9. Appendix A

9.1 Proof of Lemma 1.1

Proof: Let $\mathbf{X} = [\underline{x}, \overline{x}]$, $\mathbf{Y} = [\underline{y}, \overline{y}]$, and $\mathbf{Z} = [\underline{z}, \overline{z}]$

(i) $(\mathbf{X} \oplus \mathbf{Y}) \ominus_{gH} (\mathbf{Y} \oplus \mathbf{Z})$
$= [\min\{(\underline{x}+\underline{y})-(\underline{y}+\underline{z}), (\overline{x}+\overline{y})-(\overline{y}+\overline{z})\}, \max\{(\underline{x}+\underline{y})-(\underline{y}+\underline{z}), (\overline{x}+\overline{y})-(\overline{y}+\overline{z})\}$
$= [\min\{\underline{x}-\underline{z}, \overline{x}-\overline{z}\}, \max\{\underline{x}-\underline{z}, \overline{x}-\overline{z}\} = \mathbf{X} \ominus_{gH} \mathbf{Z}$.

(ii) Since, $\mathbf{X} = \mathbf{Y} \oplus \mathbf{Z}$,

$$[\underline{x}, \overline{x}] = [\underline{y}, \overline{y}] \oplus [\underline{z}, \overline{z}] \implies \underline{x} = \underline{y} + \underline{z},\ \overline{x} = \overline{y} + \overline{z} \implies \underline{x} - \underline{y} = \underline{z},\ \overline{x} - \overline{y} = \overline{z}.$$

Hence, $\mathbf{X} \ominus_{gH} \mathbf{Y} = \mathbf{Z}$.

$\square$

10. Appendix B

10.1 Proof of Lemma 1.2

Proof: Let $\mathbf{W} = [\underline{w}, \overline{w}]$, $\mathbf{Y} = [\underline{y}, \overline{y}]$ and $\mathbf{Z} = [\underline{z}, \overline{z}]$. From the gH-difference, we have the following four possible cases:

(i) Given that $\epsilon \preceq (\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w} - \underline{y} - \underline{z}, \overline{w} - \overline{y} - \overline{z}]$. Since $\underline{w} - \underline{y} \geq \underline{z} + \epsilon$ and $\overline{w} - \overline{y} \geq \overline{z} + \epsilon$, we have $\underline{z} + \epsilon \leq \overline{z} + \epsilon \leq \overline{w} - \overline{y}$. This implies $\underline{z} + \epsilon \leq \min\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}$. Also, $\overline{z} + \epsilon \leq \overline{w} - \overline{y} \leq \max\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}$. Clearly we have $[\underline{z} + \epsilon, \overline{z} + \epsilon] \preceq [\min\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}, \max\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}]$ and hence $\mathbf{Z} \oplus \epsilon \preceq \mathbf{W} \ominus_{gH} \mathbf{Y}$.

(ii) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w} - \overline{y} - \overline{z}, \underline{w} - \underline{y} - \underline{z}]$. Thus, the proof is straightforward and identical to **Case** (i).

(iii) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w} - \overline{y} - \underline{z}, \underline{w} - \underline{y} - \overline{z}]$. Since $\overline{w} - \overline{y} \geq \underline{z} + \epsilon, \underline{w} - \underline{y} \geq \overline{z} + \epsilon$, we have $\underline{z} + \epsilon \leq \overline{z} + \epsilon \leq \underline{w} - \underline{y}$. This implies $\underline{z} + \epsilon \leq \min\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}$. Also, $\overline{z} + \epsilon \leq \underline{w} - \underline{y} \leq \max\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}$. Clearly we have $[\underline{z} + \epsilon, \overline{z} + \epsilon] \preceq [\min\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}, \max\{\underline{w} - \underline{y}, \overline{w} - \overline{y}\}]$ and hence $\mathbf{Z} \oplus \epsilon \preceq \mathbf{W} \ominus_{gH} \mathbf{Y}$.

(iv) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w} - \underline{y} - \overline{z}, \overline{w} - \overline{y} - \underline{z}]$. Thus, the proof is identical to **Case** (iii).

□

10.2 Proof of Lemma 1.3

Proof: Let $\mathbf{X} = [\underline{x}, \overline{x}], \mathbf{Y} = [\underline{y}, \overline{y}]$, $\mathbf{Z} = [\underline{z}, \overline{z}]$ and $\mathbf{W} = [\underline{w}, \overline{w}]$. Then,

$$\begin{aligned}
&(\mathbf{X} \oplus \mathbf{Y}) \ominus_{gH} (\mathbf{Z} \oplus \mathbf{W}) \\
&= [\min\{\underline{x} + \underline{y} - \underline{z} - \underline{w}, \overline{x} + \overline{y} - \overline{z} - \overline{w}\}, \max\{\underline{x} + \underline{y} - \underline{z} - \underline{w}, \overline{x} + \overline{y} - \overline{z} - \overline{w}\}] \\
&= [\min\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\}, \max\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\}]. \quad (10.1)
\end{aligned}$$

We have

$$\min\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\} \geq \min\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\} + \min\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\} \quad (10.2)$$

$$\text{and } \max\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\} \leq \max\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\} + \max\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}. \quad (10.3)$$

By (10.2) and (10.3), from (10.1), we write

$$\begin{aligned}
&(\mathbf{X} \oplus \mathbf{Y}) \ominus_{gH} (\mathbf{Z} \oplus \mathbf{W}) \\
&= [\min\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\}, \max\{\underline{x} - \underline{z} + \underline{y} - \underline{w}, \overline{x} - \overline{z} + \overline{y} - \overline{w}\}] \\
&\subseteq [\min\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\} + \min\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}, \max\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\} + \max\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}] \\
&= [\min\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\}, \max\{\underline{x} - \underline{z}, \overline{x} - \overline{z}\}] + [\min\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}, \max\{\underline{y} - \underline{w}, \overline{y} - \overline{w}\}] \\
&= (\mathbf{X} \ominus_{gH} \mathbf{Z}) \oplus (\mathbf{Y} \ominus_{gH} \mathbf{W}).
\end{aligned}$$

□

10.3 Proof of Lemma 1.4

Proof: Let $\mathbf{W} = [\underline{w}, \overline{w}], \mathbf{Y} = [\underline{y}, \overline{y}]$ and $\mathbf{Z} = [\underline{z}, \overline{z}]$. Then, $-1 \odot \mathbf{W} = [-\overline{w}, -\underline{w}], -1 \odot \mathbf{Y} = [-\overline{y}, -\underline{y}], -1 \odot \mathbf{Z} = [-\overline{z}, -\underline{z}]$.

From Definition of gH-difference of two intervals, we have

either $-1 \odot \mathbf{W} \ominus_{gH} -1 \odot \mathbf{Y} = [\overline{y} - \overline{w}, \underline{y} - \underline{w}]$ or $-1 \odot \mathbf{W} \ominus_{gH} -1 \odot \mathbf{Y} = [\underline{y} - \underline{w}, \overline{y} - \overline{w}]$.

