for $x \in (0,1)$ and $\Psi \in \mathfrak{C}(X)$, we know that the following relation holds

$$|\Psi(t) - \Psi(x)| \leq \Omega_1(\Psi, \delta_m) \left(\frac{|t-x|}{\delta_m} + 1 \right).$$

Hence, as $\Psi' \in \mathfrak{C}(X)$, the above relation yields

$$\left| \int_x^t (\Psi'(u) - \Psi'(x))du \right| \leq \Omega_1(\Psi', \delta_m) \left(\frac{(t-x)^2}{\delta_m} + |t-x| \right).$$

So, by (2.8), we must have

$$\begin{aligned} |\mathcal{K}_m^*(\Psi; x) - \Psi(x)| \leq & |\Psi'(x)||\zeta_m(x)| \\ & + \Omega_1(\Psi', \delta_m) \left\{ \frac{1}{\delta_m} \mathcal{K}_m^*((t-x)^2; x) + \mathcal{K}_m^*(|t-x|; x) \right\}. \end{aligned}$$

Applying Cauchy-Schwarz inequality to the right hand side of above inequality, we obtain

$$\begin{aligned} |\mathcal{K}_m^*(\Psi; x) - \Psi(x)| \leq & |\Psi'(x)||\zeta_m(x)| \\ & + \Omega_1(\Psi', \delta_m) \left\{ \frac{1}{\delta_m} \sqrt{\mathcal{K}_m^*((t-x)^2; x)} + 1 \right\} \sqrt{\mathcal{K}_m^*((t-x)^2; x)}. \end{aligned}$$

By choosing $\delta_m = \sqrt{\mu_m(x)}$, we get the desired assertion. □

Theorem 2.7. *Let $\Psi \in \mathfrak{C}(X)$ and $\Phi \in \mathfrak{C}_B^2(X)$. Then there exists a positive constant M such that for each $m \in \mathbb{N}$, we have*

$$\left|\mathcal{K}_m^*(\Psi;x) - \Psi(x) - \frac{3}{2}\Phi'(x)\left(\frac{1-2x}{n+1}\right)\right| \leq M\Omega_2(\Psi, \sqrt{\tilde{\mu}_m(x)}),$$

where $\tilde{\mu}_m(x) = \left[\left\{\frac{11-3m}{2(m+1)^2}\right\}x^2 + \left\{\frac{3m-11}{2(2m+1)^2}\right\}x + \frac{7}{6(m+1)^2}\right]$.

Proof. By Taylor's expansion, for $\Phi \in \mathfrak{C}_B^2(X)$, we must have

$$\Phi(t) = \Phi(x) + \Phi'(x)(t-x) + \int_x^t (t-z)\Phi''(z)dz.$$

Applying the operators $\mathcal{K}_m^*$ on both sides of the above equation, we obtain

$$\mathcal{K}_m^*(\Phi;x) = \Phi(x) + 3\Phi'(x)\frac{\left(\frac{1}{2}-x\right)}{m+1} + \mathcal{K}_m^*\left(\int_x^t (t-z)\Phi''(z)dz;x\right).$$

Then by taking the modulus on both sides, we have

$$\begin{aligned}
&\left|\mathcal{K}_m^*(\Phi;x) - \Phi(x) - 3\Phi'(x)\frac{\left(\frac{1}{2}-x\right)}{m+1}\right| \\
\leq &\mathcal{K}_m^*((t-x)^2;x)\frac{\|\Phi''\|}{2} \\
= &\left[\left\{\frac{11-3m}{2(m+1)^2}\right\}x^2 + \left\{\frac{3m-11}{2(2m+1)^2}\right\}x + \frac{7}{6(m+1)^2}\right]\|\Phi''\|. \qquad (2.9)
\end{aligned}$$

Now, it is easy to observe that, for $\Psi \in \mathfrak{C}(X)$, we get

$$\begin{aligned}
\left|\mathcal{K}_m^*(\Psi;x) - \Psi(x) - 3\Phi'(x)\frac{\left(\frac{1}{2}-x\right)}{m+1}\right| \leq &\left|\mathcal{K}_m^*(\Psi-\Phi;x) - (\Psi-\Phi)(x)\right| \\
&+ \left|\mathcal{K}_m^*(\Phi;x) - \Phi(x) - 3\Phi'(x)\frac{\left(\frac{1}{2}-x\right)}{m+1}\right|.
\end{aligned}$$

Using (2.9) and Lemma 2.3, we can deduce

$$\left|\mathcal{K}_m^*(\Psi;x)-\Psi(x)-3\Phi'(x)\frac{\left(\frac{1}{2}-x\right)}{m+1}\right|$$
$$\leq \|\Psi-\Phi\|+\left[\left\{\frac{11-3m}{2(m+1)^2}\right\}x^2+\left\{\frac{3m-11}{2(2m+1)^2}\right\}x+\frac{7}{6(m+1)^2}\right]\|\Phi''\|.$$

Thus, taking infimum over all $\Phi \in \mathfrak{C}_B^2(X)$ on the right side of above inequality, it yields

$$\left|\mathcal{K}_m^*(\Psi;x)-\Psi(x)-3\Phi'(x)\frac{\left(\frac{1}{2}-x\right)}{m+1}\right| \leq K_2(\Psi,\tilde{\mu}_m(x)).$$

Our desired theorem is proved by using the relation (1.9). ☐

Next, we present a local approximation result for the Lipschitz class of functions.

Theorem 2.8. *For any* $\eta \in (0,1]$ *and* $\Psi \in Lip_{M,\eta}^{a,b}(X)$, *we have*

$$|\mathcal{K}_m^*(\Psi;x)-\Psi(x)| \leq M\left(\frac{\mu_m(x)}{ax^2+bx}\right)^{\eta/2},$$

where $\mu_m(x)$ *is as in* (2.4) *and* $M>0$ *depends on* Ψ.

Proof. It is easy to observe that for $\eta = 1$,

$$|\mathcal{K}_m^*(\Psi;x)-\Psi(x)| \leq \mathcal{K}_m^*(|\Psi(t)-\Psi(x)|;x) \leq M\left\{\mathcal{K}_m^*\left(\frac{|t-x|}{\sqrt{t+ax^2+bx}};x\right)\right\}.$$

Using the fact that $\frac{1}{\sqrt{t+ax^2+bx}} \leq \frac{1}{\sqrt{ax^2+bx}}$ and the Cauchy-Schwarz inequality, the above inequality gives

$$|\mathcal{K}_m^*(\Psi;x)-\Psi(x)| \leq \frac{M}{\sqrt{ax^2+bx}}\mathcal{K}_m^*(|t-x|;x)$$

$$\leq \frac{M}{\sqrt{ax^2+bx}}\left\{\mathcal{K}_m^*((t-x)^2;x)\right\}^{1/2} = M\left(\frac{\mu_m(x)}{ax^2+bx}\right)^{1/2}.$$

Hence the result holds for $\eta = 1$. Next, we consider the case when $\eta \in (0,1)$. It is easy to observe that

$$\begin{aligned}|\mathcal{K}_m^*(\Psi;x)-\Psi(x)| \leq& \mathcal{K}_m^*(|\Psi(t)-\Psi(x)|;x)\\ =&\frac{m+1}{m}\sum_{j=0}^{m}\binom{m}{j}(j-mx)^2x^{j-1}(1-x)^{m-j-1}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}|\Psi(t)-\Psi(x)|dt \quad (2.10)\end{aligned}$$

Setting $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$ and then applying the Hölder inequality to (2.10), we obtain

$$\begin{aligned}&|\mathcal{K}_m^*(\Psi;x)-\Psi(x)|\\ &\leq \left[\frac{m+1}{m}\sum_{j=0}^{m}\binom{m}{j}(j-mx)^2x^{j-1}(1-x)^{m-j-1}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}|\Psi(t)-\Psi(x)|^{2/\eta}dt\right]^{\eta/2}\\ &\qquad\times\left[\frac{m+1}{m}\sum_{j=0}^{m}\binom{m}{j}(j-mx)^2x^{j-1}(1-x)^{m-j-1}dt\right]^{(2-\eta)/2}\\ &\leq M\left[\frac{m+1}{m}\sum_{j=0}^{m}\binom{m}{j}(j-mx)^2x^{j-1}(1-x)^{m-j-1}\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}\frac{|t-x|^2}{(t+ax^2+bx)}dt\right]^{\eta/2}\\ &\leq \frac{M}{(ax^2+bx)^{\eta/2}}\left[\mathcal{K}_m^*((t-x)^2;x)\right]^{\eta/2}\\ &= M\left(\frac{\mu_m(x)}{ax^2+bx}\right)^{\eta/2}.\end{aligned}$$

This completes the proof. □

We'll now establish a global approximation result for the newly defined operators (2.2) in terms of first and second-order Ditzian-Totik uniform modulus of smoothness, where the admissible weight function is taken as $\xi(x) = [x(1-x)]^{1/2}$.

