Chapter 4

Variational and Numerical Approximations for Higher-Order Fractional Sturm–Liouville Problems

In this chapter, we study the higher-order FSLP. First, we present the introduction of the FSLP in Section 4.1. Section 4.2 discusses some preliminaries which are required for the present chapter. Section 4.3 gives the main result and shows the existence of eigenvalues and orthogonal eigenfunctions of higher-order FSLP. Using $\Phi_j(t) = t^{j+1}(1-t)^2$ for $j = 1, 2, 3, ...$ polynomials as a basis function, we approximate the problem and obtain the eigenvalues and corresponding orthogonal eigenfunctions in Section 4.4. The theoretical convergence of the numerical method are presented in Section 4.5. In section 4.6, we discuss the numerical results and demonstrate the validation of the method. Section 4.7 concludes the chapter.

4.1 Introduction

Here, our focus is the FSLPs, a class of differential equations involving fractional derivatives (FDs). In [50], authors established the fundamental theory of FSLPs by applying the methods of fractional variational analysis and investigated the applications of fractional calculus of variations (FCVs) for FSLPs. During the past few years, many researchers have expanded the theory of calculus of variations using FC and developed the idea of FCVs. Numerous research articles have been published on FCVs, and we are mentioning some of them here, [1, 31, 5, 56, 74, 66, 65, 75, 105, 106, 76]. Further, the variational approach is found useful for solving FSLPs [50, 39, 51, 99, 100, 101].

Our aim here is to present a variational and numerical method to solve the higher-order FSLPs. In literature, most of the FSLPs are defined for the second-order fractional derivatives. However, in a recent study [52], authors discussed the higher-order FSLPs and developed a numerical method to solve it using power series. Later, in 2017, Syam et al. [107] considered a regular fractional 2α order SLPs where $\alpha \in (1/2, 1]$ and developed a numerical method by using reproducing kernal method (RKM). Furthermore, Al-Mdallal et al. [108] considered the fourth-order FSLPs for $\alpha \in (3, 4]$ with variable coefficient and used fractional series solution to find the eigenfunctions. Motivated by all these works, we consider fourth-order FSLPs with mixed FDs, right Riemann-Liouville (R-L), and left Caputo fractional derivatives with $\alpha \in (0, 1)$. Here, we present a variational and numerical method to solve the higher-order FSLPs. By using fractional variational analysis, we reformulate the FSLPs in form of fractional variational isoperimetric problems and using the polynomials $\Phi_j(t) = t^{j+1}(1-t)^2$ for $j = 1, 2, 3, ...$ as a basis function, we approximate the isoperimetric variational problem and obtain its eigenvalues and eigenfunctions.

4.2 Preliminaries

Here, we discuss only some preliminaries that will be used to prove the valuable results of the chapter.

Definition 4.1. [9] A function $y(t)$ is said to be Hölder continuous for $t \in [c,d]$ iff

$$\sup_{t_1,t_2\in[c,d],t_1\neq t_2} \frac{|y(t_1)-y(t_2)|}{|t_1-t_2|^{\beta}} < \infty, \tag{4.1}$$

where $\beta \in (0,1]$ is called exponent. We use $C_H^{\beta}[c,d]$ for denoting the Hölder continuous function.

Theorem 4.2. *(**Weierstrass Approximation Theorem**)*([109]) *Let* $y \in C([c,d],\mathbb{R})$. *Then the function* y *can be approximated uniformly by the sequence of polynomials* P_n *on* $[c,d]$.

4.3 Higher-Order Regular Fractional Sturm-Liouville Problem

In this section, we have studied the higher-order FSLP using fractional variational principles. We will prove that the countably infinite sequence of eigenvalues and the corresponding sequence of eigenfunctions exist for higher-order FSLPs. Moreover, we will also demonstrate the smallest (first) eigenvalue is the minimizer of the functional and eigenfunctions are orthogonal to each other for different eigenvalues. Now, we present the theory of higher-order FSLP and establish the result for $\alpha \in (0,1)$. First, we prove the lemmas that help establish this chapter's main results.

Lemma 4.3. *Let* $\alpha \in (0,1)$ *and* $\rho(t) \in C[0,1]$. *If*

$$\int_0^1 \rho(t) D\, {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha v(t)\, dt = 0, \tag{4.2}$$

for each $v \in C^2[0,1]$ *such that* $D\, {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha v \in C[0,1]$ *and fulfilling boundary conditions*

$$v(0) = {}_0I_t^{1-\alpha} \, {}_0^C D_t^\alpha v(1) = 0, \tag{4.3}$$

$${}_0^C D_t^\alpha \, {}_0^C D_t^\alpha v(t)\big|_{t=0} = {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha v(t)\big|_{t=1} = 0, \tag{4.4}$$

then $\rho(t) = A_0 + A_1 t$, *where* A_0, A_1 *are real constants.*

Proof. Let us take a function $v(t)$ as follows

$$v(t) = {}_0I_t^{1+2\alpha}(\rho(t) - A_0 - A_1 t), \tag{4.5}$$

set the conditions

$${}_0I_t^{2}(\rho(t) - A_0 - A_1 t)\big|_{t=1} = 0, \tag{4.6}$$

and

$${}_0I_t^{1}(\rho(t) - A_0 - A_1 t)\big|_{t=1} = 0. \tag{4.7}$$

Observe that function $v(t)$ is continuous and satisfies Eqs. (4.3) and (4.4), because

$$v(0) = 0, \qquad {}_0I_t^{1-\alpha} \, {}_0^C D_t^\alpha v(t)\big|_{t=1} = {}_0I_t^{2}(\rho(t) - A_0 - A_1 t)\big|_{t=1} = 0,$$

and

$$
\begin{aligned}
{}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\big|_{t=0} = {}_0^CD_t^\alpha\, {}_0D_t^\alpha v(t)\big|_{t=0} &= {}_0I_t^{1-\alpha}\, {}_0I_t^\alpha(\rho(t)-A_0-A_1t)\big|_{t=0} \\
&= {}_0I_t^1(\rho(t)-A_0-A_1t)\big|_{t=0} = 0, \\
{}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\big|_{t=1} = {}_0^CD_t^\alpha\, {}_0D_t^\alpha v(t)\big|_{t=1} &= {}_0I_t^{1-\alpha}\, {}_0I_t^\alpha(\rho(t)-A_0-A_1t)\big|_{t=1} \\
&= {}_0I_t^1(\rho(t)-A_0-A_1t)\big|_{t=1} = 0.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
v'(t) &= {}_0I_t^{2\alpha}\,(\rho(t)-A_0-A_1t) \in C[0,1], \\
D_0^C D_t^\alpha\, {}_0^CD_t^\alpha v(t) &= \rho(t)-A_0-A_1t \in C[0,1].
\end{aligned} \tag{4.8}
$$