Then, one of the following holds true:

(a) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\overline{y} - \overline{w} + \overline{z}, \underline{y} - \underline{w} + \underline{z}]$

(b) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\underline{y} - \underline{w} + \underline{z}, \overline{y} - \overline{w} + \overline{z}]$

(c) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\underline{y} - \underline{w} + \overline{z}, \overline{y} - \overline{w} + \underline{z}]$

(d) $((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z}) = [\overline{y} - \overline{w} + \underline{z}, \underline{y} - \underline{w} + \overline{z}]$.

From this, we have

(a) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\underline{w} - \underline{y} - \underline{z}, \overline{w} - \overline{y} - \overline{z}]$

(b) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\overline{w} - \overline{y} - \overline{z}, \underline{w} - \underline{y} - \underline{z}]$

(c) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\overline{w} - \overline{y} - \underline{z}, \underline{w} - \underline{y} - \overline{z}]$

(d) $\mathbf{0} \ominus_{gH} \{((-1 \odot \mathbf{W}) \ominus_{gH} (-1 \odot \mathbf{Y})) \ominus_{gH} (-1 \odot \mathbf{Z})\} = [\underline{w} - \underline{y} - \overline{z}, \overline{w} - \overline{y} - \underline{z}]$.

On the other hand,

(a) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w} - \underline{y} - \underline{z}, \overline{w} - \overline{y} - \overline{z}]$

(b) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w} - \overline{y} - \overline{z}, \underline{w} - \underline{y} - \underline{z}]$

(c) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{w} - \overline{y} - \underline{z}, \underline{w} - \underline{y} - \overline{z}]$

(d) $(\mathbf{W} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{w} - \underline{y} - \overline{z}, \overline{w} - \overline{y} - \underline{z}]$.

Hence, the desired result follows. □

10.4 Proof of Lemma 1.5

Proof: Let $\mathbf{X} = [\underline{x}, \overline{x}], \mathbf{Y} = [\underline{y}, \overline{y}]$ and $\mathbf{Z} = [\underline{z}, \overline{z}]$.

(i) Let us consider the following four representations:

(a) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{x} - \underline{y} - \underline{z}, \overline{x} - \overline{y} - \overline{z}]$,

(b) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\underline{x} - \underline{y} - \overline{z}, \overline{x} - \overline{y} - \underline{z}]$,

(c) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{x} - \overline{y} - \underline{z}, \underline{x} - \underline{y} - \overline{z}]$,

(d) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z} = [\overline{x} - \overline{y} - \overline{z}, \underline{x} - \underline{y} - \underline{z}]$.

- **Case** 1. Given that $\mathbf{0} \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$. Then we have

$$\begin{aligned}
& 0 \leq \underline{x} - \underline{y} \text{ and } 0 \leq \overline{x} - \overline{y} \\
\implies & 0 - \underline{z} \leq \underline{x} - \underline{y} - \underline{z} \text{ and } 0 - \overline{z} \leq \overline{x} - \overline{y} - \overline{z} \\
\implies & [0 - \underline{z}, 0 - \overline{z}] \preceq [\underline{x} - \underline{y} - \underline{z}, \overline{x} - \overline{y} - \overline{z}]. \qquad (10.4)
\end{aligned}$$

So, from (10.4), we have $\mathbf{0} \ominus_{gH} \mathbf{Z} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z}$.

- **Case** 2. Similarly, we will arrive at this conclusion (10.4). So, from (10.4), we have $\mathbf{0} \ominus_{gH} \mathbf{Z} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{Z}$.
- **Case** 3. This case can be proved by using the same steps as **Case 1**.
- **Case** 4. This case can be proved by using the same steps as **Case 2**.

(ii) Let $\mathbf{W} = [\underline{w}, \overline{w}]$. By the definition of gH-difference, there may be the following four cases.

(a) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\underline{x} - \underline{y} - \underline{w}, \overline{x} - \overline{y} - \overline{w}]$

(b) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\underline{x} - \underline{y} - \overline{w}, \overline{x} - \overline{y} - \underline{w}]$

(c) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\overline{x} - \overline{y} - \underline{w}, \underline{x} - \underline{y} - \overline{w}]$

(d) $(\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W} = [\overline{x} - \overline{y} - \overline{w}, \underline{x} - \underline{y} - \underline{w}]$.

The following two cases are needed to consider for the representation of these above four cases.

- **Case** 1. Since $\mathbf{Z} \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$, we have

$$\begin{aligned}
& \underline{z} \leq \underline{x} - \underline{y} \text{ and } \overline{z} \leq \overline{x} - \overline{y} \\
\implies & \underline{z} - \underline{w} \leq \underline{x} - \underline{y} - \underline{w} \text{ and } \overline{z} - \overline{w} \leq \overline{x} - \overline{y} - \overline{w} \\
\implies & \text{either } [\underline{z} - \underline{w}, \overline{z} - \overline{w}] \preceq [\underline{x} - \underline{y} - \underline{w}, \overline{x} - \overline{y} - \overline{w}] \quad (10.5) \\
& \text{or } [\overline{z} - \overline{w}, \underline{z} - \underline{w}] \preceq [\overline{x} - \overline{y} - \overline{w}, \underline{z} - \underline{y} - \underline{w}] \quad (10.6)
\end{aligned}$$

From (10.5) and (10.6), we have $\mathbf{Z} \ominus_{gH} \mathbf{W} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W}$.

- **Case** 2. Similarly, at the last step, we have

$$\begin{aligned}
& \text{either } [\underline{z} - \underline{w}, \overline{z} - \overline{w}] \preceq [\overline{x} - \overline{y} - \underline{w}, \underline{x} - \underline{y} - \overline{w}] \quad (10.7) \\
& \text{or } [\overline{z} - \overline{w}, \underline{z} - \underline{w}] \preceq [\underline{x} - \underline{y} - \overline{w}, \overline{x} - \overline{y} - \underline{w}] \quad (10.8)
\end{aligned}$$

From (10.7) and (10.8), we have $\mathbf{Z} \ominus_{gH} \mathbf{W} \preceq (\mathbf{X} \ominus_{gH} \mathbf{Y}) \ominus_{gH} \mathbf{W}$.

(iii) Given that $\mathbf{X} \ominus_{gH} \mathbf{Y} \preceq [L, L]$. From the formula of gH-difference of intervals,

$$\begin{aligned}
& \underline{x} - \underline{y} \leq L \quad \text{and} \quad \overline{x} - \overline{y} \leq L \\
\implies & -L \leq \underline{y} - \underline{x}, -L \leq \overline{y} - \overline{x} \\
\implies & \text{ either } \; [-L, -L] \preceq [\underline{y} - \underline{x}, \overline{y} - \overline{x}] \\
& \text{or } [-L, -L] \preceq [\overline{y} - \overline{x}, \underline{y} - \underline{x}].
\end{aligned}$$

Hence, $[-L, -L] \preceq \mathbf{Y} \ominus_{gH} \mathbf{X}$.