Theorem 2.9. *For any $\Psi \in \mathfrak{C}(X)$, $\Phi \in \mathfrak{C}^2_B(X)$ and $x \in (0,1)$ there exists $M > 0$ such that*

$$|\mathcal{K}^*_m(\Psi;x) - \Psi(x)| \leq M\Omega^\xi_2\left(\Psi, \frac{\tau_m(x)}{2\xi(x)}\right) + \Omega^\xi_1\left(\Psi, \frac{\zeta_m(x)}{\xi(x)}\right),$$

where $\tau_m(x) = (\mu_m(x) + \zeta^2_m(x))^{1/2}$ and $\delta > 0$.

Proof. For $\Psi \in \mathfrak{C}(X)$, we define the auxiliary operator

$$\widetilde{\mathcal{K}}^*_m(\Psi;x) = \mathcal{K}^*_m(\Psi;x) + \Psi(x) - \Psi\left(\frac{3}{2(m+1)} + \frac{(m-2)x}{m+1}\right). \tag{2.11}$$

Clearly, the above defined auxiliary operator preserves linear as well as constant functions.

Let $u = \lambda x + (1-\lambda)t$, $\lambda \in [0,1]$. As ξ^2 is concave on X, we must have $\xi^2(u) \geq \lambda\xi^2(x) + (1-\lambda)\xi^2(t)$ and hence we have

$$\frac{|t-u|}{\xi^2(u)} \leq \frac{\lambda|x-t|}{\lambda\xi^2(x) + (1-\lambda)\xi^2(t)} \leq \frac{|t-x|}{\xi^2(x)}.$$

Also, using Lemma 2.3, for the operators (2.11) we obtain

$$\begin{aligned}|\widetilde{\mathcal{K}}^*_m(\Psi;x) - \Psi(x)| &\leq |\widetilde{\mathcal{K}}^*_m(\Psi-\Phi;x)| \ + \ |\widetilde{\mathcal{K}}^*_m(\Phi;x) - \Phi(x)| \ + \ |\Psi(x) - \Phi(x)| \\ &\leq 4\|\Psi - \Phi\| \ + \ |\widetilde{\mathcal{K}}^*_m(\Phi;x) - \Phi(x)|,\end{aligned} \tag{2.12}$$

where $\Phi \in \mathcal{W}^2_\xi(X)$.

By applying Taylor's formula, for $\Phi \in \mathcal{W}^2_\xi(X)$, we get the following relation:

$$\begin{aligned}&|\widetilde{\mathcal{K}}^*_m(\Phi;x) - \Phi(x)| \\ &\leq \mathcal{K}^*_m\left(\int_x^t |t-u||\Phi''(u)|du; x\right) + \left|\int_x^{x+\zeta_m(x)} |x + \zeta_m(x) - u||\Phi''(u)|du\right|\end{aligned}$$

$$\begin{aligned}
&\leq \|\xi^2\Phi''\|\mathcal{K}_m^*\left(\left|\int_x^t \frac{|t-u|}{\xi^2(u)}du\right|;x\right) + \|\xi^2\Phi''\|\left|\int_x^{x+\zeta_m(x)} \frac{|x+\zeta_m(x)-u|}{\xi^2(u)}du\right| \\
&\leq \xi^{-2}(x)\|\xi^2\Phi''\|\mathcal{K}_m^*((t-x)^2;x) + \xi^{-2}(x)\|\xi^2\Phi''\|\zeta_m^2(x) \\
&= \xi^{-2}(x)\|\xi^2\Phi''\|[\mu_m(x)+\zeta_m^2(x)].
\end{aligned}$$

Using the above inequality and the equation (2.12) yields

$$|\widetilde{\mathcal{K}}_m^*(\Psi;x)-\Psi(x)| \leq 4\|\Psi-\Phi\| + \xi^{-2}(x)\|\xi^2\Phi''\|[\mu_m(x)+\zeta_m^2(x)].$$

Now, by taking the infimum over all $\Phi \in \mathcal{W}_\xi^2(X)$ and then using (1.15), we get that there exists $M > 0$ such that

$$|\widetilde{\mathcal{K}}_m^*(\Psi;x)-\Psi(x)| \leq M\Omega_2^\xi\left(\Psi, \frac{\sqrt{\mu_m(x)+\zeta_m^2(x)}}{2\xi(x)}\right).$$

But, by the definition of first-order Ditzian-Totik uniform modulus of smoothness, we must have

$$\begin{aligned}
|\Psi(x+\zeta_m(x))-\Psi(x)| =& \left|\Psi\left(x+\xi(x)\frac{\zeta_m(x)}{\xi(x)}\right)-\Psi(x)\right| \\
\leq& \Omega_1^\xi\left(\Psi, \frac{\zeta_m(x)}{\xi(x)}\right).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
|\mathcal{K}_m^*(\Psi;x)-\Psi(x)| \leq& |\widetilde{\mathcal{K}}_m^*(\Psi;x)-\Psi(x)| + |\Psi(x+\zeta_m(x))-\Psi(x)| \\
\leq& M\Omega_2^\xi\left(\Psi, \frac{\sqrt{\mu_m(x)+\zeta_m^2(x)}}{2\xi(x)}\right) + \Omega_1^\xi\left(\Psi, \frac{\zeta_m(x)}{\xi(x)}\right),
\end{aligned}$$

which is the required result. □

2.1.4 Voronovskaja-type Asymptotic Results

We establish a Voronovskaja-type asymptotic result for the newly defined operators to study their speed of convergence.

Theorem 2.10. *Let Ψ be an integrable function on X such that Ψ'' exists at a fixed point $x \in (0,1)$. Then the following holds:*

$$\lim_{m\to\infty} m\{\mathcal{K}_m^*(\Psi;x) - \Psi(x)\} = \left(\frac{3}{2} - 3x\right)\Psi'(x) + \frac{3}{2}x(1-x)\Psi''(x).$$

Proof. By the Taylor expansion,

$$\Psi(t) = \Psi(x) + \Psi'(x)(t-x) + \frac{1}{2}\Psi''(x)(t-x)^2 + \Lambda(t,x)(t-x)^2, \tag{2.13}$$

where $\Lambda(t,x) \in \mathfrak{C}(X)$ and satisfies $\lim\limits_{t\to x}\Lambda(t,x) = 0$.
For $m \in \mathbb{N}$, applying the operators $\mathcal{K}_m^*$ on both sides of (2.13), we obtain

$$\begin{aligned}\mathcal{K}_m^*(\Psi;x) =& \Psi(x) + \Psi'(x)\mathcal{K}_m^*((t-x);x) \\ &+ \frac{1}{2}\Psi''(x)\mathcal{K}_m^*((t-x)^2;x) + \mathcal{K}_m^*(\Lambda(t,x)(t-x)^2;x)\end{aligned}$$

and then applying limit $m \to \infty$ on both sides, we get

$$\begin{aligned}\lim_{m\to\infty} m\{\mathcal{K}_m^*(\Psi;x) - \Psi(x)\} =& m\Psi'(x)\lim_{m\to\infty}\mathcal{K}_m^*((t-x);x) + \frac{m}{2}\Psi''(x)\lim_{m\to\infty}\mathcal{K}_m^*((t-x)^2;x) \\ &+ m\lim_{m\to\infty}\mathcal{K}_m^*(\Lambda(t,x)(t-x)^2;x).\end{aligned}$$

In the view of Lemma 2.2, we can easily obtain that

$$\begin{aligned}\lim_{m\to\infty} m\{\mathcal{K}_m^*(\Psi;x) - \Psi(x)\} =& \left(\frac{3}{2} - 3x\right)\Psi'(x) + \frac{3}{2}x(1-x)\Psi''(x) \\ &+ m\lim_{m\to\infty}\mathcal{K}_m^*(\Lambda(t,x)(t-x)^2;x).\end{aligned} \tag{2.14}$$

Now applying Cauchy-Schwarz inequality, it is easy to see that

$$m \lim_{m\to\infty} \mathcal{K}_m^*(\Lambda(t,x)(t-x)^2;x) \leq \sqrt{\mathcal{K}_m^*(\Lambda^2(t,x);x)}\sqrt{m^2\mathcal{K}_m^*((t-x)^4;x)}. \qquad (2.15)$$

Also, Theorem 2.4 ensures that

$$\lim_{m\to\infty} \mathcal{K}_m^*(\Lambda^2(t,x);x) = \Lambda^2(x,x) = 0. \qquad (2.16)$$

Lastly, in view of the fact that $\mathcal{K}_m^*((t-x)^4;x)$ is of order m^{-2} and then applying (2.15) and (2.16) in (2.14) we get the desired result. □

Next, we present a quantitative Voronovskaja-type result for the newly defined operators.

