Chapter 3

Multiple Delayed Partial State Feedback Sliding Mode Control of Uncertain Nonlinear Systems

This chapter discusses the stabilization of uncertain SISO (Single Input Single Output) dynamical systems using sliding mode control with multiple delayed partial state feedback. It is shown that delayed partial state information can stabilize a large class of SISO uncertain dynamical systems. Artificial stabilizing delay is introduced in the partial state feedback to make the reduced sliding mode dynamics of a given system asymptotically stable. It is demonstrated that despite the order of dynamical systems, using the information of just three states together with their delayed terms can asymptotically stabilize the uncertain SISO dynamical system. A numerical example of a ball and wheel system is provided to demonstrate the proposed method's efficacy.

3.1 Introduction

In many physical systems, the time delay is a source of instability. But in some systems appearance of time-delay has stabilizing effect [14–17]. To understand the effect of stabilizing delay, consider a double integrator system

$$\begin{cases} \ddot{z}(t) &= u(t), \quad z(t) \in \mathbb{R}, \\ y(t) &= z_1(t). \end{cases}$$

This double integrator system cannot be stabilized by using non-delayed static output feedback. However, this system can be stabilizable using delayed static output feedback $u(t) = k_1 z_1(t) + k_2 z_1(t-h),\ h > 0,\ k_1 = -9.61, k_2 = 4.47, h = 0.43$. The concept of artificial delay is becoming useful and was applied to sliding mode control [12] and for delay-induced consensus in multi-agent systems [20]. A series of papers on stabilization of the systems using artificial delay have been published, e.g., [6, 18].

To stabilize the uncertain systems, we need a robust controller, which mitigates the disturbances/uncertainties. To the best of the author's knowledge, there is no straightforward method to stabilize an uncertain dynamical system when complete state information is not available. This chapter presents a new approach for stabilizing uncertain dynamical systems using sliding mode control with multiple delayed partial state feedback.

The attributes of the sliding mode approach are its inherent robustness, finite-time convergence, flexibility in gain tuning, and reduced-order dynamics. Also, the control design is straightforward compared to other robust control strategies existing in the literature [25–27]. In order to apply sliding mode control, three state information and its delayed terms are required to stabilize the uncertain systems. To design a sliding surface, explicit information of first and last states and their delayed states are used. However, for synthesizing a control signal, one extra intermediate state information and delayed terms are also required.

The motivation behind the chapter is that the stabilization problem of uncertain systems, complete state information is required. But in practice, all the state information is not available due to unavailability and expensive sensors. Therefore, an alternative technique is needed, which can stabilize the system. Also, robustness analysis is required.

The contribution of the chapter is as follows:

(i) The delayed sliding surface is designed and guaranteed its finite-time convergence.

(ii) A delayed partial state feedback sliding mode control is constructed.

(iii) Sufficient conditions have been provided for the stability analysis of the closed-loop system and ensured its asymptotic stability.

(iv) Robustness against matched perturbation is provided.

(v) An application of the proposed control is demonstrated and provided that the proposed control can also be utilized for designing an observer to reconstruct the missing states.

A brief outline of the chapter is as follows. Section 3.2 introduces the problem formulation. Section 3.3 provides the stability analysis. Section 3.4 presents an application of the proposed control approach. Section 3.5 presents a simulation results of the ball and wheel system. Finally, the chapter is summarised with Section 3.6.

3.2 Problem Formulation

Consider a dynamical system of the following form

$$\begin{cases} \dot{\Theta}(t) & = F(\Theta, \eta) + G(\Theta, \eta)v(t) \\ y(t) & = h(\Theta), \end{cases} \tag{3.1}$$

where $\Theta(t) \in \mathbb{D} \subset \mathbb{R}^n$ are the states of the system. $F : \mathbb{D} \times \mathbb{R} \longrightarrow \mathbb{R}^n$ and $G : \mathbb{D} \times \mathbb{R} \longrightarrow \mathbb{R}^n$ are locally lipschitz and known continuous function, $y(t) \in \mathbb{R}$ is the output signal, $v(t) \in \mathbb{R}$ is the control signal, $h : \mathbb{D} \longrightarrow \mathbb{R}$ and $\eta(t) \in \mathbb{R}$ represents the external matched perturbations/disturbances.