Also,

$$
\begin{aligned}
\int_0^1 (\rho(t)-A_0-A_1t) D\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\, dt &= \int_0^1 (-A_0-A_1t) D\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\, dt \\
&= -A_0\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\big|_{t=0}^{t=1} - A_1t\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\big|_{t=0}^{t=1} \\
&\quad + A_1\, {}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha v(t)\big|_{t=0}^{t=1} = 0.
\end{aligned}
$$

On the other hand

$$
D_0^C D_t^\alpha\, {}_0^CD_t^\alpha v(t) = D_0^C D_t^\alpha\, {}_0^CD_t^\alpha\, {}_0I_t^{1+2\alpha}(\rho(t)-A_0-A_1t) = \rho(t)-A_0-A_1t,
$$

and

$$
0 = \int_0^1 (\rho(t)-A_0-A_1t) D\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\, dt = \int_0^1 (\rho(t)-A_0-A_1t)^2\, dt.
$$

Thus function $\rho(t)$ is

$$\rho(t) = A_0 + A_1 t.$$

□

Lemma 4.4. *1. Let $\rho_j(t) \in C[0,1]$, $j = 1,2,3$. For $\alpha \in (0,1)$, if*

$$\int_0^1 \big(\rho_1(t)v(t) + \rho_2(t)\,{}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v(t) + \rho_3(t)\, D\, {}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v(t)\big)\, dt = 0, \tag{4.9}$$

for every $v(t) \in C^2[0,1]$ such that $D\,{}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v(t) \in C[0,1]$ satisfying BCs (4.3), (4.4) then $\rho_3(t) \in C^1[0,1]$.

2. Let $\rho_j(t) \in C[0,1]$, $j = 1,2$. For $\alpha \in (0,1)$, if

$$\int_0^1 \big(\rho_1(t)v(t) + \rho_2(t)\,{}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v(t)\big)\, dt = 0, \tag{4.10}$$

for every $v(t) \in C^2[0,1]$ such that $D\,{}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v(t) \in C[0,1]$ satisfying BCs (4.3), (4.4) then

$$\rho_1(t) + {}_t^C D_1^\alpha\, {}_t^C D_1^\alpha \rho_2(t) = 0.$$

Proof. As $v(t)$ fulfills the BCs (4.3), (4.4) and holds the following relations

$${}_0I_t^\alpha\, {}_0I_t^\alpha\, {}_tI_1^1\, D\, {}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v(t) = -v(t),$$

and

$${}_tI_1^1\, D\, {}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v(t) = -{}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v(t).$$

Therefore using above relations, integral (4.9) becomes

$$\begin{aligned}
&\int_0^1 \left(\rho_1(t)v(t) + \rho_2(t){}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t) + \rho_3(t)D\,{}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\right)\, dt \\
&= \int_0^1 \left(-{}_0I_t^1\, {}_tI_1^\alpha\, {}_tI_1^\alpha \rho_1(t) - {}_0I_t^1 \rho_2(t) + \rho_3(t)\right) D\,{}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\, dt = 0.
\end{aligned}$$

Denote,

$$\rho(t) = -{}_0I_t^1\, {}_tI_1^\alpha\, {}_tI_1^\alpha \rho_1(t) - {}_0I_t^1 \rho_2(t) + \rho_3(t).$$

Clearly, $\rho(t) \in C[0,1]$. Thus by using the Lemma 4.3, $\exists$ constants A_0 and A_1 such that

$$\begin{aligned}
-\,{}_0I_t^1\, {}_tI_1^\alpha\, {}_tI_1^\alpha \rho_1(t) - {}_0I_t^1 \rho_2(t) + \rho_3(t) &= A_0 + A_1 t, \\
\rho_3(t) &= {}_0I_t^1\, {}_tI_1^\alpha\, {}_tI_1^\alpha \rho_1(t) + {}_0I_t^1 \rho_2(t) + A_0 + A_1 t. \qquad (4.11)
\end{aligned}$$

Clearly from Eq. (4.11) $\rho_3(t) \in C^1[0,1]$.

The second part of the proof is similar to part (1). Now, Eq. (4.10) can be written as

$$\begin{aligned}
&\int_0^1 \left(\rho_1(t)v(t) + \rho_2(t){}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\right)\, dt \\
&= \int_0^1 \left(-{}_0I_t^1\, {}_tI_1^\alpha\, {}_tI_1^\alpha \rho_1(t) - {}_0I_t^1 \rho_2(t)\right) D\,{}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t)\, dt = 0.
\end{aligned}$$

Denote,

$$\rho(t) = -{}_0I_t^1\, {}_tI_1^\alpha\, {}_tI_1^\alpha \rho_1(t) - {}_0I_t^1 \rho_2(t),$$

we can observe, $\rho(t) \in C[0,1]$. Thus, by using Lemma 4.3, $\exists$ constants A_0 and A_1 such that

$$ {}_0I_t^1 \, {}_tI_1^\alpha \, {}_tI_1^\alpha \rho_1(t) + {}_0I_t^1 \rho_2(t) = A_0 + A_1 t. $$

Therefore $\rho_1(t), \rho_2(t)$ fulfill equation

$$ \rho_1(t) + {}_t^C D_1^\alpha \, {}_t^C D_1^\alpha \rho_2(t) = 0. $$

$\square$

Lemma 4.5. *Let* $g(t) \in C^2[0,1] \cap C_H^2[-1,1]$ *be a even function in* $[-1,1]$ *such that* $g'' \in L^2[0,1]$. *Let* $g_N(t)$ *be the* N*-th sum of the Fourier series of* $g(t)$*, then the following convergence are valid for* $t \in [0,1]$ *and order* $0 < \alpha < 1$.

$$ \lim_{N\to\infty} \left\| {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha g_N(t) - {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha g(t) \right\|_{L^1} = 0. \tag{4.12} $$