Theorem 2.11. *For* $\Psi \in \mathfrak{C}_B^2(X)$, $x \in (0,1)$ *and for sufficiently large* m, *the following inequality holds:*

$$\left| m\left(\mathcal{K}_m^*(\Psi;x) - \Psi(x)\right) - m\zeta_m(x)\Psi'(x) - m\frac{\mu_m(x)}{2}\Psi''(x)\right| = O(1)\Omega_1\left(\Psi''; \frac{1}{\sqrt{m}}\right),$$

as $m \to \infty$.

Proof. By the Taylor series expansion,

$$\Psi(t) = \Psi(x) + \Psi'(x)(t-x) + \frac{\Psi''(x)}{2!}(y-x)^2 + \Lambda(t,x), \qquad (2.17)$$

where for y being a number between t and x, $\Lambda(t,x) = \frac{\Psi''(y)-\Psi''(x)}{2!}(t-x)^2$. Now, using the following property of 1st order modulus of continuity,

$$|\Psi''(y) - \Psi''(x)| \leq 4(1+x^2)(1+\delta^2)^2\left(1 + \frac{(t-x)^4}{\delta^4}\right)\Omega_1(\Psi'';\delta),$$

and then combining it with the value of $\Lambda(t,x)$, we get

$$|\Lambda(t,x)| \leq 2(1+x^2)(1+\delta^2)^2\left(1+\frac{(t-x)^4}{\delta^4}\right)(t-x)^2\Omega_1(\Psi'';\delta).$$

But then restricting $\delta<1$, the above expression is reduced to

$$|\Lambda(t,x)| \leq 8(1+x^2)\left(1+\frac{(t-x)^4}{\delta^4}\right)(t-x)^2\Omega_1(\Psi'';\delta). \tag{2.18}$$

But then applying the operators $\mathcal{K}_m^*$ on both sides of equation (2.17), we get,

$$\left|\mathcal{K}_m^*(\Psi;x)-\Psi(x)-\zeta_m(x)\Psi'(x)-\frac{\mu_m(x)}{2}\Psi''(x)\right| \leq \mathcal{K}_m^*\big(|\Lambda(t,x)|;x\big). \tag{2.19}$$

Again, applying $\mathcal{K}_m^*$ on both side of (2.18), we have

$$\mathcal{K}_m^*\big(|\Lambda(t,x)|;x\big) \leq 8(1+x^2)\Omega_1(\Psi'';\delta)\left(\mathcal{K}_m^*((t-x)^2;x)+\frac{1}{\delta^4}\mathcal{K}_m^*((t-x)^6;x)\right)$$

But, it is easy to notice that $\mathcal{K}_m^*((t-x)^2;x)$ is of $O(1/m)$ and $\mathcal{K}_m^*((t-x)^6;x)$ is of $O(1/m^3)$. Hence, the above expression becomes,

$$\mathcal{K}_m^*\big(|\Lambda(t,x)|;x\big) \leq 8(1+x^2)\Omega_1(\Psi'';\delta)\left[O\left(\frac{1}{m}\right)+\frac{1}{\delta^4}O\left(\frac{1}{m^3}\right)\right]. \tag{2.20}$$

Now, from (2.19) and (2.20), we get

$$\begin{aligned}&\left|m(\mathcal{K}_m^*(\Psi;x)-\Psi(x))-m\zeta_m(x)\Psi'(x)-m\frac{\mu_m(x)}{2}\Psi''(x)\right|\\ \leq& 8(1+x^2)\Omega_1(\Psi'';\delta)\left[mO\left(\frac{1}{m}\right)+\frac{m}{\delta^4}O\left(\frac{1}{m^3}\right)\right]\end{aligned}$$

Lastly, we choose $\delta=\frac{1}{\sqrt{m}}$, to get the desired result. $\square$

As an immediate consequence of Theorem 2.11, we have the following result.

Corollary 2.12. *For* $\Psi \in \mathfrak{C}_B^2(X)$

$$\lim_{m\to\infty} m\left|\mathcal{K}_m^*(\Psi;x)-\Psi(x)-\zeta_m(x)\Psi'(x)-\frac{\mu_m(x)}{2}\Psi''(x)\right|=0,$$

where $\zeta_m(x)$ *and* $\mu_m(x)$ *are given in* (2.3) *and* (2.4) *respectively.*

We end this subsection by establishing a Grüss-Voronovskaja type result for the special class of the newly defined sequence of operators.

Theorem 2.13. *For* $\Psi,\Phi \in \mathfrak{C}_B^2(X)$ *and* $x \in (0,1)$,

$$\lim_{m\to\infty} m\{\mathcal{K}_m^*(\Psi\Phi;x)-\mathcal{K}_m^*(\Psi;x)\mathcal{K}_m^*(\Phi;x)\}=3x(1-x)\Psi'(x)\Phi'(x).$$

Proof. For $\Psi,\Phi \in \mathfrak{C}_B^2(X)$, we have the following relation:

$$\begin{aligned}
&\mathcal{K}_m^*(\Psi\Phi;x)-\mathcal{K}_m^*(\Psi;x)\mathcal{K}_m^*(\Phi;x)\\
=&\mathcal{K}_m^*(\Psi\Phi;x)-\Psi(x)\Phi(x)-(\Psi\Phi)'(x)\zeta_m(x)-(\Psi\Phi)''(x)\frac{\mu_m(x)}{2}\\
&-\Phi(x)\left[\mathcal{K}_m^*(\Psi;x)-\Psi(x)-\Psi'(x)\zeta_m(x)-\Psi''(x)\frac{\mu_m(x)}{2}\right]\\
&-\mathcal{K}_m^*(\Psi;x)\left[\mathcal{K}_m^*(\Phi;x)-\Phi(x)-\Phi'(x)\zeta_m(x)-\Phi''(x)\frac{\mu_m(x)}{2}\right]\\
&+\frac{\mu_m(x)}{2}[\Psi(x)\Phi''(x)+2\Psi'(x)\Phi'(x)-\Phi''(x)\mathcal{K}_m^*(\Psi;x)]\\
&+\zeta_m(x)[\Psi(x)\Phi'(x)-\Phi'(x)\mathcal{K}_m^*(\Phi;x)].
\end{aligned}$$

Multiplying both sides by m and then taking the limit, we get

$$\begin{aligned}
&\lim_{m\to\infty} m\{\mathcal{K}_m^*(\Psi\Phi;x)-\mathcal{K}_m^*(\Psi;x)\mathcal{K}_m^*(\Phi;x)\}\\
=&\lim_{m\to\infty} m\left\{\mathcal{K}_m^*(\Psi\Phi;x)-\Psi(x)\Phi(x)-(\Psi\Phi)'(x)\zeta_m(x)-(\Psi\Phi)''(x)\frac{\mu_m(x)}{2}\right\}\\
&-\Phi(x)\lim_{m\to\infty} m\left[\mathcal{K}_m^*(\Psi;x)-\Psi(x)-\Psi'(x)\zeta_m(x)-\Psi''(x)\frac{\mu_m(x)}{2}\right]
\end{aligned}$$

$$
\begin{aligned}
&- \lim_{m\to\infty} \mathcal{K}_m^*(\Psi;x) m \left[\mathcal{K}_m^*(\Phi;x) - \Phi(x) - \Phi'(x)\zeta_m(x) - \Phi''(x)\frac{\mu_m(x)}{2} \right] \\
&+ \lim_{m\to\infty} [m\mu_m(x)] \Psi'(x)\Phi'(x) + \lim_{m\to\infty} \frac{m\mu_m(x)}{2} [\Psi(x)\Phi''(x) - \Phi''(x)\mathcal{K}_m^*(\Psi;x)] \\
&+ \lim_{m\to\infty} [m\zeta_m(x)] [\Psi(x)\Phi'(x) - \Phi'(x)\mathcal{K}_m^*(\Phi;x)].
\end{aligned}
$$

Lastly, using Theorem 2.4, Corollary 2.12 and the facts that $\lim_{m\to\infty} m\zeta_m(x) = \frac{3}{2} - 3x$ and $\lim_{m\to\infty} m\mu_m(x) = 3x(1-x)$, we get the desired assertion. □

2.1.5 Graphical Examples

In this section, we present some numerical examples for the Kantorovich variant of Bernstein type operators (2.2) in order to show their approximation properties.

In Figures 2.1 and 2.2, we draw the results of Kantorovich variant of Bernstein type operators for the test functions $\Psi(x) = e^{-x}\sin(\frac{3\pi x}{2})$ and $x^2 log(1+x)$ respectively. Clearly, the proposed operator converges to these test functions.

