Chapter 1

Introduction

The Weinstein transform is a valuable tool utilized in various fields of mathematics and engineering, particularly in solving problems related to partial differential equations, waveforms, signal processing, fluid mechanics, and other related areas. In image processing, Boussakta and Holt [10] showed that the Weinstein transform can be used for feature extraction, such as edge detection, texture analysis, and object recognition. In fluid mechanics, the Weinstein transform can be utilized to study the behavior of fluid flow and solve complex equations governing fluid motion [1, 9]. Overall, the Weinstein transform serves as a versatile tool in mathematical analysis and problem-solving, offering insights and solutions to a wide range of problems in diverse fields. In mathematical terms, Weinstein transform is an integral transform whose integral representation is exactly the product of an absolutely integrable function and a Weinstein kernel generated by the product of a complex exponential function and a generalized Bessel function of the first kind. Regarding the Weinstein transform theory, several good research works came to light, and many authors made important contributions to explore the theory associated with the Weinstein transform. In this connections, Mehraj [30] proved the Paley-Wiener theorem for the Weinstein transform and Mejjaoli et al. [31, 32] found many important observations regarding the Weinstein wavelet transform. Salhi [33] investigated many properties associated with the uncertainty principle. Mohamed, H.B [35] examined Weinstein-Sobolev spaces of exponential type and applications, Nahia and Salem [58, 59] exposed a mean value property and discussed spherical harmonics. Salem and Nasr [56] discussed Heisenberg-type inequalities associated with the Weinstein Transform. Saoudi [60] introduced the Weinstein-Wigner transform and the Weinstein Weyl-transform. Furhter, Saoudi, and Nefzi [61] proved the boundedness and compactness of localization operators for the Weinstein-Wigner transform.

The theory of Ultradistributions have been introduced by Beurling [5], Björck [8], and Roumieu [53], which is the generalization of Schwartz distributions. A unification of Beurling-Björck theory and Roumieu theory has been made by Komatsu [29]. Later on, the Hankel transform of ultradistributions in Roumieu setting has been done by Pathak and Pandey [42]. Motivated by the results of authors [42], Pathak and Shrestha [46] defined the Beurling type ultradistribution space H_ω^μ and discussed important properties .

Sobolev space is a useful and powerful tool which is frequently used in nonlinear analysis, differential geometry, physics, and other areas of the mathematical sciences. This theory is useful for solving the problems of partial differential equations. Pahk and Kang [40] extended the concept of Sobolev space to the generalized distribution spaces of Beurling-Björck type [6, 8], and investigated the Sobolev imbedding theorem, the Rellich's compactness theorem by exploiting the weight function ω. Pathak and Pandey [44] introduced the Sobolev space of type $G_\mu^{p,s}$ and discussed its various properties by using the theory of distributional Hankel transform. The aforesaid authors introduced the Hankel potential $\mathcal{H}_\mu^s$ and showed that the Hankel potential $\mathcal{H}_\mu^s$ is a continuous linear mapping of the Zemanian space $\mathcal{H}_\mu$ into itself. Later on, Pathak and Shrestha [45] defind the space of type $G_{\omega,\mu}^{p,s}$ and discussed many results. Pathak [43] considered the generalized Sobolev space $H_\omega^w(\mathbb{R}^n)$ which is a generalization of the Sobolev space $H^s(\mathbb{R}^n)$ and developed a multiresolution analysis for the generalised Sobolev space.

Pseudo-differential operators are the generalization of partial differential operators. Exploiting the Fourier transform theory, algebra and calculus of pseudo-differential operators have been developed by Kohn and Nirenberg [26] and many properties have been done. Later on, Hörmander [22, 23], Kato [25], Rhuzhansky and Turunen [54, 55], Treves [80], Wong [88], made important contributions in this area by taking the concept of the Fourier transform. Pathak [47] introduced generalized Sobolev spaces and pseudo-differential operators on the space of ultradistributions by exploiting the Fourier transform theory. Utilizing the Hankel transform theory, Pathak and Pandey [48] developed a class of pseudo-differential operators associated with Bessel operators and their various properties. Later on, Pathak and Upadhyay [51] investigated pseudo-differential operators associated with the homogeneous class of symbols involving the Hankel transform. Pathak and Upadhyay [52] proved the L_μ^p boundedness result of pseudo-differential operators associated with the Bessel operator and applied these results in Sobolev type spaces. Recently, Upadhyay and Sartaj [84] introduced the theory of Pseudo-differential operator on space $S(\mathbb{R}^{n+1})$ and discussed its properties by considering the theory of the Weinstein transform. Therefore, our main objective in this thesis is to study various properties of distributions, generalized distributions, Sobolev spaces, and pseudo-differential operators associated with the Weinstein transform.