By considering a diffeomorphism [21,22], one can write (3.1) into the following normal form

$$\begin{cases} \dot{z}_i(t) & = z_{i+1}(t),\ i = 1, \cdots, n-1 \\ \dot{z}_n(t) & = f(\Theta, z) + g(\Theta, z)v(t) + d(t), \end{cases} \tag{3.2}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ are known smooth functions and $d(t)$ represents matched perturbations/disturbances, which is considered to be bounded and piecewise continuous in t, i.e. $|d(t)| < d_0,\ \forall t \geq 0$.

Applying $v(t) := \dfrac{1}{g(\Theta, z)}(u(t) - f(\Theta, z))$ to system (3.2) yields the following perturbed chain of integrators

$$\begin{cases} \dot{z}_i(t) & = z_{i+1}(t),\ i = 1, \cdots, n-1 \\ \dot{z}_n(t) & = u(t) + d(t), \end{cases} \tag{3.3}$$

where $u(t)$ is the virtual control input to be designed. The asymptotic stabilization of system (3.2) about the origin is equivalent to stabilization of system (3.3).

Consider a sliding surface with the partial state information with delayed states $z_n(t), z_1(t), z_1(t-h_i),\quad i = 0, 1, \cdots, n-2$ as:

$$s(t) := z_n(t) - \overline{K}_{n-1} z_1(t - h_{n-2}) - \overline{K}_{n-2} z_1(t - h_{n-3}) - \cdots - \overline{K}_1 z_1(t - h_0), \tag{3.4}$$

where $0 = h_0 < h_1 < .... < h_{n-2}$.

Remark 1 *Since we are using partial state feedback, we assume that the information of states z_1, z_2 and z_n is available to us. Indeed, this controller is not suitable for the case where the higher-order derivative is not available. But, one can find higher-order derivative terms with the help of a differentiator. Based on this fact, we have used the higher-order derivative term in the design process of the proposed approach.*

After differentiation of the sliding surface $s(t)$, we have

$$\begin{aligned} \dot{s}(t) &= \dot{z}_n(t) - \overline{K}_{n-1}\dot{z}_1(t-h_{n-2}) - \overline{K}_{n-2}\dot{z}_1(t-h_{n-3}) - \cdots - \overline{K}_1\dot{z}_1(t-h_0), \\ \dot{s}(t) &= u(t) + d(t) - \overline{K}_{n-1}z_2(t-h_{n-2}) - \overline{K}_{n-2}z_2(t-h_{n-3}) - \cdots - \overline{K}_1 z_2(t-h_0). \end{aligned} \tag{3.5}$$

Define, the delayed partial state feedback sliding mode control as

$$u(t) := -\lambda\text{sign}(s) + \overline{K}_{n-1}z_2(t-h_{n-2}) + \overline{K}_{n-2}z_2(t-h_{n-3}) + \cdots + \overline{K}_1 z_2(t-h_0). \tag{3.6}$$

Therefore, (3.5) becomes

$$\dot{s}(t) = -\lambda\text{sign}(s) + d(t), \quad \lambda > |d(t)|. \tag{3.7}$$

The closed-loop system can be represented as

$$\begin{cases} \dot{z}_i(t) &= z_{i+1}(t), \ i = 1, \cdots, n-2 \\ \dot{z}_{n-1}(t) &= s(t) + \overline{K}_{n-1}z_1(t-h_{n-2}) + \overline{K}_{n-2}z_1(t-h_{n-3}) + \cdots + \overline{K}_1 z_1(t-h_0) \\ \dot{s}(t) &= -\lambda\text{sign}(s) + d(t). \end{cases} \tag{3.8}$$

Consider a Lyapunov function $V = \dfrac{s^2(t)}{2}$ and tuning parameter $\lambda > |d(t)|$ of (3.7), then it is possible to show $s(t) = 0$ in finite-time [28]. At $s(t) = 0$, $\forall t \geq t_0$, system (3.8) can be written as

$$\begin{cases} \dot{z}_i(t) &= z_{i+1}(t), \ i = 1, \cdots, n-2 \\ \dot{z}_{n-1}(t) &= \psi(t), \end{cases} \tag{3.9}$$

where

$$\psi(t) = \overline{K}_{n-1}z_1(t-h_{n-2}) + \overline{K}_{n-2}z_1(t-h_{n-3}) + \cdots + \overline{K}_1 z_1(t-h_0). \tag{3.10}$$

The system (3.9) can also be written as

$$\dot{z}(t) = Az(t) + B\psi(t), \tag{3.11}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Here, the objective is to design the control $\psi(t)$ such that the system (3.9) becomes asymptotically stable. One can write (3.10) in the following form

$$\psi(t) = \sum_{\alpha=1}^{n-1} \overline{K}_\alpha z_1(t - h_{\alpha-1}). \tag{3.12}$$