Proof. Using the fact that if $g(t) \in C^2[0,1]$, ${}_0^C D_t^\alpha g(t) \in C[0,1]$. For $t > 0$, we have

$$ \begin{aligned} D \, {}_0^C D_t^\alpha g_N(t) &= {}_0D_t^\alpha g_N'(t) = {}_0^C D_t^\alpha g_N'(t) + \frac{g_N'(0) t^{-\alpha}}{\Gamma(1-\alpha)}, \\ D \, {}_0^C D_t^\alpha g(t) &= {}_0D_t^\alpha g'(t) = {}_0^C D_t^\alpha g'(t) + \frac{g'(0) t^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned} $$

Therefore, we can estimate the $\left\| {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha g_N(t) - {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha g(t) \right\|_{L^1}$ norm in $[0,1]$

$$ \begin{aligned} \left\| {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha g_N(t) - {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha g(t) \right\|_{L^1} &= \left\| {}_0^C D_t^\alpha \left({}_0^C D_t^\alpha \left(g_N(t) - g(t) \right) \right) \right\|_{L^1} \\ &= \left\| {}_0I_t^{1-\alpha} D \left({}_0^C D_t^\alpha \left(g_N(t) - g(t) \right) \right) \right\|_{L^1} \\ &= \Bigg\| {}_0I_t^{1-\alpha} \Bigg({}_0^C D_t^\alpha \left(g_N'(t) - g'(t) \right) \\ &\quad + \left(g_N'(0) - g'(0) \right) \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \Bigg) \Bigg\|_{L^1} \end{aligned} $$

$$
\begin{aligned}
&\le \left\| {}_0I_t^{1-\alpha}\left({}_0^C D_t^{\alpha}\left(g'_N(t)-g'(t)\right)\right)\right\|_{L^1} \\
&\quad + \left|g'_N(0)-g'(0)\right| \left\| {}_0I_t^{1-\alpha}\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right)\right\|_{L^1} \\
&\le \left\| {}_0I_t^{2-2\alpha}\left(g''_N(t)-g''(t)\right)\right\|_{L^1} \\
&\quad + \frac{\left|g'_N(0)-g'(0)\right|}{\Gamma(2-\alpha)\Gamma(1-\alpha)} \left\| t^{-\alpha}\right\|_{L^1} \\
&\le \frac{1}{\Gamma(3-2\alpha)} \left\| g''_N(t)-g''(t)\right\|_{L^1} \\
&\quad + \frac{1}{(\Gamma(2-\alpha))^2}\left|g'_N(0)-g'(0)\right| \\
&\le \frac{\sqrt{2}}{\Gamma(3-2\alpha)} \left\| g''_N(t)-g''(t)\right\|_{L^2} \\
&\quad + \frac{1}{(\Gamma(2-\alpha))^2}\left|g'_N(0)-g'(0)\right|.
\end{aligned}
$$

By the assumption we have in $[-1,1]$ and hence in $[0,1]$:

$$
\lim_{N\to\infty} \left\| g''_N(t)-g''(t)\right\|_{L^2} = 0, \qquad \lim_{N\to\infty} \left|g'_N(0)-g'(0)\right| = 0.
$$

Hence, we conclude that Eq. (4.12) is valid. □

4.3.1 Eigenvalues for Higher-Order Fractional Sturm-Liouville Problem

Here, we consider the fractional beam problem as the higher-order FSLP, which has infinite increasing eigenvalues, and for each eigenvalue, there is a unique eigenfunction. As α approaches to 1, the fractional beam problem reduces to the classical beam problem. The higher-order regular fractional Sturm–Liouville equation for fractional order $0 < \alpha < 1$, with fixed BCs is

$$
{}_tD_1^{\alpha}\,{}_tD_1^{\alpha}\,{}_0^C D_t^{\alpha}\,{}_0^C D_t^{\alpha} y(t) = \lambda y(t), \tag{4.13}
$$

$$y(0) = y(1) = y'(0) = y'(1) = 0. \tag{4.14}$$

Theorem 4.6. *The higher-order FSLP (4.13)-(4.14) has an infinitely many monotonic increasing eigenvalues* $\lambda^{(1)}, \lambda^{(2)}, ...,$ *and for every eigenvalue* $\lambda^{(N)}$ *there correspond an unique (prior to a constant multiple) eigenfunction* $y^{(N)}$ *which forms an orthogonal set in* $L^2[0,1]$.

Proof. The proof of the theorem is given in 6 steps and follow as the [50, 39].

Step 1. Observe that Eq. (4.13) is the Euler-Lagrange equation corresponding to the problem of minimizing the functional

$$J(y) = \int_0^1 \left({}_0^C D_t^\alpha \, {}_0^C D_t^\alpha y(t)\right)^2 \, dt, \tag{4.15}$$

with isoperimetric constraint,

$$F(y) = \int_0^1 (y(t))^2 \, dt = 1, \tag{4.16}$$

and BCs Eq. (4.14). Clearly,

$$J(y) = \int_0^1 \left({}_0^C D_t^\alpha \, {}_0^C D_t^\alpha y(t)\right)^2 \, dt \geq 0. \tag{4.17}$$

Therefore, $J(y)$ is bounded from below. By applying the Ritz method, we approximate a solution of (4.13)-(4.14) using the following polynomial function of the form $\Phi_j(t) = t^{j+1}(1-t)^2, j = 1, 2, ..., N$ which satisfies the BCs (4.14). Using the Gram-Schmidt orthogonalisation process (for better approximation), we orthonormalise our basis function $\Phi_j(t)$ and get an orthonormal basis $\{\chi_1, \chi_2, \chi_3, ..., \chi_N, ...\}$. Now approximating the functional by using the Ritz method,

$$y_N(t)=\sum_{j=1}^{N}c_j\chi_j(t), \tag{4.18}$$

and from Eq. (4.18), $y_N(0)=y_N(1)=y'_N(0)=y'_N(1)=0$. Now, using Eq. (4.18) in (4.15) and (4.16), we get

$$\widetilde{J}(c_1,c_2,...,c_N)=\widetilde{J}([c])=\sum_{i,j=1}^{N}c_ic_j\int\limits_0^1\left({}_0^CD_t^\alpha\ {}_0^CD_t^\alpha\chi_i(t)\right)\left({}_0^CD_t^\alpha\ {}_0^CD_t^\alpha\chi_j(t)\right)\,dt, \tag{4.19}$$

subject to the condition

$$\widetilde{I}(c_1,c_2,...,c_N)=\widetilde{I}([c])=\sum_{j=1}^{N}c_j^2=1. \tag{4.20}$$

Since $\widetilde{J}(c_1,c_2,...,c_N)$ is continuous and the surface of $N-$dimensional sphere (4.20) is compact, so $\widetilde{J}(c_1,c_2,...,c_N)$ has a minimum, say $\lambda_N^{(1)}$, at some point $[c^{(1)}]=(c_1^{(1)},c_2^{(1)},...,c_N^{(1)})$. If we continue this process, we get a sequence $\lambda_1^{(1)},\lambda_2^{(1)},...$ for $N=1,2,....$ Since $\lambda_{N+1}^{(1)}\leq\lambda_N^{(1)}$ and $J[y]$ is bounded below, so its limit exist *i.e.*

$$\lim_{N\to\infty}\lambda_N^{(1)}=\lambda^{(1)}.$$

Step 2. Now, in this step, we will show that sequence $(y_N^{(1)})_{N\in\mathbb{N}}$ contain a uniformly convergent subsequence.

