

## CHAPTER 5

# BOUNDARY INTEGRAL EQUATIONS FORMULATION FOR STRAIN AND TEMPERATURE-RATE DEPENDENT THERMOELASTICITY

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### 5.1 Introduction<sup>1</sup>

Investigations on coupled thermoelasticity are carried out through various methods, including experimental approaches, theoretical analysis, or by numerical solutions. Although analytical analysis is of great significance for fundamental research, but for most of the practical problems it is usually impossible to find the solution in the closed form in terms of elementary or special functions. Generally, such problems become more complex that involve complicated geometry or boundary conditions and also to physical complexity, which leads to non-linear behaviour. The analysis in such cases is therefore dependent on only numerical techniques such as the finite-element method (FEM), the boundary element method (BEM), finite difference method etc. In recent decades, the boundary element method (BEM) is considered as an appealing methodology and has emerged as an efficient alternative to the more conventional finite element method (FEM) in engineering analysis. This method discretizes the boundary of the domain, which lowers down the dimension of computational domain by one, which leads to much easier mesh generation as comparison to finite element method and other domain based method. This method provides substantial reduction in modeling effort in

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terms of computer time and storage for obtaining the solutions with the same accuracy (Brebbia (1980), Brebbia and Walker (2016) and Fenner (1982)). The boundary integral equations play important roles in BEM. The first boundary integral equation formulation (BIEM) for classical potential theory and its numerical treatment were done by Jaswon (1963) and Symm (1963) (see also Jaswon and Symm (1977)). Later on, this work was extended by Rizzo (1967) and Cruse (1969; 1973) for elastostatic case. Rizzo and Shippy (1977) described the features of BIE for three-dimensional linear, homogeneous isotropic and steady-state thermoelasticity and also discussed the numerical implementation for BE method. Subsequently, comprehensive boundary integral formulation for uncoupled and coupled thermoelasticity theories were discussed by Sladek (1983; 1984), Chen and Dargush (1995). Later on, Anwar and Sheriff (1988a; 1994), Prasad and Das (2013), Semwal and Mukhopadhyay (2014) also developed BIE formulation for thermoelastic media in various contexts. El Karamany (2004), Elhagary (2019) presented the BIE formulation and illustrated its implementation for the generalized thermoviscoelasticity.

The present Chapter is concerned with the formulation of boundary integral equations of strain and temperature rate dependent thermoelasticity theory (MGL) for the homogeneous isotropic thermoelastic medium having mixed type thermal and mechanical boundary conditions. Here the fundamental solutions in the Laplace transform domain are derived analytically when the body is subjected to two different situations: one is for concentrated heat source and another for concentrated body force in particular direction. Accordingly, we obtain a reciprocal relation between the field variables for these two different systems of thermomechanical causes. Further, the integral formulation of field variables is carried out in terms of boundary conditions by using this reciprocity relation. In the last section, we illustrate the implementation of our BI formulation and discuss the aspects for numerical implementation through boundary element method.

## 5.2 Basic Equations and Problem Formulation

We consider the system of governing equations and constitutive relations for generalized thermoelasticity theory in the context of modified Green-Lindsay model (Yu et al. (2018)) in three-dimensional cartesian-coordinate system. Here, the study is consolidated for homogenous, isotropic thermoelastic medium enclosing the volume  $V$  with smooth boundary surface  $D$  in the presence of body force  $b$  and heat supply  $R$ . Then the basic equations can be written as follows:

**The equation of motion in presence of body force:**

$$\sigma_{ij,j} + b_i = \rho \frac{\partial^2 u_i}{\partial t^2}. \quad (5.2.1)$$

**Stress-strain-temperature relation:**

$$\sigma_{ij} = \left(1 + \tau_1 \frac{\partial}{\partial t}\right) (2\mu e_{ij} + (\lambda e_{kk} - \gamma T) \delta_{ij}) \quad (5.2.2)$$

**Heat conduction equation in presence of heat source:**

$$kT_{,ii} = \frac{\partial}{\partial t} \left(1 + \tau_0 \frac{\partial}{\partial t}\right) (\rho c_E T + \gamma \theta_0 e_{kk}) - Q \quad (5.2.3)$$

**Strain-displacement relation:**

$$e_{ij} = \frac{u_{i,j} + u_{j,i}}{2} \quad (5.2.4)$$

On combining Eq. (5.2.1) and Eq. (5.2.2), we obtain

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \left(1 + \tau_1 \frac{\partial}{\partial t}\right) (2\mu e_{ij} + (\lambda e_{kk} - \gamma T) \delta_{ij})_{,j} + b_i \quad (5.2.5)$$

Eliminating  $e_{ij}$  from Eq. (5.2.5) by using Eq. (5.2.4), we obtain the equation of motion as

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \left( 1 + \tau_1 \frac{\partial}{\partial t} \right) (\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - \gamma T_{,i}) + b_i \quad (5.2.6)$$

### Mixed Initial Boundary Value Problem

Now, consider  $\bar{D}$  as the closure of an open, bounded, connected domain with boundary,  $\partial D$ , enclosing an homogeneous and isotropic thermoelastic material. Let  $D$  denote the interior of  $\bar{D}$ . Let us consider two disjoint partitions  $(D_1, D_2)$  and  $(D_3, D_4)$  of surface  $D$  such that different mechanical and thermal conditions are applied on boundary surface, which can be described mathematically as follows:

For a mixed initial and boundary value problem, the field equations and constitutive relations are given by Eqs. (5.2.2, 5.2.3) defined on  $D \times [0, \infty)$  together with the following initial conditions and boundary conditions:

**Thermal boundary conditions:** The temperature and temperature gradient are specified on portion  $(D_1, D_2)$  as follows:

$$\left. \begin{aligned} T &= T_0(x, t), \text{ when } x \in D_1, t > 0 \\ T_{,n} &= T_{,j}n_j = T_{n0}(x, t), \text{ when } x \in D_2, t > 0. \end{aligned} \right\} \quad (5.2.7)$$

**Mechanical boundary conditions:** The traction and displacement are specified on portion  $(D_3, D_4)$  as follows:

$$\left. \begin{aligned} u_i &= u_{i0}(x, t), \text{ when } x \in D_3, t > 0 \\ \sigma_{ij}n_j &= p_i(x, t), \text{ when } x \in D_4, t > 0 \end{aligned} \right\} \quad (5.2.8)$$