1.1 The Fourier Transform

In the present section, from [15, 88, 90], definitions and properties of the Fourier transform and the Hankel transform of the Haimo-type are given. These definitions and properties played an important role in defining the Weinstein transform.

Definition 1.1.1. The Fourier transform of $\phi \in L^1(\mathbb{R}^n)$ is defined by

$$\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x,\, \xi\rangle}\phi(x)dx, \qquad \xi \in \mathbb{R}^n, \tag{1.1.1}$$

where $\langle x,\, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$.

If $\phi \in L^1(\mathbb{R}^n)$ and $\hat{\phi} \in L^1(\mathbb{R}^n)$, then the inversion formula of the Fourier transform is given by

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x',\, \xi'\rangle}\hat{\phi}(\xi)\, d\xi, \quad a.e. \;\; \xi \in \mathbb{R}^n. \tag{1.1.2}$$

Definition 1.1.2. Let $\phi \in L^1(\mathbb{R}^n)$ and $\psi \in L^1(\mathbb{R}^n)$. Then the convolution of ϕ, ψ is given by

$$(\phi * \psi)(x) = \int_{\mathbb{R}^n} \phi(x-y)\psi(y)dy, \qquad x \in \mathbb{R}^n. \tag{1.1.3}$$

Properties of the Fourier transform:

1. If $\phi \in L^1(\mathbb{R}^n)$, then the function $\hat{\phi}$ is continuous on $\mathbb{R}^n$ and

$$\|\hat{\phi}\|_{L^\infty(\mathbb{R}^n)} \leq \|\phi\|_{L^1(\mathbb{R}^n)}.$$

2. **Riemann-Lebesgue Lemma:** Let $\phi \in L^1(\mathbb{R}^n)$. Then

   (i) $\hat{\phi}$ is a continuous function on $\mathbb{R}^n$,

(ii) $\lim_{|\xi|\to\infty} \hat{\phi}(\xi) = 0$,

(iii) $\phi_k \to \phi$ in $L^1(\mathbb{R}^n)$ implies $\hat{\phi}_k \to \hat{\phi}$ uniformly on $\mathbb{R}^n$.

3. Let $\phi, \psi \in L^1(\mathbb{R}^n)$, then $\phi * \psi \in L^1(\mathbb{R}^n)$ and we have

$$(\phi \hat{*} \psi) = (2\pi)^{-\frac{n}{2}} \hat{\phi}\hat{\psi}. \tag{1.1.4}$$

4. Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $\phi \in L^p(\mathbb{R}^n)$ and $\psi \in L^q(\mathbb{R}^n)$, the function $\phi * \psi$ belongs to $L^r(\mathbb{R}^n)$ and

$$\|\phi * \psi\|_{L^r(\mathbb{R}^n)} \leq \|\phi\|_{L^p(\mathbb{R}^n)} \|\psi\|_{L^q(\mathbb{R}^n)}. \tag{1.1.5}$$

1.2 The Hankel Transform

Definition 1.2.1. Space $L^p_\sigma(0, \infty)$ is defined as the space of all real-valued measurable functions ϕ defined on $(0, \infty)$ with norm

$$\|\phi\|_{L^p_\sigma(0,\infty)} = \left(\int_0^\infty |\phi(x)|^p d\sigma(x) \right)^{1/p}, \quad 1 \leq p < \infty \tag{1.2.1}$$

where for fixed $\mu > 0, d\sigma(x)$ is defined by

$$d\sigma(x) = \frac{x^{2\mu}}{2^{\mu-1/2}\Gamma(\mu + 1/2)} dx. \tag{1.2.2}$$

Definition 1.2.2. The Hankel transform of $\phi \in L^1_\sigma(0, \infty)$ is defined by

$$(h_\mu \phi)(\xi) = \int_0^\infty J_\mu(x\xi)\phi(x) d\sigma(x), \quad \xi \in (0, \infty) \tag{1.2.3}$$