(iv) Given that $[-\gamma, -\gamma] \preceq \mathbf{X} \ominus_{gH} \mathbf{Y}$. From the formula of gH-difference of intervals,

$$\begin{aligned}
& -\gamma \leq \underline{x} - \underline{y} \quad \text{and} \quad -\gamma \leq \overline{x} - \overline{y} \\
\implies & \underline{y} - \gamma \leq \underline{x} \text{ and } \overline{y} - \gamma \leq \overline{x} \\
\implies & [\underline{y} - \gamma, \overline{y} - \gamma] \preceq [\underline{x}, \overline{x}].
\end{aligned}$$

Hence, $\mathbf{Y} \ominus_{gH} [\gamma, \gamma] \preceq \mathbf{X}$.

(v) Given that $\mathbf{Z} \preceq \mathbf{X} \oplus \mathbf{Y}$. Then,

$$\begin{aligned}
& [\underline{z}, \overline{z}] \preceq [\underline{x}, \overline{x}] \oplus [\underline{y}, \overline{y}] \\
\implies & \underline{z} \leq \underline{x} + \underline{y}, \overline{z} \leq \overline{x} + \overline{y} \\
\implies & \underline{z} - \underline{y} \leq \underline{x}, \overline{z} - \overline{y} \leq \overline{x} \\
\implies & [\underline{z} - \underline{y}, \overline{z} - \overline{y}] \preceq [\underline{x}, \overline{x}].
\end{aligned}$$

Hence, $\mathbf{Z} \ominus_{gH} \mathbf{Y} \preceq \mathbf{X}$.

$\square$

10.5 Proof of Lemma 3.1

Proof: Let $y^\top \odot \widehat{\mathbf{C}} = \mathbf{D}$ and $\mathbf{D} = [\underline{d}, \overline{d}]$. Note that

$$\|\mathbf{D}\|_{I(\mathbb{R})} = \max\{|\underline{d}|, |\overline{d}|\}. \tag{10.9}$$

On the other hand,

$$\begin{aligned}
\|\mathbf{D}\|_{I(\mathbb{R})} &= \|y_1 \odot \mathbf{C}_1 \oplus y_2 \odot \mathbf{C}_2 \oplus \cdots \oplus y_n \odot \mathbf{C}_n\|_{I(\mathbb{R})} \\
&\leq \|y_1 \odot \mathbf{C}_1\|_{I(\mathbb{R})} + \|y_2 \odot \mathbf{C}_2\|_{I(\mathbb{R})} + \cdots + y_n \odot \mathbf{C}_n\|_{I(\mathbb{R})} \\
&= |y_1|\|\mathbf{C}_1\|_{I(\mathbb{R})} \oplus |y_2|\|\mathbf{C}_2\|_{I(\mathbb{R})} + \cdots + |y_n|\|\mathbf{C}_n\|_{I(\mathbb{R})} \\
&\leq \|y\| \sum_{i=1}^{n} \|\mathbf{C}_i\|_{I(\mathbb{R})} \\
&= \|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n}.
\end{aligned} \tag{10.10}$$

Then, taking into account (10.9) and (10.10), we obtain

$$\begin{aligned}
& |\underline{d}| \leq \|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \text{ and } |\overline{d}| \leq \|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \\
\Longrightarrow \quad & -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq \underline{d} \text{ and } -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq \overline{d} \\
\Longrightarrow \quad & -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq |\underline{d}| \text{ and } -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq |\overline{d}| \\
\Longrightarrow \quad & -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq \max\{|\underline{d}|, |\overline{d}|\} \\
\Longrightarrow \quad & -\|y\|\|\widehat{\mathbf{C}}\|_{I(\mathbb{R})^n} \leq \|\mathbf{D}\|_{I(\mathbb{R})}
\end{aligned}$$

Thus, we arrived at the desired result. $\square$

11. Appendix C

11.1 Proof of Lemma 1.6

Proof: Let $\mathbf{Q} = [\underline{q}, \overline{q}], \mathbf{C} = [\underline{c}, \overline{c}], \mathbf{R} = [\underline{r}, \overline{r}]$. From Definition of gH-difference of two intervals, we have

$$\begin{aligned}
&\inf\{\mathbf{Q}, \mathbf{R} \oplus \mathbf{C}\} \ominus_{gH} \mathbf{C} \\
=&\inf\{[\underline{q}, \overline{q}], [\underline{r} + \underline{c}, \overline{r} + \overline{c}]\} \ominus_{gH} [\underline{c}, \overline{c}] \\
=&[\inf\{\underline{q}, \underline{r} + \underline{c}\}, \inf\{\overline{q}, \overline{r} + \overline{c}\}] \ominus_{gH} [\underline{c}, \overline{c}] \\
=&[\min\{\inf\{\underline{q}, \underline{r} + \underline{c}\} - \underline{c}, \inf\{\overline{q}, \overline{r} + \overline{c}\} - \overline{c}\}, \max\{\inf\{\underline{q}, \underline{r} + \underline{c}\} - \underline{c}, \inf\{\overline{q}, \overline{r} + \overline{c}\} - \overline{c}\}],
\end{aligned} \tag{11.1}$$

and

$$\begin{aligned}
&\inf\{\mathbf{Q} \ominus_{gH} \mathbf{C}, \mathbf{R}\} \\
=&\inf\{[\min\{\underline{q} - \underline{c}, \overline{q} - \overline{c}\}, \max\{\underline{q} - \underline{c}, \overline{q} - \overline{c}\}], [\underline{r}, \overline{r}]\} \\
=&[\inf\{\min\{\underline{q} - \underline{c}, \overline{q} - \overline{c}\}, \underline{r}\}, \inf\{\max\{\underline{q} - \underline{c}, \overline{q} - \overline{c}\}, \overline{r}\}].
\end{aligned} \tag{11.2}$$

In order to show $\inf\{\mathbf{Q}, \mathbf{R} \oplus \mathbf{C}\} \ominus_{gH} \mathbf{C} \subseteq \inf\{\mathbf{Q} \ominus_{gH} \mathbf{C}, \mathbf{R}\}$, we first need to show that

$$\inf\{\min\{\underline{q} - \underline{c}, \overline{q} - \overline{c}\}, \underline{r}\} \leq \min\{\inf\{\underline{q}, \underline{r} + \underline{c}\} - \underline{c}, \inf\{\overline{q}, \overline{r} + \overline{c}\} - \overline{c}\}, \tag{11.3}$$

and

$$\max\{\inf\{\underline{q}, \underline{r} + \underline{c}\} - \underline{c}, \inf\{\overline{q}, \overline{r} + \overline{c}\} - \overline{c}\} \leq \inf\{\max\{\underline{q} - \underline{c}, \overline{q} - \overline{c}\}, \overline{r}\}. \tag{11.4}$$