**Initial conditions:**

$$u_i(x, 0) = \frac{\partial u_i}{\partial t}(x, 0) = T(x, 0) = \frac{\partial T}{\partial t}(x, 0) = 0, \quad x \in D. \quad (5.2.9)$$

From Eqs. (5.2.2), we obtain the component of traction vector in the form

$$p_i = \left(1 + \tau_1 \frac{\partial}{\partial t}\right) (\mu u_{i,j} n_j + \mu u_{k,i} n_k + \lambda n_i u_{k,k} - \gamma T n_i) \quad (5.2.10)$$

### 5.3 Formulation of the Problem in Laplace Transform Domain

In this part, we apply Laplace transform to Eqs. (5.2.2, 5.2.3, 5.2.6) and take into account that the medium described here is quiescent initially. Therefore, we obtain

$$\bar{\sigma}_{ij} = (1 + \tau_1 s) (2\mu \bar{e}_{ij} + (\lambda \bar{e}_{kk} - \gamma \bar{T}) \delta_{ij}) \quad (5.3.1)$$

$$k \bar{T}_{,ii} = s (1 + \tau_0 s) (\rho c_E \bar{T} + \gamma \theta_0 \bar{e}_{kk}) - \bar{Q} \quad (5.3.2)$$

$$\rho s^2 \bar{u}_i = (1 + \tau_1 s) (\mu \bar{u}_{i,jj} + (\lambda + \mu) \bar{u}_{j,ji} - \gamma \bar{T}_{,i}) + \bar{b}_i \quad (5.3.3)$$

where,  $s$  is the Laplace transform parameter.

Now, we disintegrate displacement and body forces into sum of an irrotational vector field and a solenoidal vector field by using Helmholtz representation theorem, which is given as

$$u = \text{grad}(\phi) + \text{curl}(\psi) \quad (5.3.4)$$

$$b = \rho (\text{grad}(X) + \text{curl}(Y)) \quad (5.3.5)$$

$$\text{div}(\psi) = \text{div}(Y) = 0 \quad (5.3.6)$$

where  $\phi$ ,  $X$  are scalar fields (scalar potential) and  $\psi$ ,  $Y$  are the vector fields (vector potential).

Substituting  $\bar{u}$  and  $\bar{b}$  from Eqs. (5.3.4) and (5.3.5) into Eq. (5.3.3), we get

$$\begin{aligned} \rho s^2 (\text{grad}(\bar{\phi}) + \text{curl}(\bar{\psi})) &= (1 + \tau_1 s) \{ (\lambda + 2\mu) \text{grad}(\nabla^2 \bar{\phi}) + \text{curl}(\nabla^2 \bar{\psi}) - \gamma \text{grad}(\bar{T}) \} \\ &+ \rho (\text{grad}(\bar{X}) + \text{curl}(\bar{Y})) \end{aligned} \quad (5.3.7)$$

Now, we apply divergence to the above equation to obtain

$$\nabla^2 \bar{\phi} - \frac{s^2}{c_1^2} \bar{\phi} - m \bar{T} = \frac{-\bar{X}}{c_1^2} \quad (5.3.8)$$

where  $c_1^2 = \frac{(1+\tau_1 s)(\lambda+2\mu)}{\rho}$ ,  $m = \frac{\gamma}{\lambda+2\mu}$ .

In a similar manner, operating curl on Eq. (5.3.9), we get

$$\left( \nabla^2 - \frac{s^2}{c_2^2} \right) \bar{\psi}_i = \frac{-\bar{Y}_i}{c_2^2} \quad (5.3.9)$$

where  $c_2^2 = \frac{1+\tau_1 s}{\rho}$ .

Substituting  $\bar{u}$  from Eq. (5.3.4) into Eq. (5.3.2), we obtain

$$k \nabla^2 \bar{T} - m_2 \bar{T} - s(1 + \tau_0 s) \gamma \theta_0 \nabla^2 \bar{\phi} = -\bar{Q} \quad (5.3.10)$$

where  $m_2 = s(1 + \tau_0 s) \rho c_E$ .

## 5.4 Fundamental Solutions in the Laplace Transform Domain

In this subsection, we find out the fundamental solutions for the two different cases and relate the physical components obtained in both cases. In the first case, we assume that highly intensive heat is generated for very short period of time at a point in the absence of body force. In the second case, it is assumed that intensive concentrated body force is acting along the coordinate axis at a point in the absence of external heat supply. Hence, we proceed as follows:

**Case-I:** We assume that heat is generated for a very short period of time and is of large magnitude at the point  $x = \xi$  in the absence of external body forces, i.e.

$$Q = \delta(x - \xi) \delta(t), \quad b = 0 \quad \text{at } x = \xi \quad (5.4.1)$$

We assume  $\bar{u}'_i(x, \xi, s)$  and  $\bar{T}'(x, \xi, s)$  are the fundamental solutions for displacement and temperature in the present case.

Now, substituting  $R$  and  $b$  from Eq. (5.4.1) into Eqs. (5.3.8), (5.3.9) and (5.3.10), we obtain

$$\left( \nabla^2 - \frac{s^2}{c_1^2} \right) \bar{\phi}' - m\bar{T}' = 0 \quad (5.4.2)$$

$$\left( \nabla^2 - \frac{s^2}{c_2^2} \right) \bar{\psi}'_i = 0 \quad (5.4.3)$$

$$(k\nabla^2 - m_2) \bar{T}' - s(1 + \tau_0 s) \gamma \theta_0 \nabla^2 \bar{\phi}' = -\delta(x - \xi) \quad (5.4.4)$$

Under the homogenous initial conditions, Eq. (5.4.3) yields

$$\bar{\psi}'_i = 0 \quad (5.4.5)$$

Eliminating  $\bar{T}'$  from Eq. (5.4.2) by using Eq. (5.4.4), we obtain

$$\left\{ \nabla^4 - \left( \frac{s^2}{c_1^2} + \frac{m_2}{k} + \frac{ms(1 + \tau_0 s) \gamma \theta_0}{k} \right) \nabla^2 + \frac{s^2 m_2}{c_1^2 k} \right\} \bar{\phi}' = \frac{-m\delta(x - \xi)}{k} \quad (5.4.6)$$

Now, above equation can be re-written as

$$(\nabla^2 - k_1^2) (\nabla^2 - k_2^2) \bar{\phi}' = \frac{-m\delta(x - \xi)}{k} \quad (5.4.7)$$

where  $k_1^2$  and  $k_2^2$  are the roots of the following equation:

$$z^2 - \left( \frac{m_3}{k} (\rho c_E + m\gamma\theta_0) + \frac{s^2}{c_1^2} \right) z + \frac{s^2 m_2}{c_1^2 k} = 0$$

where  $m_3 = s(1 + \tau_0 s)$ .