where j is the Bessel function for $\mu \geq -\frac{1}{2}$ which is given by

$$j(x) = 2^{\mu-1/2}\Gamma(\mu+1/2)x^{1/2-\mu}J_{\mu-1/2}(x), \tag{1.2.4}$$

Let $\phi \in L^1_\sigma(0,\infty)$ and $h_\mu\phi \in L^1_\sigma(0,\infty)$, then the inversion formula of the Hankel transform is defined by

$$\phi(x) = \int_0^\infty j(x\xi)(h_\mu\phi)(\xi)d\sigma(\xi). \tag{1.2.5}$$

Definition 1.2.3. Let $\phi \in L^1_\sigma(0,\infty)$. Then translation of $\phi(x)$ is defined by

$$(\tau_x\phi)(y) = \int_0^\infty \phi(z)D(x,y,z)d\sigma(z), \quad 0 < x, y < \infty, \tag{1.2.6}$$

where $D(x,y,z)$ is given by

$$D(x,y,z) = \int_0^\infty j(xt)j(yt)j(zt)d\sigma(t). \tag{1.2.7}$$

Definition 1.2.4. Let $\phi, \psi \in L^1_\sigma(0,\infty)$. Then the Hankel convolution of ϕ, ψ is given by

$$(\phi\#\psi)(x) = \int_0^\infty (\tau_x\phi)(y)\psi(y)d\sigma(y), \quad 0 < x < \infty. \tag{1.2.8}$$

1.3 Space $S_\omega(\mathbb{R}^n)$ and the Weinstein Operator

In this section, from [8, 22], the definition of the $S_\omega(\mathbb{R}^n)$ space is given, and then from [58, 59, 81], the definitions and properties of the Weinstein operator are discussed, which are used in the next subsequent chapters.

Definition 1.3.1. Let ω be a real-valued continuous function defined on $\mathbb{R}^n$, continuous at origin, then the class of all ω is denoted by M, which satisfies the following conditions:

$$0 = \omega(0) = \lim_{x \to 0} \omega(x) \leq \omega(x + y) \leq \omega(x) + \omega(y), \ (\forall\, x,\, y \in \mathbb{R}^n). \tag{1.3.1}$$

$$J_n(\omega) = \int_{|y| \geq 1} \frac{\omega(y)}{\mid y \mid^n}\, d\mu_\beta(y) < \infty. \tag{1.3.2}$$

$$a + b\log(1 + \mid y \mid) \leq \omega(y),\ \forall\, y \in \mathbb{R}^n_+,\ a \in \mathbb{R}, \text{ and } b > 0. \tag{1.3.3}$$

Definition 1.3.2. Let $\omega \in M$ and $\lambda \geq 0$. Then for each multi-index α the space $S_\omega(\mathbb{R}^n)$ is defined to be the set of all complex-valued infinitely differentiable function ϕ on $\mathbb{R}^n$ such that

$$p'_{\alpha,\lambda}(\phi) = \sup_{x \in \mathbb{R}^n} e^{\lambda\omega(x)} \mid D_x^\alpha \phi(x) \mid < \infty \tag{1.3.4}$$

and

$$\pi'_{\alpha,\lambda}(\phi) = \sup_{\xi \in \mathbb{R}^n} e^{\lambda\omega(\xi)} \mid D_\xi^\alpha (\mathcal{F}_w \phi)(\xi) \mid < \infty. \tag{1.3.5}$$

$S_\omega(\mathbb{R}^n)$ is called the Schwartz space and it forms a vector space over the field of complex numbers.

Definition 1.3.3. $S'_\omega(\mathbb{R}^n)$ is defined as the set of all linear continuous functional on $S_\omega(\mathbb{R}^n)$. We say that $S'_\omega(\mathbb{R}^n)$ is the dual of $S_\omega(\mathbb{R}^n)$.

Definition 1.3.4. The Fourier transform of $\phi \in S_\omega(\mathbb{R}^n)$ is defined by

$$\hat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x,\, \xi\rangle} \phi(x) dx, \qquad \xi \in \mathbb{R}^n, \tag{1.3.6}$$

where $\langle x,\, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$.

