

Chapter 2

Direct Projective Modules and Semi-Simple Direct Projective Modules

In 1976, W.K Nicolson [34] defined the concept of direct projective modules. A module M is said to be direct projective if, for every direct summand, N of M that satisfies $M/A \cong N \leq^{\oplus} M$ implies $A \leq^{\oplus} M$. In this chapter, we study some properties of direct projective modules in respect of SSP and SIP properties and characterize such modules in terms of endoregular and Hopfian modules. Further, we introduce the concept of Semi-simple direct projective modules, which is a generalization of Direct projective modules, and study their properties.

2.1 Direct Projective Modules

Proposition 2.1.1. *Every direct summand of the direct projective module is a direct projective module.*

Proof:- Let N be a direct summand of M , where M is a direct projective module. Then if N is an indecomposable submodule, it is direct projective trivially. In another case, let $N/P \cong S \leq^{\oplus} N$ where P is a submodule of N . We must prove that P is a direct summand of N . Since N is a direct summand, $\exists N'$ complement of N such that $M = N \oplus N'$. Consider $N/P \cong S \leq^{\oplus} N$ and taking direct sum with N' we get $(N \oplus N')/P \cong S \oplus N' \leq^{\oplus} N \oplus N'$ which is equivalent to $M/P \cong S \oplus N' \leq^{\oplus} M$. Due to M being a direct projective module, P is a direct summand of M . But P is a submodule of N . Therefore P is a direct summand of N .

However, the sum of two direct projective modules need not be a direct projective module. For example, \mathbb{Z} -module $\mathbb{Z} \oplus \mathbb{Z}_2$, here both \mathbb{Z} and \mathbb{Z}_2 are direct projective modules over \mathbb{Z} , but their direct sum is not a direct projective module. Similarly, the submodule of a direct projective module need not be a direct projective module. For example, \mathbb{Z} -module $\mathbb{Q} \oplus \mathbb{Z}_2$ is a direct projective module, but its submodule $\mathbb{Z} \oplus \mathbb{Z}_2$ is not a direct projective module.

Definition 2.1.2. *An R -module M is called a relatively direct projective to R -module N , if for every R -homomorphism $f : M \rightarrow N$ with $\text{Img}(f) \leq^{\oplus} N$, then $\text{Ker}(f) \leq^{\oplus} M$.*

Proposition 2.1.3. *Let $M = \bigoplus_{i=1}^n M_i$ be a direct sum of direct projective modules. Then M is direct projective if M_i is relatively direct projective to M_j for all $1 \leq i, j \leq n$.*

Proof:- Follows from the definition of relatively direct projective modules.

Corollary 2.1.4. *Let $M = \bigoplus_{i=1}^n M_i$ be an R -module, and M_i is relatively direct projective to M_j for all $1 \leq i, j \leq n$. If M has SSP property, then M has SIP property.*

Proof:- Follows from the definition of relatively direct projective modules and SSP property.

The following lemma gives another condition for a module to be direct projective.

Lemma 2.1.5. *Let M be a direct projective R -module, then for each direct summand N of M , every epimorphism $f : M \rightarrow N$ splits. In particular, every surjective homomorphism of M to its direct summand splits.*

Proof:- Let M be a direct projective module and N be a direct summand of M . If $f : M \rightarrow N$ is an epimorphism, then from the Fundamental Theorem of module homomorphism $M/\text{Ker}(f) \cong \text{Img}(f) \cong N \leq^{\oplus} M$. Since M is a direct projective module, $\text{Ker}(f)$ is a direct summand of M . Hence epimorphism f splits.

Next, we will study the endomorphism ring of direct projective modules. Specifically, we explore the concept of endoregular modules whose endomorphism rings are Von Neumann regular. We note that every endoregular module is also a direct projective module. However, the converse is not always true, for example, \mathbb{Z}_9 over the ring \mathbb{Z} , which is a direct projective module but not an endoregular module. For further study of these modules, we will examine various conditions under which a direct projective module satisfies the properties of an endoregular module.