Let

$$y_N^{(1)}(t)=\sum_{j=1}^{N}c_j^{(1)}\chi_j(t), \tag{4.21}$$

achieves the minimum $\lambda_N^{(1)}$. For convenience, we write y_N rather than $y_N^{(1)}$. Since, the sequence

$$\lambda_N^{(1)} = \int\limits_0^1 \left({}_0^C D_t^\alpha \, {}_0^C D_t^\alpha y_N(t)\right)^2 \, dt,$$

is convergent, therefore it is bounded $\forall\, N \in \mathbb{N}$ such that,

$$\int\limits_0^1 \left({}_0^C D_t^\alpha \, {}_0^C D_t^\alpha y_N(t)\right)^2 \, dt \leq M. \tag{4.22}$$

Here M is positive real number.

Using Eqs. (1.9) and condition $y_N(0) = y_N'(0) = 0$, we have

$$\begin{aligned}
\|y_N(t)\|_{L^2(0,1)} &= \|{}_0I_t^\alpha \, {}_0I_t^\alpha \, {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha y_N(t)\|_{L^2(0,1)} \\
&= \|{}_0I_t^{2\alpha} \, {}_0^C D_t^\alpha \, {}_0^C D_t^\alpha y_N(t)\|_{L^2(0,1)} \;\; [[9], \text{Lemma } 2.3] \\
&= \left\| \frac{1}{\Gamma(2\alpha)} \int_0^t (t-x)^{2\alpha-1} \, {}_0^C D_x^\alpha \, {}_0^C D_x^\alpha y_N(x) \, dx \right\|_{L^2(0,1)} \\
&= \frac{1}{\Gamma(2\alpha)} \left\| t^{2\alpha-1} * {}_0^C D_x^\alpha \, {}_0^C D_x^\alpha y_N(t) \right\|_{L^2(0,1)},
\end{aligned}$$

where $'*'$ represents the convolution of the functions. Now from using the Eq. (4.22) and Young's inequality [[110], Theorem 3.1], we get the following

$$\begin{aligned}
\|y_N(t)\|_{L^2(0,1)} &\leq \frac{1}{\Gamma(2\alpha)} \left(\int_0^t \left|(t-x)^{2\alpha-1}\right| dx \right) \left(\int_0^t \left|{}_0^C D_x^\alpha \, {}_0^C D_x^\alpha y_N(x)\right|^2 dx \right)^{\frac{1}{2}} \\
&\leq \frac{1}{\Gamma(2\alpha)} \frac{1}{2\alpha} \left(\int_0^1 \left|{}_0^C D_x^\alpha \, {}_0^C D_x^\alpha y_N(x)\right|^2 dx \right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{M}}{\Gamma(2\alpha+1)}.
\end{aligned} \tag{4.23}$$

Therefore, $(y_N(t))_{N\in\mathbb{N}}$ is uniformly bounded under L^2-norm. We have for any $0 < t_1 < t_2 \le 1$,

$$\begin{aligned}
\Big|y_N(t_2) - y_N(t_1)\Big| &= \Big|{}_0I^{\alpha}_{t_2}\,{}_0I^{\alpha}_{t_2}\,{}^C_0D^{\alpha}_{t_2}\,{}^C_0D^{\alpha}_{t_2}y_N(t_2) - {}_0I^{\alpha}_{t_1}\,{}_0I^{\alpha}_{t_1}\,{}^C_0D^{\alpha}_{t_1}\,{}^C_0D^{\alpha}_{t_1}y_N(t_1)\Big| \\
&= \Big|{}_0I^{2\alpha}_{t_2}\,{}^C_0D^{\alpha}_{t_2}\,{}^C_0D^{\alpha}_{t_2}y_N(t_2) - {}_0I^{2\alpha}_{t_1}\,{}^C_0D^{\alpha}_{t_1}\,{}^C_0D^{\alpha}_{t_1}y_N(t_1)\Big| \\
&= \frac{1}{\Gamma(2\alpha)}\Bigg|\int_0^{t_2}(t_2-u)^{2\alpha-1}\,{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\,du \\
&\quad - \int_0^{t_1}(t_1-u)^{2\alpha-1}\,{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\,du\Bigg| \\
&= \frac{1}{\Gamma(2\alpha)}\Bigg|\int_{t_1}^{t_2}(t_2-u)^{2\alpha-1}\,{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\,du \\
&\quad - \int_0^{t_1}\left((t_1-u)^{2\alpha-1}-(t_2-u)^{2\alpha-1}\right)\,{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\,du\Bigg| \\
&\le \frac{1}{\Gamma(2\alpha)}\sup_{0<t_1\le u<t_2\le 1}\Big|{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\Big|\int_{t_1}^{t_2}\Big|(t_2-u)^{2\alpha-1}\Big|\,du \\
&\quad + \frac{1}{\Gamma(2\alpha)}\Bigg[\sup_{0\le u<t_1<1}\Big|{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\Big| \\
&\quad \int_0^{t_1}\Big|\left((t_1-u)^{2\alpha-1}-(t_2-u)^{2\alpha-1}\right)\Big|\,du\Bigg] \\
&= \frac{1}{\Gamma(2\alpha+1)}(t_2-t_1)^{2\alpha}\sup_{0<t_1\le u<t_2\le 1}\Big|{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\Big| \\
&\quad + \frac{1}{\Gamma(2\alpha+1)}\sup_{0\le u<t_1<1}\Big|{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\Big|\left[(t_2-t_1)^{2\alpha}+t_1^{2\alpha}-t_2^{2\alpha}\right] \\
&\le \frac{1}{\Gamma(2\alpha+1)}(M_1+M_2)(t_2-t_1)^{2\alpha}, \qquad (4.24)
\end{aligned}$$

where, $M_1 = \sup_{0<t_1\le u<t_2\le 1}\left|{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\right|$, and $M_2 = \sup_{0\le u<t_1<1}\left|{}^C_0D^{\alpha}_{u}\,{}^C_0D^{\alpha}_{u}y_N(u)\right|$.