If $\phi \in S_\omega(\mathbb{R}^n)$ and $\hat{\phi} \in S_\omega(\mathbb{R}^n)$, then the inversion formula of the Fourier transform is given by

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x', \xi' \rangle} \hat{\phi}(\xi)\, d\xi, \quad a.e. \;\; \xi \in \mathbb{R}^n. \tag{1.3.7}$$

Definition 1.3.5. Let $\phi \in S_\omega(\mathbb{R}^n)$ and $\psi \in S_\omega(\mathbb{R}^n)$. Then the convolution of ϕ, ψ is given by

$$(\phi * \psi)(x) = \int_{\mathbb{R}^n} \phi(x-y)\psi(y)dy, \qquad x \in \mathbb{R}^n. \tag{1.3.8}$$

Properties of the Fourier transform on $S_\omega(\mathbb{R}^n)$

- **Parseval formula:** For f and $g \in S_\omega\big(\mathbb{R}^n\big)$, we have

$$\int_{\mathbb{R}^n_+} f(x)\overline{g(x)}\, d\mu_\beta(x) = \int_{\mathbb{R}^n_+} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d(\xi). \tag{1.3.9}$$

- **Inversion formula:** For $f \in L^1(\mathbb{R}^n)$, if $\hat{f} \in L^1(\mathbb{R}^n)$ then, we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\langle x', \xi' \rangle}\, d(\xi), \;\; a.e. \tag{1.3.10}$$

- For $f \in S_\omega\big(\mathbb{R}^n\big)$, we have

$$\mathcal{F}(D^\alpha f)(\xi) = (\xi)^\alpha (\hat{f})(\xi), \;\; \forall \xi \in \mathbb{R}^n. \tag{1.3.11}$$

Theorem 1.3.6. *Let $\phi, \psi \in S_\omega\big(\mathbb{R}^n\big)$, then $(\phi \#_\beta \psi) \in S_\omega\big(\mathbb{R}^n\big)$ and*

$$\mathcal{F}(\phi * \psi) = \mathcal{F}(\phi)\mathcal{F}(\psi)\,. \tag{1.3.12}$$

Theorem 1.3.7. *Let $1 \le p, q, r \le \infty$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $(f * g) \in L^r(\mathbb{R}^n)$ and*

$$||f * g||_{L^r(\mathbb{R}^n)} \le ||f||_{L^p(\mathbb{R}^n)} ||g||_{L^q(\mathbb{R}^n)}. \tag{1.3.13}$$

Definition 1.3.8. Let f and g be $S_\omega(\mathbb{R}^n)$ and α be any multi-index, then

$$D^\alpha(fg) = \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \big(D^\gamma f\big)\big(D^{\alpha-\gamma} g\big). \quad \alpha \in \mathbb{N}_0^n. \tag{1.3.14}$$

The Weinstein Operator

The $L^p_\beta(\mathbb{R}^{n+1}_+)$ - norm of a Lebesgue measurable function ϕ is given by

$$\|\phi\|_{L^p_\beta(\mathbb{R}^{n+1}_+)} = \left(\int_{\mathbb{R}^{n+1}_+} |\phi(x)|^p d\mu_\beta(x) \right)^{1/p} < \infty, \quad 1 \leq p < \infty \tag{1.3.15}$$

$$\|\phi\|_{L^\infty_\beta(\mathbb{R}^{n+1}_+)} = esssup_{x\in\mathbb{R}^{n+1}_+} |\phi(x)| < \infty, \tag{1.3.16}$$

and

$$d\mu_\beta(x) = A_\beta x_{n+1}^{2\beta+1} dx, \tag{1.3.17}$$

where

$$A_\beta = \frac{1}{(2\pi)^{\frac{n}{2}} 2^\beta \Gamma(\beta+1)}, \tag{1.3.18}$$

and dx is the Lebesgue measure on $\mathbb{R}^{n+1}_+$.

The Weinstein operator $\Delta^n_{W,\beta}$ on $\mathbb{R}^{n+1}_+$ for (n+1) variables is defined by

$$\Delta^n_{W,\beta} = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\beta+1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}} = \Delta^n_W + L_\beta, \quad \beta > -1/2\,, \tag{1.3.19}$$

where Δ^n_W is the Laplacian operator for first the n variables

$$\Delta^n_W = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \tag{1.3.20}$$

and L_β is the Bessel operator for the last variable on $(0,\infty)$, which is given by

$$L_\beta f = \frac{\partial^2}{\partial x_{j+1}^2} f + \frac{2\beta+1}{x_{n+1}} \frac{\partial}{\partial x_{n+1}} f\,. \tag{1.3.21}$$

Bessel Function: The Bessel function J_ν is known to satisfy a general differential equations. For example, it is known and can be easily verify that $u = x^\alpha J_\nu(\lambda x)$ is a solution of the differential equation

$$u'' + \left(\frac{1-2\alpha}{x}\right)u' + \left(\lambda^2 + \frac{\alpha^2-\nu^2}{x^2}\right)u = 0.$$

A normalized Bessel function is a Bessel function that has been scaled by a constant factor to satisfy certain conditions, such as orthonormality or unit integral over a given range. Also, the normalization factor for Bessel functions is typically defined to ensure that the function satisfies specific properties, such as orthonormality or having a unit integral over a defined range.

