Chapter 3

A generalization of a result of Vemuri

Here we prove a generalization of the Theorem 1.4.

3.1 Statement of the main result

Theorem 3.1. *Let* $a, b \in (0, \infty)$ *and suppose* $ab < 1$*. If* $f \in E(a, b)$ *then*

$$\langle f, \varphi_n \rangle = O\left[n^{-\frac{1}{4}} \left(\frac{a+b-2ab}{a+b+2ab}\right)^{n/4}\right]$$

for $n = 1, 2, \ldots,$ *and this estimate is sharp.*

3.2 Proof of the main result

Suppose $a, b > 0$, $ab < 1$ and $f \in E(a,b)$. Let $\mu = \frac{1-a}{1+a}$ and $\nu = \frac{1-b}{1+b}$. Observe that $\mu, \nu \in (-1,1)$, $\mu+\nu \in (0,2)$, and hence $(\mu+\nu-1)^2 < 1$. Therefore

$$\mu+\nu-2\mu\nu = \frac{1}{2}\left[1-(\mu+\nu-1)^2+(\mu-\nu)^2\right] > 0,$$

and hence

$$\frac{a+b-2ab}{a+b+2ab} = \frac{\mu+\nu-2\mu\nu}{2-\mu-\nu} > 0.$$

Define

$$A(a,b) = \sqrt{\frac{a+b-2ab}{a+b+2ab}}. \tag{3.1}$$

Lemma 3.2. *We have $A = A(a,b) \in (0,1)$ and there exist unique numbers $\theta_0 = \theta_0(a,b), \theta_1 = \theta_1(a,b) \in (0, \pi/2)$, and $\tau = \tau(a,b) \in (-\pi/4, \pi/4)$ with the following properties.*

a. $\theta_0 < \tau + \pi/4 < \theta_1$.

b. $A\sin(2\theta_0 - 2\tau) = \mu + (1-\mu)\sin^2\theta_0$,
$2A\cos(2\theta_0 - 2\tau) = (1-\mu)\sin 2\theta_0$.

c. $A\sin(2\theta_1 - 2\tau) = \nu + (1-\nu)\cos^2\theta_1$,
$2A\cos(2\theta_1 - 2\tau) = -(1-\nu)\sin 2\theta_1$.

Proof. Clearly $A \in (0,1)$ by equation (3.1). Since $(\mu+\nu-2\mu\nu)(2-\mu-\nu)-(\nu-\mu)^2 = 2(1-\mu)(1-\nu)(\mu+\nu) > 0$,

$$\frac{\nu-\mu}{\sqrt{(\mu+\nu-2\mu\nu)(2-\mu-\nu)}} \in (-1,1).$$

Therefore there exists a unique $\tau \in (-\pi/4, \pi/4)$ such that

$$\sin 2\tau = \frac{\nu - \mu}{\sqrt{(\mu + \nu - 2\mu\nu)(2 - \mu - \nu)}}.$$

Observe that $\frac{2A\cos 2\tau}{1+\mu} > 0$ and

$$\left(\frac{(1-\mu) - 2A\sin 2\tau}{1+\mu}\right)^2 + \left(\frac{2A\cos 2\tau}{1+\mu}\right)^2 = 1.$$

Therefore there exists a unique $\theta_0 \in (0, \pi/2)$ such that

$$\cos 2\theta_0 = \frac{(1-\mu) - 2A\sin 2\tau}{1+\mu}, \quad \text{and} \quad \sin 2\theta_0 = \frac{2A\cos 2\tau}{1+\mu},$$

and (b) follows.

Similarly $\frac{2A\cos 2\tau}{1+\nu} > 0$ and

$$\left(\frac{(\nu-1) - 2A\sin 2\tau}{1+\nu}\right)^2 + \left(\frac{2A\cos 2\tau}{1+\nu}\right)^2 = 1.$$

Therefore there exists a unique $\theta_1 \in (0, \pi/2)$ such that

$$\cos 2\theta_1 = \frac{(\nu-1) - 2A\sin 2\tau}{1+\nu}, \quad \text{and} \quad \sin 2\theta_1 = \frac{2A\cos 2\tau}{1+\nu},$$

and (c) follows.

From (b), it follows that $\cos(2\theta_0 - 2\tau) > 0$. Since $\theta_0 \in (0, \pi/2)$ and $\tau \in (-\pi/4, \pi/4)$, it follows that $2\theta_0 - 2\tau \in (-\pi/2, 3\pi/2)$. These two facts together imply that $\theta_0 - \tau < \pi/4$, whence $\theta_0 < \tau + \pi/4$.

