Chapter 4

An asymptotic expansion of the constant term of certain automorphic function

In this chapter, we study the asymptotic behaviour of certain Lambert series associated to normalized Hecke eigenforms over $SL_2(\mathbb{Z})$.

4.1 Introduction

Recall that the Zagier's conjecture (mentioned in the introduction of Chapter 2) proposes a connection between the Lambert series associated to Ramanujan tau function $\tau(n)$ and non-trivial zeros of the Riemann zeta function $\zeta(s)$. This was proved by Hafner and Stopple [26] by studying certain heat kernel associated to Ramanujan tau function and utilizing Rankin-Selberg method for the automorphic functions which are of rapid decay. It is well-known that the Ramanujan delta function $\Delta(\tau) = \sum_{n\geq 1} \tau(n)e(n\tau)$ is a normalized Hecke eigenform of weight 12 over $SL_2(\mathbb{Z})$. Recently, Chakraborty, Kanemitsu and Maji [14] extended the result of Hafner and Stopple [26] to normalized Hecke eigenforms and studied the constant term of the automorphic form $v^k|f(\tau)|^2 = v^k f(\tau)\overline{f(\tau)}$, i.e., the following Lambert series:

$$v^k \sum_{n=1}^{\infty} |a_f(n)|^2 \exp\left(-4\pi n v\right), \tag{4.1}$$

where $a_f(n)$ denotes the n-th Fourier coefficient of a normalized Hecke eigenform f of weight k over $SL_2(\mathbb{Z})$. In this chapter we extend their result for two normalized Hecke eigenforms of different weights.

4.2 Statement of results

In this section, we state the main results of this chapter. Before stating the main result, we provide necessary backgrounds which are essential to state the main result of this chapter. We first associate certain Dirichlet series to normalized Hecke eigenforms. Let f and g be two normalized Hecke eigenforms for the group $SL_2\left(\mathbb{Z}\right)$ of weight k_1 and k_2 respectively, with their Fourier series expansion given by

$$f\left(\tau\right)=\sum_{n=1}^{\infty}a_f\left(n\right)e(n\tau)\ , \qquad g\left(\tau\right)=\sum_{n=1}^{\infty}a_g\left(n\right)e(n\tau). \tag{4.2}$$

Then the Rankin-Selberg type Dirichlet series associated to f and g is defined [49, p. 144] as

$$L\left(f\otimes g,s\right):=\zeta(2s)\sum_{n=1}^{\infty}\frac{a_f\left(n\right)\overline{a_g\left(n\right)}}{n^{s+\frac{k_1+k_2}{2}-1}}. \tag{4.3}$$

The series in (4.3) is absolutely convergent for $\Re(s)>1$. Next, we define an arithmetical function $A_{f,g}(n)$ satisfying the generating function

$$\frac{L(f\otimes g,s)}{\zeta(2s-1)}=\sum_{n=1}^{\infty}\frac{A_{f,g}(n)}{n^s}. \tag{4.4}$$

The above series is absolutely and uniformly convergent for $\Re(s)>1$. As previously noted in the introduction, our focus lies in examining specific Lambert series associated to normalized Hecke eigenforms. More precisely, for $v>0$, we study the constant term of the function $f(\tau)\overline{g(\tau)}$, which is given by the following Lambert series:

$$\sum_{n=1}^{\infty} a_f(n)\,\overline{a_g(n)}\exp(-4\pi n v), \tag{4.5}$$

where $a_f(n)$ and $a_g(n)$ are defined as in (4.2).