Lemma 2.1.6. *[26, Proposition 1.1] Let $S = \text{End}_R(M)$ be an endomorphism ring of an R -module M . Then M is an endoregular module if and only if for every $f \in S$, $\text{Img}(f)$ and $\text{Ker}(f)$ are direct summands of M .*

Proposition 2.1.7. *Let M be a direct projective module, and $S = \text{End}_R(M)$ is an endomorphism ring of M . For any $f \in S$ if $\text{Img}(f)$ is a direct summand of M , then $\text{Ker}(f)$ is a direct summand of M .*

Proof:- By the Fundamental theorem of module homomorphism $M/\text{Ker}(f) \cong \text{Img}(f) \leq^{\oplus} M$. Hence $\text{Ker}(f)$ is a direct summand of M .

Proposition 2.1.8. *The following statements are equivalent for an R -module M :*

1. M is a direct projective and dual Rickart module;
2. M is an endoregular module.

Proof:- (1) \Rightarrow (2) If M is a dual Rickart, hence for every $f \in \text{End}_R(M)$ we have $\text{Img}(f)$ is direct summand of M . By the above proposition, since M is direct projective $\text{Ker}(f)$ is the direct summand of M . Therefore M is an endoregular module.

(2) \Rightarrow (1) The converse is true by the definition of Endoregular modules.

Corollary 2.1.9. *The following statements are equivalent for an R -module M :*

1. M is a direct projective, and $M \oplus M$ has SSP property;
2. M is an endoregular module.

Proof:- (1) \Rightarrow (2) Since, $M \oplus M$ has SSP property therefore for every homomorphism $f : M \rightarrow M$, $\text{Img}(f)$ is a direct summand of M . Hence M is a dual Rickart module. So from Proposition 2.1.8, we get that M is endoregular.

(2) \Rightarrow (1) Follows from definition.

Corollary 2.1.10. *The following statements are equivalent for an R -module M :-*

1. $M \oplus M$ has D_3 condition and SSP property;
2. M is an endoregular module.

Proof:- If $M \oplus M$ satisfies D_3 condition, then M is a direct projective module. Hence followed from the Corollary 2.1.9

Proposition 2.1.11. *Every direct projective module with summand sum property also has summand intersection property.*

Proof:- Let M be a direct projective module with SSP property. Consider K and T as two direct summands of M such as $M = K \oplus T$. Since M has SSP property for every homomorphism $f : K \rightarrow T$, $Img(f)$ is a direct summand of T . Also, M is a direct projective module and satisfies $M/Ker(f) \cong Img(f) \leq^{\oplus} M$. Therefore $Ker(f)$ is a direct summand of K , hence M has SIP property.

Proposition 2.1.12. *Let M be a dual Rickart module. Then M is a direct projective module if and only if $End_R(M)$ is a SIP ring.*

Proof:- Let M be direct projective module and for every $f \in End_R(M)$, we have $M/Ker(f) \cong Img(f)$. Since M is a dual Rickart module, therefore $Img(f)$ is a direct summand. Since M is a direct projective module so $Ker(f)$ is a direct summand of M . Therefore, its $End_R(M)$ is Von Neumann's regular ring hence, $End_R(M)$ is a SIP ring. Conversely, if $End_R(M)$ is a SIP ring, then for every $f \in End_R(M)$, we have $Ker(f)$ as a direct summand of M . Therefore M is a direct projective module.

Corollary 2.1.13. *Let M be a dual Rickart module and $S = End_R(M)$ be its endomorphism ring. Then M is a direct projective module if and only if $Mat_2(S)$ is a SIP ring.*

Proof:- Follows from the fact that if $S = \text{End}_R(M)$ is a von Neumann Regular ring then, From [27, Proposition 3.16] $\text{Mat}_2(S)$ is also Von Neumann Regular ring.