The sum and product of the roots of above quadratic equation can be written as:

$$k_1^2 + k_2^2 = \frac{m_3}{k} (\rho c_E + m\gamma\theta_0) + \frac{s^2}{c_1^2} \quad (5.4.8)$$

$$k_1^2 \cdot k_2^2 = \frac{s^2 m_2}{c_1^2 k} \quad (5.4.9)$$

The solution of inhomogeneous Helmholtz equation in space is given by

$$\frac{1}{(\nabla^2 - a^2)} \delta(r) = \frac{-e^{-ar}}{4\pi r} \quad (5.4.10)$$

Using above formula, the solution of Eq. (5.4.7) can be written in the form:

$$\bar{\phi}' = \frac{m(e^{-k_1 r} - e^{-k_2 r})}{4\pi r k (k_1^2 - k_2^2)} \quad (5.4.11)$$

where  $r = \|x - \xi\|_2$

Now, we substitute the expression of  $\bar{\psi}'_i$  and  $\bar{\phi}'$  from Eqs. (5.4.5) and (5.4.11), respec-

tively into Eq. (5.3.4) and we obtain the solution of the component of displacement in Laplace transform domain as

$$\begin{aligned}\bar{u}'_i &= \bar{\phi}'_{,i} \\ &= \frac{m}{4\pi k(k_1^2 - k_2^2)} \left\{ \frac{e^{-k_1 r} - e^{-k_2 r}}{r} \right\}_{,i} \\ \bar{u}'_i(x, \xi, s) &= -\frac{r_{,i} g_1}{kr}\end{aligned}\quad (5.4.12)$$

$$\text{where } g_1 = \frac{m}{4\pi(k_1^2 - k_2^2)} \sum_{n=1}^2 (-1)^{n-1} \left(k_n + \frac{1}{r}\right) e^{-k_n r}$$

Using the relation  $r_{,ij} = \frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^2}$ , we have

$$\bar{u}'_{i,j} = \frac{1}{k} \left\{ g_1 \left( -\frac{\delta_{ij}}{r^2} + \frac{3r_i r_j}{r^2} \right) + \frac{r_i r_j}{r} g_3 \right\}\quad (5.4.13)$$

$$\text{where } g_3 = \frac{m}{4\pi(k_1^2 - k_2^2)} \sum_{n=1}^2 (-1)^{n-1} \left(k_n + \frac{1}{r}\right) e^{-k_n r}$$

From Eqs. (5.4.2) and (5.4.11), we obtain the solution for  $\bar{T}'$  in Laplace transform domain in the following form:

$$\bar{T}'(x, \xi, s) = \frac{g_2}{kr}\quad (5.4.14)$$

$$\text{where } g_2 = \frac{1}{4\pi(k_1^2 - k_2^2)} \sum_{n=1}^2 (-1)^{n-1} \left(k_n^2 - \frac{s^2}{c_1^2}\right) e^{-k_n r}$$

Next, by applying Laplace transform over the traction vector given by Eq. (5.2.10), we have

$$\bar{p}'_i = (1 + \tau_1 s) \left[ \mu (\bar{u}'_{i,j} + \bar{u}'_{j,i}) n_j + \left( \lambda \bar{u}'_{k,k} - \gamma \bar{T}' \right) n_i \right]\quad (5.4.15)$$

Using Eqs. (5.4.11) and (5.4.14) in Eq. (5.4.15), we have

$$\bar{p}'_i = \left( \frac{1 + \tau_1 s}{kr} \right) \left[ (\lambda g_3 + \gamma g_2) n_i + 2\mu \left\{ r_{,i} r_{,j} \left( g_3 + \frac{3g_1}{r} \right) \right\} n_j \right]\quad (5.4.16)$$

From Eq. (5.4.14), we obtain temperature gradient in the form:

$$\bar{T}'_{,n} = \left( \frac{g_2}{kr} \right)_{,i} n_i = -\frac{r_{,i} g_4}{kr} n_i \quad (5.4.17)$$

$$\text{where } g_4 = \frac{1}{4\pi(k_1^2 - k_2^2)} \sum_{n=1}^2 (-1)^{n-1} \left( k_n^2 - \frac{s^2}{c_1^2} \right) \left( k_n - \frac{1}{r} \right) e^{-k_n r}$$

**Case-II:** In this case, we consider external heat supply  $Q = 0$  and instantaneous concentrated body force  $b_i = b_i^j = \delta(x - \xi) \delta_{ij} \delta(t)$  is acted at the point  $x = \xi$  in the direction of  $x_j$ -axis. We assume that this force produces the displacement  $u_i = u_i^j(x, \xi, t)$  and temperature  $T = T^j(x, \xi, t)$ .