Theorem 4.1. *Let f and g be two normalized Hecke eigenforms for $SL_2(\mathbb{Z})$ of weight k_1 and k_2, respectively with $k_1 \leq k_2$ and with their Fourier series expansions given by (4.2). Then under the assumption of the simplicity hypothesis of the non-trivial zeros of $\zeta(s)$, for $v > 0$ we have*

$$\begin{aligned}
&(4\pi v)^{\frac{k_1+k_2}{2}} \sum_{n=1}^{\infty} a_f(n)\,\overline{a_g(n)}\exp(-4\pi n v)\\
=&\sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2}+\frac{k_1+k_2}{2}-1\right) L\left(f\otimes g, \frac{\rho}{2}\right)}{\zeta'(\rho)\,(4\pi v)^{\frac{\rho}{2}-1}} + 4\sqrt{\pi}\sum_{n=1}^{\infty} A_{f,g}(n)\frac{(k_1-k_2)\,\Gamma\left(\frac{k_1+k_2}{2}\right)}{8\pi^2 n} + R\\
&+4\sqrt{\pi}\sum_{n=1}^{\infty} A_{f,g}(n)\left(\frac{4\pi n}{v}\right)^{\frac{k_1+k_2}{2}-1} G_{2,3}^{2,1}\left(\begin{array}{c} 1-\frac{k_1+k_2}{2}, \frac{1}{2}-\frac{k_1+k_2}{2} \\ 0, 1-k_1, 1-k_2 \end{array} \middle| \frac{4\pi n}{v}\right),
\end{aligned}$$

where the sum over ρ runs through non-trivial zeros of $\zeta(s)$ satisfying the bracketing condition specified in (2.3) and

$$R = \begin{cases} \frac{3}{\pi}(4\pi)^k \langle f, f\rangle & \text{if} \quad f = g, k = k_1 = k_2, \\ 0 & \text{if} \quad f \neq g. \end{cases}$$

Next, we state a few immediate corollaries of the Theorem 4.1. Letting $f = g$ in the above Theorem we immediately obtain the identity of Chakraborty et. al. [14, Theorem 1.1].

Corollary 4.1. *Let f be a normalized Hecke eigenform for $SL_2(\mathbb{Z})$ of weight k with the Fourier series expansion $f(\tau) = \sum\limits_{n=1}^{\infty} a_f(n) e(n\tau)$. Then under the assumption of the simplicity hypothesis of the non-trivial zeros of $\zeta(s)$ and (2.3), for $v > 0$, we have*

$$(4\pi v)^k \sum_{n=1}^{\infty} a_f^2(n) \exp(-4\pi n v) = \sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2}+k-1\right) L\left(f \otimes f, \frac{\rho}{2}\right)}{\zeta'(\rho)(4\pi v)^{\frac{\rho}{2}-1}} + \frac{3(4\pi)^k}{\pi}\langle f, f\rangle$$
$$+ 4\sqrt{\pi} \sum_{n=1}^{\infty} A_{f,f}(n) \left(\frac{4\pi n}{v}\right)^{k-1} \exp\left(-\frac{4n\pi}{v}\right) U\left(-\frac{1}{2}, k; \frac{4n\pi}{v}\right),$$

where $U(a, b; z)$ denotes the confluent hypergeometric function of the second kind.

Letting $v \to 0^+$, we obtain the following asymptotic result which corroborates with Zagier's prediction (2.13).

Corollary 4.2. *Let us assume the given conditions as in Corollary 4.1. For $v \to 0^+$, we have*

$$v^k \sum_{n=1}^{\infty} a_f^2(n) \exp(-4\pi n v) \sim \frac{3}{\pi}\langle f, f\rangle + \sum_{\rho} v^{1-\frac{\rho}{2}} A_{\rho}, \tag{4.6}$$

where $A_\rho = \frac{\Gamma\left(\frac{\rho}{2}+k-1\right) L\left(f\otimes f, \frac{\rho}{2}\right)}{\zeta'(\rho)(4\pi)^{\frac{\rho}{2}-1+k}}$ is some complex number depends on ρ.

In particular, letting $f(z) = \Delta(z)$ in Corollary 4.1, we obtain the below result of Hafner and Stopple [26].