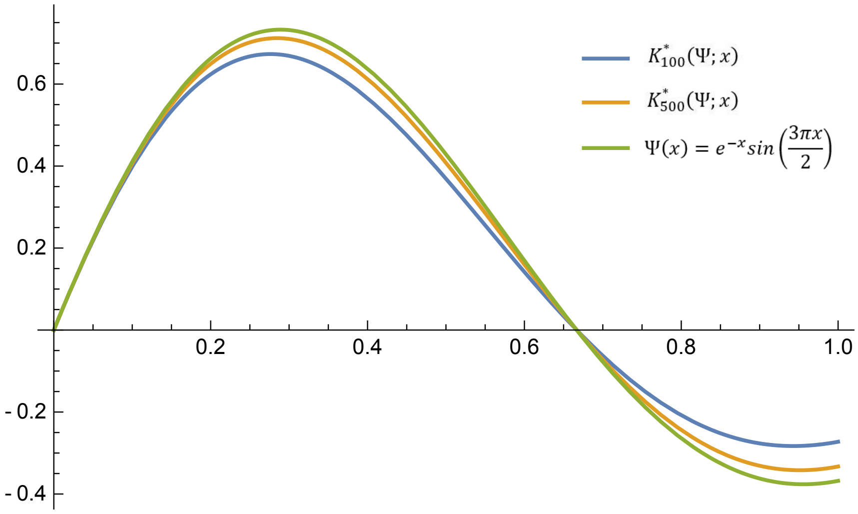

FIGURE 2.1: Convergence of Kantorovich variant of Bernstein type operators $\mathcal{K}_m^*(\Psi;x)$ to the test function $\Psi(x) = e^{-x}\sin(\frac{3\pi x}{2})$ for $m = 100, 500$.

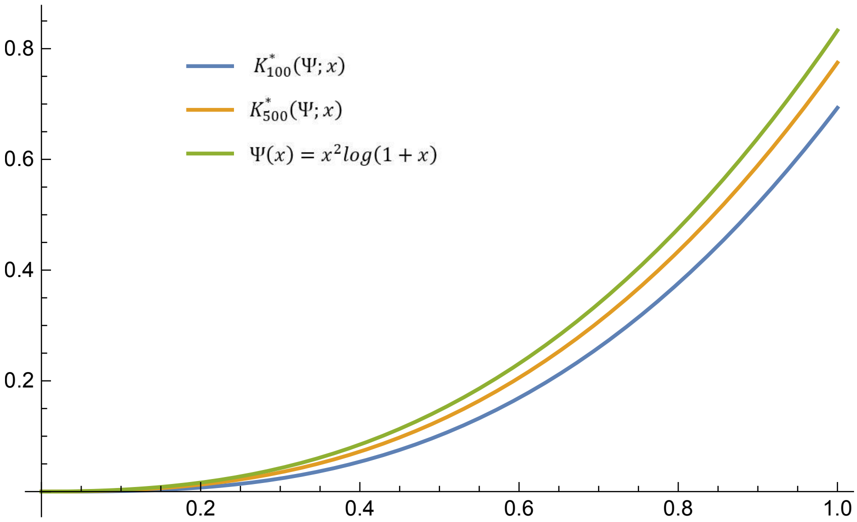

FIGURE 2.2: Convergence of Kantorovich variant of Bernstein type operators $\mathcal{K}^*_m(\Psi;x)$ to the test function $\Psi(x)=x^2log(1+x)$ for $m=100,500$.

2.2 Blending type approximation by λ-Bernstein-Beta type operators

2.2.1 Introduction

In 2010, Ye et al. [113] defined a new Bézier bases $\tilde{p}_{m,j}(\lambda;x),\ j=0,1,...,m$ with the shape parameter $\lambda\in[-1,1]$ by

$$\begin{cases}\tilde{p}_{m,0}(\lambda;x)=p_{m,0}(x)-\frac{\lambda}{m+1}p_{m+1,1}(x),\\ \tilde{p}_{m,j}(\lambda;x)=p_{m,j}(x)+\lambda\left(\frac{m-2j+1}{m^2-1}p_{m+1,j}(x)-\frac{m-2j-1}{m^2-1}p_{m+1,j+1}(x)\right),\quad(1\leq j\leq m-1),\\ \tilde{p}_{m,m}(\lambda;x)=p_{m,m}(x)-\frac{\lambda}{m+1}p_{m+1,m}(x),\end{cases} \tag{2.21}$$

where $p_{m,j}(x)=\binom{m}{j}x^j(1-x)^{m-j}$.

Referring to equation (2.21), it's essential to highlight that incorporation of the shape parameter λ provides us with increased modeling flexibility for the positive linear operators. Due to the above property, Cai et al. [40] have explored a modification of the Bernstein polynomials as follows

$$P_{m,\lambda}(\Psi;x)=\sum_{j=0}^{m}\tilde{p}_{m,j}(\lambda;x)\Psi\left(\frac{j}{m}\right), \tag{2.22}$$

where $\tilde{p}_{m,j}(\lambda;x)$ defined in (2.21). In particular, when $\lambda=0$ the operators (2.22) boils down to the Bernstein polynomials (1.1). Further, some analysis of important approximation properties of (2.22) has been conducted and results regarding their rate of convergence have been established. Also, to ensure that some continuous functions and different values of λ lead to a better convergence speed compared to the classical Bernstein polynomials.

Further, Cai [38] introduced generalized λ-Bernstein operators by developing Kantorovich-type λ-Bernstein operators and their Bézier variant, and analyzed these operators in terms of several approximation properties. Later, Acu et al. [15] introduced λ-Bernstein-Kantorovich operators and discussed various approximation properties and asymptotic type results. In this context, we highlight the works of authors who introduced modifications to the λ-Bernstein operators and established their convergence in the papers, see [37, 75, 101, 102, 107].

It is worth noting that, many Beta-type generalizations like the Stancu-Beta operator, Beta operator of the first kind, q-Stancu-Beta operator, (p,q)-Bernstein-Beta operator, etc. are contributed to this field of Approximation theory, see [2, 17, 26, 41, 64, 88, 91, 92].

In recent years, the Beta-type generalization of many operators has became quite a popular area of research. Motivated by the above-mentioned works, we introduce a

Beta-type generalization of the operators (2.22) as follows

$$\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)=\sum_{j=0}^{m}\tilde{p}_{m,k}(\lambda;x)\int_0^1\frac{t^{j\nu+\rho(x)-1}(1-t)^{(m-j)\nu+\rho(x)-1}}{\beta(l\nu+\rho(x),(m-j)\nu+\rho(x))}\Psi(t)dt, \qquad (2.23)$$

where the maps Ψ and ρ are continuous on X, $\lambda\in[-1,1]$, $x\in X$ and $\tilde{p}_{m,j}(\lambda;x)$ is given in (2.21). Also, $\nu>0$ and $\beta(q,r)$ is the beta function defined by

$$\beta(q,r)=\int_0^1 t^{q-1}(1-t)^{r-1}dt, \quad q,r>0.$$

One can easily observe that the operators $\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)$ are linear and positive.

Remark 2.14. Some Special cases:

1. When $\rho(x)=\nu=1$, we obtain the λ-Bernstein-Durrmeyer operators

   $$D_{m,\lambda}(\Psi,x)=(m+1)\sum_{j=0}^{m}\tilde{p}_{m,j}(\lambda;x)\int_0^1 p_{m,j}(t)\Psi(t)dt.$$

   In addition, if $\lambda=0$, then we get the classical Bernstein-Durrmeyer operators

   $$D_{m}(\Psi,x)=(m+1)\sum_{j=0}^{m}p_{m,j}(x)\int_0^1 p_{m,j}(t)\Psi(t)dt.$$

2. When $\rho(x)=1$, the above defined operators get reduced to the operators defined in [13].

During this discussion, our primary concern is to analyze some of the essential approximation properties of the operators (2.23). Also, we examine their speed of convergence by establishing the Voronovskaja-type results. At last, we propose a direct estimation result for absolutely continuous maps on X, whose derivatives are equivalent to some function of bounded variation.

Now onwards, we equip the space $\mathfrak{C}(X)$ with the sup norm: $\|\Psi\| = \sup\{|\Psi(x)| : x \in X\}$.

2.2.2 Auxiliary Results

This section discusses some basic results that will be used to establish the main results.