The corresponding Eqs. (5.3.4), (5.3.5), (5.3.8), (5.3.9) and (5.3.10) can be re-written as

$$\bar{u}^j = \text{grad}(\bar{\phi}^j) + \text{curl}(\bar{\psi}^j) \quad (5.4.18)$$

$$\bar{b}^j = \rho \left( \text{grad}(\bar{X}^j) + \text{curl}(\bar{Y}^j) \right) \quad (5.4.19)$$

$$\left( \nabla^2 - \frac{s^2}{c_1^2} \right) \bar{\phi}^j - m \bar{T}^j = \frac{-\bar{X}^j}{c_1^2} \quad (5.4.20)$$

$$\left( \nabla^2 - \frac{s^2}{c_2^2} \right) \bar{\psi}_i^j = \frac{-\bar{Y}_i^j}{c_2^2} \quad (5.4.21)$$

$$k \nabla^2 \bar{T}^j - m_2 \bar{T}^j - s(1 + \tau_0 s) \gamma \theta_0 \nabla^2 \bar{\phi}^j = -\bar{Q} = 0 \quad (5.4.22)$$

Eliminating  $\bar{T}^j$  from Eq. (5.4.20) by using Eq. (5.4.22), we get

$$(\nabla^2 - k_1^2) (\nabla^2 - k_2^2) \bar{\phi}^j = - \left( \nabla^2 - \frac{m_2}{k} \right) \frac{\bar{X}^j}{c_1^2} \quad (5.4.23)$$

where,  $k_1$  and  $k_2$  satisfy Eqs. (5.4.8) and (5.4.9).

Taking Laplace transform on body force, we get

$$\bar{b}_i^j = \delta(x - \xi) \delta_{ij} \quad (5.4.24)$$

From Eqs. (5.4.19) and (5.4.24), we obtain the potential functions as

$$\bar{Y}_k^j = \frac{1}{4\pi\rho} \epsilon_{iqk} \left( \frac{\delta_{qj}}{r} \right)_{,i} \quad (5.4.25)$$

$$\bar{X}^j = -\frac{1}{4\pi\rho} \left( \frac{\delta_{ij}}{r} \right)_{,i} \quad (5.4.26)$$

Substituting  $\bar{Y}_k^j$  from Eq. (5.4.25) into Eq. (5.4.23) and by using the following well-known relation

$$\frac{1}{(\nabla^2 - a^2)} \left( \frac{1}{r} \right)_{,i} = -\frac{1}{a^2} [(1 + ar) e^{-ar} - 1] \frac{r_{,i}}{r^2},$$

we find

$$\begin{aligned} \bar{\phi}^j &= \left\{ \frac{1}{4\pi s^2 \rho} + \frac{1}{4\pi c_1^2 \rho (k_2^2 - k_1^2)} \sum_{n=1}^2 (-1)^{n-1} \frac{(k_n^2 - \frac{m_2}{k})(1 + k_n r)}{k_n^2} e^{-k_n r} \right\} \frac{r_{,i}}{r^2} \delta_{ij} \\ &= (\Gamma + E) \frac{r_{,i}}{r^2} \delta_{ij} \end{aligned} \quad (5.4.27)$$

where  $\Gamma = \frac{1}{4\pi s^2 \rho}$ ,  $E = \frac{1}{4\pi c_1^2 \rho (k_2^2 - k_1^2)} \sum_{n=1}^2 (-1)^{n-1} \frac{(k_n^2 - \frac{m_2}{k})(1 + k_n r)}{k_n^2} e^{-k_n r}$ ,

$$B_n = \frac{1}{4\pi c_1^2 \rho (k_2^2 - k_1^2)} \frac{(k_n^2 - \frac{m_2}{k})}{k_n^2}, \quad n = 1, 2$$

Now, we determine the solution for temperature by using Eqs. (5.4.20), (5.4.26) and (5.4.27) in the form

$$\bar{T}^j = \frac{1}{4\pi m c_1^2 \rho (k_2^2 - k_1^2)} V \frac{r_{,j}}{r^2} \quad (5.4.28)$$

$$\text{where } V = \sum_{n=1}^2 (-1)^{n-1} \frac{(k_n^2 - \frac{m_2}{k})(1 + k_n r) (k_n^2 - \frac{s^2}{c_1^2})}{k_n^2} e^{-k_n r}$$

Solving Eq. (5.4.21) by using Eq. (5.4.25), we obtain the potential function as

$$\bar{\psi}_k^j = \epsilon_{ijk} \frac{1}{4\pi s^2 \rho} \left( \left( 1 + \frac{sr}{c_2} \right) e^{-\frac{sr}{c_2}} - 1 \right) \frac{r_{,i}}{r^2} \quad (5.4.29)$$

Substituting the expressions of potential functions from Eqs. (5.4.27) and (5.4.29) into Eq. (5.4.18), we get the solutions for displacement component as

$$\bar{u}_i^j = \frac{1}{r} (\delta_{ij}U_1 + U_2 r_{,i} r_{,j}) \quad (5.4.30)$$

$$\text{where } U_1 = \frac{1}{4\pi\rho} \left[ \left( \frac{1}{s^2 r^2} + \frac{1}{s r c_2} + \frac{1}{c_2^2} \right) e^{-\frac{s r}{c_2}} \right] + \sum_{n=1}^2 (-1)^{n-1} B_n \frac{(1+k_n r)}{r^2} e^{-k_n r}$$

$$U_2 = - \left\{ \frac{1}{4\pi\rho} \left[ \left( \frac{3}{s^2 r^2} + \frac{3}{s r c_2} + \frac{1}{c_2^2} \right) e^{-\frac{s r}{c_2}} \right] + \sum_{n=1}^2 (-1)^{n-1} B_n \left( \frac{3}{r^2} + \frac{3k_n}{r} + k_n^2 \right) e^{-k_n r} \right\}$$

Now, from Eqs. (5.4.12) and (5.4.28), we obtain the relation between the displacement in case-I and temperature in case-II in Laplace transform domain as

$$\bar{T}^j(x, \xi, s) = \frac{s(1 + \tau_0 s)}{1 + \tau_1 s} \bar{u}_j^i(x, \xi, s) \quad (5.4.31)$$

## 5.5 Reciprocal Relation for Thermoelasticity

The reciprocity theorem is used to derive various methods of integrating the elasticity equations in terms of Green's function. It has significant practical applications in finding the numerical solution of engineering problems (Nowacki (1975)) as reciprocity theorem states the relation between two sets of thermoelastic loadings and the corresponding thermoelastic configurations. Maizel (1951) developed the Maxwell-Betti reciprocity theorem for the static problems in the theory of thermoelasticity. Later, the reciprocity theorem was extended to uncoupled thermoelasticity, coupled thermoelasticity and coupled thermoelasticity for anisotropic homogeneous material by Predeleanu (1959), Ionescu-Cazimir (1964) and Nowacki (1975), respectively. Ieşan (1967) presented the first reciprocal relation without using the Laplace transform. Convolution type reciprocity theorems were also derived by Ieşan (1966; 1974). Scalia (1990) used a method to deduce reciprocity relations without using the Laplace transform and without incorporation of the initial data in the field equations. An exhaustive treatment of the variational principles in thermoelasticity is available in the books by Lebon (1980),