Corollary 4.3. *For $v > 0$, we have*

$$(4\pi v)^{12} \sum_{n=1}^{\infty} \tau^2(n) \exp(-4\pi n v) = \sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2}+11\right) L\left(\Delta \otimes \Delta, \frac{\rho}{2}\right)}{\zeta'(\rho)(4\pi v)^{\frac{\rho}{2}-1}} + \frac{3}{\pi}(4\pi)^{12}\langle \Delta, \Delta\rangle$$
$$4\sqrt{\pi} \sum_{n=1}^{\infty} A_{\Delta,\Delta}(n) \left(\frac{4\pi n}{v}\right)^{11} \exp\left(-\frac{4n\pi}{v}\right) U\left(-\frac{1}{2}, 12; \frac{4n\pi}{v}\right).$$

4.3 Preparatory results

In this section, we gather several key results that will be crucial in the proof of our main result. First we complete the Dirichlet series given in (4.3) as follows

$$\Lambda\left(f\otimes g,s\right):=(2\pi)^{-2s}\,\Gamma\left(s-\frac{k_1-k_2}{2}\right)\Gamma\left(s+\frac{k_1+k_2}{2}-1\right)L\left(f\otimes g,s\right). \tag{4.7}$$

Next two results are special cases of results of [49].

Theorem 4.2. *[49, p. 144, Theorem 2.2] Let f and g be normalized Hecke eigenforms of weight k_1 and k_2 (with $k_1 \leq k_2$) for the group $SL_2(\mathbb{Z})$. Then $\Lambda\left(f\otimes g,s\right)$ can be continued to the whole s-plane which is entire if $f \neq g$ and analytic except for simple poles at $s=1$ and $s=0$ if $f=g$. Moreover, $\Lambda\left(f\otimes g,s\right)$ satisfies the functional equation*

$$\Lambda\left(f\otimes g,s\right)=\Lambda\left(\bar{f}\otimes\bar{g},1-s\right)=\Lambda\left(f\otimes g,1-s\right). \tag{4.8}$$

Theorem 4.3. *[49, p. 148, Theorem 3.2] The convoluted L-series $L(f\otimes g,s)$ is entire if $f\neq g$, and analytic except for a simple pole at $s=1$ with residue $\frac{4^k\pi^{k+1}}{2\Gamma(k)}\langle f,g\rangle$ if $f=g$ (so that $k=k_1=k_2$).*

4.4 Proof of results

In this section, we provide a proof of the Theorem 4.1 and corollaries 4.1, 4.2 and 4.3.

Proof of Theorem 4.1. Using the definition of Mellin transform of $\Gamma(s)$, we see that for $\Re(s) > \frac{k_1+k_2}{2}$,

$$\Gamma(s)\sum_{n=1}^{\infty}\frac{a_f(n)\,\overline{a_g(n)}}{n^{s+\frac{k_1+k_2}{2}-1}} = \int_0^{\infty}\sum_{n=1}^{\infty}a_f(n)\overline{a_g(n)}\exp(-nv)v^{s-1}dv. \tag{4.9}$$

Using the inverse Mellin transform in (4.9), we obtain

$$\sum_{n=1}^{\infty}a_f(n)\overline{a_g(n)}\exp(-4\pi nv) = \frac{1}{2\pi i}\int_{(c)}\Gamma(s)\sum_{n\geq 1}\frac{a_f(n)\overline{a_g(n)}}{n^s}(4\pi v)^{-s}ds, \tag{4.10}$$

where c is a positive constant with $c > \frac{k_1+k_2}{2}$. Throughout the chapter, the symbol (c) denotes the line integral $c - i\infty$ to $c + i\infty$. Replace s by $s + \frac{k_1+k_2}{2} - 1$ in (4.10) to obtain

$$(4\pi v)^{\frac{k_1+k_2}{2}-1}\sum_{n=1}^{\infty}a_f(n)\overline{a_g(n)}\exp(-4\pi nv) = \frac{1}{2\pi i}\int_{(c')}\frac{\Gamma\left(s+\frac{k_1+k_2}{2}-1\right)L(f\otimes g,s)}{\zeta(2s)(4\pi v)^s}ds, \tag{4.11}$$