Hence, $y_N(t)$ is equicontinuous. So, by Ascoli's theorem a uniformly convergent subsequence $(y_{N_n})_{n\in\mathbb{N}}$ of $(y_N)_{N\in\mathbb{N}}$ exists. Therefore, we can get a function $y^{(1)} \in C[0,1]$,

$$y^{(1)} = \lim_{n\to\infty} y_{N_n}.$$

Step 3. By using the Lagrange multipliers rule at $[c] = [c^{(1)}]$ we obtain

$$0 = \frac{\partial}{\partial c_j}\left[\widetilde{J}[c] - \lambda_N^{(1)}\widetilde{I}[c]\right]\Big|_{[c]=[c^{(1)}]}, \quad j = 1, 2, ..., N.$$

Multiplying above by arbitrary constant β^j on every equations and taking the sum over j from 1 to N, we get

$$0 = \sum_{j=1}^{N}\beta^j\frac{\partial}{\partial c_j}\left[\widetilde{J}[c] - \lambda_N^{(1)}\widetilde{I}[c]\right]\Big|_{[c]=[c^{(1)}]}. \tag{4.25}$$

By introducing

$$v_N(t) = \sum_{j=1}^{N}\beta^j\chi_j(t),$$

Eq. (4.25) can be rewritten as

$$0 = \int_0^1 \left[{}_0^C D_t^\alpha\, {}_0^C D_t^\alpha y_N(t)\, {}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v_N(t) - \lambda_N^{(1)} y_N v_N\right] dt. \tag{4.26}$$

Using the fact that $y_N'(0) = 0$ and differentiation properties

$$D\, {}_0^C D_t^\alpha y_N(t) = {}_0D_t^\alpha y_N'(t) = {}_0^C D_t^\alpha y_N'(t) + \frac{y_N'(0)t^{-\alpha}}{\Gamma(1-\alpha)} = {}_0^C D_t^\alpha y_N'(t),$$

we write (4.26) as

$$\begin{aligned}
0 &= \int_0^1 \left[{}_0I_t^{1-\alpha}\, D\left({}_0^C D_t^\alpha y_N(t)\right)\, {}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v_N(t) - \lambda_N^{(1)} y_N v_N\right] dt\\
0 &= \int_0^1 \left[{}_0I_t^{1-\alpha}\, {}_0^C D_t^\alpha y_N'(t)\, {}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v_N(t) - \lambda_N^{(1)} y_N v_N\right] dt\\
0 &= \int_0^1 \left[{}_0I_t^{1-\alpha}\, {}_0^C D_t^\alpha y_N'(t)\, {}_0^C D_t^\alpha\, {}_0^C D_t^\alpha v_N(t)\right] dt - \int_0^1 \lambda_N^{(1)} y_N v_N\, dt = I_N.
\end{aligned} \tag{4.27}$$

Since $v(t)$ fulfills the assumption of Lemma 4.3. Thus by using Lemma 4.5 with Lemma 2.1 of [39], we obtain in the limit

$$0 = \int_0^1 \left[{}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha y'^{(1)}(t)\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t) \right] dt - \int_0^1 \lambda^{(1)} y^{(1)} v\, dt = I. \tag{4.28}$$

We analyze the convergence of integral (4.27) as

$$\begin{aligned}\left|I_N - I\right| \leq \int_0^1 \Big| {}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha y_N'(t)\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v_N(t) - {}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha y'^{(1)}(t) \\ {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t) \Big| \, dt + \int_0^1 \left| -\lambda_N^{(1)} y_N v_N + \lambda^{(1)} y^{(1)} v \right| dt.\end{aligned} \tag{4.29}$$

For the first integral in the Eq. (4.29), we get

$$\begin{aligned}&\int_0^1 \left| {}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha y_N'(t)\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v_N(t) - {}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha y'^{(1)}(t)\, {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t) \right| dt \\ &\leq \int_0^1 \left| {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t) \left({}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha y_N'(t) - {}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha y'^{(1)}(t) \right) \right| dt \\ &\quad + \int_0^1 \left| {}_0I_t^{1-\alpha}\, {}_0^CD_t^\alpha y_N'(t) \left({}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v_N(t) - {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t) \right) \right| dt \\ &\leq \left\| {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha v(t) \right\|_{L^2} \cdot \left\| {}_0I_t^{2-2\alpha} \left(y_N''(t) - y''^{(1)}(t) \right) \right\|_{L^2} \\ &\quad + \left\| {}_0I_t^{2-2\alpha} y_N''(t) \right\|_{L^2} \cdot \left\| {}_0^CD_t^\alpha\, {}_0^CD_t^\alpha \left(v_N(t) - v(t) \right) \right\|_{L^2}.\end{aligned} \tag{4.30}$$

Finally, from the last integral in Eq. (4.29) we get

$$\begin{aligned}\int_0^1 \left| -\lambda_N^{(1)} y_N v_N + \lambda^{(1)} y^{(1)} v \right| dt &= \int_0^1 \Big| -\lambda_N^{(1)} y_N \left(v_N - v \right) \\ &\quad + v \left(\lambda^{(1)} y^{(1)} - \lambda_N^{(1)} y_N \right) \Big| \, dt \\ &\leq \Lambda \left(M_3 \|v_N - v\| + \|v\| \|y_N - y^{(1)}\| \right) \\ &\quad + \|y^{(1)} v\| \left| \lambda_N^{(1)} - \lambda^{(1)} \right|,\end{aligned} \tag{4.31}$$

where, constants $M_3 = \sup_{N\in\mathbb{N}} \|y_N\|$ and $\Lambda = \sup_{N\in\mathbb{N}} \left|\lambda_N^{(1)}\right|$. From Eqs. (4.30) and (4.31) we conclude that

$$0 = \lim_{N\to\infty} I_N = I.$$

For steps 4, 5, and 6, we will follow the proof described in the [50], we can show that $y^{(1)}(t)$ is also a solution of FSLP (4.13)-(4.14) and $(y_N^{(1)})_{N\in\mathbb{N}}$ converges uniformly to its limit $y^{(1)}$ and get eigen-solution $y^{(2)}$ and the correspond eigenvalue $\lambda^{(2)}$ and we obtain strict inequality

$$\lambda^{(1)} < \lambda^{(2)}.$$

Eventually, if we continue the above process, with identical changes, we can find eigenvalues $\lambda^{(3)}, \lambda^{(4)}, \ldots$ and corresponding eigenfunctions $y^{(3)}, y^{(4)}, \ldots$. ☐

4.3.2 The Lowest (First) Eigenvalue

Definition 4.7. Rayleigh quotient of FSLP (4.13)-(4.14) can be defined as

$$R(y) = \frac{J(y)}{F(y)}.$$

Here $J(y)$ and $F(y)$ is given by Eqs. (4.33) and (4.32) respectively.