To show (11.3), we first have that

$$\begin{aligned}
&\inf\{\min\{\underline{q} - \underline{c}, \overline{q} - \overline{c}\}, \underline{r}\} \\
\leq\ & \min\{\underline{q} - \underline{c}, \underline{r}\} \\
=\ & \inf\{\underline{q} - \underline{c}, \underline{r}\} \\
\leq\ & \inf\{\underline{q}, \underline{r} + \underline{c}\} - \inf\{\underline{c}, \underline{c}\} \\
=\ & \inf\{\underline{q}, \underline{r} + \underline{c}\} - \underline{c}.
\end{aligned} \tag{11.5}$$

Similarly,

$$\inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\} \le \inf\{\overline{q},\overline{r}+\overline{c}\}-\overline{c}. \tag{11.6}$$

Combining (11.5) and (11.6), we have

$$\begin{aligned} & \inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\} \\ \le\ & \min\{\inf\{\underline{q},\underline{r}+\underline{c}\}-\underline{c},\inf\{\overline{q},\overline{r}+\overline{c}\}-\overline{c}\}. \end{aligned} \tag{11.7}$$

Note that

$$\inf\{\underline{q},\underline{r}+\underline{c}\}-\underline{c} \le \underline{q}-\underline{c} \le \max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\} \tag{11.8}$$

and

$$\inf\{\underline{q},\underline{r}+\underline{c}\}-\underline{c} \le \underline{r} \le \overline{r}. \tag{11.9}$$

From (11.8) and (11.9), we have

$$\inf\{\underline{q},\underline{r}+\underline{c}\}-\underline{c} \le \inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}. \tag{11.10}$$

Similarly,

$$\inf\{\overline{q},\overline{r}+\overline{c}\}-\overline{c} \le \inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}. \tag{11.11}$$

Combining (11.10) and (11.11), we have

$$\max\{\inf\{\underline{q},\underline{r}+\underline{c}\}-\underline{c},\inf\{\overline{q},\overline{r}+\overline{c}\}-\overline{c}\} \le \inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}. \tag{11.12}$$

Accumulating (11.7) and (11.12), we have

$$\begin{aligned} & \inf\{\mathbf{Q},\mathbf{R}\oplus\mathbf{C}\}\ominus_{gH}\mathbf{C} \\ =\ & [\min\{\inf\{\underline{q},\underline{r}+\underline{c}\}-\underline{c},\inf\{\overline{q},\overline{r}+\overline{c}\}-\overline{c}\},\max\{\inf\{\underline{q},\underline{r}+\underline{c}\}-\underline{c},\inf\{\overline{q},\overline{r}+\overline{c}\}-\overline{c}\}] \\ \subseteq\ & [\inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\},\inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}] \\ =\ & \inf\{\mathbf{Q}\ominus_{gH}\mathbf{C},\mathbf{R}\}. \end{aligned}$$

□

11.2 Proof of Lemma 1.7

Proof: Let $\mathbf{Q} = [\underline{q}, \overline{q}], \mathbf{C} = [\underline{c}, \overline{c}], \mathbf{R} = [\underline{r}, \overline{r}]$. From Definition of gH-difference of two intervals, we have

$$\begin{aligned}
&\inf\{\mathbf{Q} \ominus_{gH} \mathbf{C}, \mathbf{R}\} \ominus_{gH} \mathbf{R} \\
=\ &\inf\{[\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}], [\underline{r}, \overline{r}]\} \ominus_{gH} [\underline{r}, \overline{r}] \\
=\ &[\min\{\inf\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \underline{r}\} - \underline{r}, \inf\{\max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \overline{r}\} - \overline{r}\}, \\
&\max\{\inf\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \underline{r}\} - \underline{r}, \inf\{\max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \overline{r}\} - \overline{r}\}]
\end{aligned}$$

On the other hand, from the similar gH-difference definition,

$$\begin{aligned}
&\inf\{(\mathbf{Q} \ominus_{gH} \mathbf{C}) \ominus_{gH} \mathbf{R}, 0\} \\
=\ &[\inf\{\min\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r}, \max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \overline{r}\}, 0\}, \\
&\quad \inf\{\max\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r}, \max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}\} - \overline{r}, 0\}]
\end{aligned}$$

Next, we can have two following inequalities:

$$\begin{aligned}
&\inf\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \underline{r}\} - \underline{r} \\
=\ &\inf\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \underline{r}\} - \inf\{\underline{r}, \underline{r}\} \\
\leq\ &\inf\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r}, 0\} \\
\leq\ &\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r} \\
\leq\ &\max\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r}, \max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \overline{r}\}. \qquad (11.13)
\end{aligned}$$

and

$$\begin{aligned}
&\inf\{\max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \underline{r}\} - \underline{r} \\
=\ &\inf\{\max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \underline{r}\} - \inf\{\underline{r}, \underline{r}\} \\
\leq\ &\inf\{\max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r}, 0\} \\
\leq\ &\max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r} \\
\leq\ &\max\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r}, \max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \overline{r}\}. \qquad (11.14)
\end{aligned}$$

By combining (11.13) and (11.14), we have

$$\begin{aligned}
&\max\{\inf\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \underline{r}\} - \underline{r}, \inf\{\max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\}, \overline{r}\} - \overline{r}\} \\
\leq\ &\max\{\min\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \underline{r}, \max\{\underline{q}-\underline{c}, \overline{q}-\overline{c}\} - \overline{r}\}. \qquad (11.15)
\end{aligned}$$

Next, we can make other two inequalities:

$$\begin{aligned} & \inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\}-\underline{r} \\ =\ & \inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\}-\inf\{\underline{r},\underline{r}\} \\ \leq\ & \inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\underline{r},0\} \\ \leq\ & 0. \end{aligned} \tag{11.16}$$

and

$$\begin{aligned} & \inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}\}-\overline{r} \\ =\ & \inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}-\inf\{\overline{r},\overline{r}\} \\ \leq\ & \inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\overline{r},0\} \\ \leq\ & 0. \end{aligned} \tag{11.17}$$

By combining (11.16) and (11.17), we have

$$\max\{\inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\}-\underline{r},\inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}-\overline{r}\}\leq 0. \tag{11.18}$$

Therefore, accumulating (11.15) and (11.18), we have

$$\begin{aligned} & \max\{\inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\}-\underline{r},\inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}-\overline{r}\} \\ \leq & \inf\{\max\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\underline{r},\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\overline{r}\}\},0\}. \end{aligned} \tag{11.19}$$

Now, it is remaining to show that

$$\begin{aligned} & \inf\{\min\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\underline{r},\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}\}-\overline{r},0\} \\ \leq\ & \min\{\inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\}-\underline{r},\inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}-\overline{r}\}. \end{aligned} \tag{11.20}$$