where $c' > 1$. Now recall Theorem 4.2 which states that the series $L(f\otimes g, s)$ can be extended to an entire function for $f \neq g$ and has simple poles at $s = 1$ for $f = g$. Further the poles of the integrand given in (4.11) are contributed by the non-trivial zeros of $\zeta(2s)$ for $0 < s < \frac{1}{2}$ inside the contour $\mathcal{C}$ consisting of the end points $c' - iT, c' + iT, d' + iT, d' - iT$ for $d' \in (-1, 0)$ and T is some large positive constant. Now Cauchy's residue theorem yields

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma\left(s+\frac{k_1+k_2}{2}-1\right)L\left(f\otimes g,s\right)\left(4\pi v\right)^{-s}}{\zeta(2s)}ds = R_1 + \sum_{|\Im(\rho)|<T}R_\rho, \tag{4.12}$$

$$\text{where } R_1 = \begin{cases}\frac{3}{\pi}\frac{(4\pi)^{k-1}}{v}\langle f,f\rangle & \text{if} f = g, k = k_1 = k_2,\\ 0 & \text{if } f \neq g.\end{cases} \tag{4.13}$$

Here the term R_1 denotes the residual term corresponding to simple pole at $s = 1$ and R_ρ denotes the residual term at the non-trivial zeros ρ of $\zeta(s)$ on the critical strip $0 < \Re(s) < \frac{1}{2}$ with $|\Im(\rho)| < T$. Here, we have used that $\zeta(2) = \pi^2/6$. Now assuming the simplicity hypothesis, which states that the non-trivial zeros of the Riemann zeta function $\zeta(s)$ are simple, leads to the following evaluation of R_ρ :

$$R_\rho = \frac{\Gamma\left(\frac{\rho}{2} + \frac{k_1+k_2}{2} - 1\right) L(f \otimes g, \frac{\rho}{2})(4\pi v)^{\frac{-\rho}{2}}}{\zeta'(\rho)}. \tag{4.14}$$

By taking $T \to \infty$ in (4.12) and using Stirling's bound (1.5) for $\Gamma(s)$ along with Lemma 1.2, we see that both horizontal integrals vanish. Finally, using (4.12) in (4.11), we obtain

$$\begin{aligned} &(4\pi v)^{\frac{k_1+k_2}{2}-1} \sum_{n=1}^{\infty} a_f(n)\overline{a_g(n)} \exp(-4\pi n v) \\ &= \frac{1}{2\pi i}\int_{(d')} \frac{\Gamma\left(s + \frac{k_1+k_2}{2} - 1\right) L(f \otimes g, s)(4\pi v)^{-s}}{\zeta(2s)} ds + R_1 + \sum_\rho R_\rho, \end{aligned} \tag{4.15}$$

where the sum over ρ runs through the non-trivial zeros of $\zeta(s)$. Note the existence of infinitely many zeros of $\zeta(s)$ from Hardy's theorem [66, p. 257] implies that the sum over ρ is an infinite sum. However, the convergence of this series is complex due to our constrained understanding of the lower bound for $\zeta'(s)$. Assuming convergence of the sum in (4.15) we proceed further. Our next aim is to simplify the following integral:

$$V_k(v) := \frac{1}{2\pi i}\int_{(d')} \frac{\Gamma\left(s + \frac{k_1+k_2}{2} - 1\right) L(f \otimes g, s)(4\pi v)^{-s}}{\zeta(2s)} ds. \tag{4.16}$$

Using functional equation and completed L-function of $\Lambda(f\otimes g, s)$ given in (4.7) and (4.8) respectively, we obtain

$$V_k(v) = \frac{1}{2\pi i}\int_{(d')} \frac{\Gamma\left(1-s-\frac{k_1-k_2}{2}\right)\Gamma\left(\frac{k_1+k_2}{2}-s\right)L(f\otimes g, 1-s)}{(2\pi)^{2-4s}\Gamma\left(s-\frac{k_1-k_2}{2}\right)\zeta(2s)(4\pi v)^s}ds. \tag{4.17}$$