Similarly, it follows from (c) that $\theta_1 > \tau + \pi/4$. Hence (a) follows. $\square$

Remark 3.3. Observe that $A(b,a) = A(a,b)$, $\tau(b,a) = -\tau(a,b)$, $\theta_0(b,a) = \frac{\pi}{2} - \theta_1(a,b)$, and $\theta_1(b,a) = \frac{\pi}{2} - \theta_0(a,b)$.

3.3 Gaussian estimates of the Bargmann transform

Lemma 3.4. *Let $a, b \in (0,\infty)$ and suppose $ab < 1$. Let $m = \min\{a,b\}$ and let A, τ, θ_0 and θ_1 be as in Lemma 3.2. If $f \in E(a,b)$ then*

$$|Bf(w)| \le C\sqrt{\frac{2}{1+m}}\exp\left(A\frac{r^2}{4}\sin\left(2\theta - 2\tau\right)\right), \quad (w = re^{i\theta}), \tag{3.2}$$

for $\theta_0 \le \theta \le \theta_1$, $\theta_0 + \pi \le \theta \le \theta_1 + \pi$ (the sector between a blue and red line in the figure below), and

$$|Bf(w)| \le C\sqrt{\frac{2}{1+m}}\exp\left(A\frac{r^2}{4}\sin\left(-2\theta - 2\tau\right)\right), \quad (w = re^{i\theta}), \tag{3.3}$$

for $2\pi - \theta_1 \le \theta \le 2\pi - \theta_0$, and $\pi - \theta_1 \le \theta \le \pi - \theta_0$ (the sector between a green and black line in the figure below).

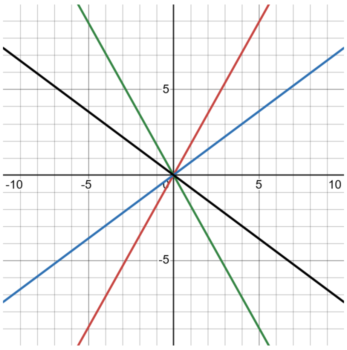

Figure showing the regions where (3.2), and (3.3) holds for $a = \frac{1}{2}$, and $b = \frac{1}{3}$.

Proof. Suppose $a, b > 0$, $ab < 1$ and $f \in E(a, b)$. Then there exists $C > 0$ such that $|f(x)| \leq Cg_a(x)$, $x \in \mathbb{R}$, and $\left|\widehat{f}(\xi)\right| \leq Cg_b(\xi)$, $\xi \in \mathbb{R}$. Write $w = u + iv = re^{i\theta}$. By arguments analogous to those used in ([13, Theorem 2.1]), we have

$$|Bf(w)| \leq C\sqrt{\frac{2}{1+a}} \exp \frac{(\mu + (1-\mu)\sin^2\theta)r^2}{4}, \tag{3.4}$$

and

$$|Bf(w)| \leq C\sqrt{\frac{2}{1+a}} \exp \frac{(\nu + (1-\nu)\cos^2\theta)r^2}{4}. \tag{3.5}$$

Define

$$F(re^{i\theta}) = \exp\left(iA\frac{r^2}{4}e^{2i(\theta-\tau)}\right) Bf(w),$$

where A and τ are as in Lemma 3.2. From equation (3.4) and Lemma 3.2(b), it follows that

$$\begin{aligned} \left|F(re^{i\theta_0})\right| &\leq \exp\left(-A\frac{r^2}{4}\sin(2\theta_0 - 2\tau)\right)\left|Bf(re^{i\theta_0})\right| \\ &\leq C\sqrt{\frac{2}{1+a}} \exp\left(\frac{r^2}{4}\left[-A\sin(2\theta_0 - 2\tau) + (\mu + (1-\mu)\sin^2\theta_0)\right]\right) \\ &= C\sqrt{\frac{2}{1+a}}. \end{aligned}$$

Similarly, from equation (3.5) and Lemma 3.2(c), it follows that

$$\left|F(re^{i\theta_1})\right| \leq C\sqrt{\frac{2}{1+b}}.$$

Then F is entire, bounded by $\sqrt{2}Ce^{|w|^2/2}$ everywhere (by equation (3.4) and the definition of F), and by $C\sqrt{\frac{2}{1+m}}$ on the rays $\theta = \theta_0$ and $\theta = \theta_1$. It follows from the Phragmén-Lindelöf principle that

$$|F(w)| \leq C\sqrt{\frac{2}{1+m}}$$

for $\theta_0 \le \theta \le \theta_1$ (note that $\theta_0 < \theta_1$ by Lemma 3.2(a)). Therefore

$$|Bf(w)| \le C\sqrt{\frac{2}{1+m}}\exp\left(A\frac{r^2}{4}\sin\left(2\theta - 2\tau\right)\right)$$

for $\theta_0 \le \theta \le \theta_1$.