Lemma 2.15. [40] *The operators* (2.22) *satisfy the following relations*

$$
\begin{aligned}
P_{m,\lambda}(e_0;x) =& 1,\\
P_{m,\lambda}(e_1;x) =& x + \lambda\left\{\frac{1-2x+x^{m+1}-(1-x)^{m+1}}{m(m-1)}\right\},\\
P_{m,\lambda}(e_2;x) =& x^2 + \frac{x(1-x)}{m} + \lambda\left\{\frac{2x-4x^2+2x^{m+1}}{m(m-1)} + \frac{x^{m+1}+(1-x)^{m+1}-1}{m^2(m-1)}\right\},\\
P_{m,\lambda}(e_3;x) =& x^3 + \frac{3x^2(1-x)}{m} + \frac{2x^3-3x^2+x}{m^2} + \lambda\left[\frac{-6x^3+6x^{m+1}}{m^2} + \frac{3x^2-3x^{m+1}}{m(m-1)}\right.\\
&\left.+\frac{-9x^2+9x^{m+1}}{m^2(m-1)} + \frac{-4x+4x^{m+1}}{m^3(m-1)} + \frac{(1-x^{m+1}-(1-x)^{m+1})(m+3)}{m^3(m^2-1)}\right].
\end{aligned}
$$

Lemma 2.16. *For the newly defined operators* (2.23)*, the usual moments are calculated as:*

$$
\begin{aligned}
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_0;x) =& 1,\\
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_1;x) =& \frac{1}{m\nu+2\rho(x)}\left[\{m\nu x+\rho(x)\} + \lambda\nu\left\{\frac{1-2x+x^{m+1}-(1-x)^{m+1}}{m-1}\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_2;x) =& \frac{1}{\{m\nu+2\rho(x)\}(m\nu+2\rho(x)+1)}\Big[\{m^2\nu^2x^2 + m\nu^2x(1-x) + mx\nu(2\rho(x)+1)\\
&+\rho(x)(\rho(x)+1)\} + \frac{\lambda\nu}{m-1}\Big\{m\nu(2x-4x^2+2x^{m+1})\\
&+\nu\left(x^{m+1}+(1-x)^{m+1}-1\right) + (2\rho(x)+1)(1-2x+x^{m+1}-(1-x)^{m+1})\Big\}\Big].
\end{aligned}
$$

Proof. From the operators (2.23), we have the following relations

$$
\begin{aligned}
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_0;x) =& P_{m,\lambda}(e_0;x), \qquad (2.24)\\
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_1;x) =& \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda;x) \int_0^1 \frac{t^{j\nu+\rho(x)}(1-t)^{(m-j)\nu+\rho(x)-1}}{\beta(j\nu+\rho(x),(m-j)\nu+\rho(x))}dt\\
=& \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda;x) \frac{\beta(j\nu+\rho(x)+1,(m-j)\nu+\rho(x))}{\beta(j\nu+\rho(x),(m-j)\nu+\rho(x))}\\
=& \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda;x) \left(\frac{j\nu+\rho(x)}{m\nu+2\rho(x)}\right)\\
=& \frac{m\nu}{m\nu+2\rho(x)} P_{m,\lambda}(e_1;x) + \frac{\rho(x)}{m\nu+2\rho(x)} P_{m,\lambda}(e_0;x), \qquad (2.25)\\
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_2;x) =& \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda;x) \int_0^1 \frac{t^{j\nu+\rho(x)+1}(1-t)^{(m-j)\nu+\rho(x)-1}}{\beta(j\nu+\rho(x),(m-j)\nu+\rho(x))}dt\\
=& \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda;x) \frac{\beta(j\nu+\rho(x)+2,(m-j)\nu+\rho(x))}{\beta(j\nu+\rho(x),(m-j)\nu+\rho(x))}\\
=& \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda;x) \left(\frac{j\nu+\rho(x)}{m\nu+2\rho(x)}\right)\left(\frac{j\nu+\rho(x)+1}{m\nu+2\rho(x)+1}\right)\\
=& \frac{m^2\nu^2 P_{m,\lambda}(e_2;x) + m\nu(2\rho(x)+1)P_{m,\lambda}(e_1;x) + \rho(x)(1+\rho(x))P_{m,\lambda}(e_0;x)}{(m\nu+2\rho(x))(m\nu+2\rho(x)+1)}.
\end{aligned}
$$

$$(2.26)$$

To conclude our desired assertions, we use the Lemma 2.15 in (2.24), (2.25) and (2.26). ☐

Lemma 2.17. *The operators* (2.23) *have the following central moments*

$$
\begin{aligned}
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_{0,x};x) =& 1,\\
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_{1,x};x) =& \frac{1}{m\nu+2\rho(x)}\left[\rho(x)(1-2x)+\lambda\nu\left\{\frac{1-2x+x^{m+1}-(1-x)^{m+1}}{m-1}\right\}\right]\\
&= \alpha^{\rho,\nu}_{m,\lambda}(x), \qquad (2.27)\\
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_{2,x};x) =& \frac{m(x\nu^2-x^2\nu^2+x\nu-x^2\nu)}{(m\nu+2\rho(x))(m\nu+2\rho(x)+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2(2x^2\rho^2(x)+x^2\rho(x)-2x\rho^2(x)-x\rho(x))+\rho^2(x)+\rho(x)}{(m\nu+2\rho(x))(m\nu+2\rho(x)+1)}\\
&+\frac{\lambda\nu\left\{2m\nu x(1-x)(x^m+(1-x)^m)+\nu(x^{m+1}+(1-x)^{m+1}-1)\right\}}{(m-1)(m\nu+2\rho(x))(m\nu+2\rho(x)+1)}\\
&+\frac{\lambda\nu\left\{(2\rho(x)+1)(1-4x+4x^2-2x^{m+2}+2x(1-x)^{m+1})\right\}}{(m-1)(m\nu+2\rho(x))(m\nu+2\rho(x)+1)}\\
&+\frac{\lambda\nu\left\{(2\rho(x)+1)(x^{m+1}-(1-x)^{m+1})\right\}}{(m-1)(m\nu+2\rho(x))(m\nu+2\rho(x)+1)}\\
&=\beta^{\rho,\nu}_{m,\lambda}(x). \qquad (2.28)
\end{aligned}
$$

Remark 2.18. The calculated central moments in Lemma 2.17 satisfy the following limiting conditions

$$
\begin{aligned}
\lim_{m\to\infty} m\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_{1,x};x) =&\frac{(1-2x)\rho(x)}{\nu},\\
\lim_{m\to\infty} m\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_{2,x};x) =&x(1-x)\left(1+\frac{1}{\nu}\right).
\end{aligned}
$$

Lemma 2.19. *The inequality* $\|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;y)\| \leq \|\Psi\|$ *holds for any* $\Psi\in\mathfrak{C}(X)$.

Proof. In view of the operator (2.23), Lemma 2.16, and the norm defined for $\mathfrak{C}(X)$, we immediately conclude the result. □

Theorem 2.20. *For any* $\Psi\in\mathfrak{C}(X)$ *and* $\lambda\in[-1,1]$,

$$
\lim_{m\to\infty}\|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)-\Psi(x)\|=0.
$$

Proof. From Lemma 2.16, one can easily derive that

$$
\lim_{m\to\infty}\|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(e_i;y)-e_i(y)\|=0, \quad \text{for i=0,1,2.}
$$

Hence the result follows from Korovkin's Theorem [74]. □

2.2.3 Local and Global Approximation Results

In this section, we estimate the rate of convergence of the newly defined operators (2.22) in terms of the modulus of continuity and with the help of the relation (1.9)

Theorem 2.21. *For $x \in X$ and $\Psi \in \mathfrak{C}_B^1(X)$, we have the following relation*

$$|\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}(\Psi;x) - \Psi(x)| \leq \left|\alpha_{m,\lambda}^{\rho;\nu}(x)\right||\Psi'(x)| + 2\sqrt{\beta_{m,\lambda}^{\rho;\nu}(x)}\Omega_1\left(\Psi', \sqrt{\beta_{m,\lambda}^{\rho;\nu}(x)}\right),$$

where $\alpha_{m,\lambda}^{\rho;\nu}(x)$ and $\beta_{m,\lambda}^{\rho;\nu}(x)$ are defined in (2.27).

Proof. Being similar to the proof of Theorem 2.6, hence we omit it. ☐

Theorem 2.22. *Let $\Psi \in \mathfrak{C}(X)$ and $\Phi \in \mathfrak{C}_B^2(X)$. Then $\exists$ a constant $C > 0$ obeying the following relation for each $m \in \mathbb{N}$*

$$\left|\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}(\Psi;x) - \Psi(x) - \alpha_{m,\lambda}^{\rho,\nu}(x)\Phi'(y)\right| \leq C\Omega_2\left(\Psi, \sqrt{\beta_{m,\lambda}^{\rho,\nu}(x)}\right).$$

Proof. Similar to that of Theorem 2.7. So, we omit it. ☐

Next, we present a local approximation result for the Lipschitz-type class of functions.

