

**PSEUDO-DIFFERENTIAL OPERATORS INVOLVING
THE WEINSTEIN TRANSFORM**



**Thesis submitted in partial fulfillment
for the Award of Degree
*Doctor of Philosophy***

by

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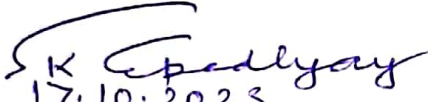
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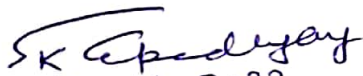
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Abbreviations

- $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$
- $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty) = \{(x_1, x_2, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$
- $\mathbb{N}_0^{n+1} = \{(a_1, a_2, \dots, a_n, a_{n+1}) \in \mathbb{N}^{n+1} : a_j \in \mathbb{N}_0, \forall j = 1, 2, \dots, n+1\}$
- $x = (x', x_{n+1}) = (x_1, x_2, \dots, x_n, x_{n+1}) \in \mathbb{R}_+^{n+1}$
- $-x = (-x', x_{n+1}) = (-x_1, -x_2, \dots, -x_n, x_{n+1}) \in \mathbb{R}_+^{n+1}$
- $\langle x', y' \rangle = \sum_{j=1}^n x_j y_j$
- $\|x\|^2 = \sum_{j=1}^{n+1} x_j^2$
- $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} x_{n+1}^{\alpha_{n+1}}$, for $x \in \mathbb{R}_+^{n+1}$ and $\alpha \in \mathbb{N}_0^{n+1}$
- $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} D_{n+1}^{\alpha_{n+1}}$, for $\alpha \in \mathbb{N}_0^{n+1}$
- $D_\xi^\nu = \frac{\partial^{|\nu|}}{\partial \xi_1^{\nu_1} \dots \partial \xi_n^{\nu_n} \partial \xi_{n+1}^{\nu_{n+1}}}$, for $\nu \in \mathbb{N}_0^{n+1}$ and $|\nu| = \nu_1 + \dots + \nu_n + \nu_{n+1}$
- $\beta! = \beta_1! \beta_2! \dots \beta_{n+1}!$, for $\beta \in \mathbb{N}_0^{n+1}$
- $\binom{\gamma}{\beta} = \binom{\gamma_1}{\beta_1} \binom{\gamma_2}{\beta_2} \dots \binom{\gamma_n}{\beta_n} \binom{\gamma_{n+1}}{\beta_{n+1}}$, for $\gamma, \beta \in \mathbb{N}_0^{n+1}$ such that $\beta \leq \gamma$
- $C^\infty(\mathbb{R}_+^{n+1})$, the space of infinitely differentiable functions on \mathbb{R}_+^{n+1}
- \hat{J}_α , the normalized Bessel function of the first kind
- $C^k(\mathbb{R}_+^{n+1})$, the space of k -times differentiable functions on \mathbb{R}_+^{n+1}
- $C_c^\infty(\mathbb{R}_+^{n+1})$, the space of C^∞ -functions on \mathbb{R}_+^{n+1} with compact support
- $S_*(\mathbb{R}_+^{n+1})$, the Schwartz space on \mathbb{R}_+^{n+1}
- $S'_*(\mathbb{R}_+^{n+1})$, the dual of the Schwartz space $S_*(\mathbb{R}_+^{n+1})$