Using the change of variable $1-s=w$, we obtain

$$V_k(v) = \frac{\pi}{v}\frac{1}{2\pi i}\int_{(d)} \frac{\Gamma\left(w-\frac{k_1-k_2}{2}\right)\Gamma\left(w-1+\frac{k_1+k_2}{2}\right)L(f\otimes g, w)}{\Gamma\left(1-w-\frac{k_1-k_2}{2}\right)\zeta(2(1-w))}\left(\frac{v}{4\pi^3}\right)^w dw, \tag{4.18}$$

where $1<d<2$. Next, Theorem 1.2 implies that

$$\zeta(2(1-w)) = \frac{\pi^{\frac{3}{2}-2w}\Gamma\left(w-\frac{1}{2}\right)\zeta(2w-1)}{\Gamma(1-w)}. \tag{4.19}$$

Substituting (4.19) in (4.18) we obtain

$$V_k(v) = \frac{1}{2\pi i}\frac{1}{\sqrt{\pi v}}\int_{(d)} \frac{\Gamma\left(w-\frac{k_1-k_2}{2}\right)\Gamma\left(w+\frac{k_1+k_2}{2}-1\right)\Gamma(1-w)L(f\otimes g, w)}{\Gamma\left(1-w-\frac{k_1-k_2}{2}\right)\Gamma\left(w-\frac{1}{2}\right)\zeta(2w-1)}\left(\frac{v}{4\pi}\right)^w dw. \tag{4.20}$$

It is easy to verify that $\Re(2w-1)>1$, as $1<\Re(w)=d<2$. This allows us to represent the infinite series expansions for $L(f\otimes g, w)$ and $1/\zeta(2w-1)$, since both the series are absolutely and uniformly convergent within the region where $\Re(w)>1$ and satisfies the generating function given in (4.4). By replacing the series expansion (4.4) in (4.20) and interchanging the order of summation and integration, we obtain

$$V_k(v) = \frac{1}{\sqrt{\pi v}}\sum_{n=1}^{\infty} A_{f,g}(n)I_k(n,v), \tag{4.21}$$

where

$$I_k(n,v) := \frac{1}{2\pi i}\int\limits_{(d)} \frac{\Gamma(w-\frac{k_1-k_2}{2})\Gamma\left(w+\frac{k_1+k_2}{2}-1\right)\Gamma(1-w)}{\Gamma\left(1-w-\frac{k_1-k_2}{2}\right)\Gamma\left(w-\frac{1}{2}\right)}\left(\frac{v}{4n\pi}\right)^{w} dw. \tag{4.22}$$

Next, substituting the variable $w+\frac{k_1+k_2}{2}-1$ by w_1 in (4.22), we obtain

$$I_k(n,v) = \left(\frac{4n\pi}{v}\right)^{\frac{k_1+k_2}{2}-1}\int\limits_{(d_1)} \frac{\Gamma(w_1-k_1+1)\Gamma(w_1)\Gamma\left(\frac{k_1+k_2}{2}-w_1\right)}{2\pi i\;\Gamma\left(k_2-w_1\right)\Gamma\left(w_1+\frac{1}{2}-\frac{k_1+k_2}{2}\right)}\left(\frac{v}{4n\pi}\right)^{w_1} dw_1, \tag{4.23}$$

where $\frac{k_1+k_2}{2} < d_1 < \frac{k_1+k_2}{2}+1$. Note that the line of integration does not separate the poles of $\Gamma\left(\frac{k_1+k_2}{2}-w_1\right)$ from the poles of $\Gamma(w_1)\Gamma(w_1-k_1+1)$. To separate these poles, we define a new path of integration by choosing the contour $\mathcal{C}'$ with line segments $[d_1-it, d_1+it], [d_1+it, d_2+it], [d_2+it, d_2-it], [d_2-it, d_1-it]$ for some large real number t and $k_1-1 < d_2 < \frac{k_1+k_2}{2}$. Using Cauchy's residue theorem, we obtain