For this, we consider the following inequalities:

$$\begin{aligned} & \inf\{\min\{\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\underline{r},\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\overline{r}\}\},0\} \\ \leq & \inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\},0\} \\ \leq & \inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\}-\inf\{\underline{r},\overline{r}\} \\ = & \inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\}-\underline{r}. \end{aligned} \tag{11.21}$$

and

$$\begin{aligned}
&\inf\{\min\{\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\underline{r},\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\overline{r}\}\},0\}\\
\leq&\inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\},0\}\\
\leq&\inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}-\inf\{\overline{r},\overline{r}\}\\
=&\inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}-\overline{r} \qquad (11.22)
\end{aligned}$$

From (11.21) and (11.22), we have

$$\begin{aligned}
&\inf\{\min\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}-\underline{r},\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\}\}-\overline{r},0\}\\
\leq\ &\min\{\inf\{\min\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\underline{r}\}-\underline{r},\inf\{\max\{\underline{q}-\underline{c},\overline{q}-\overline{c}\},\overline{r}\}-\overline{r}\}. \qquad (11.23)
\end{aligned}$$

Again, from (11.19) and (11.23), we have

$$\inf\{\mathbf{Q}\ominus_{gH}\mathbf{C},\mathbf{R}\}\ominus_{gH}\mathbf{R}\subseteq\inf\{(\mathbf{Q}\ominus_{gH}\mathbf{C})\ominus_{gH}\mathbf{R},0\}.$$

□

11.3 Proof of Lemma 1.8

Proof: Let $\mathbf{Q}=[\underline{q},\overline{q}],\mathbf{R}=[\underline{r},\overline{r}],\mathbf{S}=[\underline{s},\overline{s}]$. Then, we have

$$\begin{aligned}
&-\epsilon\ \preceq(\mathbf{Q}\ominus_{gH}\mathbf{R})\ominus_{gH}\mathbf{S}\\
\Longrightarrow\ &-\epsilon\preceq[\min\{\underline{q}-\underline{r},\overline{q}-\overline{r}\},\max\underline{q}-\underline{r},\overline{q}-\overline{r}]\ominus_{gH}[\underline{s},\overline{s}]\\
\Longrightarrow\ &-\epsilon\preceq[\min\{\min\{\underline{q}-\underline{r},\overline{q}-\overline{r}\}-\underline{s},\max\{\underline{q}-\underline{r},\overline{q}-\overline{r}\}-\overline{s}\},\\
&\max\{\min\{\underline{q}-\underline{r},\overline{q}-\overline{r}\}-\underline{s},\max\{\underline{q}-\underline{r},\overline{q}-\overline{r}\}-\overline{s}\}]\\
\Longrightarrow\ &-\epsilon\leq\min\{\underline{q}-\underline{r},\overline{q}-\overline{r}\}-\underline{s}\ \textbf{and}\ -\epsilon\leq\max\{\underline{q}-\underline{r},\overline{q}-\overline{r}\}-\overline{s}\\
\Longrightarrow\ &[\underline{s},\overline{s}]\preceq[\min\{\underline{q}-\underline{r},\overline{q}-\overline{r}\},\max\{\underline{q}-\underline{r},\overline{q}-\overline{r}\}]\oplus\epsilon\\
\Longrightarrow\ &\mathbf{S}\preceq(\mathbf{Q}\ominus_{gH}\mathbf{R})\oplus\epsilon.
\end{aligned}$$

□

11.4 Proof of Lemma 1.9

Proof: Let $\mathbf{\Upsilon}=[\underline{\gamma},\overline{\gamma}],\mathbf{\Upsilon_1}=[\underline{\gamma}_{\mathbf{1}},\overline{\gamma}_{\mathbf{1}}],\mathbf{\Upsilon_2}=[\underline{\gamma}_{\mathbf{2}},\overline{\gamma}_{\mathbf{2}}]$.

(i) We note that

$$\begin{aligned}
&\liminf_{z\to\bar z}\{-1\odot \boldsymbol{\Upsilon}(z)\}\\
=\ &\liminf_{z\to\bar z}\{-1\odot [\underline{\gamma}(z),\overline{\gamma}(z)]\}\\
=\ &\liminf_{z\to\bar z}[-\overline{\gamma}(z),-\underline{\gamma}(z)]\\
=\ &[\liminf_{z\to\bar z}(-\overline{\gamma}(z)),\liminf_{z\to\bar z}(-\underline{\gamma}(z))]\\
=\ &[-\limsup_{z\to\bar z}\overline{\gamma}(z),-\limsup_{z\to\bar z}\underline{\gamma}(z)]\\
=\ &-1\odot[\limsup_{z\to\bar z}\underline{\gamma}(z),\limsup_{z\to\bar z}\overline{\gamma}(z)]\\
=\ &-1\odot\limsup_{z\to\bar z}\boldsymbol{\Upsilon}(z).
\end{aligned}$$

(ii) Note that

$$\begin{aligned}
&\liminf_{z\to\bar z}\{\boldsymbol{\Upsilon_1}(z)\ominus_{gH}\boldsymbol{\Upsilon}_2(z)\}\\
=\ &\liminf_{z\to\bar z}\{[\min\{\underline{\gamma_1}(z)-\underline{\gamma}_2(z),\overline{\gamma_1}(z)-\overline{\gamma_2}(z)\},\max\{\underline{\gamma_1}(z)-\underline{\gamma}_2(z),\overline{\gamma_1}(z)-\overline{\gamma_2}(z)\}].
\end{aligned} \tag{11.24}$$

provided $\lim_{z\to\bar z}\boldsymbol{\Upsilon_2}(z)$. We know if $\lim_{z\to\bar z}\underline{\gamma}_2(z)$ exists, then

$$\liminf_{z\to\bar z}\{\underline{\gamma}_1(z)-\underline{\gamma}_2(z)\}=\liminf_{z\to\bar z}\{\underline{\gamma}_1(z)+(-\underline{\gamma}_2(z))\}=\liminf_{z\to\bar z}\underline{\gamma}_1(z)-\lim_{z\to\bar z}\underline{\gamma}_2(z). \tag{11.25}$$

Similarly, if $\lim_{z\to\bar z}\overline{\gamma}_2(z)$ exists, then

$$\liminf_{z\to\bar z}\{\overline{\gamma}_1(z)-\overline{\gamma}_2(z)\}=\liminf_{z\to\bar z}\overline{\gamma}_1(z)-\lim_{z\to\bar z}\overline{\gamma}_2(z). \tag{11.26}$$