Observe that $\mathcal{F}^k f \in E(a,b)$ or $E(b,a)$ according as k is even or odd. Firstly

$$|Bf(-w)| = \left|B\mathcal{F}^2 f(w)\right| \le C\sqrt{\frac{2}{1+m}}\exp\left(A\frac{r^2}{4}\sin\left(2\theta - 2\tau\right)\right)$$

for $\theta_0 \le \theta \le \theta_1$, or

$$|Bf(w)| \le C\sqrt{\frac{2}{1+m}}\exp\left(A\frac{r^2}{4}\sin\left(2\theta - 2\tau\right)\right)$$

for $\theta_0 + \pi \le \theta \le \theta_1 + \pi$. Also

$$|Bf(-iw)| = |B\mathcal{F}f(w)| \le C\sqrt{\frac{2}{1+m}}\exp\left(A(b,a)\frac{r^2}{4}\sin\left(2\theta - 2\tau(b,a)\right)\right)$$

for $\theta_0(b,a) \le \theta \le \theta_1(b,a)$. By this and Remark 3.3 we have

$$|Bf(w)| \le C\sqrt{\frac{2}{1+m}}\exp\left(A\frac{r^2}{4}\sin\left(-2\theta - 2\tau\right)\right)$$

for $2\pi - \theta_1 \le \theta \le 2\pi - \theta_0$. Repeating the first part of the argument with $\mathcal{F}f$ in place of f shows that (3.3) also holds for $\pi - \theta_1 \le \theta \le \pi - \theta_0$. □

Now define $\gamma_n(t) = \sqrt{\frac{2n}{A}}e^{it}$ for $0 \le t \le 2\pi$. By the Cauchy integral formula for derivatives, we have $Bf(w) = \sum_{n=1}^{\infty} c_n w^n$ where

$$c_n = \frac{1}{2\pi i}\int_{\gamma_n}\frac{Bf(w)}{w^{n+1}}\,dw.$$

Therefore

$$\begin{aligned}|c_n| &\leq \frac{1}{2\pi}\int_{\gamma_n}\frac{|Bf(w)|}{|w|^{n+1}}\,|dw| \\ &= \frac{1}{2\pi}\left(\frac{A}{2n}\right)^{n/2}\int_0^{2\pi}\left|Bf\left(\sqrt{\frac{2n}{A}}e^{it}\right)\right|\,dt \\ &= \frac{1}{2\pi}\left(\frac{A}{2n}\right)^{n/2}\sum_{k=1}^{4}\int_{\frac{(k-1)\pi}{2}}^{\frac{k\pi}{2}}\left|Bf\left(\sqrt{\frac{2n}{A}}e^{it}\right)\right|\,dt.\end{aligned} \tag{3.6}$$

By Lemma 3.4, and inequalities (3.4) and (3.5) we have

$$\int_0^{\pi/2}\left|Bf\left(\sqrt{\frac{2n}{A}}e^{it}\right)\right|\,dt \leq C\sqrt{\frac{2}{1+m}}(I_n+J_n+K_n)$$

where

$$\begin{aligned}I_n &= \int_0^{\theta_0}\exp\left(\frac{(\mu+(1-\mu)\sin^2 t)\,n}{2A}\right)dt, \\ J_n &= \int_{\theta_0}^{\theta_1}\exp\left(\frac{n}{2}\sin(2t-2\tau)\right)dt, \quad \text{and} \\ K_n &= \int_{\theta_1}^{\frac{\pi}{2}}\exp\left(\frac{(\nu+(1-\nu)\cos^2 t)\,n}{2A}\right)dt.\end{aligned}$$

By Theorem 1.12, we have

$$\begin{aligned}J_n &= \int_{\theta_0}^{\theta_1}\exp\left(\frac{n}{2}\right)\exp\left[i\frac{n}{2}\left(i\left(1-\sin(2t-2\tau)\right)\right)\right]dt \\ &= O\left(n^{-1/2}e^{n/2}\right).\end{aligned}$$