Carlson (1973), Hetnarski and Ignaczak (2016), and Hetnarski et al. (2009). Recently, the convolution type variational principles and reciprocal relations on different theories of thermoelasticity are reported by Chiriță and Ciarletta (2010), Mukhopadhyay et al. (2011b), Kothari and Mukhopadhyay (2013), and Kumari and Mukhopadhyay (2017). Further, Shivay and Mukhopadhyay (2019) presented Somigliano and Green's theorem based on reciprocity theorem in the context of the generalized thermoelasticity model with single delay. Moreover, Jangid and Mukhopadhyay (2020) focused on variational and reciprocal principles for modified temperature-rate dependent two-temperature thermoelasticity theory. The Betti reciprocity theorem constitutes one of the most interesting theorems and it is used to deduce various methods of integrating the thermoelasticity equation by means of the Green's function. In this subsection, we will derive a reciprocity relation of type Maxwell-Betti between two systems of causes and effects. Let us consider the body is exposed to two independent systems of thermoelastic causes  $I = \{b^{(1)}, R^{(1)}\}$  and  $I' = \{b^{(2)}, R^{(2)}\}$ .

After using Laplace transform, the stress-strain temperature relation given by Eq. (5.2.2) can be written as

$$\bar{\sigma}_{ij}^{(1)} = (1 + \tau_1 s) \left( 2\mu \bar{e}_{ij}^{(1)} + \left( \lambda \bar{e}_{kk}^{(1)} - \gamma \bar{T}^{(1)} \right) \delta_{ij} \right) \quad (5.5.1)$$

$$\bar{\sigma}_{ij}^{(2)} = (1 + \tau_1 s) \left( 2\mu \bar{e}_{ij}^{(2)} + \left( \lambda \bar{e}_{kk}^{(2)} - \gamma \bar{T}^{(2)} \right) \delta_{ij} \right) \quad (5.5.2)$$

We multiply Eqs. (5.5.1) and (5.5.2) by  $\bar{e}_{ij}^{(2)}$  and  $\bar{e}_{ij}^{(1)}$ , respectively and combine the results to obtain

$$\bar{\sigma}_{ij}^{(1)} \bar{e}_{ij}^{(2)} - \bar{\sigma}_{ij}^{(2)} \bar{e}_{ij}^{(1)} = (1 + \tau_1 s) \gamma \left( \bar{e}_{kk}^{(1)} \bar{T}^{(2)} - \bar{e}_{kk}^{(2)} \bar{T}^{(1)} \right) \quad (5.5.3)$$

Now, integrating Eq. (5.5.3) over the volume  $D$ , we get

$$\int_V \left\{ \bar{\sigma}_{ij}^{(1)} \bar{e}_{ij}^{(2)} - \bar{\sigma}_{ij}^{(2)} \bar{e}_{ij}^{(1)} \right\} dV = (1 + \tau_1 s) \gamma \int_V \left( \bar{e}_{kk}^{(1)} \bar{T}^{(2)} - \bar{e}_{kk}^{(2)} \bar{T}^{(1)} \right) dV \quad (5.5.4)$$

In a similar manner, stress-displacement relation given by Eq. (5.2.1) yields

$$\int_D \left\{ \bar{\sigma}_{ij,j}^{(2)} \bar{u}_i^{(1)} - \bar{\sigma}_{ij,j}^{(1)} \bar{u}_i^{(2)} \right\} dV = \int_D \left\{ \bar{b}_i^{(1)} \bar{u}_i^{(2)} - \bar{b}_i^{(2)} \bar{u}_i^{(1)} \right\} dV \quad (5.5.5)$$

Applying Gauss divergence theorem to Eq. (5.5.5) and combining the result with Eq. (5.5.4), we have

$$\begin{aligned} (1 + \tau_1 s) \gamma \int_V \left( \bar{e}_{kk}^{(1)} \bar{T}^{(2)} - \bar{e}_{kk}^{(2)} \bar{T}^{(1)} \right) dV &= \int_V \left\{ \bar{b}_i^{(1)} \bar{u}_i^{(2)} - \bar{b}_i^{(2)} \bar{u}_i^{(1)} \right\} dV \\ &+ \int_D \left\{ \bar{\sigma}_{ij}^{(1)} \bar{u}_i^{(2)} - \bar{\sigma}_{ij}^{(2)} \bar{u}_i^{(1)} \right\} n_j dS \end{aligned} \quad (5.5.6)$$

In deriving Eq. (5.5.6), we have used the following well known relation from vector calculus

$$u \cdot \text{div} (A^T) = \text{div} (Au) - A^T \cdot \text{grad} (u), \quad \text{where } A \in \mathbb{M}_{n \times n}$$

After using the boundary conditions given by Eq. (5.2.9) in Eq. (5.5.6), we get the relation

$$\begin{aligned} (1 + \tau_1 s) \gamma \int_V \left( \bar{e}_{kk}^{(1)} \bar{\theta}^{(2)} - \bar{e}_{kk}^{(2)} \bar{\theta}^{(1)} \right) dV &= \int_V \left\{ \bar{b}_i^{(1)} \bar{u}_i^{(2)} - \bar{b}_i^{(2)} \bar{u}_i^{(1)} \right\} dV + \int_{D_3} \left\{ \bar{\sigma}_{ij}^{(1)} \bar{u}_{i0}^{(2)} - \bar{\sigma}_{ij}^{(2)} \bar{u}_{i0}^{(1)} \right\} n_j dS \\ &+ \int_{D_4} \left\{ \bar{p}_j^{(1)} \bar{u}_j^{(2)} - \bar{p}_j^{(2)} \bar{u}_j^{(1)} \right\} dS \end{aligned} \quad (5.5.7)$$

The reciprocal relation given by Eq. (5.5.7) contain only causes of surface traction and the body force. In a similar manner, we will derive in the following, another part of reciprocal relation by using heat conduction equation associated with both the system of causes  $I$  and  $I'$  with thermal boundary condition.