With the help of (11.25), (11.26), from (11.24), we have

$$\begin{aligned}
&[\min\{\liminf_{z\to\bar z}\underline{\gamma}_1(z)-\lim_{z\to\bar z}\underline{\gamma}_2(z),\liminf_{z\to\bar z}\overline{\gamma}_1(z)-\lim_{z\to\bar z}\overline{\gamma}_2(z)\},\\
&\max\{\liminf_{z\to\bar z}\underline{\gamma}_1(z)-\lim_{z\to\bar z}\underline{\gamma}_2(z),\liminf_{z\to\bar z}\overline{\gamma}_1(z)-\lim_{z\to\bar z}\overline{\gamma}_2(z)\}]\\
=\ &\liminf_{z\to\bar z}\boldsymbol{\Upsilon_1}(z)\ominus_{gH}\lim_{z\to\bar z}\boldsymbol{\Upsilon_2}(z).
\end{aligned}$$

□

12. Appendix D

Example 12.0.1 *We consider a ZDT1-type [194] SOP involving a set of* 100 *objective functions, given by*

$$F(x) := \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where, for each $i \in [100]$, $f^i : \mathbb{R}^n \to \mathbb{R}^2$ *is given by*

$$f^i(x) = \begin{pmatrix} f_1(x) + (0.05 + 0.05 * (\cos^{16}(\frac{4\pi i}{100}))) \cos(\frac{2\pi i}{100}) \\ g(x)h(f_1(x), g(x)) + (0.05 + 0.05 * (\cos^{16}(\frac{4\pi i}{100}))) \sin(\frac{2\pi i}{100}) \end{pmatrix}$$

with

$$\begin{aligned} f_1(x) &= x_1 \\ g(x) &= 1 + 9\sum_{i=2}^{n} x_i \\ h(f_1, g) &= 1 - \sqrt{\frac{f_1}{g}}. \end{aligned}$$

We consider 4 *different versions of this example for* $n = 2, 5, 8, 10$. *For randomly sampling 100 initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$, *where* $L = [0, 0, 0 \ldots, 0]$ *and* $U = [1, 1, 1, \ldots, 1]$.

Example 12.0.2 *We consider ZDT4-type [194] SOP involving n-variables and a set of* 100 *objective functions, given by*

$$F(x) = \{f^1(x), f^2(x), \ldots, f^{100}(x)\}.$$

where, for each $i \in [100]$, *the objective function of the SOP,* $f^i : \mathbb{R}^n \to \mathbb{R}^2, n = 10$, *is given by*

$$f^i(x) = \begin{pmatrix} f_1(x) + (1 + (\cos(\frac{4i\pi}{100}))^{16} \cos(\frac{2i\pi}{100})) \\ g(x)h(f_1(x)g(x)) + (1 + (\cos(\frac{4i\pi}{100}))^{16} \sin(\frac{2i\pi}{100})), \end{pmatrix}$$

with

$$\begin{aligned} f_1(x) &= x_1 \\ g(x) &= 1 + 10(n-1) + \sum_{i=2}^{n} (x_i^2 - 10\cos(4\pi x_i)) \end{aligned}$$

$$h_1(f_1, g) = 1 - \sqrt{\frac{f_1}{g}},$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [0, 0, \ldots, 0]$*, and* $U = [1, 1, \ldots, 1]$.

Example 12.0.3 *Motivated from Ex.4.3 in [93], we consider a SOP involving a set of* 100 *objective functions, given by*

$$F(x) := \{f^1(x), f^2(x), \ldots, f^{125}(x)\},$$

where, for each $i \in [125]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^4$ *is given by*

$$f^i(x) = g(x) + (x_1 + \frac{1}{10}\sin(20\phi_{1i})\sin(20\phi_{2i}))\begin{pmatrix} \cos(\phi_{1i}) \\ \sin(\phi_{1i})\cos(\phi_{2i}) \\ \sin(\phi_{1i})\sin(\phi_{2i})\cos(\phi_{3i}) \\ \sin(\phi_{1i})\sin(\phi_{2i})\sin(\phi_{3i}) \end{pmatrix}.$$

with

$$g(x) = 100(x_1^2 + x_2^2 - 9)^2 \begin{pmatrix} |x_1| \\ |x_2| \\ |x_1 + x_2| \\ |x_1| \end{pmatrix}.$$

For randomly sampling 100 initial points, we consider the box region $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [-10, -10]$ *and* $U = [10, 10]$*, and the set* $\{(\phi_{1i}, \phi_{2i}, \phi_{3i}) : i \in [125]\}$ *is an enumeration of the set* $\{\frac{2\pi}{5}(i_1 - 1) : i_1 \in [5]\} \times \{\frac{2\pi}{5}(i_2 - 1) : i_2 \in [5]\} \times \{\frac{2\pi}{5}(i_3 - 1) : i_3 \in [5]\}$.

Example 12.0.4 *We consider DTLZ1 [37] type SOP involving a set of* 100 *objective functions, given by*

$$F(x) := \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where, for each $i \in [100]$*,* $f^i : \mathbb{R}^6 \to \mathbb{R}^4$*, is given by*

$$f^i(x) = h(x) + \begin{pmatrix} \cos(\phi_i)\sin(\psi_i) \\ \sin(\phi_i)\sin(\psi_i) \\ (\cos(\psi_i) + \log(\tan(\frac{(\psi_i)}{2}))) + 0.2 * \phi_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with

$$h(x) = \begin{bmatrix} (1+g(x_M))x_1x_2\ldots x_{M-1} \\ (1+g(x_M))x_1x_2\ldots(1-x_{M-1}) \\ \vdots \\ (1+g(x_M))\frac{1}{2}x_1(1-x_2) \\ \frac{1}{2}(1-x_1)(1+g(x_M)). \end{bmatrix}$$

and

$$g(x_M) = 100(|x_M| + \sum_{x_i \in x_M}(x_i - 0.5)^2 - \cos(20 * \pi * (x_i - 0.5))).$$

Here, last $k = n - M + 1$ *variables are represented as* x_M *and the set* $\{(\phi_i, \psi_i) : i \in [100]\}$ *is an enumeration of the set* $\{\frac{\pi}{5}(j-1) : j \in [10]\} \times \{\frac{\pi}{5}(\ell - 1) : \ell \in [10]\}$*. For randomly sampling* 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [0, 0, \ldots, 0]$*, and* $U = [1, 1, \ldots, 1]$*.*

Example 12.0.5 *We consider DTLZ3 [37] type SOP, involving a set of* 100 *objective functions, given by*

$$F(x) := \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where for each $i \in [100]$*,* $f^i : \mathbb{R}^n \to \mathbb{R}^m, n = 5, m = 4$*, is given by*

$$f^i(x) = h(x) + \begin{pmatrix} \text{sech}(\phi_i)\cos(\phi_2) \\ \text{sech}(\phi_i)\sin(\psi_i) \\ \phi_i - tanh(\phi_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with

$$h(x) = \begin{bmatrix} (1+g(x_M))\cos(\frac{x_1\pi}{2})\ldots\cos(\frac{x_{M-2}\pi}{2})\cos(\frac{x_{M-1}\pi}{2}) \\ (1+g(x_M))\cos(\frac{x_1\pi}{2})\ldots\cos(\frac{x_{M-2}\pi}{2})\sin(\frac{x_{M-1}\pi}{2}) \\ (1+g(x_M))\cos(\frac{x_1\pi}{2})\cos(\frac{x_1\pi}{2}) \\ \vdots \\ (1+g(x_M))\sin(\frac{x_1\pi}{2}). \end{bmatrix}$$

and

$$g(x_M) = 100 * (|x_M| + \sum_{x_i \in x_M} (x_i - 0.5)^2 - \cos(20 * \pi * (x_i - 0.5)))$$
$$0 \leq x_i \leq 1, \ \textit{for}\ i = 1, 2, \ldots, n$$