Theorem 2.23. *For any $\eta \in (0,1]$ and $\Psi \in Lip_{M,\eta}^{a,b}(X)$, we must have*

$$|\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}(\Psi;x) - \Psi(x)| \leq M\left(\frac{\beta_{m,\lambda}^{\rho,\nu}(x)}{ax^2 + bx}\right)^{\eta/2},$$

where $\beta_{m,\lambda}^{\rho;\nu}(x)$ is given in (2.28) and $M > 0$ depends on Ψ.

Proof. Similar to that of Theorem 2.8. So, we omit it. ☐

We'll now establish a global approximation result for the newly defined operators (2.23) in terms of first and second-order Ditzian-Totik uniform modulus of smoothness, where the admissible weight function is taken as $\xi(x) = [x(1-x)]^{1/2}$.

Theorem 2.24. *For any $\Psi \in \mathfrak{C}(X)$ and $x \in X$, $\exists\ M > 0$, such that*

$$|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi; x) - \Psi(x)| \leq M\Omega^{\xi}_2\left(\Psi, \frac{\tau^{\rho,\nu}_{m,\lambda}(x)}{2\xi(x)}\right) + \Omega^{\xi}_1\left(\Psi, \frac{\alpha^{\rho,\nu}_{m,\lambda}(x)}{\xi(x)}\right), \tag{2.29}$$

where $\tau^{\rho,\nu}_{m,\lambda}(x) = \left\{\beta^{\rho,\nu}_{m,\lambda}(x) + \left(\alpha^{\rho,\nu}_{m,\lambda}(x)\right)^2\right\}^{1/2}$.

Proof. Being similar to that of Theorem 2.9, so we skip it. □

2.2.4 Voronovskaja-type Asymptotic Results

This section presents Voronovskaja-type asymptotic results to study the speed of convergence of the operators (2.23).

Theorem 2.25. *Assume that an integrable function Ψ on X such that Ψ'' exists at some point $x \in X$. Then we have the following relation*

$$\lim_{m\to\infty} m\{\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi; x) - \Psi(x)\} = \frac{(1-2x)\rho(x)}{\nu}\Psi'(x) + x(1-x)\left(1 + \frac{1}{\nu}\right)\Psi''(x).$$

Proof. By the well-known Taylor's expansion,

$$\Psi(t) = \Psi(x) + \Psi'(x)(t-x) + \frac{1}{2}\Psi''(x)(t-x)^2 + \Lambda(t, x)(t-x)^2, \tag{2.30}$$

where $\Lambda(t, x) \in \mathfrak{C}(X)$ and it satisfies $\lim\limits_{t\to x} \Lambda(t, x) = 0$.

For $m \in \mathbb{N}$, we apply the operators $\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}$ on (2.30) to obtain

$$\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi; x) = \Psi(x) + \Psi'(x)\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x); x) + \frac{1}{2}\Psi''(x)\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^2; x)$$

$$+\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Lambda(t,x)(t-x)^2;x),$$

and of course applying limit $m \to \infty$ on both sides, we get

$$\begin{aligned}&\lim_{m\to\infty} m\{\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)-\Psi(x)\}\\&=\Psi'(x)\lim_{m\to\infty} m\,\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x);x)+\frac{\Psi''(x)}{2}\lim_{m\to\infty} m\,\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^2;x)\\&+\lim_{m\to\infty} m\,\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Lambda(t,x)(t-x)^2;x).\end{aligned}$$

Next, using Remark 2.18, we can easily obtain

$$\begin{aligned}\lim_{m\to\infty} m\{\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)-\Psi(x)\}=&\frac{(1-2x)\rho(x)}{\nu}\Psi'(x)+x(1-x)\left(1+\frac{1}{\nu}\right)\Psi''(x)\\&+\lim_{m\to\infty} m\,\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Lambda(t,x)(t-x)^2;x).\end{aligned}\tag{2.31}$$

Hence, from the Cauchy-Schwarz inequality, we get

$$\lim_{m\to\infty} m\,\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Lambda(t,x)(t-x)^2;x)\leq\sqrt{\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Lambda^2(t,x);x)}\sqrt{m^2\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^4;x)}.\tag{2.32}$$

Also, Theorem 2.20 ensures that

$$\lim_{m\to\infty}\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Lambda^2(t,x);x)=\Lambda^2(x,x)=0.\tag{2.33}$$

Lastly, in view of the fact that $\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^4;x)$ is of order m^{-2} and then applying (2.32) and (2.33) in (2.31), we conclude the proposed result. □

Next, we proceed for a quantitative Voronovskaja-type estimation for the newly defined operators using the 1st-order Ditzian-Totik modulus of smoothness.

Theorem 2.26. *For $\Psi \in \mathfrak{C}_B^2(X)$, $x \in X$ and sufficiently large m, there exists $M > 0$ satisfying the following relation*

$$\left| m\left(\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}(\Psi;x) - \Psi(x)\right) - m\alpha_{m,\lambda}^{\rho,\nu}(x)\Psi'(x) - m\frac{\beta_{m,\lambda}^{\rho,\nu}(x)}{2}\Psi''(x)\right| \leq \frac{M}{m}\xi^2(x)\Omega_1^{\xi}\left(\Psi'', \frac{1}{\sqrt{m}}\right),$$

where $\alpha_{m,\lambda}^{\rho,\nu}(x)$ and $\beta_{m,\lambda}^{\rho,\nu}(x)$ are given in (2.27) *and* (2.28) *respectively.*

Proof. For $\Psi \in \mathfrak{C}_B^2(X)$ and $x \in X$. Taylor's series ensures us the following

$$\Psi(t) - \Psi(x) - (t-x)\Psi'(x) = \int_x^t (t-z)\Psi''(z)dz.$$

This leads to the following relation

$$\Psi(t) - \Psi(x) - (t-x)\Psi'(x) - \frac{\Psi''(x)}{2}(t-x)^2 = \int_x^t (t-z)[\Psi''(z) - \Psi''(x)]dz.$$

Now, applying the operators (2.23) on both sides, we obtain

$$\begin{aligned}&\left|\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}(\Psi;x) - \Psi(x) - \widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}(t-x;x)\Psi'(x) - \frac{\Psi''(x)}{2}\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}((t-x)^2;x)\right| \\ &\leq \widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}\left(\left|\int_x^t |t-z||\Psi''(z) - \Psi''(x)|dz\right|;x\right). \qquad (2.34)\end{aligned}$$

It is easy to notice from [56, p.337] that the following inequality is true for any $\Phi \in \mathcal{W}_\xi^2(X)$:

$$\left|\int_x^t |t-z||\Psi''(z) - \Psi''(x)|dz\right| \leq 2\|\Psi'' - \Phi\|(t-x)^2 + 2\|\xi\Phi'\|\xi^{-1}(x)|t-x|^3. \qquad (2.35)$$

Also, using the facts that $\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}((t-x)^2;x)$ is of order m^{-1} and $\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}((t-x)^4;x)$ is of order m^{-2}, for sufficiently large m, we can get a constant $M > 0$, such that

$$\widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}((t-x)^2;x) \leq \frac{M}{2m}\xi^2(x) \quad \text{and} \quad \widetilde{\mathfrak{P}}_{m,\lambda}^{\rho,\nu}((t-x)^4;x) \leq \frac{M}{2m^2}\xi^4(x). \qquad (2.36)$$

In view of (2.35), (2.36) and the well-known Cauchy-Schwarz inequality, (2.34) yields

$$\begin{aligned}
&\left|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)-\Psi(x)-\alpha^{\rho,\nu}_{m,\lambda}(x)\Psi'(x)-\frac{\beta^{\rho,\nu}_{m,\lambda}(x)}{2}\Psi''(x)\right|\\
&\leq 2\|\Psi''-\Phi\|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^2;x)+2\|\xi\Phi'\|\xi^{-1}(x)\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(|t-x|^3;x)\\
&\leq \frac{M}{m}\xi^2(x)\|\Psi''-\Phi\|+2\|\xi\Phi'\|\xi^{-1}(x)\sqrt{\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^2;x)}\sqrt{\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^4;x)}\\
&\leq \frac{M}{m}\xi^2(x)\left(\|\Psi''-\Phi\|+m^{-1/2}\|\xi\Phi'\|\right).
\end{aligned}$$

Lastly, taking infimum over all $\Phi\in\mathcal{W}^2_\xi(X)$, we get the desired assertion. ☐

In light of the above theorem, one can draw the following conclusion.

Corollary 2.27. *For* $\Psi\in\mathfrak{C}^2_B(X)$

$$\lim_{m\to\infty} m\left|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)-\Psi(x)-\alpha^{\rho,\nu}_{m,\lambda}\Psi'(x)-\frac{\beta^{\rho,\nu}_{m,\lambda}}{2}\Psi''(x)\right|=0.$$

We complete this section by showing a Grüss-Voronovskaja type result for a particular class of the newly defined sequence of operators.