From [6], we have

$$z_1(t - h_{\alpha-1}) = \sum_{\beta=1}^{n-1} \frac{1}{(\beta-1)!}(-h_{\alpha-1})^{\beta-1} z_\beta(t) + W_\alpha(z_{nt}), \quad \alpha = 2, \cdots, n-1, \tag{3.13}$$

where

$$W_\alpha(z_{nt}) = \frac{(-1)^n}{(n-1)!} \int_{t-h_{\alpha-1}}^{t} (s - t + h_i)^{n-1} z_n(s) ds. \tag{3.14}$$

Assume $h_0 = 0$. After substituting (3.12) into (3.11), the resulting closed-loop system with artificial delay is obtained as

$$\dot{z}(t) = Az(t) + \sum_{\alpha=1}^{n-1} B\overline{K}_\alpha z_1(t - h_{\alpha-1}). \tag{3.15}$$

From (3.13), we have

$$\sum_{\alpha=1}^{n-1} \overline{K}_\alpha z_1(t - h_{\alpha-1}) = \sum_{\beta=1}^{n-1} K_\beta z_\beta(t) + \sum_{\alpha=2}^{n-1} \overline{K}_\alpha W_\alpha(z_{nt}) \tag{3.16}$$

where

$$K_1 = \sum_{i=1}^{n-1} \overline{K}_i, \ K_j = \frac{(-1)^j}{j!} \sum_{\alpha=2}^{n-1} h_{(\alpha-1)}^j \overline{K}_\alpha, \ j = 1, \cdots, n-2. \tag{3.17}$$

From (3.17) with $\overline{K} = \begin{bmatrix} \overline{K}_1 & \overline{K}_2 & \cdots & \overline{K}_{n-1} \end{bmatrix}$, we obtain $K = \overline{K}E$,

$$E = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ I & -h_1 I & \frac{h_1^2}{2} I & \cdots & \frac{(-h_1)^{n-2}}{(n-2)!} I \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I & -h_{n-2} I & \frac{h_{n-2}^2}{2} I & \cdots & \frac{(-h_{n-2})^{n-2}}{(n-2)!} I \end{bmatrix}. \tag{3.18}$$

It is straightforward to see that matrix E is invertible due to presence of different delays in (3.18).

Substitution of (3.16) into (3.15), yields the following closed-loop system

$$\dot{z}(t) = Fz(t) + \sum_{\alpha=2}^{n-1} B\overline{K}_\alpha W_\alpha(z_{nt}),$$
$$F = A + BK = A + B\overline{K}E, \tag{3.19}$$

where E is defined earlier in (3.18).

3.3 Stability Analysis

In this section, we have presented the mathematical guaranty of the proposed approach. To fulfill this objective, we have considered Lyapunov-Krasovskii's functional and formulated the LMIs. The feasibility of LMIs shows the effectiveness of the proposed method.

Lemma 7 *(a) Suppose* $\overline{K} = \begin{bmatrix} \overline{K}_1 & \overline{K}_2 & \cdots & \overline{K}_{n-1} \end{bmatrix}$ *and a constant known delay* $0 = h_0 < h_1 < \cdots < h_{n-2}$ *such that the matrix* $F = A + B\overline{K}E$ *is Hurwitz. If there exist* $0 < P \in \mathbb{R}^{n\times n}$ *and* $0 < R_i \in \mathbb{R}$, $i = 2, \cdots, n-1$, *that satisfies the LMI*

$$\chi = \begin{bmatrix} \varphi & \phi & \cdots & \phi & F^T[0_{1\times(n-1)}\ I]^T\hat{R} \\ * & -(n!)^2R_2 & \cdots & 0 & B^T\hat{R} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & -(n!)^2R_{n-1} & B^T\hat{R} \\ * & * & \cdots & * & -\hat{R} \end{bmatrix} < 0 \tag{3.20}$$

where $\varphi = PF^T + FP \in \mathbb{R}^{n\times n}$, $\phi = PB \in \mathbb{R}^{n\times 1}$,

$$\hat{R} = \sum_{\alpha=1}^{n-1} h_{(\alpha-1)}^{2n}\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha \in \mathbb{R}, \tag{3.21}$$

then system (3.16) is asymptotically stable.

(b) Consider $K \in \mathbb{R}^{1\times(n-1)}$ *such that* $F = A + BK$ *is Hurwitz,* $\overline{K} = E^{-1}K$ *and* $h_\alpha = \alpha h$ $(\alpha = 0, \cdots, n-2)$ *with some* $h > 0$. *Therefore LMI (3.20) is always feasible for small enough* h. *In other words, we can say (3.16) is stabilizable by the delay feedback (3.12) with small enough* $h_\alpha = \alpha h$.