Proposition 2.1.14. *Let M be an injective module over a hereditary ring R . Then M is a direct projective module whenever M is a dual Rickart module.*

Proof:- Let M be an injective module over a hereditary ring R , every quotient module of M is injective. Since M is dual Rickart therefore for each $f \in \text{End}_R(M)$, we have $\text{Img}(f) \leq^\oplus M$. Therefore by the Fundamental Theorem of module homomorphism $M/\text{Ker}(f) \cong \text{Img}(f) \leq^\oplus M$. Since $M/\text{Ker}(f)$ is injective, this implies $\text{Ker}(f)$ is a direct summand of M for each $f \in \text{End}_R(M)$, M is a direct projective module.

An R -module M is said to satisfy $(**)$ property [47], if each non-zero endomorphism of M is an epimorphism. Using this property, we find equivalent conditions among direct projective modules, Endoregular modules, and Hopfian modules.

Proposition 2.1.15. *Let M be an R -module with $(**)$ property. Then every direct summand of M has $(**)$ property.*

Proof:- We prove it by contradiction. Let N be a direct summand of M such that $M = N \oplus N'$ where N' is the complement of N . We assume that N doesn't have $(**)$ property, therefore $\exists f \in \text{End}_R(N)$ such that $\text{Img}(f) \neq N$. Since M has $(**)$ property implies for each $h \in \text{End}_R(M)$, we have $\text{Img}(h) = M$. In particular if $h = f \oplus f'$ this implies $h(M) = (f \oplus f')M$ here $f' \in \text{End}_R(N')$ and since $\text{Img}(f) \neq N$, hence $h(M) \neq M$ implies M doesn't have $(**)$ property, which is contraction to our assumption. Hence N has $(**)$ property.

Proposition 2.1.16. *Let M be an R -module. Then the following statements are equivalent:*

1. M is a direct projective module with $(**)$ property;
2. M is a Hopfian module with $(**)$ property;
3. M is an indecomposable Endoregular module.

Proof:- (1) \Rightarrow (2) Let M be a direct projective module with $(**)$ property, then for every non zero endomorphism f in S , $Img(f) = M \leq^{\oplus} M$. Using Lemma 2.5, we get that $Ker(f) \leq^{\oplus} M$. Therefore $Ker(f) = \{0\} \leq^{\oplus} M$. Hence f is a monomorphism, so M is a Hopfian module.

(2) \Rightarrow (3) Let f be a non-zero endomorphism in $S = End_R(M)$. Since M has $(**)$ property, therefore f is an epimorphism. Since M is a Hopfian module, therefore, f becomes an automorphism. Hence, $Ker(f) = \{0\}$ and $Img(f) = M$. Therefore $Ker(f)$ and $Img(f)$ are direct summands of M . Therefore, S is von-Neumann regular ring and M is an Endoregular module. Since we know that for every module with $(**)$ property, the endomorphism ring S is a domain and the module is a torsion-free module. This implies M has no non-trivial idempotents, so M is indecomposable.

(3) \Rightarrow (1) Since M is an Endoregular module therefore M is a direct projective module, also M is indecomposable implies for every non zero $f \in S$, we get $Img(f) = M$. Hence M satisfies the $(**)$ property.

Now we use the fact that for an abelian endoregular R -module M , we have

$$M = Ker(s) \oplus Img(s) \text{ for all } s \in End_R(M).$$

Proposition 2.1.17. *If M is an abelian endoregular module with $(**)$ property. Then $End_R(M)$ is a division ring.*

Proof:- Let M be an abelian endoregular module with $(**)$ property, then each non-zero endomorphism is an epimorphism. We can write module as $M = Ker(s) \oplus Img(s)$ for all $s \in End_R(M)$. Since each $s \in End_R(M)$ is an epimorphism, therefore

$\text{Img}(s) = M$ and $\text{Ker}(s) \cong \{0\}$. Therefore each epimorphism becomes an automorphism. Thus, every non-zero endomorphism is invertible, hence $\text{End}_R(M)$ is a division ring.