Theorem 2.28. *For* $\Psi,\Phi\in\mathfrak{C}^2_B(X)$ *and* $x\in X$,

$$\lim_{m\to\infty} m\{\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi\Phi;x)-\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Phi;x)\}=\left(1+\frac{1}{\nu}\right)x(1-x)\Psi'(x)\Phi'(x).$$

Proof. The same line of proof as that of Theorem 2.13 so, we omit it. ☐

2.2.5 A direct Estimation

The final step of our discussion is to calculate the error while approximating the functions from a special class, namely $BV'(X)$, which consists of all continuous maps with derivatives of bounded variation.

By setting $\tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t) = \sum_{j=0}^{m} \tilde{p}_{m,j}(\lambda;x)\frac{t^{j\nu+\rho(x)-1}(1-t)^{(m-j)\nu+\rho(x)-1}}{\beta(j\nu+\rho(x),(m-j)\nu+\rho(x))}$ as the kernel, we can easily rewrite our operator (2.23) as

$$\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x) = \int_0^1 \tilde{\mathcal{K}}_{m,\lambda}(x,t)\Psi(t)dt. \tag{2.37}$$

Lemma 2.29. *For $x \in (0,1]$ and large positive integer m, we obtain the following inequalities*

1. *If $0 \leq y < x$, then*

$$\zeta^{\rho,\nu}_{m,\lambda}(x,y) = \int_0^y \tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt \leq \frac{\beta^{\rho,\nu}_{m,\lambda}(x)}{(x-y)^2}.$$

2. *If $x < z < 1$, then*

$$1 - \zeta^{\rho,\nu}_{m,\lambda}(x,z) = \int_z^1 \tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt \leq \frac{\beta^{\rho,\nu}_{m,\lambda}(x)}{(z-x)^2}.$$

Proof. Using (2.37) and (2.28) we observe that, for $0 \leq y < x$,

$$\begin{aligned}\zeta^{\rho,\nu}_{m,\lambda}(x,y) = \int_0^y \tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt &\leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 \tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt \\ &= \frac{\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^2;x)}{(x-y)^2} = \frac{\beta^{\rho,\nu}_{m,\lambda}(x)}{(x-y)^2}.\end{aligned}$$

For the second part, the proof is similar. □

Theorem 2.30. *Let $\Psi \in BV'(0,1)$ and $x \in (0,1)$. If $m \in \mathbb{N}$ is sufficiently large, then the following inequality is estimated:*

$$\begin{aligned}\left|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x) - \Psi(x)\right| \leq &\frac{1}{2}\{\Psi'(x+) + \Psi'(x-)\}\alpha^{\rho,\nu}_{m,\lambda}(x) + \frac{1}{2}\sqrt{\beta^{\rho,\nu}_{m,\lambda}(x)}\{\Psi'(x+) - \Psi'(x-)\} \\ &+ x^{-1}\beta^{\rho,\nu}_{m,\lambda}(x)\sum_{j=1}^{[\sqrt{m}]} \mathfrak{T}^{x}_{x-\frac{x}{j}}(\Psi'_x) + \frac{x}{\sqrt{m}}\mathfrak{T}^{x}_{x-\frac{x}{\sqrt{m}}}(\Psi'_x)\end{aligned}$$

$$+(1-x)^{-1}\beta^{\rho,\nu}_{m,\lambda}(x)\sum_{j=1}^{[\sqrt{m}]}\mathfrak{T}_x^{x+\frac{(1-x)}{j}}(\Psi'_x)+\frac{(1-x)}{\sqrt{m}}\mathfrak{T}_x^{x+\frac{(1-x)}{\sqrt{m}}}(\Psi'_x),$$

where the function Ψ'_x *is defined as*

$$\Psi'_x(t)=\begin{cases}\Psi'(t)-\Psi'(x-), & 0\leq t<x,\\ \Psi'(t)-\Psi'(x+), & x<t\leq 1,\\ 0, & t=x.\end{cases}$$

Proof. We know that the operator (2.23) preserves the constant functions and hence

$$\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x)-\Psi(x)=\int_0^1(\Psi(t)-\Psi(x))\tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt=\int_0^1\tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)\left(\int_x^t\Psi'(z)dz\right)dt. \tag{2.38}$$

Also, for $\Psi\in BV'(0,1)$, we can write

$$\begin{aligned}\Psi'(z)=&\frac{1}{2}\{\Psi'(x+)+\Psi'(x-)\}+\Psi'_x(z)+\frac{1}{2}\{\Psi'(x+)-\Psi'(x-)\}sgn(z-x)\\ &+[\Psi'(z)-\frac{1}{2}\{\Psi'(x+)+\Psi'(x-)\}]\delta_x^z,\end{aligned} \tag{2.39}$$

where δ_x^z is the Kronecker delta function and defined as

$$\delta_x^z=\begin{cases}1, & z=x\\ 0, & z\neq x\end{cases}$$

and $sgn(x)$ is the signum function defined by

$$sgn(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0, & x=0.\end{cases}$$

It is easily observed that

$$\int_0^1 \left(\int_x^t \left[\Psi'(z) - \frac{1}{2}\{\Psi'(x+) + \Psi'(x-)\}\right] \delta_x^z dz\right) \tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt = 0. \tag{2.40}$$

Now, using relation (2.37), we have

$$\int_0^1 \left(\int_x^t \frac{1}{2}\{\Psi'(x+) + \Psi'(x-)\}dz\right) \tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt = \frac{1}{2}\{\Psi'(x+) + \Psi'(x-)\}\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(t-x;x). \tag{2.41}$$

Also, by some manipulations and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\int_0^1 \left(\int_x^t \frac{1}{2}\{\Psi'(x+) - \Psi'(x-)\}sgn(z-x)dz\right) \tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt \\
&\quad\leq \frac{1}{2}\{\Psi'(x+) - \Psi'(x-)\}\int_0^1 |t-x|\tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt \\
&\quad= \frac{1}{2}\{\Psi'(x+) - \Psi'(x-)\}\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(|t-x|;x) \\
&\quad\leq \frac{1}{2}\{\Psi'(x+) - \Psi'(x-)\}\left[\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^2;x)\right]^{1/2}.
\end{aligned} \tag{2.42}$$

Now, using (2.39)-(2.42) in (2.38), we have

$$\begin{aligned}
\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x) - \Psi(x) \leq & \frac{1}{2}\{\Psi'(x+) + \Psi'(x-)\}\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(t-x;x) \\
& + \frac{1}{2}\{\Psi'(x+) - \Psi'(x-)\}\left[\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}((t-x)^2;x)\right]^{1/2} \\
& + \int_0^1 \left(\int_x^t \Psi_x'(z)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt \\
= & \frac{1}{2}\{\Psi'(x+) + \Psi'(x-)\}\alpha^{\rho,\nu}_{m,\lambda}(x) + \frac{1}{2}\sqrt{\beta^{\rho,\nu}_{m,\lambda}(x)}\{\Psi'(x+) - \Psi'(x-)\} \\
& + \int_0^x \left(\int_x^t \Psi_x'(z)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt + \int_x^1 \left(\int_x^t \Psi_x'(z)dz\right)\tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt.
\end{aligned}$$

This yields

$$\begin{aligned}
\left|\widetilde{\mathfrak{P}}^{\rho,\nu}_{m,\lambda}(\Psi;x) - \Psi(x)\right| \leq & \frac{1}{2}\{\Psi'(x+) + \Psi'(x-)\}\alpha^{\rho,\nu}_{m,\lambda}(x) \\
& + \frac{1}{2}\sqrt{\beta^{\rho,\nu}_{m,\lambda}(x)}\{\Psi'(x+) - \Psi'(x-)\} + I_1 + I_2,
\end{aligned} \tag{2.43}$$

where $I_1 = \left|\int_0^x \left(\int_x^t \Psi_x'(z)dz\right) \tilde{\mathcal{K}}_{m,\lambda}^{\rho,\nu}(x,t)dt\right|$, and $I_2 = \left|\int_x^1 \left(\int_x^t \Psi_x'(z)dz\right) \tilde{\mathcal{K}}_{m,\lambda}^{\rho,\nu}(x,t)dt\right|$. To establish our claim, we need to estimate the integrals I_1 and I_2. It is noticed that for $t < x$, we have $\tilde{\mathcal{K}}_{m,\lambda}^{\rho,\nu}(x,t)dt = d_t\zeta_{m,\lambda}^{\rho,\nu}(x,t)$, where d_t denotes the differential operator with respect to t.