Theorem 4.8. *If $y^{(1)}(t)$ is the eigenfunction and satisfies the integral constraint*

$$F(y) = \int_0^1 (y(t))^2 \, dt = 1, \tag{4.32}$$

corresponding to the first eigenvalue $\lambda^{(1)}$ of the FSLP (4.13)-(4.14), and let function ${}_tD_1^{\alpha}\,{}_tD_1^{\alpha}\,{}_0^CD_t^{\alpha}\,{}_0^CD_t^{\alpha}y(t) \in C[0,1]$, then $y^{(1)}(t)$ is the required solution of the following *functional,*

$$J(y) = \int_0^1 \left({}_0^C D_t^\alpha \ {}_0^C D_t^\alpha y(t)\right)^2 \, dt, \tag{4.33}$$

in $C[0,1]$ *with* ${}_0^C D_t^\alpha \ {}_0^C D_t^\alpha y(t)$ *and* ${}_t D_1^\alpha {}_t D_1^\alpha \ {}_0^C D_t^\alpha \ {}_0^C D_t^\alpha y(t)$ *are continuous in* $[0,1]$, *subject to the integral constraint and*

$$y(0) = y(1) = y'(0) = y'(1) = 0. \tag{4.34}$$

This implies,

$$J(y^{(1)}(t)) = \lambda^{(1)}.$$

Theorem 4.9. *Let us consider that* $y(t) \in C[0,1]$ *with* ${}_0^C D_t^\alpha \ {}_0^C D_t^\alpha y(t) \in C[0,1]$ *and fulfills the BCs (4.34). Since* y *is non-trivial so it is minimizer of Rayleigh quotient* R *for the FSLP (4.13)-(4.14). Furthermore, if function* ${}_t D_1^\alpha {}_t D_1^\alpha \ {}_0^C D_t^\alpha \ {}_0^C D_t^\alpha y(t) \in C[0,1] \implies R(y) = \lambda^{(1)}$.

Remark 4.10. The proofs of Theorem (4.8) and Theorem (4.9) can be obtained in a similar way as in Ref. [50, 39]. So, from Theorems (4.8) and (4.9), we obtain that the minimizer of the functional is the lowest (first) eigenvalue, and it is equal to the Rayleigh quotient R at y.

4.4 Numerical Approximation to Variational Form of Higher-Order Fractional Sturm-Liouville Problem

This section presents a numerical scheme for a variational form of FSLP given by Eqs. (4.33-4.32), satisfying the BCs (4.14). The approximate solution of Eqs. (4.33) and (4.32) is as follows:

$$y(t) \approx y_N(t) = \sum_{j=1}^{N} c_j \chi_j(t), \tag{4.35}$$

where $\chi_j(t), j = 1, ..., N$ are the orthonormal basis obtained by using the Gram-Schmidt orthogonalisation process (for better approximation). We orthonormalise $\Phi_j(t) = t^{j+1}(1-t)^2$ which satisfies BCs (4.14), and c_j are unknown coefficients to be determined.

Substituting Eq. (4.35) into Eq. (4.32), the functional can be expressed as

$$J(y) \approx C^T A C, \tag{4.36}$$

where matrix A is

$$A_{ij} = \int_0^1 \left({}_0^C D_t^{\alpha} \; {}_0^C D_t^{\alpha} \chi_j(t) \right) \left({}_0^C D_t^{\alpha} \; {}_0^C D_t^{\alpha} \chi_j(t) \right) \, dt, \quad A = [A_{ij}], \tag{4.37}$$

and

$$C = [c_1, c_2, ..., c_N]^T. \tag{4.38}$$

Similarly, by substituting Eq. (4.35) into Eq. (4.32), the integral constraint can be written as,

$$F \approx C^T B C = 1, \tag{4.39}$$

where matrix B is given by

$$B_{ij} = \int_0^1 \chi_j(t)\chi_j(t)\ dt, \quad B = [B_{ij}]. \tag{4.40}$$

Now, the Lagrangian can then be written as,

$$L = C^T AC - \lambda(C^T BC - 1). \tag{4.41}$$

Now Differentiating the Eq. (4.41) with respect to C and putting it to zero, we obtain

$$AC = \lambda BC, \tag{4.42}$$

which is an eigenvalue problem. Here A and B are positive definite matrices, therefore the eigenvalues λ_i, $i = 1, 2, ..., N$ of Eq. (4.42) are positive real numbers. We arrange the λ_i such that $\lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_N$. The eigenvectors C_i, $i = 1, 2, ..., N$ of this problem are such that

$$C_i^T BC_j = \lambda_i \delta_{ij}, \tag{4.43}$$

where δ_{ij} is the Kronecker delta function. Thus, any arbitrary vector C can be expressed as

$$C = \sum_{i=1}^{N} b_i C_i, \tag{4.44}$$

where b_is are scalars which belong to the real numbers. Now, substituting the Eq. (4.44) in to the Eq. (4.32), the integral constraint becomes

$$F(y) = \sum_{i=1}^{N} b_i^2 = 1. \tag{4.45}$$

Following further steps from [59], we can find that the eigenfunction corresponding to the lowest eigenvalue is the minimizer of the functional $J(y)$. It should be noted that C_i, $i = 1, 2, ...N$ are orthonormal vectors. The orthonormal vectors are orthonormal functions for the analytical solution.

4.5 Convergence Analysis

Here, we show the theoretical convergence of the numerical method proposed in Section 4.4. We will exhibit that the approximate solution converges to the exact solution, *i.e.* $|J(y_N(t)) - J(y(t))| \to 0$, where $y_N(t)$ is the approximate solutions corresponding N basis functions, and $y(t)$ is the exact solution, minimizer of the functional J. Now, we define our functional spaces and give some required theorems and lemmas.