- *For* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n, \lambda_{n+1}) \in \mathbb{C}_+^{n+1}$, the system

$$\begin{cases} \frac{\partial^2 u}{\partial x_k^2}(x) = -\lambda_k^2 u(x), \;\; if \;\; 1 \le k \le n \\ L_\beta u(x) = -\lambda_{n+1}^2 u(x), \beta > \frac{-1}{2} \\ u(0) = 1, \frac{\partial u}{\partial x_{n+1}}(0) = 0, \frac{\partial u}{\partial x_k}(0) = -i\lambda_k, \;\; if \;\; 1 \le k \le n \end{cases} \tag{1.3.22}$$

has a unique solution denoted by $W_{\beta,n}(x,\,\lambda)$ and is given by

$$W_{\beta,n}(x,\,\lambda) = e^{-i\langle x',\,\lambda'\rangle} J_\beta(x_{n+1}\lambda_{n+1}),\; \forall\, \lambda \in \mathbb{C}_+^{n+1}. \tag{1.3.23}$$

where $x = (x', x_{n+1}) = (x_1, x_2, ...x_n, x_{n+1}) \in \mathbb{C}_+^{n+1}$, and J_β is the normlized Bessel function of index β and is defined by

$$J_\beta(x) = \Gamma(\beta+1)\sum_{k=0}^{\infty}\frac{(-1)^k(x)^{2k}}{2^k k!\Gamma(\beta+k+1)} \ . \tag{1.3.24}$$

The system (1.3.22) is equivalent to the system

$$\begin{cases} \Delta_{W,\beta}^n u(x) = \big(\Delta_W^n + L_\beta\big)u(x) = -\|\lambda\|^2 u(x), \quad \beta > -1/2 \\ u(0) = 1, \ \frac{\partial u}{\partial x_{n+1}}(0) = 0, \ \frac{\partial u}{\partial x_k}(0) = -i\lambda_k \ \text{ for } \ 1 \le k \le n \end{cases}$$

The integral representation of J_β is

$$J_\beta(x\xi) = \frac{2\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})}\int_0^1 (1-t^2)^{\beta-\frac{1}{2}} \ \cos(x\xi t)dt, \quad \beta > -\frac{1}{2}. \tag{1.3.25}$$

Using the Euler's formula and properties of the definite integral (1.3.25) can be written as

$$J_\beta(x\xi) = \frac{\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})}\int_{-1}^1 (1-t^2)^{\beta-\frac{1}{2}} \ e^{ix\xi t}dt, \quad \beta > -\frac{1}{2}. \tag{1.3.26}$$

Properties of $J_\beta(x\xi)$

1. $\forall \ x \in \mathbb{R}, \xi \in \mathbb{C}, \quad \big|J_\beta(x\xi)\big| \le e^{|Im(\xi)||x|}.$

2. $\forall \ x \in \mathbb{R}, \xi \in \mathbb{C}$ *and* $\nu \in \mathbb{N}, \quad \Big|\frac{d^\nu}{d\xi^\nu}J_\beta(x\xi)\Big| \le |x|^\nu e^{|Im(\xi)||x|}.$

3. $\forall \ x \in \mathbb{R}, \quad \big|J_\beta(x\xi)\big| \le 1$, if and only if $\xi \in \mathbb{R}$.