Also

$$\begin{aligned}I_n &\leq \theta_0\exp\left(\frac{(\mu+(1-\mu)\sin^2\theta_0)\,n}{2A}\right) \\ &= \theta_0\exp\left(\frac{n}{2}\sin(2\theta_0-2\tau)\right) \qquad \text{(by Lemma 3.2(b))} \\ &\leq \left(\frac{\theta_0}{\tau+\frac{\pi}{4}-\theta_0}\right)J_n.\end{aligned}$$

Similarly, $K_n \leq \left(\frac{\frac{\pi}{2}-\theta_1}{\theta_1-\tau-\frac{\pi}{4}}\right)J_n$ by Lemma 3.2(c). Therefore

$$\int_0^{\pi/2}\left|Bf\left(\sqrt{\frac{2n}{A}}e^{it}\right)\right|\,dt = O\left(n^{-1/2}e^{n/2}\right).$$

The other three integrals in (3.6) are also $O\left(n^{-1/2}e^{n/2}\right)$ by Lemma 3.4, and the fact that the right hand sides of inequalities (3.4) and (3.5) do not change when we replace θ by $\pi-\theta$ or $2\pi-\theta$. We conclude from equation (3.6) that

$$c_n = O\left[n^{-1/2}\left(\frac{Ae}{2n}\right)^{n/2}\right].$$

It follows from equation (1.7), Stirling's formula, and equation (3.1) that

$$\langle f,\varphi_n\rangle = O\left(n^{-\frac{1}{4}}A^{n/2}\right) = O\left[n^{-\frac{1}{4}}\left(\frac{a+b-2ab}{a+b+2ab}\right)^{n/4}\right].$$

Now we will show that this estimate is sharp. Let

$$f(x) = \exp\left[-\left(\frac{1+iAe^{-2i\tau}}{1-iAe^{-2i\tau}}\right)\frac{x^2}{2}\right]$$

where A and τ are as in Lemma 3.2. We claim that $f\in E(a,b)$ and

$$|\langle f,\varphi_n\rangle| \sim \left(\frac{2}{\pi^3}\right)^{1/4} n^{-1/4}\left(\frac{a+b-2ab}{a+b+2ab}\right)^{n/4}, \qquad n=0,2,4,\dots.$$

Indeed,

$$\begin{aligned}
|f(x)| &= \exp\left[-\mathrm{Re}\left(\frac{1+iAe^{-2i\tau}}{1-iAe^{-2i\tau}}\right)\frac{x^2}{2}\right] = e^{-\frac{ax^2}{2}}, \qquad \text{and}\\
\left|\hat{f}(\xi)\right| &= \sqrt{2\pi}\left(\frac{b}{a}\right)^{1/4}\exp\left[-\mathrm{Re}\left(\frac{1-iAe^{-2i\tau}}{1+iAe^{-2i\tau}}\right)\frac{\xi^2}{2}\right]\\
&= \sqrt{2\pi}\left(\frac{b}{a}\right)^{1/4} e^{-\frac{b\xi^2}{2}}.
\end{aligned}$$

It follows that $f \in E(a,b)$. However,

$$\begin{aligned} Bf(w) &= \frac{e^{-w^2/4}}{\sqrt{\pi}} \int e^{xw} e^{-x^2/2} \exp\left[- \left(\frac{1 + iAe^{-2i\tau}}{1 - iAe^{-2i\tau}} \right) \frac{x^2}{2} \right] dx \\ &= \frac{1}{\sqrt{\pi}} \exp\left(-iAe^{-2i\tau} \frac{w^2}{4} \right) \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-i)^n A^n e^{-2in\tau} w^{2n}}{4^n n!}. \end{aligned}$$

Therefore $\langle f, \varphi_n \rangle = 0$ for n odd and

$$|\langle f, \varphi_n \rangle| = \frac{2^{-n/2} A^{n/2} \sqrt{n!}}{\sqrt{\pi} (\frac{n}{2})!}, \quad n = 0, 2, 4, \dots.$$

Therefore, by Stirling's formula, we have

$$|\langle f, \varphi_n \rangle| \sim \left(\frac{2}{\pi^3} \right)^{1/4} n^{-1/4} \left(\frac{a + b - 2ab}{a + b + 2ab} \right)^{n/4}, \quad n = 0, 2, 4, \dots.$$

□