Consider the Banach space provided with the norm $\|.\|$. Let, $\Theta = [0, 1]$.

$$\begin{aligned} B(\Theta) =& \{y(t) \in C^2(\Theta) \mid y(0) = 0, y(1) = 0, y'(0) = 0, y'(1) = 0\} \\ G^N[\Theta] =& B(\Theta) \cap \left\langle \ \{\Phi_j\}_{j=1}^N \right\rangle \\ \|y\| =& \|y\|_\infty + \|y'\|_\infty + \|y''\|_\infty \,, \qquad [61] \end{aligned}$$

where $\left\langle \ \{\Phi_j\}_{j=1}^N \right\rangle$ is the subspace of $C^2(\Theta)$ generated by the polynomials of degree at most N. Now, we state a lemma which shows that the polynomial functions are dense in norm space $B(\Theta)$.

Lemma 4.11. *Let* $y(t) \in B(\Theta)$. *There exist a sequence of polynomials* $\{\Phi_j(t)\}_{j\in\mathbb{N}} \subset B(\Theta)$ *such that* $y_N = \sum_{j=1}^{N} c_j \Phi_j \xrightarrow{\|.\|} y$ *when* $N \to +\infty$.

Proof. Since $y(t) \in C^2([0,1],\mathbb{R})$, then $y(t) \in C([0,1],\mathbb{R})$. According to Theorem 4.2 there exist a sequence of polynomials $\{\Phi_j(t)\}_{j\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \left\| \sum_{j=1}^{N} c_j \Phi_j(t) - y(t) \right\| = 0. \tag{4.46}$$

Thus $\displaystyle\sum_{j=1}^{N} c_j\Phi_j(t) \xrightarrow{\|\cdot\|} y(t)$, implying that $\{\Phi_j(t), j = 1,2,3,...\}$ is dense in norm space $B(\Theta)$ and forms a basis. ☐

Remark 4.12. Hence from the Eq. (4.46) $\lim_{N\to\infty} y_N(t) = \lim_{N\to\infty} \displaystyle\sum_{j=1}^{N} c_j\Phi_j(t) = y(t)$ demonstrate that y_N converges to y.

Lemma 4.13. *The functional J is continuous on its space $(C^2(\Theta),\ \|.\|)$.*

Proof. Let $y^*(t) \in C^2(\Theta)$. Consider $r > 0$ and,
$I = \Theta \times \displaystyle\prod_{k=0}^{2}[-l_k - r, l_k + r]$,
where, $l_0 = \|y^*\|_\infty$, $l_1 = \|{}_0^C D_t^\alpha y^*\|_\infty$, $l_2 = \|{}_0^C D_t^\alpha\ {}_0^C D_t^\alpha y^*\|_\infty$. We have,
$Y^* = (t, y^*(t), {}_0^C D_t^\alpha\ {}_0^C D_t^\alpha y^*) \in I,\ \forall\ t \in \Theta$.
If $\delta > 0$ and $\|y - y^*\| < \delta$. Then $\|y - y^*\|_\infty < \delta$, and

$$\begin{aligned} \left\|{}_0^C D_t^\alpha\ {}_0^C D_t^\alpha y - {}_0^C D_t^\alpha\ {}_0^C D_t^\alpha y^*\right\|_\infty &= \left\|{}_0^C D_t^\alpha\ {}_0^C D_t^\alpha (y - y^*)\right\|_\infty \\ &= \left\|{}_0I_t^{1-\alpha}\, D\left({}_0^C D_t^\alpha (y - y^*)\right)\right\|_\infty \\ &= \left\|{}_0I_t^{1-\alpha}\, {}_0^C D_t^\alpha (y' - y'^*)\right\|_\infty . \end{aligned}$$

From the Lemma 1.6,

$$\begin{aligned} \left\|{}_0^C D_t^\alpha\ {}_0^C D_t^\alpha y - {}_0^C D_t^\alpha\ {}_0^C D_t^\alpha y^*\right\|_\infty &\le \frac{\|(y - y^*)''\|_\infty t^{2-2\alpha}}{\Gamma(3-2\alpha)} \\ &\le c\delta. \end{aligned} \tag{4.47}$$

Therefore, for sufficiently small δ, we have $Y = \big(t, y(t), {}_0^C D_t^\alpha\ {}_0^C D_t^\alpha y(t)\big) \in I$, $\forall\, t \in \Theta$. Let $T(Y) = {}_0^C D_t^\alpha\ {}_0^C D_t^\alpha y(t)$. Now T is the Lagrange function of the functional J, and T is continuously differentiable on the compact set I, which implies that T is uniformly continuous on I. Hence, for the small enough values of δ, for any given $\epsilon > 0$, $\big|T(Y) - T(Y^*)\big| < \epsilon$ and $\big|\ J(y) - J(y^*)\ \big| = \big|\int_0^1 [T(Y) - T(Y^*)]\, dt\big| < \epsilon$. □

Theorem 4.14. *Let us assume that* $y^*(t)$ *minimizes the functional* J *on* $B(\Theta)$*, and* $y_N(t)$ *minimizes the functional* J *on* $G^N(\Theta)$*. So,* $\lim_{N\to\infty} J[y_N(t)] = J[y(t)]$.

Proof. According to Lemma 4.13, for every $\epsilon > 0$, $\exists\ y(t) \in B(\Theta)$ such that $J(y(t)) \le J(y^*(t)) + \epsilon$. Now, from Lemmas 4.11 and 4.13, there exist a $\delta > 0$ and $M \in \mathbb{Z}^+$ such that when $N > M$, $\|y_N(t) - y(t)\| < \delta$, $\big|J(y_N(t)) - J(y(t))\big| \le \epsilon$, where $y_N(t) \in G^N(\Theta)$. Hence, we get $J(y^*(t)) \le J(y_N(t)) \le J(y(t)) + \epsilon \le J(y^*(t)) + 2\epsilon$.

Since, $\epsilon > 0$ is arbitrary, $\lim_{N\to\infty} J[y_N(t)] = J[y^*(t)]$ ([61]). □

4.6 Numerical Results and Discussions

The numerical scheme developed in Section 4.4 is applied to solve the variational form of FSLP defined by Eqs. (4.32) and (4.34). The convergence results of $y_N(t)$ obtained for a fixed value of $\alpha = 1$ and different N are shown in Figure 4.1. As N increases, the results become stable. Figure 4.2 shows the eigenfunction for various values of α. As α tends to 1, the numerical solution recovers to the analytical solution corresponding to the integer-order problem. For $\alpha = 1$, the problem is defined by Eqs. (4.13) and (4.14) becomes an integer-order problem whose eigenvalues are given by the equation

$$\cos(m)\cosh(m) - 1 = 0, m \in \mathbb{R}^+, \tag{4.48}$$

where, eigenvalues $\lambda_j = m_j^4, j \in \mathbb{N}$ and m_j are the roots of the Eq. (4.48).

