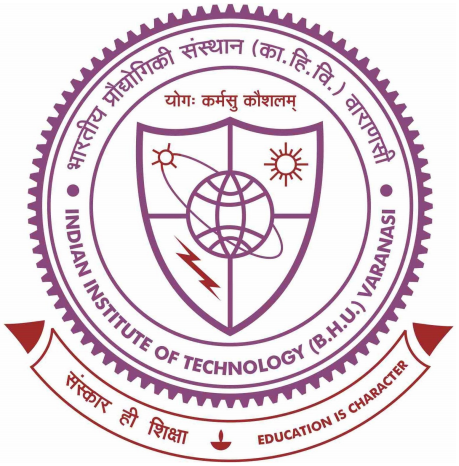
Approximation of Functions by Positive Linear Operators and their Applications in Integral Equations



Thesis submitted in partial fulfillment

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by

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Chapter 7
Conclusions and Future Remarks

This chapter concludes the thesis and presents some possible future work in the direction of the work done in this thesis.

7.1 Conclusion

This section serves as the conclusion of the entire thesis, summarizing the important findings and concluding remarks from the overall investigation.

Chapter 1 has introduced the fundamental concepts and literature related to the work presented in this thesis.

Chapter 2 has presented a study on the approximation properties of two Bernstein-type operators; one is about a new Kantorovich-type modification and another is about a Beta-type generalization. The error estimation using the modulus of continuity and Ditzian-Totik moduli of smoothness is presented for both operators. Also, the exact order of approximation is obtained using Voronovskaja as well as Grüss-Voronovskaja type results. At the end, a direct estimation of the error for the class of functions having derivatives of bounded variation is obtained.

Chapter 3 has proposed a Stancu-type integral generalization of a modified version of the Jain operators. The convergence is confirmed using a Korovkin-type result as

well as a weighted approximation result. This chapter has also explored a quantitative upper estimation for the error in approximation using a weighted modulus of continuity. Additionally, the error bound is also obtained using the usual modulus of continuity and for Lipschitz-class of functions.

Chapter 4 has investigated a Korovkin as well as an intermediate Voronovskaja-type theorem for a new Kantorovich-type operator in general probability measure space. Also, it contains the error of approximation of the operator using the 1st and 2nd-order modulus of continuity. At the end, a direct estimation of the error for a special class of functions, namely, the class of functions having derivatives of bounded variation is presented.

Chapter 5 has discussed the complex-type generalization of the $\alpha-$Bernstein-Durrmeyer operators. A recurrence relation to calculate the moments has also been established. Further a quantitative upper estimation for the error of approximation is also calculated. Towards the end of the chapter, the exact order of approximation of the newly defined operators as well as their derivatives is established using the Big-Θ notation with the help of the Voronovskaja-type result.

Chapter 6 has initiated solving Integral equations using the $\lambda-$Bernstein operators. The Volterra Integral Equations of both 1st and 2nd kind have been solved numerically using the $\lambda-$Bernstein approximation technique. The upper bound of the error is also estimated. At the end, the relevance of using the $\lambda-$Bernstein operators over the usual Bernstein operators is shown with some numerical examples, where for some test problems, some values of the shape parameter λ yields less error than the tradition Bernstein approximation technique.

7.2 Some Future Directions

The investigation of positive linear operators is an active field in mathematics, offering a wide range of future possibilities that span multiple areas of theoretical and applied sciences. Below are some of the potential directions and opportunities for future research:

- Future studies may concentrate on creating new types of positive linear operators that provide improved approximation properties, particularly in higher dimensions or for more complex function spaces. This also involves enhancing the error analysis and convergence rates associated with such operators.
- Expanding the theory to include multivariate functions and investigating the effective use of positive linear operators in this area is another direction. This is especially relevant for applications in image processing and machine learning, where multivariate data is frequently encountered.
- Further research can explore how positive linear operators can be used in discretization methods for numerical solutions of integral and differential equations, particularly in ensuring stability and accuracy. As done in the last chapter, we can use the $\lambda-$Bernstein approximation technique to solve other types of integral as well as integro-differential equations.
- In recent years, the study of exponential-type operators and Sampling Kantorovich type operators has become a popular area of research as they have a wide range of applications. Some of them can be seen in [24, 46, 47, 48, 49]. In the future, we can also work on these types of operators, their approximation properties as well as their order of approximation.

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