After using Laplace transform to Eq. (5.2.1) one can find for two systems

$$k \bar{T}_{,ii}^{(1)} = s (1 + \tau_0 s) \left( \rho c_E \bar{T}^{(1)} + \gamma \theta_0 \bar{e}_{kk}^{(1)} \right) - \bar{Q}^{(1)} \quad (5.5.8)$$

$$k \bar{T}_{,ii}^{(2)} = s (1 + \tau_0 s) \left( \rho c_E \bar{T}^{(2)} + \gamma \theta_0 \bar{e}_{kk}^{(2)} \right) - \bar{Q}^{(2)} \quad (5.5.9)$$

We multiply Eq. (5.5.8) and (5.5.9) by  $\bar{T}^{(2)}$  and  $\bar{T}^{(1)}$ , respectively and combine the results, after that, integrate over the volume  $D$  to obtain

$$k \int_V \left\{ \bar{T}^{(2)} \bar{T}_{,ii}^{(1)} - \bar{T}^{(1)} \bar{T}_{,ii}^{(2)} \right\} dV = s(1 + \tau_0 s) \gamma \theta_0 \int_V \left\{ \bar{T}^{(2)} \bar{e}_{kk}^{(1)} - \bar{T}^{(1)} \bar{e}_{kk}^{(2)} \right\} dV - \int_V \left\{ \bar{T}^{(2)} \bar{R}^{(1)} - \bar{T}^{(1)} \bar{R}^{(2)} \right\} dV \quad (5.5.10)$$

Applying the Gauss divergence theorem to left hand side and combining the result with Eqs. (5.2.6) and (5.5.7), we find

$$\begin{aligned} & s(1 + \tau_0 s) \theta_0 \int_V \left\{ \bar{b}_i^{(1)} \bar{u}_i^{(2)} - \bar{b}_i^{(2)} \bar{u}_i^{(1)} \right\} dV - (1 + \tau_1 s) \int_V \left\{ \bar{T}^{(2)} \bar{Q}^{(1)} - \bar{T}^{(1)} \bar{Q}^{(2)} \right\} dV \\ & = (1 + \tau_1 s) k \left[ \int_{D_1} \left\{ \bar{T}_0^{(2)} \bar{T}_{,n}^{(1)} - \bar{T}_0^{(1)} \bar{T}_{,n}^{(2)} \right\} dS + \int_{D_2} \left\{ \bar{T}^{(2)} \bar{T}_{,n0}^{(1)} - \bar{T}^{(1)} \bar{T}_{,n0}^{(2)} \right\} dS \right] \\ & + s(1 + \tau_0 s) \theta_0 \left[ \int_{D_4} \left\{ \bar{p}_j^{(1)} \bar{u}_j^{(2)} - \bar{p}_j^{(2)} \bar{u}_j^{(1)} \right\} dS + \int_{D_3} \left\{ \bar{\sigma}_{ij}^{(1)} \bar{u}_{i0}^{(2)} - \bar{\sigma}_{ij}^{(2)} \bar{u}_{i0}^{(1)} \right\} n_j dS \right] \end{aligned} \quad (5.5.11)$$

Here, Eq. (5.5.11) shows the relation between temperature and displacement in the interior of the body in terms of displacement, temperature, tractions and temperature gradient which are prescribed on the boundary surface.

The above relation can be written in particular simple form for the infinite thermoelastic medium as

$$s(1 + \tau_0 s) \theta_0 \int_V \left( \bar{b}_i^{(1)} \bar{u}_i^{(2)} - \bar{b}_i^{(2)} \bar{u}_i^{(1)} \right) dV = (1 + \tau_1 s) \int_V \left( \bar{Q}^{(1)} \bar{T}^{(2)} - \bar{Q}^{(2)} \bar{T}^{(1)} \right) dV \quad (5.5.12)$$

## 5.6 BIE Formulation for MGL Model

In this section, we shall derive the BI representation for the temperature and displacement variables in terms of boundary conditions and the derived fundamental solutions by applying them in the reciprocal relation as obtained in previous section. We assume  $u_i(x, \xi, s)$ ,  $T(x, \xi, s)$ ,  $p_i(x, \xi, s)$  and  $\sigma_{ij}(x, \xi, s)$  are the solutions for Eqs. (5.2.3) and (5.2.6) such that they satisfy the boundary conditions given by Eqs. (5.2.7) and (5.2.8). Using the fundamental solution of case-I, Eq. (5.5.11) can be written as

$$\begin{aligned}
 (1 + \tau_1 s) L(x) \bar{T} &= (1 + \tau_1 s) \int_V \overline{RT}' dV - s(1 + \tau_0 s) \theta_0 \int_V \bar{b}_i \bar{u}'_i dV \\
 &\quad - (1 + \tau_1 s) k \left[ \int_{D_1} \{ \bar{T}_0 \bar{T}'_{,n} - \bar{T}'_0 \bar{T}_{,n} \} dS + \int_{D_2} \{ \overline{TT}'_{n0} - \overline{T}'_0 \bar{T}_{n0} \} dS \right] \\
 &\quad - s(1 + \tau_0 s) \theta_0 \left[ \int_{D_4} \{ \bar{p}'_j \bar{u}_j - \bar{p}_j \bar{u}'_j \} dS + \int_{D_3} \{ \bar{\sigma}'_{ij} \bar{u}_{i0} - \bar{\sigma}_{ij} \bar{u}'_{i0} \} n_j dS \right] \quad (5.6.1)
 \end{aligned}$$

where  $L(x) = \int_V \delta(x - \xi) dV(\xi) = \begin{cases} 1, & \text{if } x \in V \\ 0, & \text{if } x \notin V \cup D \\ \frac{1}{2}, & \text{if } x \in D \end{cases}$