Now, applying integration by parts, we get

$$\begin{aligned}
I_1 =&\left|\int_0^x \left(\int_x^t \Psi_x'(z)dz\right) d_t\zeta_{m,\lambda}^{\rho,\nu}(x,t)\right|\\
=&\left|\left[\left(\int_x^t \Psi_x'(z)dz\right)\zeta_{m,\lambda}^{\rho,\nu}(x,t)\right]_0^x - \int_0^x \Psi_x'(t)\zeta_{m,\lambda}^{\rho,\nu}(x,t)dt\right|\\
=&\left|\int_0^x \zeta_{m,\lambda}^{\rho,\nu}(x,t)\Psi_x'(t)dt\right|\\
\leq&\int_0^y |\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t)|dt + \int_y^x |\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t)|dt.
\end{aligned}$$

Substituting $y = x - \frac{x}{\sqrt{m}}$,

$$I_1 \leq \int_0^{x-\frac{x}{\sqrt{m}}} |\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t)|dt + \int_{x-\frac{x}{\sqrt{m}}}^x |\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t)|dt.$$

Now, by noting the facts that $\Psi_x'(x) = 0$ and $|\zeta_{m,\lambda}^{\rho,\nu}(x,t)| \leq 1$, it follows

$$\begin{aligned}
\int_{x-\frac{x}{\sqrt{m}}}^x |\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t)|dt =& \int_{x-\frac{x}{\sqrt{m}}}^x |\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t) - \Psi_x'(x)|dt\\
\leq& \int_{x-\frac{x}{\sqrt{m}}}^x \mathfrak{T}_t^x(\Psi_x')dt\\
\leq& \frac{x}{\sqrt{m}}\mathfrak{T}_{x-\frac{x}{\sqrt{m}}}^x(\Psi_x').
\end{aligned} \tag{2.44}$$

Also, by using Lemma 2.29 and the substitution $t = x - \frac{x}{z}$, we have

$$\begin{aligned}
\int_0^{x-\frac{x}{\sqrt{m}}} |\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t)|dt \leq&\beta_{m,\lambda}^{\rho,\nu}(x)\int_0^{x-\frac{x}{\sqrt{m}}} \frac{|\Psi_x'(t)|}{(x-t)^2}dt\\
=&\beta_{m,\lambda}^{\rho,\nu}(x)\int_0^{x-\frac{x}{\sqrt{m}}} \frac{|\Psi_x'(t) - \Psi_x'(x)|}{(x-t)^2}dt
\end{aligned}$$

$$
\begin{aligned}
&\leq x^{-1}\beta^{\rho,\nu}_{m,\lambda}(x)\int_1^{\sqrt{m}}\mathfrak{T}^x_{x-\frac{x}{z}}(\Psi'_x)dz\\
&\leq x^{-1}\beta^{\rho,\nu}_{m,\lambda}(x)\sum_{j=1}^{[\sqrt{m}]}\int_j^{j+1}\mathfrak{T}^x_{x-\frac{x}{z}}(\Psi'_x)dz\\
&\leq x^{-1}\beta^{\rho,\nu}_{m,\lambda}(x)\sum_{j=1}^{[\sqrt{m}]}\mathfrak{T}^x_{x-\frac{x}{j}}(\Psi'_x). \qquad (2.45)
\end{aligned}
$$

Combining (2.44) and (2.45), we get

$$
I_1 \leq x^{-1}\beta^{\rho,\nu}_{m,\lambda}(x)\sum_{j=1}^{[\sqrt{m}]}\mathfrak{T}^x_{x-\frac{x}{j}}(\Psi'_x)+\frac{x}{\sqrt{m}}\mathfrak{T}^x_{x-\frac{x}{\sqrt{m}}}(\Psi'_x). \qquad (2.46)
$$

Further,, we note the fact that for $t > x$, the relation $\tilde{\mathcal{K}}^{\rho,\nu}_{m,\lambda}(x,t)dt = d_t(1-\zeta^{\rho,\nu}_{m,\lambda}(x,t))$ holds and hence by applying the by parts rule of integration on I_2, it yields

$$
\begin{aligned}
I_2 &= \left|\int_x^1\left(\int_x^t \Psi'_x(z)dz\right)d_t(1-\zeta^{\rho,\nu}_{m,\lambda}(x,t))\right|\\
&= \left|\left[\left(\int_x^t \Psi'_x(z)dz\right)(1-\zeta^{\rho,\nu}_{m,\lambda}(x,t))\right]_x^1 - \int_x^1 \Psi'_x(t)(1-\zeta^{\rho,\nu}_{m,\lambda}(x,t))dt\right|\\
&= \left|\int_x^1 (1-\zeta^{\rho,\nu}_{m,\lambda}(x,t))\Psi'_x(t)dt\right|\\
&\leq \int_x^w |1-\zeta^{\rho,\nu}_{m,\lambda}(x,t)||\Psi'_x(t)|dt + \int_w^1 |1-\zeta^{\rho,\nu}_{m,\lambda}(x,t)||\Psi'_x(t)|dt.
\end{aligned}
$$

Substituting $w = x + \frac{(1-x)}{\sqrt{m}}$, in the above inequality we get

$$
I_2 \leq \int_x^{x+\frac{(1-x)}{\sqrt{m}}} |1-\zeta^{\rho,\nu}_{m,\lambda}(x,t)||\Psi'_x(t)|dt + \int_{x+\frac{(1-x)}{\sqrt{m}}}^1 |1-\zeta^{\rho,\nu}_{m,\lambda}(x,t)||\Psi'_x(t)|dt.
$$

By using Lemma 2.29 and the substitution $t = x + \frac{(1-x)}{z}$, we have

$$
\int_{x+\frac{(1-x)}{\sqrt{m}}}^1 |1-\zeta^{\rho,\nu}_{m,\lambda}(x,t)||\Psi'_x(t)|dt \leq \beta^{\rho,\nu}_{m,\lambda}(x)\int_{x+\frac{(1-x)}{\sqrt{m}}}^1 \frac{|\Psi'_x(t)|}{(t-x)^2}dt
$$

$$\begin{aligned}
&=\beta_{m,\lambda}^{\rho,\nu}(x)\int_{x+\frac{(1-x)}{\sqrt{m}}}^{1}\frac{|\Psi_x'(t)-\Psi_x'(x)|}{(t-x)^2}dt\\
&\leq(x-1)^{-1}\beta_{m,\lambda}^{\rho,\nu}(x)\int_{\sqrt{m}}^{1}\mathfrak{T}_x^{x+\frac{(1-x)}{z}}(\Psi_x')dz\\
&=(1-x)^{-1}\beta_{m,\lambda}^{\rho,\nu}(x)\int_{1}^{\sqrt{m}}\mathfrak{T}_x^{x+\frac{(1-x)}{z}}(\Psi_x')dz\\
&\leq(1-x)^{-1}\beta_{m,\lambda}^{\rho,\nu}(x)\sum_{j=1}^{[\sqrt{m}]}\int_{j}^{j+1}\mathfrak{T}_x^{x+\frac{(1-x)}{z}}(\Psi_x')dz\\
&\leq(1-x)^{-1}\beta_{m,\lambda}^{\rho,\nu}(x)\sum_{j=1}^{[\sqrt{m}]}\mathfrak{T}_x^{x+\frac{(1-x)}{j}}(\Psi_x').
\end{aligned}$$

Using the fact $|1-\zeta_{m,\lambda}^{\rho,\nu}(x,t)|\leq 1$, we can deduce

$$\begin{aligned}
\int_{x}^{x+\frac{(1-x)}{\sqrt{m}}}|1-\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t)|dt &= \int_{x}^{x+\frac{(1-x)}{\sqrt{m}}}|1-\zeta_{m,\lambda}^{\rho,\nu}(x,t)||\Psi_x'(t)-\Psi_x'(x)|dt\\
&\leq\int_{x}^{x+\frac{(1-x)}{\sqrt{m}}}\mathfrak{T}_x^{t}(\Psi_x')dt\leq\frac{(1-x)}{\sqrt{m}}\mathfrak{T}_x^{x+\frac{(1-x)}{\sqrt{m}}}(\Psi_x').
\end{aligned}$$

This leaves us with the following inequality

$$I_2\leq(1-x)^{-1}\beta_{m,\lambda}^{\rho,\nu}(x)\sum_{j=1}^{[\sqrt{m}]}\mathfrak{T}_x^{x+\frac{(1-x)}{j}}(\Psi_x')+\frac{(1-x)}{\sqrt{m}}\mathfrak{T}_x^{x+\frac{(1-x)}{\sqrt{m}}}(\Psi_x'). \tag{2.47}$$

Finally, substituting (2.46) and (2.47) in (2.43), we get the desired estimation.

□

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