Proof: (a) Assume $V_1(z) = z^T(t)Pz(t)(P > 0)$. Then

$$\dot{V}_1(z) = 2z^T(t)P\left(Fz(t) + \sum_{\alpha=2}^{n-1} B\overline{K}_\alpha W_\alpha(z_{nt})\right). \tag{3.22}$$

To eliminate the term $W_\alpha(z_{nt})$ from (3.22), assume the functional

$$V_2(z_t) = \sum_{\alpha=2}^{n-1} h_{\alpha-1}^n \int_{t-h_{\alpha-1}}^{t} (s - t + h_{\alpha-1})^n z_n^T(s)\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha z_n(s)ds,$$

where $R_\alpha > 0$.

Therefore,

$$\dot{V}_2(z_t) = \sum_{\alpha=2}^{n-1} \Big\{ h_{\alpha-1}^{2n} z_n^T(t)\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha z_n(t) \tag{3.23}$$

$$- nh_{\alpha-1}^n \int_{t-h_{\alpha-1}}^{t} (s - t + h_{\alpha-1})^{n-1} z_n^T(s)\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha z_n(s)ds \Big\}. \tag{3.24}$$

From Jensen's inequality (2.20) for $W_\alpha(z_{nt})$, we get

$$W_\alpha^T(z_{nt})\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha W_\alpha(z_{nt}) \le \frac{h_{\alpha-1}^n}{(n-1)!n!} \int_{t-h_{\alpha-1}}^{t} (s - t + h_{\alpha-1})^{n-1} z_n^T(s)\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha z_n(s)ds$$

or

$$-nh_{\alpha-1}^n \int_{t-h_{\alpha-1}}^{t} (s - t + h_{\alpha-1})^{n-1} z_n^T(s)\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha z_n(s)ds \le -(n!)^2 W_\alpha^T(z_{nt})\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha W_\alpha(z_{nt}). \tag{3.25}$$

From (3.25), (3.23) and (3.21), one can write

$$\dot{V}_2(z_t) \le z_n^T(t)\hat{R}_\alpha z_n(t) - \sum_{\alpha=2}^{n-1} (n!)^2 W_\alpha^T(z_{nt})\overline{K}_\alpha^T R_\alpha \overline{K}_\alpha W_\alpha(z_{nt}). \tag{3.26}$$

Now, consider a Lyapunov functional $V(z_t) = V_1(z) + V_2(z_t)$. Using (3.22) and (3.26) we get

$$\begin{aligned}\dot{V}(z_t) &\le z^T(t)(PF^T + FP)z(t) + z_n^T(t)\hat{R}z_n(t) + 2z^T(t)\phi\sum_{\alpha=2}^{n-1} \overline{K}_\alpha W_\alpha(z_{nt}) \\ &\quad - \sum_{\alpha=2}^{n-1}(n!)^2 W_\alpha^T(z_{nt})R_\alpha W_\alpha(z_{nt}) \\ &= \gamma'(t)\Xi\gamma(t) + z_n^T(t)\hat{R}z_n(t).\end{aligned} \tag{3.27}$$

Here, Ξ coincides with χ, where the last row and the last column are cancelled and

$$\gamma(t) = col\begin{bmatrix} z(t) & \overline{K}_2 W_2(z_{nt}) & \cdots & \overline{K}_{n-1} W_{n-1}(z_{nt}) \end{bmatrix}.$$

From (3.19),

$$z_n(t) = \begin{bmatrix} 0_{1\times(n-1)} & 1 \end{bmatrix} Fz(t) + \sum_{\alpha=1}^{n-1} B\overline{K}_\alpha W_\alpha(z_n t), \tag{3.28}$$

substituting (3.28) into the last term of (3.27) and applying Schur complement, we obtain that $\frac{d}{dt}V(z_t) \leq -c|z(t)|^2$ for some $c > 0$. If $\chi < 0$ then (3.19) is asymptotically stable. Hence, the proof of (a) is over.

(b) From [6], we know $E^{-1} = O(h^{-n+2}) \implies \overline{K} = O(h^{-n+2})$. Assume $P > 0$ is a solution of Lyapunov equation $F^T P + PF = -I_n$. Consider $R_\alpha = h^{-1}I$. Therefore, $\hat{R} = O(h^{-1})$. By applying Schur complement on $\chi < 0$, we obtain $F^T P + PF + O(h) < 0$ that holds for small enough h. The proof of (b) is completed.