In a similar manner, using Case-II in Eq. (5.5.11), we have

$$\begin{aligned}
 s(1 + \tau_0 s) \theta_0 L(x) \bar{u}_k(x, s) &= s(1 + \tau_0 s) \theta_0 \int_V \bar{b}_i \bar{u}_i^{(k)} dV - (1 + \tau_1 s) \int_V \overline{QT}^{(k)} dV \\
 &\quad + (1 + \tau_1 s) k \left[ \int_{D_1} \{ \bar{T}_0 \bar{T}^{(k)}_{,n} - \bar{T}^{(k)}_0 \bar{T}_{,n} \} dS + \int_{D_2} \{ \overline{TT}^{(k)}_{n0} - \overline{T}^{(k)}_0 \bar{T}_{n0} \} dS \right] \\
 &\quad + s(1 + \tau_0 s) \theta_0 \left[ \int_{D_4} \{ \bar{p}_j^{(k)} \bar{u}_j - \bar{p}_j \bar{u}_j^{(k)} \} dS + \int_{D_3} \{ \bar{\sigma}_{ij}^{(k)} \bar{u}_{i0} - \bar{\sigma}_{ij} \bar{u}_{i0}^{(k)} \} n_j dS \right] \quad (5.6.2)
 \end{aligned}$$

Taking the inverse Laplace Transform of Eqs. (5.6.1) and (5.6.2) by using convolution theorem, we arrive at

$$L(x) \ell_1(T) = \Gamma_1(x, t) \quad (5.6.3)$$

$$L(x) \ell_2(u_j(x, t)) = \Gamma_2(x, t) \quad (5.6.4)$$

where

$$\ell_1 = 1 + \tau_1 \frac{\partial}{\partial t} \quad \text{and} \quad \ell_2 = \frac{\partial}{\partial t} \left( 1 + \tau_0 \frac{\partial}{\partial t} \right), \quad (5.6.5)$$

$$\begin{aligned}
 \Gamma_1(x, t) = & \int_V \int_0^t [Q(\xi, t - \tau) \ell_1 T'(\xi, x, t) - b_i(\xi, t - \tau) \ell_2 u'_i(\xi, x, t)] d\tau dV(\xi) \\
 & - k \int_{D_1} \int_0^t [\theta_0(\xi, t - \tau) \ell_1 T'_{,n}(\xi, x, t) - \theta_{,n}(\xi, t - \tau) \ell_1 T'_0(\xi, x, t)] d\tau dS(\xi) \\
 & - k \int_{D_2} \int_0^t [\theta(\xi, t - \tau) \ell_1 T'_{n0}(\xi, x, t) - \theta_{n0}(\xi, t - \tau) \ell_1 T'(\xi, x, t)] d\tau dS(\xi) \\
 & - \theta_0 \int_{D_3} \int_0^t [u_{i0}(\xi, t - \tau) \ell_2 \bar{\sigma}'_{ij}(\xi, x, t) - \bar{\sigma}_{ij}(\xi, t - \tau) \ell_2 u'_{i0}(\xi, x, t)] n_j d\tau dS(\xi) \\
 & - \theta_0 \int_{D_4} \int_0^t [u_i(\xi, t - \tau) \ell_2 \bar{p}'_i(\xi, x, t) - \bar{p}_i(\xi, t - \tau) \ell_2 u'_i(\xi, x, t)] d\tau dS(\xi) \quad (5.6.6)
 \end{aligned}$$

$$\begin{aligned}
 \Gamma_2^{(k)}(x, t) = & - \int_V \int_0^t \left[ \frac{Q(\xi, t - \tau) \ell_1 T^{(k)}(\xi, x, t)}{T_0} - b_i(\xi, t - \tau) \ell_2 u_i^{(k)}(\xi, x, t) \right] d\tau dV(\xi) \\
 & + \frac{k}{T_0} \int_{D_1} \int_0^t [T_0(\xi, t - \tau) \ell_1 T_{,n}^{(k)}(\xi, x, t) - T_{,n}(\xi, t - \tau) \ell_1 T_0^{(k)}(\xi, x, t)] d\tau dS(\xi) \\
 & + \frac{k}{\theta_0} \int_{D_2} \int_0^t [T(\xi, t - \tau) \ell_1 T_{n0}^{(k)}(\xi, x, t) - T_{n0}(\xi, t - \tau) \ell_1 T^{(k)}(\xi, x, t)] d\tau dS(\xi) \\
 & + \int_{D_3} \int_0^t [u_{i0}(\xi, t - \tau) \ell_2 \bar{\sigma}_{ij}^{(k)}(\xi, x, t) - \bar{\sigma}_{ij}(\xi, t - \tau) \ell_2 u_{i0}^{(k)}(\xi, x, t)] n_j d\tau dS(\xi) \\
 & + \int_{D_4} \int_0^t [u_i(\xi, t - \tau) \ell_2 \bar{p}_i^{(k)}(\xi, x, t) - \bar{p}_i(\xi, t - \tau) \ell_2 u_i^{(k)}(\xi, x, t)] d\tau dS(\xi) \quad (5.6.7)
 \end{aligned}$$

After solving Eqs. (5.6.3) and (5.6.4), we can write

$$L(x) T(x, t) = \frac{1}{\tau_1} e^{\frac{-t}{\tau_1}} \int_0^t e^{\frac{t}{\tau_1}} \Gamma_1(x, \tau) d\tau \quad (5.6.8)$$

$$L(x) u_k(x, t) = \int_0^t \Gamma_2^{(k)} d\tau - e^{\frac{-t}{\tau_0}} \int_0^t e^{\frac{t}{\tau_0}} \Gamma_2^{(k)}(x, \tau) d\tau \quad (5.6.9)$$

Let us assume that the point  $\xi$  is on boundary and taking  $x \rightarrow \xi$ , the final solution for

temperature and displacement can be expressed as follows:

$$T(\xi, t) = \frac{2}{\tau_1} e^{\frac{-t}{\tau_1}} \int_0^t e^{\frac{t}{\tau_1}} \Gamma_1(\xi, \tau) d\tau \quad (5.6.10)$$

$$u_k(\xi, t) = 2 \int_0^t \Gamma_2^{(k)} d\tau - 2e^{\frac{-t}{\tau_0}} \int_0^t e^{\frac{t}{\tau_0}} \Gamma_2^{(k)}(x, \tau) d\tau \quad (5.6.11)$$

Eqs. (5.6.10) and (5.6.11) represent the boundary integral equations formulation for displacement and temperature variable in terms of boundary conditions and the above solution can be used to set up a linear system of equations for the BE method.