The function $(x, \ \xi) \to W_{\beta,n}(x, \ \xi)$ is called the Weinstein kernal, and satisfied the following properties:

Proposition 1.3.9. *For all* $(x,\xi) \in \mathbb{C}_+^{n+1} \times \mathbb{C}_+^{n+1}$, *we have*

$$W_{\beta,n}(\xi,x) = W_{\beta,n}(x,\xi) = e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}). \tag{1.3.27}$$

$$W_{\beta,n}(-x,\xi) = W_{\beta,n}(x,-\xi) = e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}). \tag{1.3.28}$$

$$W_{\beta,n}(0,\xi) = 1. \tag{1.3.29}$$

$$\forall\ \nu \in \mathbb{N}_0^{n+1}, \left|D_\xi^\nu W_{\beta,n}(x,\xi)\right| \leq \|x\|^{|\nu|}\ e^{\|x\|\mathrm{Im}(\|\xi\|)}. \tag{1.3.30}$$

$$\forall\ (x,\xi) \in \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}, \left|W_{\beta,n}(x,\xi)\right| \leq 1. \tag{1.3.31}$$

1.4 The Weinstein Transform

In this section, from [32, 58, 59, 81, 84], the definitions and properties of the Weinstein transform are discussed. After that, the definition and various properties of the Weinstein convolution are given.

Definition 1.4.1. The Weinstein transform of $\phi \in L_\beta^1(\mathbb{R}_+^{n+1})$ is defined by

$$(\mathcal{F}_w\phi)(\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\phi(x)d\mu_\beta(x), \qquad \xi \in \mathbb{R}_+^{n+1} \tag{1.4.1}$$

where $d\mu_\beta(x)$ is the measure on $\mathbb{R}_+^{n+1}$, which is given by (1.3.17).

Let $\phi \in L_\beta^1(\mathbb{R}_+^{n+1})$ such that $\mathcal{F}_w\phi \in L_\beta^1(\mathbb{R}_+^{n+1})$, then inversion formula of the Weinstein transform is given by

$$\phi(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})(\mathcal{F}_w\phi)(\xi)d\mu_\beta(\xi), \quad a.e.\ \ \xi \in \mathbb{R}_+^{n+1}. \tag{1.4.2}$$

Properties of the Weinstein transform

- **Parseval formula:** For f and $g \in S_\omega\big(\mathbb{R}^{n+1}_+\big)$, we have

$$\int_{\mathbb{R}^{n+1}_+} f(x)\overline{g(x)}\, d\mu_\beta(x) = \int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_w f)(\xi)\overline{(\mathcal{F}_w g)(\xi)}\, d\mu_\beta(\xi). \tag{1.4.3}$$

- **Inversion formula:** For $f \in L^1_\beta(\mathbb{R}^{n+1}_+)$, if $\mathcal{F}_w f \in L^1_\beta(\mathbb{R}^{n+1}_+)$ then, we have

$$f(x) = \int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_w f)(\xi) e^{i\langle x', \xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\, d\mu_\beta(\xi),\ \ a.e. \tag{1.4.4}$$

- For $f \in S_\omega\big(\mathbb{R}^{n+1}_+\big)$, we have

$$\mathcal{F}_w((\Delta^n_{W,\beta})^\alpha f)(\xi) = (-||\xi||^2)^\alpha (\mathcal{F}_w f)(\xi),\ \forall \xi \in \mathbb{R}^{n+1}_+. \tag{1.4.5}$$

- For $f \in S_\omega\big(\mathbb{R}^{n+1}_+\big)$, inverse of the Weinstein transform is given by

$$\mathcal{F}_w^{-1} f(x) = \mathcal{F}_w f(-x)\ \ \forall x\ \in \mathbb{R}^{n+1}_+. \tag{1.4.6}$$

Definition 1.4.2. The translation operator $\tau^\beta_y,\ y \in \mathbb{R}^{n+1}_+$ associated with the Weinstein operator $\Delta^n_{W,\beta}$ is defined by

$$\phi(x,y) = \tau^\beta_y \phi(x) = \int_{\mathbb{R}^{n+1}_+} \phi(z)\mathcal{D}_\beta(x,y,z)\, d\mu_\beta(z), \quad x, z \in \mathbb{R}^{n+1}_+. \tag{1.4.7}$$

From [84], $\mathcal{D}_\beta(x,y,z)$ is the basic function which is defined by

$$\begin{aligned}\mathcal{D}_\beta(x,y,z) = \int_{\mathbb{R}^{n+1}_+} & e^{-i\langle x', \xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) e^{i\langle y', \xi'\rangle} J_\beta(y_{n+1}\xi_{n+1}) e^{-i\langle z', \xi'\rangle} \\ & \times J_\beta(z_{n+1}\xi_{n+1})\, d\mu_\beta(\xi).\end{aligned} \tag{1.4.8}$$

and for $x, y, \xi \in \mathbb{R}_+^{n+1}$ the following formula holds

$$e^{-\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) e^{\langle y',\xi'\rangle} J_\beta(y_{n+1}\xi_{n+1}) = \int_{\mathbb{R}_+^{n+1}} e^{\langle z',\xi'\rangle} J_\beta(z_{n+1}\xi_{n+1}) \mathcal{D}_\beta(x,y,z) d\mu_\beta(z) \tag{1.4.9}$$