Remark 2 *From the above lemma, it can be observed that the straightforward advantage of the proposed method begins with the fourth order systems. In other words, the advantage of the proposed approach will appear for higher-order systems.*

3.4 Application

3.4.1 Reconstruction of Missing States (Observer Design)

The proposed control can also be utilized for the observer design. It can estimate the unknown states of the system.

Consider a chain of integrators dynamics

$$\begin{cases} \dot{z}_i(t) &= z_{i+1}(t), \; i = 1, \cdots, n-1 \\ \dot{z}_n(t) &= -\lambda \text{sign}(z_n(t) - \overline{K}_1 z_1(t-h_0) - \cdots - \overline{K}_{n-1} z_1(t-h_{n-2})) + \phi(t) + d(t), \end{cases} \tag{3.29}$$

where

$$\begin{aligned} \phi(t) &:= \phi(z_2(t-h_0), \cdots, z_2(t-h_{n-2})) \\ &:= \overline{K}_1 z_2(t-h_0) + \cdots + \overline{K}_{n-1} z_2(t-h_{n-2}). \end{aligned}$$

The above represents the closed-loop system after substituting the sliding mode control (3.6). It is essential to mention here that (3.29) depicts an autonomous time-delay system, which can be used for designing an observer also. In the next part of this note, we will demonstrate the constructive procedure to recover missing information based on autonomous system (3.29).

Assumption 1 *Only $z_1(t)$ with its multiple delayed terms and $z_n(t)$ state information is available.*

Case A:

$$\begin{cases} \dot{z}_i(t) &= z_{i+1}(t), \ i = 1, \cdots, n-1 \\ \dot{z}_n(t) &= -\lambda sign(z_n(t) - \overline{K}_1 z_1(t-h_0) - \cdots - \overline{K}_{n-1} z_1(t-h_{n-2})) + \phi(t) + d(t), \end{cases} \tag{3.30}$$

where $\lambda > |d(t)| + |\phi(t)|$. The above inequality is valid only if all the states $(z_1(t), \cdots, z_n(t))$ lie in the compact (closed and bounded) set. Inequality $\lambda > |d(t)| + |\phi(t)|$ is always satisfied for large value of λ. Therefore, from the above discussion all the states $(z_1(t), \cdots, z_n(t))$ will converge to zero asymptotically.

Case B: We can construct an observer as follows

$$\begin{cases} \dot{\hat{z}}_i(t) &= \hat{z}_{i+1}(t), \ i = 1, \cdots, n-1 \\ \dot{\hat{z}}_n(t) &= u(t) + X. \end{cases} \tag{3.31}$$

Consider the error as $e_i(t) := z_i(t) - \hat{z}_i(t)$. Then using (3.3) and (3.31), the chain of integrators form of error can be represented as

$$\begin{cases} \dot{e}_1(t) &= e_{i+1}(t), \ i = 1, \cdots, n-1 \\ \dot{e}_n(t) &= -X + d(t), \end{cases} \tag{3.32}$$

where $X = \lambda sign(e_n(t) - \overline{K}_1 e_1(t-h_0) - \cdots - \overline{K}_{n-1} e_1(t-h_{n-2}))$. For $\lambda > |d(t)|$ the error states $(e_1(t), \cdots, e_n(t))$ will asymptotically converge to the origin, which means that estimation of unkown states, such as, $(z_3(t), \cdots, z_{n-1}(t))$ are obtained.

Remark 3 *Inequality $\lambda > |\phi(t)| + |d(t)|$ can be relaxed to $\lambda > |d(t)|$ if information of z_2 is also available. In this case, X is modified by*

$$X = \lambda sign(e_n(t) - \overline{K}_1 e_1(t-h_0) - \cdots - \overline{K}_{n-1} e_1(t-h_{n-2})) - \phi(t).$$

3.5 Simulation Results

Consider the ball and wheel system [23], and the dynamics of which can be written in the form (3.1) as:

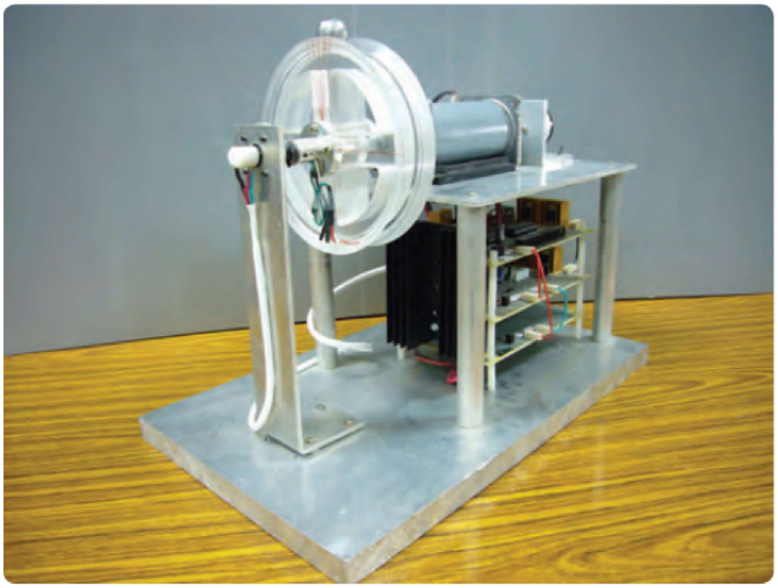

Figure 3.1: Ball and Wheel System [23]

$$\begin{cases} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= \alpha z_4(t) + \beta \text{sin} z_1(t) + \gamma(u(t) + \eta(t)) \\ \dot{z}_3(t) &= z_4(t) \\ \dot{z}_4(t) &= l z_4(t) + m \text{sin} z_1(t) + n(u(t) + \eta(t)), \end{cases} \tag{3.33}$$

where $z_1(t)$ and $z_2(t)$ are the angular displacement and the angular velocity of the ball, respectviely, $z_3(t)$ and $z_4(t)$ are the angular displacement and the angular velocity of the wheel, respectviely, $\eta(t) = 0.5\text{sin}(3t)$ and $\alpha, \beta, \gamma, l, m, n$ are defined by

$$\alpha = -\frac{2r_w k_m^2}{R_a(7I_w + 2r_w^2 m_b)(r_b + r_w)}, \ \beta = \frac{g(5I_w + 2r_w^2 m_b)}{(7I_w + 2r_w^2 m_b)(r_b + r_w),}$$
$$\gamma = \frac{2r_w k_m}{R_a(7I_w + 2r_w^2 m_b)(r_b + r_w)}, \ l = -\frac{7k_m^2}{R_a(7I_w + 2r_w^2 m_b),}$$
$$m = \frac{2g r_w m_b}{(7I_w + 2r_w^2 m_b)}, \ n = \frac{7k_m}{R_a(7I_w + 2r_w^2 m_b)}.$$

Note that $\alpha n = \gamma l$.

Table 3.1: Design parameters for ball and wheel system

Parameter	Definition	Value	SI Unit
I_w	Moment of inertia of the wheel	1.71×10^{-3}	$kg - m^2$
r_w	Radius of the wheel	0.075	m
m_b	Mass of the ball	0.042	kg
r_b	Radius of the ball	0.011	m
R_a	Motor armature resistance	0.6558	Ω
K_m	Motor constant	0.0662	N-m/A
g	Gravity acceleration	9.81	m/s^2

By diffeomorphism, the following variables are defined

$$\begin{aligned}
x_1(t) &= nz_1(t) - \gamma z_3(t) \\
x_2(t) &= nz_2(t) - \gamma z_4(t) \\
x_3(t) &= (\beta n - \gamma m)\sin z_1(t) \\
x_4(t) &= (\beta n - \gamma m)z_2(t)\cos z_1(t).
\end{aligned}$$

Then, system (3.33) is equivalently represented as

$$\begin{cases}
\dot{x}_1(t) & = x_2(t) \\
\dot{x}_2(t) & = x_3(t) \\
\dot{x}_3(t) & = x_4(t) \\
\dot{x}_4(t) & = f(x) + g(x)(u(t) + \eta(t))
\end{cases} \tag{3.34}$$

where

$$\begin{aligned}
x(t) &= \mathrm{col}\left\{x_1(t), x_2(t), x_3(t), x_4(t)\right\} \\
f(x) &= -\frac{x_4^2 x_3}{(\beta n - \gamma m)^2 \cos^2\left\{\sin^{-1}\left(\frac{x_3}{\beta n - \gamma m}\right)\right\}} - (\beta n - \gamma m)\cos\left\{\sin^{-1}\left(\frac{x_3}{\beta n - \gamma m}\right)\right\} \\
&\quad \times \left[\alpha\left(-\frac{x_2(t)}{\gamma} + \frac{n x_4}{\gamma(\beta n - \gamma m)\cos\left\{\sin^{-1}\left(\frac{x_3}{\beta n - \gamma m}\right)\right\}}\right) + \frac{\beta x_3}{(\beta n - \gamma m)}\right] \\
g(x) &= \frac{1}{\gamma(en - \gamma m)\cos\left\{\sin^{-1}\left(\frac{x_3}{\beta n - \gamma m}\right)\right\}}.
\end{aligned}$$