## 5.7 Example

In this section, we attempt to illustrate the boundary integral (BI) formulation in which we determine the field variables (temperature and displacement) for the mixed boundary conditions. Let us consider  $T(x, t)$  and  $u_k(x, t)$  be the solutions of Eqs. (5.2.3) and (5.2.6). Here, we assume that surface traction and temperature are  $p_i$  and  $T_0$ , respectively prescribed on boundary surface  $D_1 = D_4$ , whereas displacement and temperature gradient are zero on the boundary surface  $D_2 = D_3$ .

In mathematical terms, we can write these conditions as:

$$\left. \begin{aligned} u_k(z, t) = T_{,n}(z, t) = 0, \text{ where } z \in D_3 = D_2 \\ \sigma_{ij}(z, t) n_j(z) = p_i(z, t), T(z, t) = T_0(z, t), \text{ when } z \in D_1 = D_4 \end{aligned} \right\} \quad (5.7.1)$$

We assume that the fundamental solutions also satisfy the following boundary conditions and the point  $x$  belongs to the interior of domain (i.e.,  $L(x) = 1$ ), so that we have

$$p'_i(z, t) = p_i^{(k)}(z, t) = T'(z, t) = T^{(k)}(z, t) = 0, \quad \text{where } z \in D_1 = D_4 \quad (5.7.2)$$

Applying the above boundary conditions in Eqs. (5.6.1) and (5.6.2), the temperature and displacement at interior point can be written as

$$\begin{aligned} \bar{T}(x, s) &= \bar{T}^0(x, s) - k \int_{D_2} \bar{T}(y, s) \bar{T}'_{n0}(y, x, s) dS(y) \\ &\quad + \frac{s(1 + \tau_0 s) \theta_0}{1 + \tau_1 s} \int_{D_2} \bar{p}_i(y, s) \bar{u}'_{i0}(y, x, s) dS(y) \end{aligned} \quad (5.7.3)$$

$$\begin{aligned} s\bar{u}_k(x, s) &= s\bar{u}_k^0(x, s) + \frac{(1 + \tau_1 s) k}{(1 + \tau_0 s) \theta_0} \int_{D_2} \bar{T}(y, s) \bar{T}^{(k)}_{n0}(y, x, s) dS(y) \\ &\quad - s \int_{D_2} \bar{p}_i(y, s) \bar{u}^{(k)}_{i0}(y, x, s) dS(y) \end{aligned} \quad (5.7.4)$$

where

$$\begin{aligned} \bar{T}^0(x, s) &= -k \int_{D_4} \bar{T}_0(y, s) \bar{T}'_{,n}(y, x, s) dS(y) + \frac{s(1 + \tau_0 s) \theta_0}{1 + \tau_1 s} \int_{D_4} \bar{p}_i(y, s) \bar{u}'_i(y, x, s) dA(y) \\ &\quad + \int_V \bar{R}(y, s) \bar{T}'(y, x, s) dV(y) - \frac{s(1 + \tau_0 s) \theta_0}{1 + \tau_1 s} \int_V \bar{b}_i(y, s) \bar{u}'_i(y, x, s) dV(y) \end{aligned} \quad (5.7.5)$$

$$\begin{aligned} s\bar{u}_k^0(x, s) &= -\frac{(1 + \tau_1 s)}{(1 + \tau_0 s) \theta_0} \int_V \bar{R}(y, s) \bar{T}^k(y, x, s) dV(y) + s \int_V \bar{b}_i(y, s) \bar{u}^k_i(y, x, s) dV(y) \\ &\quad - k \int_{D_4} \bar{T}_0(y, s) \bar{T}'_{,n}(y, x, s) dS(y) + \frac{s(1 + \tau_0 s) \theta_0}{1 + \tau_1 s} \int_{D_4} \bar{p}_i(y, s) \bar{u}'_i(y, x, s) dS(y) \end{aligned} \quad (5.7.6)$$

Eqs. (5.7.3) and (5.7.4) represent the expressions for determining transformed temperature and displacement at interior point but traction and temperature are unknowns on the boundary surface  $D_2$ .

Therefore, taking the limit as  $x \rightarrow \zeta$  (where  $\zeta \in D_2$ ) in Eqs. (5.7.3) and (5.7.4), we get

$$\frac{\partial \bar{T}^0(\zeta, s)}{\partial n'} = k \int_{D_2} \bar{T}(y, s) \frac{\partial \bar{T}'_{n0}(y, \zeta, s)}{\partial n'} dS(y) - \frac{s(1 + \tau_0 s) \theta_0}{1 + \tau_1 s} \int_{D_2} \bar{p}_i(y, s) \frac{\partial \bar{u}'_{i0}(y, \zeta, s)}{\partial n'} dS(y) \quad (5.7.7)$$

$$s\bar{u}_k^0(\zeta, s) = -\frac{(1 + \tau_1 s)k}{(1 + \tau_0 s)\theta_0} \int_{D_2} \bar{T}(y, s) \bar{T}_{n_0}^{(k)}(y, \zeta, s) dS(y) + s \int_{D_2} \bar{p}_i(y, s) \bar{u}_{i0}^{(k)}(y, \zeta, s) dS(y) \quad (5.7.8)$$

where  $n'(\zeta)$  is normal to surface  $D_2$ .

For smooth boundary shapes, it appears difficult to find analytical solutions from Eqs. (5.7.7) and (5.7.8) for this particular mixed boundary problem. However, for numerical implementation the resulting integrals given by Eqs. (5.7.7) and (5.7.8) in Laplace transform domain can be discretized and the problem reduces to system of linear algebraic equations in Laplace transform domain. Then, we can find the solution in physical domain by applying an appropriate numerical inversion method.

## 5.8 Conclusion

In the present Chapter, some important relations under generalized thermoelasticity model by Yu et al. (2018) are established. The fundamental solutions are mentioned for modified Green Lindsay model in Laplace transform domain when the body is subjected to two different situations for smooth boundary surface. In this study, we derive the integral formulation for a homogenous isotropic thermoelastic medium with mixed type thermal and mechanical boundary conditions. Further, the boundary integral formulation in physical domain is presented by using Maxwell-Betti reciprocity relation approach.