Last $k = n - M + 1$ *variables are represented as* x_M *and the set* $\{(\phi_i, \psi_i) : i \in [100]\}$ *is an enumeration of the set* $\{\frac{\pi}{5}(j-1) : j \in [10] \times \frac{\pi}{5}(l-1) : l \in [10]\}$*, and for randomly sampling* 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [0, 0, \ldots, 0]$*, and* $U = [1, 1, \ldots, 1]$.

Example 12.0.6 *We consider FDSa [56] type SOP involving a set of* 100 *objective functions, given by*

$$F(x) := \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where, for each $i \in [100]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^3$ *is given by*

$$f^i(x) = \begin{pmatrix} G_1(x) + (1 + \cos(\phi_i)\cos(\psi_i) \\ G_2(x) + (1 + \cos(\phi_i)\sin(\psi_i)) \\ G_3(x) + sin(\phi_i) \end{pmatrix}$$

with

$$G_1(x) = \frac{1}{n^2}\sum_{i=1}^{n} i(x_i - i)^4$$
$$G_2(x) = \exp\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) + \|x\|_2^2$$
$$G_3(x) = \frac{1}{n(n+1)}\sum_{i=1}^{n} i(n - i + 1)\exp(-x_i).$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form*

$[L, U]$, where $L = [-2, -2]$, and $U = [2, 2]$.

Example 12.0.7 *We consider DTLZ5 [37] type SOP involving a set of 100-objective functions, is given by*

$$F(x) := \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where, for each $i \in [100]$, $f^i : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$f^i(x) = h(x) + \begin{pmatrix} 5\psi_i/2\pi \\ \lambda\cos(\phi_i)/10 \\ \lambda\sin(\phi_i)/10 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with

$$h(x) = \begin{bmatrix} (1 + g(x_M))\cos(\theta_1(\pi/2))\ldots\cos(\theta_{M-2}\pi/2)\cos(\theta_{M-1}\pi/2) \\ (1 + g(x_M))\cos(\theta_1(\pi/2))\ldots\cos(\theta_{M-2}\pi/2)\sin(\theta_{M-1}\pi/2) \\ (1 + g(x_M))\cos(\theta_1\pi/2), \ldots\sin(\theta_{M-2}\pi/2) \\ \vdots \\ (1 + g(x_M))\sin(\theta_1\pi/2) \end{bmatrix},$$

and

$$\begin{aligned} &\theta_i = \frac{1}{2(1 + g(x_M))}(1 + g(x_M)x_i), \ \textit{for } i = 2, 3, \ldots, (M - 1), \\ &g(x_M) = \sum_{x_i \in x_M} (x_i - 0.5)^2, \\ &0 \le x_i \le 1, \ \textit{for } i = 1, 2 \ldots, n, \end{aligned}$$

Last $k = n - M + 1$ variables are represented as x_M and the set $\{(\phi_i, \psi_i) : i \in [100]\}$ is an enumeration of the set $\{\frac{\pi}{5}(j - 1) : j \in [10]\} \times \{\frac{\pi}{5}(\ell - 1) : \ell \in [10]\}$ and for randomly sampling 100 initial points, we consider the box region $\mathcal{S}$ of the form $[L, U]$, where $L = [0, 0, \ldots, 0], U = [1, 1, \ldots, 1]$. We consider three different version of this problem where $(n, m) = (3, 3), (5, 3), (7, 5)$.

Example 12.0.8 *We consider DGO1 [44] type SOP involving a set of 100-objective* *functions, given by*

$$F(x) = \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where for each $i \in [100], f^i : \mathbb{R} \to \mathbb{R}^2$ *is given by*

$$f^i(x) = g(x) + \begin{pmatrix} \sin(2\pi i/100 + \cos(2\pi i/100)) \\ \cos(2\pi i/100 + \sin(2\pi i/100)) \end{pmatrix}$$

with

$$g(x) = \begin{pmatrix} \sin(x) \\ \sin(x + 0.7) \end{pmatrix}.$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [-10, -10], U = [13, 13]$.

Example 12.0.9 *We consider DGO2 [44] type SOP involving a set of 100 objective functions, given by*

$$F(x) = \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where, for each $i \in [100], f^i : \mathbb{R} \to \mathbb{R}^2$*, is given by*

$$f^i(x) = g(x) + \begin{pmatrix} \sin(2\pi i/100 + \cos(2\pi i/100)) \\ \cos(2\pi i/100 + \sin(4\pi i/100)) \end{pmatrix}$$

with

$$g(x) = \begin{pmatrix} x^2 \\ 9 - \sqrt{81 - x^2}. \end{pmatrix}$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [-9, -9]$*, and* $U = [9, 9]$.

Example 12.0.10 *We consider Hil type [87] SOP involving a set of 100 objective functions, given by*

$$F(x) := \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where for each $i \in [100]$*,* $f^i : \mathbb{R}^2 \to \mathbb{R}^2$ *is given by*

$$f^i(x) = g(x) + \begin{pmatrix} 10((9+\exp(\sin(4\pi i/100)) - \sin(4\pi i/100) + 2(\cos(8\pi i/100))^2)/128)\cos(2\pi i/100) \\ 10((9+\exp(\sin(4\pi i/100)) - \sin(4\pi i/100) + 2(\cos(8\pi i/100))^2)/128)\sin(2\pi i/100) \end{pmatrix}.$$

with

$$g(x) = \begin{pmatrix} \cos((2\pi/360)[45+40\sin(2\pi x_1)+25\sin(2\pi x_2)](1+0.5\cos(2\pi x_1)) \\ \sin((2\pi/360)[45+40\sin(2\pi x_1)+25\sin(2\pi x_2)](1+0.5\cos(2\pi x_1)). \end{pmatrix}$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [0, 0]$*, and* $U = [5, 5]$*.*

Example 12.0.11 *We consider JOS1a [98] type SOP involving 100 objective functions, given by*

$$F(x) = \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where for each $i \in [100], f^i : \mathbb{R}^{50} \to \mathbb{R}^2$ *is given by*

$$f^i(x) = g(x) + \begin{pmatrix} 0.1\cos(2\pi i/100) \\ 50\sin(2\pi i/100) \end{pmatrix}$$

with

$$g(x) = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n x_i^2 \\ \frac{1}{n}\sum_{i=1}^n (x_i-2)^2 \end{pmatrix}.$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [-2, -2, \ldots, -2]$*, and* $U = [2, 2, \ldots, 2]$*.*