Applying $u(t) = \frac{1}{g(x)}\left[-f(x) + \nu(t)\right]$ to system (3.34) yields the following perturbed fourth-order integrator (3.35).

$$\dot{x}(t) = Ax(t) + B(\nu(t) + d_1(t)) \tag{3.35}$$

where $\nu(t)$ is the virtual control input to be designed, $d_1(t) = d(t)g(x)$, and matrices A, B are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that, the control $u(t)$ is well defined for $-\pi/2 < z_1(t) < \pi/2$. To stabilize the system (3.35), it is assumed that just three state information with its delayed terms is available. Consider the sliding surface and the control input,

$$\begin{aligned} s(t) &= x_4(t) - \overline{K}_1 x_1(t-h_0) - \overline{K}_2 x_1(t-h_1) - \overline{K}_3 x_1(t-h_2) \\ \nu(t) &= \overline{K}_1 x_2(t-h_0) + \overline{K}_2 x_2(t-h_1) + \overline{K}_3 x_2(t-h_2) - \lambda \mathrm{sign}(s). \end{aligned} \tag{3.36}$$

Select, $\lambda > |d_1(t)|$, $|d_1(t)| = 32739$, $\lambda = 32750$. To observe the performance of the control (3.36), ball and wheel system is simulated in Matlab/Simulink based on the parameters shown in Table 3.1. The controller gain are selected as $\overline{K}_1 = -1.15$, $\overline{K}_2 = 2.1$, $\overline{K}_3 = -1$ and delay $h_0 = 0, h = h_1 = 1,\ h_2 = 2h$. The initial conditions are set to $\begin{bmatrix} -0.08 & 0 & 0.08 & 0 \end{bmatrix}$.

To show the advantage of the proposed method, we have compared the proposed results with that obtained in [6]. The control law stated in [6] for the system (3.35) is

$$\nu(t) = \overline{K}_1 x_1(t-h_0) + \overline{K}_2 x_1(t-h_1) + \overline{K}_3 x_1(t-h_2) + \overline{K}_4 x_1(t-h_3), \tag{3.37}$$

where $\overline{K}_1 = -0.0208, \overline{K}_2 = -0.3200, \overline{K}_3 = -1.180, \overline{K}_4 = -0.7$, and $h_0 = 0, h = h_1 = 0.0873, h_2 = 2h, h_3 = 3h$.

The simulation results of the angular position and the ball angular velocity response are shown at the top of Fig. 3.2. The angular position and the angular velocity response are given at the bottom of Fig. 3.2. Fig. 3.3 represents the control input of the ball and wheel system. It can be seen from the simulation results that all the states asymptotically converge to zero, which is not in the case of [6], and also the control input is taking smaller values initially for the proposed controller.

The chattering effect in control input $u(t)$ is due to the sliding mode controls discontinuous nature. In summary, the simulation results confirm that the results generated by the proposed approach are more practical than [6] for the stabilization of the system. Another advantage of the proposed controller (3.37) is that it requires lesser gains and lesser delayed data as compared to the controller (3.37).

3.6 Summary

A multiple delayed partial state feedback-based sliding mode control has been proposed. The proposed method is attractive due to its non-complex nature and straightforward implementation. However, simple and practical approaches for the robust stability analysis of uncertain dynamical systems are not available in the literature. The proposed method bridges these gaps by introducing partially state feedback sliding mode control with multiple delays. A robust stability analysis of the closed-loop system is provided. The proposed method's effectiveness is evaluated through the simulation results of the ball and wheel system.