$$\frac{1}{2\pi i}\int\limits_{\mathcal{C}'} \frac{\Gamma(w_1-k_1+1)\Gamma(w_1)\Gamma\left(\frac{k_1+k_2}{2}-w_1\right)}{\Gamma\left(k_2-w_1\right)\Gamma\left(w_1+\frac{1}{2}-\frac{k_1+k_2}{2}\right)}\left(\frac{v}{4n\pi}\right)^{w_1} dw_1 = R_{\frac{k_1+k_2}{2}}. \tag{4.24}$$

A routine calculations yield

$$R_{\frac{k_1+k_2}{2}} = -\frac{\Gamma\left(1-\frac{k_1-k_2}{2}\right)\Gamma\left(\frac{k_1+k_2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k_2-k_1}{2}\right)}\left(\frac{v}{4n\pi}\right)^{\frac{k_1+k_2}{2}} = \frac{(k_1-k_2)\Gamma\left(\frac{k_1+k_2}{2}\right)}{2\sqrt{\pi}}\left(\frac{v}{4n\pi}\right)^{\frac{k_1+k_2}{2}}.$$

Using Stirling's bound (1.5) for Gamma function, horizontal integrals of contour $\mathcal{C}'$ vanishes as $t\to\infty$ and we obtain

$$\frac{1}{2\pi i}\int_{(d_1)} \frac{\Gamma(w_1-k_1+1)\Gamma(w_1)\Gamma\left(\frac{k_1+k_2}{2}-w_1\right)}{\Gamma\left(k_2-w_1\right)\Gamma\left(w_1+\frac{1}{2}-\frac{k_1+k_2}{2}\right)}\left(\frac{v}{4n\pi}\right)^{w_1} dw_1$$

$$= \frac{1}{2\pi i}\int_{(d_2)} \frac{\Gamma(w_1-k_1+1)\Gamma(w_1)\Gamma\left(\frac{k_1+k_2}{2}-w_1\right)}{\Gamma\left(k_2-w_1\right)\Gamma\left(w_1+\frac{1}{2}-\frac{k_1+k_2}{2}\right)}\left(\frac{v}{4n\pi}\right)^{w_1} dw_1 + R_{\frac{k_1+k_2}{2}}. \tag{4.25}$$

By using the Definition 1.6 with $m=2, n=1, p=2, q=3$, and specific values $a_1 = 1-\frac{k_1+k_2}{2}, a_2 = \frac{1}{2}-\frac{k_1+k_2}{2}, b_1 = 0, b_2 = 1-k_1, b_3 = 1-k_2$, we observe that the integral mentioned in (4.25) takes the form of the following Meijer G-function:

$$\frac{1}{2\pi i}\int_{(d_2)} \frac{\Gamma(w_1-k_1+1)\Gamma(w_1)\Gamma\left(\frac{k_1+k_2}{2}-w_1\right)}{\Gamma\left(k_2-w_1\right)\Gamma\left(w_1+\frac{1}{2}-\frac{k_1+k_2}{2}\right)}\left(\frac{v}{4n\pi}\right)^{w_1} dw_1$$
$$= G^{2,1}_{2,3}\left(\begin{matrix} 1-\frac{k_1+k_2}{2}, \frac{1}{2}-\frac{k_1+k_2}{2} \\ 0, 1-k_1, 1-k_2 \end{matrix} \middle| \frac{4\pi n}{v}\right). \tag{4.26}$$