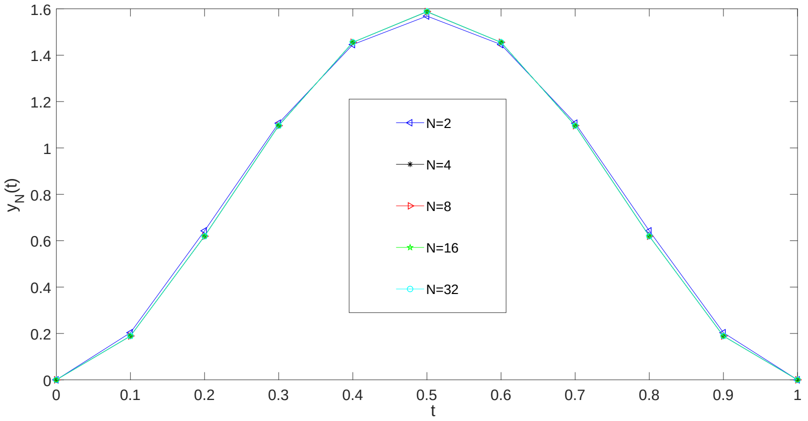

FIGURE 4.1: Plot for the approximated eigenfunction function $y_N(t)$ for fixed $\alpha = 1$ and various values of N.

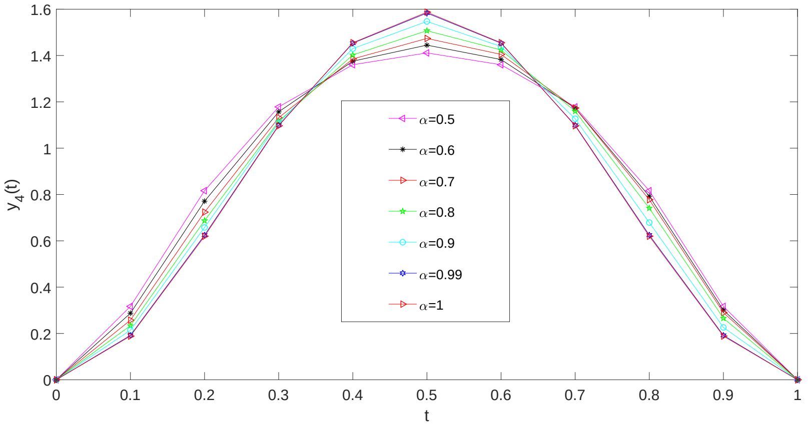

FIGURE 4.2: Plot for the approximated eigenfunction function $y_N(t)$ for fixed $N = 4$ and different values of α.

TABLE 4.1: Convergence of the first (minimum) eigenvalue λ_1

N/α	0.5	0.6	0.7	0.8	0.9	0.99	1
2	12	22.8937	44.8884	93.2075	208.9462	460.0651	504
4	10.8686	20.9553	41.7134	89.3506	207.4835	458.1676	500.5847
8	10.2376	19.7893	39.4487	86.0537	205.3466	458.0698	500.5639
16	10.0296	19.3428	38.4040	84.3429	204.5818	458.0620	500.5639
32	9.9812	19.1379	38.0171	84.0250	204.1358	458.0570	500.5639

TABLE 4.2: Absolute error calculation for fixed $\alpha = 1$ from the numerical solutions only.

t	$\lvert y_4(t) - y_2(t)\rvert$	$\lvert y_8(t) - y_4(t)\rvert$	$\lvert y_{16}(t) - y_8(t)\rvert$	$\lvert y_{32}(t) - y_{16}(t)\rvert$
0.1	1.460×10^{-2}	3.781×10^{-4}	7.250×10^{-8}	4.290×10^{-9}
0.2	2.260×10^{-2}	5.409×10^{-4}	1.048×10^{-7}	1.309×10^{-8}
0.3	1.010×10^{-2}	8.065×10^{-4}	6.150×10^{-8}	7.110×10^{-9}
0.4	9.400×10^{-3}	2.579×10^{-4}	7.350×10^{-8}	1.026×10^{-8}
0.5	1.840×10^{-2}	9.770×10^{-4}	1.559×10^{-7}	2.629×10^{-8}
0.6	9.400×10^{-3}	2.579×10^{-4}	7.350×10^{-8}	2.105×10^{-8}
0.7	1.010×10^{-2}	8.065×10^{-4}	6.150×10^{-8}	1.335×10^{-8}
0.8	2.260×10^{-2}	5.409×10^{-4}	1.048×10^{-7}	1.112×10^{-8}
0.9	1.460×10^{-2}	3.781×10^{-4}	7.250×10^{-8}	4.980×10^{-9}

Table 4.3: Numerical Convergence order (CO) and maximum absolute error (MAE) for the problem.

α	N	MAE	CO
0.6	8		
	16	1.74×10^{-3}	
	32	4.39×10^{-4}	1.987
0.9	8		
	16	1.85×10^{-4}	
	32	3.40×10^{-5}	2.444
1	8		
	16	1.56×10^{-7}	
	32	2.63×10^{-8}	2.568

Furthermore, the first (minimum) eigenvalue is shown in Table 4.1 at various fractional orders and different values of N. As N increases, the eigenvalues decrease. The first (minimum) eigenvalue for the integer-order problem is $\lambda_1 = 500.5639$. Table 4.1 shows that as N increases for $\alpha = 1$, the first (minimum) eigenvalue approaches the real value. Table 4.2 presents the absolute error of the fractional variational problem for fixed $\alpha = 1$. From Table 4.3, the maximum absolute error is 2.63×10^{-8} for $\alpha = 1$ and $N = 32$.

4.7 Conclusions

We have considered the higher-order FSLP under the BCs (4.14) for orders $\alpha \in (0, 1)$. We have demonstrated that the higher-order FSLP has infinite eigenvalues and corresponding unique eigenfunctions using the principles of FCVs. Further, it is shown that the first eigenvalue of FSLP is the minimizer of the functional. The numerical method is also presented to validate our theoretical findings. The convergence of the numerical method is discussed. The results are presented for various fractional order derivatives; as order α tends to 1, the eigenvalues recover to an analytical result of the corresponding integer-order problem. The method is easy to understand and can also be implemented for other problems.

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