Setting $\xi = 0$ in (1.4.9), then we obtain

$$\int_{\mathbb{R}_+^{n+1}} \mathcal{D}_\beta(x,y,z) d\mu_\beta(z) = 1. \tag{1.4.10}$$

- The translation operator τ_y^β satisfies the following properties:

For $\phi \in C_*\big(\mathbb{R}_+^{n+1}\big)$, for all $x, y \in \mathbb{R}_+^{n+1}$ we have

$$\tau_y^\beta \phi(x) = \tau_x^\beta \phi(y) = \phi(x,y) \ \textit{and} \ \tau_0^\beta \phi = \phi\,. \tag{1.4.11}$$

By using the generalized translation, the generalized convolution product $\phi \#_\beta \psi$ for the functions $\phi, \psi \in L^1(\mathbb{R}_+^{n+1})$ is given by

$$(\phi \#_\beta \psi)(y) = \int_{\mathbb{R}_+^{n+1}} \phi(x)(\tau_y^\beta \psi)(x)\, d\mu_\beta(x) = \int_{\mathbb{R}_+^{n+1}} \phi(x)\psi(x,y)\, d\mu_\beta(x)\,. \tag{1.4.12}$$

Theorem 1.4.3. *Let $\phi, \psi \in L_\beta^1(\mathbb{R}_+^{n+1})$, then $(\phi \#_\beta \psi) \in L_\beta^1(\mathbb{R}_+^{n+1})$ and*

$$\mathcal{F}_w(\phi \#_\beta \psi) = \mathcal{F}_w(\phi)\mathcal{F}_w(\psi)\,. \tag{1.4.13}$$

Theorem 1.4.4. *Let $1 \le p, q, r \le \infty$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If $f \in L^p(\mathbb{R}_+^{n+1})$ and $g \in L^q(\mathbb{R}_+^{n+1})$, then $(f \#_\beta g) \in L^r(\mathbb{R}_+^{n+1})$ and*

$$||f \#_\beta g||_{L^r(\mathbb{R}_+^{n+1})} \le ||f||_{L^p(\mathbb{R}_+^{n+1})} ||g||_{L^q(\mathbb{R}_+^{n+1})}. \tag{1.4.14}$$

Theorem 1.4.5. *Let $\phi, \psi \in S_\omega(\mathbb{R}^{n+1}_+)$ and α be any multi-index then, from [84], we find the following expression*

$$(\Delta^n_{W,\beta})^\alpha[\phi(x)\psi(x)] = \sum_{j=0}^{\alpha}\sum_{r=1}^{2j}\sum_{q=0}^{r}\sum_{|\rho'|\leq 2(\alpha-j)}\sum_{\delta_1,\delta_2,..\delta_n\geq 0}\binom{\alpha}{j}\binom{r}{q}\binom{\alpha-j}{\delta_1,\delta_2,..\delta_n} \times \frac{1}{\rho'!}E'_{\beta,r}x_{n+1}^{r-\alpha}\big(D_x^{\rho'+q}\phi(x)\big)\big(D_x^{\rho'+2\delta'+r-q}\psi(x)\big) \quad (1.4.15)$$

where $\frac{1}{\rho'!}E'_{\beta,r}$ *is a constant,* $\rho'+q = (\rho_1, \dots.\rho_n, q) \in \mathbb{N}_0^{n+1}$, $\rho'+2\delta'+r-q = (\rho_1+2\delta_1, \dots., \rho_n+2\delta_n,\ r-q) \in \mathbb{N}_0^{n+1}$, $|\rho'|+q = \rho_1+\dots.+\rho_n+q,\ |\rho'|+2|\delta'|+r-q = \rho_1+2\delta_1+\dots.+\rho_n+2\delta_n+r-q$ *and*

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_{x_1}^{\alpha_1}\dots.\partial x_{x_n}^{\alpha_n}\partial x_{x_{n+1}}^{\alpha_{n+1}}}\ . \quad (1.4.16)$$

***