It can be easily verified that the Meijer G-function mentioned in (4.26) satisfies all the convergence criteria are given in the Definition 1.6. Using (4.26) in (4.25) and then in (4.23) gives

$$I_k(n,v) := \left(\frac{4n\pi}{v}\right)^{\frac{k_1+k_2}{2}-1} G^{2,1}_{2,3}\left(\begin{matrix} 1-\frac{k_1+k_2}{2}, \frac{1}{2}-\frac{k_1+k_2}{2} \\ 0, 1-k_1, 1-k_2 \end{matrix} \middle| \frac{4\pi n}{v}\right)$$
$$+\left(\frac{v}{4n\pi}\right)\frac{(k_1-k_2)\Gamma(\frac{k_1+k_2}{2})}{2\sqrt{\pi}}. \tag{4.27}$$

Finally, considering the expression (4.27) in (4.21), together with (4.16), residual terms (4.13) and (4.14) in (4.15), we complete the proof of Theorem 4.1. □

Proof of Corollary 4.1. Letting $f=g$ in Theorem 4.1, the Meijer G-function

$$G^{2,1}_{2,3}\left(\begin{matrix} 1-\frac{k_1+k_2}{2}, \frac{1}{2}-\frac{k_1+k_2}{2} \\ 0, 1-k_1, 1-k_2 \end{matrix} \middle| \frac{4n\pi}{v}\right)$$

reduces to

$$G_{2,0}^{1,2}\left(\begin{array}{c} \frac{1}{2}-k \\ 0, 1-k \end{array} \Big| \frac{4n\pi}{v}\right).$$

Now applying Lemma 1.5, we can easily see that

$$G_{2,0}^{1,2}\left(\begin{array}{c} \frac{1}{2}-k \\ 0, 1-k \end{array} \Big| \frac{4n\pi}{v}\right) = \exp\left(-\frac{4n\pi}{v}\right) U\left(-\frac{1}{2}, k, \frac{4n\pi}{v}\right).$$

Utilizing this expression, one can complete the proof of this result. ☐

Proof of Corollary 4.2. Utilizing Lemma 1.4, it can be shown that,

$$\begin{aligned} &\left| A_{f,f}(n) \left(\frac{4\pi n}{v}\right)^{k-1} \exp\left(-\frac{4n\pi}{v}\right) U\left(-\frac{1}{2}, k, \frac{4n\pi}{v}\right)\right| \\ &\ll |A_{f,f}(n)| \left(\frac{4\pi n}{v}\right)^{k-\frac{3}{2}} \exp\left(-\frac{4n\pi}{v}\right) \\ &\ll_k |A_{f,f}(n)| \left(\frac{v}{n}\right)^{3/2}. \end{aligned} \tag{4.28}$$

From (4.4), we know that $\sum_{n=1}^{\infty} \frac{A_{f,g}(n)}{n^s}$ is absolutely convergent for $\Re(s) > 1$. Therefore, as $v \to 0^+$, we have

$$\sum_{n=1}^{\infty} A_{f,f}(n) \left(\frac{4\pi n}{v}\right)^{k-1} \exp\left(-\frac{4n\pi}{v}\right) U\left(-\frac{1}{2}, k, \frac{4n\pi}{v}\right) = O(y^{3/2}).$$

Using this bound in Corollary 4.1, one can finish the proof of Corollary 4.2. ☐

Proof of Corollary 4.3. The proof of this result is immediately followed by substituting $f = \Delta(z)$ in Corollary 4.1. ☐

4.5 Conclusion

In this chapter, we have studied the asymptotic behaviour of the Lambert series which is the constant term of the Fourier series expansion of two normalised Hecke eigenforms of weight k_1 and K_2 under the assumption of simplicity hypothesis for non-trivial zeros of Riemann zeta function. From the main result, Chakraborty, Kanemitsu and Maji's result has been obtained along with Zagier's prediction. Further, the result obtained can be viewed as a special case of the result obtained in Chapter 2, as the Fourier-Jacobi coefficients of Siegel cusp forms coincides with Fourier coefficients of normalised Hecke eigenforms for $n = 1$.

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