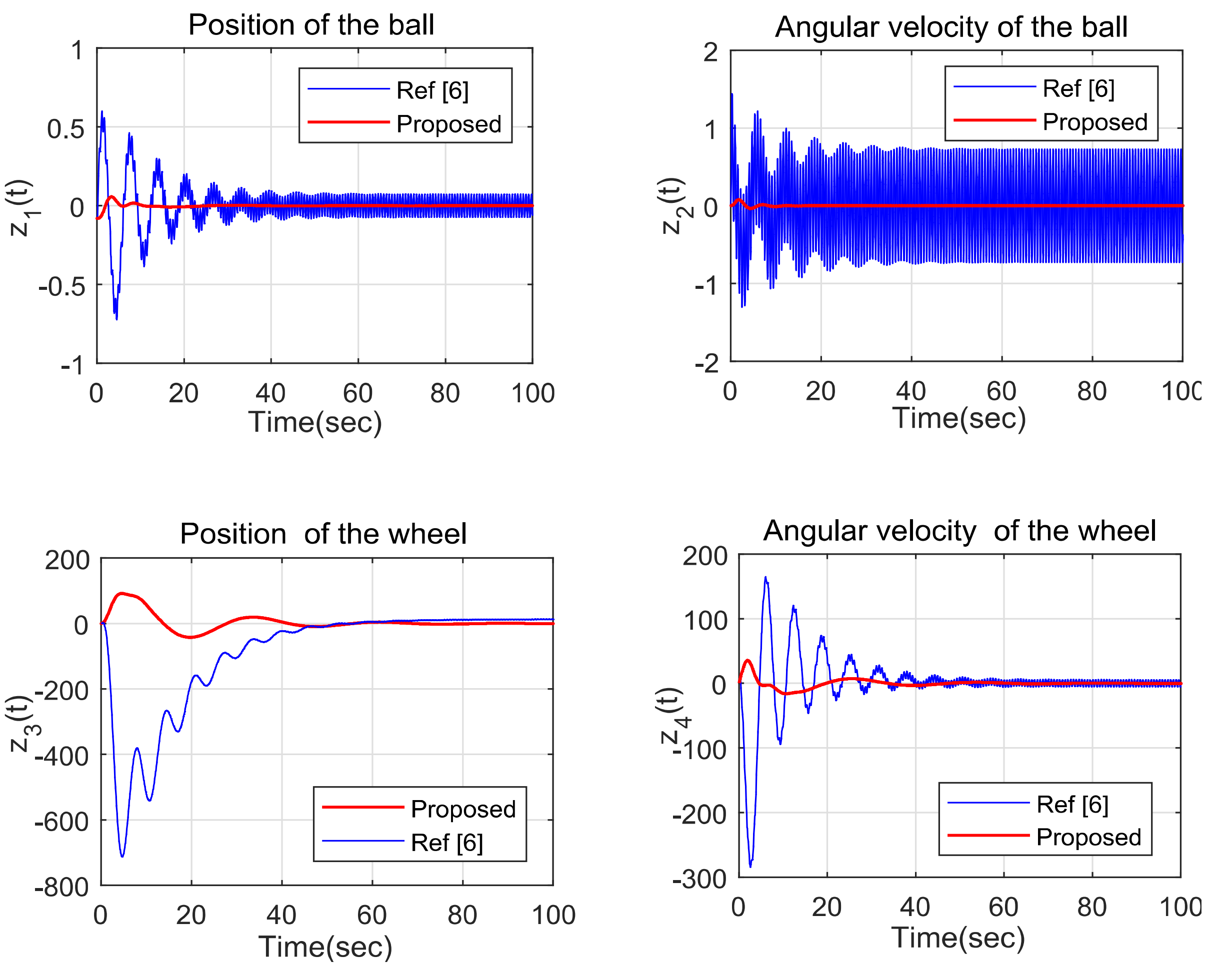

Figure 3.2: State trajectories of the ball and wheel system

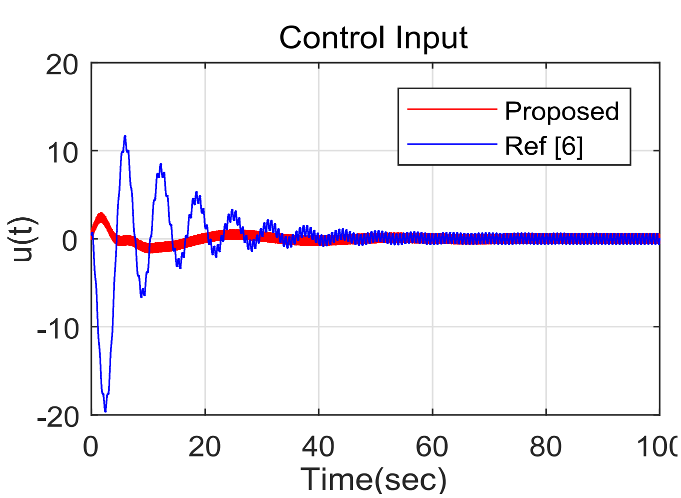

Figure 3.3: Control input of the ball and wheel system