Example 12.0.12 *We consider Rosenbrock [176] type SOP involving a set of* 100*-objective functions, given by*

$$F(x) = \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where, for each $i \in [100], f^i : \mathbb{R}^4 \to \mathbb{R}^3$ *is given by*

$$f^i(x) = \begin{pmatrix} 100(x_2 - x_1^2)^2 + (x_2-1)^2 + (r^2(\cos\phi_i \cos(\psi_i)\sin(\psi_i))) \\ 100(x_3 - x_2^2)^2 + (x_3-1)^2 + (r^2(\cos\phi_i \sin(\psi_i)\sin(\psi_i)) \\ 100(x_4 - x_3^2)^2 + (x_4-1)^2 + (r^2(\cos\phi_i \sin(\psi_i)\cos^2(\psi_i)) \end{pmatrix}.$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form*

$[L, U]$, *where* $L = [-2, -2, \ldots, -2], U = [2, 2, \ldots, 2]$. *We set* $r = 16$ *and the set* $\{(\phi_i, \psi_i) : i \in [100]\}$ *is an enumeration of the set* $\{\frac{\pi}{5}(j-1) : j \in [10]\} \times \{\frac{\pi}{5}(k-1) : k \in [10]\}$.

Example 12.0.13 *We consider Brown and Dennis [147] type SOP involving a set of 100 objective functions, given by*

$$F(x) = \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where for each $i \in [100]$, $f^i : \mathbb{R}^4 \to \mathbb{R}^5$ *is given by*

$$f^i(x) = \begin{pmatrix} (x_1 + \frac{1}{5}x_2 - \exp^{\frac{1}{5}})^2 + (x_3 + x_4 \sin(\frac{1}{5}) - \cos(\frac{1}{5}))^2 + \cos(\phi_i)\sin(\psi_i) \\ (x_1 + \frac{2}{5}x_2 - \exp^{\frac{2}{5}})^2 + (x_3 + x_4 \sin(\frac{2}{5}) - \cos(\frac{2}{5}))^2 + \sin(\phi_i)\sin(\psi_i) \\ (x_1 + \frac{3}{5}x_3 - \exp(\frac{3}{5}))^2 + (x_3 + x_4 \sin(\frac{3}{5}) - \cos(\frac{3}{5}))^2 + (\cos(\psi_i) + \log(\tan(\frac{\psi_i}{2}))) + 0.5\phi_i \end{pmatrix}.$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$, *where* $L = [-25, -5, -5, -1]$, *and* $U = [25, 5, 5, 1]$. *The set* $\{(\phi_i, \psi_i) : i \in [100]\}$ *is an enumeration of the set* $\{\frac{2\pi}{5}(j-1) : j \in [10]\} \times \{0.01 + \frac{0.98}{10}(k-1) : k \in [10]\}$.

Example 12.0.14 *We consider Trigonometric [147] type SOP involving a set of 100 objective functions, given by*

$$F(x) = \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where, for each $i \in [100], f^i : \mathbb{R}^4 \to \mathbb{R}^4$ *is given by*

$$f^i(x) = \begin{pmatrix} (1 - \cos x_1 + (1 - \cos x_1) - \sin x_1)^2 + \cos(\phi_i)\sin(\psi_i) \\ (2 - \cos(x_1 + x_2) + 2(1 - \cos x_2) - \sin x_2)^2 + \sin(\phi_i)\sin(\psi_i) \\ (3 - \cos(x_1 + x_2 + x_3) + 3(1 - \cos x_3) - \sin x_3)^2 + (\cos(\psi) + \log(\tan(\frac{\psi_i}{2}))) + 0.2\phi_i \\ (4 - \cos(x_1 + x_2 + x_3 + x_4) + 4(1 - \cos x_4) - \sin x_4) \end{pmatrix},$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$, *where* $L = [-1, -1, \ldots, -1]$, *and* $U = [1, 1, \ldots, 1]$. *The set* $\{(\phi_i, \psi_i) : i \in [100]\}$ *is an enumeration of the set* $\{\frac{2\pi}{5}(j-1) : j \in [10]\} \times \{0.01 + \frac{0.98}{10}(k-1) : k \in [10]\}$.

Example 12.0.15 *We consider Das and Dennis [36] type involving a set of 100 objective functions, given by*

$$F(x) = \{f^1(x), f^2(x), \ldots, f^{100}(x)\},$$

where, for each $i \in [100], f^i : \mathbb{R}^5 \to \mathbb{R}^2$, *is given by*

$$f^i(x) = \begin{pmatrix} (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + (\sin(\frac{2i\pi}{100}) + \cos(\frac{2i\pi}{100})) \\ (3x_1 + 2x_2 - \frac{x_3}{3} + 0.01(x_4 - x_5)^3 + (\sin(\frac{2i\pi}{100}) + \cos(\frac{2i\pi}{100})) \end{pmatrix},$$

For randomly sampling 100 *initial points, we consider the box region* $\mathcal{S}$ *of the form* $[L, U]$*, where* $L = [-20, -20, \ldots, -20]$*, and* $U = [20, 20, \ldots, 20]$.

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LIST OF PUBLICATIONS

1. Suprova Ghosh, Debdas Ghosh, and Anshika. Normal and tangent cones for set of intervals and their application in optimization with functions of interval variables. *Soft Computing*, 27 (2023) 10737–10758.

2. Suprova Ghosh, Debdas Ghosh, Adrian Petruşel and Xiaopeng Zhao. Generalized Hukuhara Weak subdifferential and its application on identifying optimality conditions for nonsmooth interval-valued functions. *Journal of Nonlinear and Variational Analysis*, 8 (2024) 333–368.

3. Debdas Ghosh and Suprova Ghosh. Sufficient optimality conditions and Duality for a nonsmooth interval-valued optimization problem with Generalized Convexity via gH-Clarke subgradients. In *Continuous Optimization and Variational Inequalities*. Chapman and Hall/CRC, 2022, pp. 219–242.

4. Debdas Ghosh, Suprova Ghosh, Christiane Tammer, Jen-Chih Yao, and Xiaopeng Zhao. Trust-region method for set optimization problems with set-valued mapping of finitely many vector-valued functions, 2024. (Communicated)

5. Debdas Ghosh, Suprova Ghosh, and Christiane Tammer. Nonmonote trust-region method for set optimization problems with set-valued mapping of finitely many vector-valued functions, 2024. (Communicated)

6. Suprova Ghosh, Debdas Ghosh, Hao-Ren Yao, and Radko Mesiar. Nonsmooth interval optimization with gH-Dini Hadamard subdifferential and its applications in control systems, 2024. (Communicated)