A Numerical Study of a Class of Diffusion Equations with Fractional Derivatives

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by

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Chapter 6

Conclusion and future work

6.1 Conclusion

The classical advection-diffusion equations are widely used in science and engineering in different areas, such as transportation in porous media, global weather prediction, transport of mass and energy, chemical transformation, etc., but the classical models are not able to capture the anomalous diffusion process. The fractional advection-diffusion equations play an important role in describing the anomalous transport dynamics in a heterogeneous media and are a useful tool for modeling many physical problems. At the same time, the fractional models become more complicated than the classical advection-diffusion models, and the classical approaches become nearly ineffective for the models with fractional derivatives. Therefore, the researcher faced the challenges of creating new techniques that could reliably handle fractional advection-diffusion models. It is found that the analytical solution to the fractional advection-diffusion equations is limited due to the complexity of the models. Hence, researchers focus on numerical solutions to these problems. Thus the current work focuses on the numerical techniques for advection-diffusion equations with Caputo fraction derivatives. In this study, different types of one and two dimensional fractional diffusion models are demonstrated. The chapter wise summary of some of the important finding and conclusions from the whole thesis is given below:

The first chapter includes some basic terms as well as a summary of the literature survey in porous media. This chapter begins with a brief overview of porous media in the context of diffusion equations. Furthermore, a brief description of the advection-reaction diffusion equation is also given, as the primary objective of the thesis is to solve a such type of diffusion equation.

In the chapter 2, the approximations of fractional time derivative and space derivative are considered with the aid of non standard finite difference schemes and Vieta-Fibonacci polynomials, respectively to solve the non-linear fractional-order space-time reaction advection-diffusion equation. The stability and convergence of the proposed scheme are established theoretically. From this study, it is found that presented scheme is unconditionally stable and the convergence order with respect to time is $O(\Delta t^{2-\alpha})$. Several numerical examples are used to show the validity of this approach, and it is observed that our solution is more accurate than the other available schemes given in [84], [93] and [1]. It is seen from the example that the accuracy of the scheme increases as the degree of approximating polynomials rises. It is also shown that shifted Vieta-Fibonacci collocation method and Chebyshev collocation method produces the same results. Finally, it is observed that the proposed approach is a simpler and efficient tool to solve the nonlinear fractional-order advection reaction diffusion equations. Moreover, this approach will be helpful to handle various problems arising in the different fields of science and industries. Further, for both the conservative and non-conservative system, the parameters α, β, and γ affects the concentration profiles, and all the concentration profiles shift towards integer-order case from fractional-order when we increase α or/and β or/and γ.

From the future point of view, the researchers can extend the proposed scheme to solve the various linear and nonlinear problems in two and three dimensions with different kinds of boundary conditions. The researcher can implement the proposed scheme on various problems from the different fields, like heat and mass transfer, fluid flow in porous media, bio-mechanics, gas dynamics, etc. Further, the researchers can develop other algorithms based on the proposed scheme by taking different polynomials like Legendre, Laguerre, Genonacchi, etc.

In chapter 3, an operational matrix of variable-order fractional derivative is derived with the aid of Vieta-Lucas polynomials for the two-dimensions problems. Based on this operational matrix of Vieta-Lucas polynomials and collocation approach, a numerical scheme is discussed to solve the two-dimension nonlinear problem involving space-time fractional reaction advection diffusion equations of variable-order in the Caputo sense. Convergence and error analysis of the proposed algorithm are also discussed analytically and it is found that the error associated with the obtained approximate solution rapidly tends to zero as number of polynomials or degree of Vieta-Lucas polynomials enhances. The validity and applicability of the proposed algorithms are demonstrated through three examples, and it is found that the proposed algorithm is efficient and sufficiently accurate.

The authors have developed an approach based on an operational matrix of Vieta-Fibonacci polynomials and the collocation method to solve two-dimension space-time variable-order reaction advection diffusion equations in the chapter 4. The fractional derivative of variable order are described in the Caputo sense. It is seen that the proposed scheme is sufficiently efficient and accurate, and the accuracy of the scheme enhances as the degree of Vieta-Fibonacci polynomials increases. It is analytically shown that the obtained approximate solution converges to the exact solution. Moreover, it is also numerically found that the rate of convergence of the scheme becomes fast when we improve the degree of Vieta-Fibonacci polynomials (m), and these facts are verified for the three considered examples.

In chapter 5, the author have successfully applied the second kind shifted Airfoil operational matrix method and collocation technique to present the numerical solutions of non-linear fractional fourth-order sub-diffusion equation of variable-order for one and two-dimensional. From the study of this chapter, it is observed that the proposed numerical technique is efficient and simple to discuss the wide class of a nonlinear sub-diffusion model with neumann boundary conditions. It is analytically found that our scheme generates convergence order of nearly $O\left(\frac{1}{N}\right)$ this fact is verified by some examples. Therefore, this study shows that the proposed scheme is sufficiently accurate and effective to solve time-space fractional non-linear fourth-order sub-diffusion equation of variable-order.

6.2 Future Work

From the science and engineering points of view, the nonlinear advection-diffusion equations are an exciting and important area of research, and looking into the future, the following are the motivations from the mathematical point of view to continue the research work in this direction:

(1) Mathematical Formulation: A correct mathematical model related to a physical problem is very important and helps to understand how to study and control the physical situation. Every mathematical model of a physical problem is formulated under some assumptions. Therefore, the area of mathematical modeling is continuously growing to present more realistic models. In this direction, the researchers can improve the existing models of advection-diffusion equations with different types of boundary conditions to present more reliable and accurate mathematical models of actual physical processes, like fluid flow in porous media, gas dynamics, biomechanics, heat, and mass transfer, etc.

(2) Numerical Computation: The fractional-order linear and non-linear problems in multidimensions lack an exact analytical solution. Numerical and approximate analytical techniques are highly helpful in resolving these kinds of equations. From the computational point of view, an efficient and accurate numerical scheme is required to solve the complicated fractional nonlinear advection-diffusion equation with different types of boundary conditions. Several numerical schemes are available in the literature, but there is still a scope to establish a more accurate and efficient numerical scheme to solve these nonlinear problems. One can also develop an operational matrix scheme with different polynomials are apply the presented schemes to handle the different nonlinear models. Artificial neural networks (ANN) have recently been used to handle different kinds of boundary value problems. The researcher can try to implement this approach to solve fractional nonlinear advection-diffusion equations with different boundary conditions.

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