Corollary 2.1.18. *If an R -module M is a direct projective module with $(**)$ property. Then every epimorphism from M to its direct summand is a projection map.*

Proof:- Let M be a direct projective module, and N be a direct summand of M with projection $\pi : M \rightarrow N$. Then any epimorphism $g : M \rightarrow N$ there exists $f \in \text{End}_R(M)$ such that $g \circ f = \pi$. Now since M has $(**)$ property implies for every $f \in \text{End}_R(M)$, we have $\text{Img}(f) = M$. Hence $g \circ f(M) = g(M) = \pi(M)$, implies every epimorphism is a projection map.

Next, we characterize direct projective modules with respect to weakly Hopfian modules. Let M be a R -module, and M is called a weakly Hopfian module if any small surjection of M is an isomorphism or $M/N \cong M$ for any small submodule N of M if and only if $N = 0$. Also, we find equivalent conditions among Hopfian, weakly Hopfian, and Dedekind finite modules. An R -module M is called Dedekind finite module if $M \oplus N \cong M$ for some module N , then $N = 0$. Recalling the fact [7] that M is Dedekind finite module if and only if $\text{End}_R(M)$ is Dedekind finite ring. A ring is called Dedekind finite, for some $a, b \in R$ if $ba = 1$ implies $ab = 1$.

Proposition 2.1.19. *The following statements are equivalent for a direct projective R -module M :*

1. M is a Hopfian module;
2. M is a weakly Hopfian module;
3. M is a Dedekind finite module.

Proof:- (1) \Rightarrow (2) Follows from the fact that each Hopfian module is a weakly Hopfian module.

(2) \Rightarrow (3) Let M be a weakly Hopfian module and for $M/N \cong M$ we get $N = 0$. Since $M \oplus N \cong M$ for small submodule N , then $N = 0$. Hence M is Dedekind finite. (3) \Rightarrow (1) Let M be any Dedekind finite module, then its endomorphism ring is a Dedekind finite ring. Consider epimorphism $g : M \rightarrow M$, then g splits since M is direct projective. Hence there exists an endomorphism f of M satisfies $g \circ f = i_M$ where i_M is an identity map. Also, M is Dedekind finite therefore $f \circ g = i_M$. Hence g is an isomorphism and M is a Hopfian module.

An R -module M is said to satisfy the exchange property if for every R -module N and any two direct sum decompositions $N = M_1 \oplus N' = \bigoplus_{i \in I} N_i$ with $M \cong M_1$, there exist submodules P_i of N_i such that $N = M' \oplus (\bigoplus_{i \in I} P_i)$. M is said to satisfy finite exchange property if this holds only for any finite index set I . A ring R is said to be an exchange ring if the module R_R satisfies the exchange property. Warfield [20] proved that a module satisfies finite exchange property if and only if its endomorphism ring satisfies exchange property.

Proposition 2.1.20. *Let R -module M be a direct-projective such that $\text{Img}(s)$ lies under a direct summand of M for each $s \in \text{End}_R(M)$. Then M satisfies the finite exchange property.*

Proof:- Let M be a direct-projective module and $S = \text{End}_R(M)$. Since $\text{Img}(s)$ lies under a direct summand of M for every $s \in \text{End}_R(M)$, then S is a semiregular ring [34]. Since each semiregular ring is an exchange ring, this implies that S is an exchange ring; hence M has the finite exchange property.

Corollary 2.1.21. *Every direct projective module with $\text{Img}(f)$ direct summand for every $f \in \text{End}_R(M)$ satisfies finite exchange property.*

Proof:- Follows from the above proposition.

2.2 Semi-Simple Direct Projective Modules

Next, we generalize direct projective modules into a new notion as semi-simple direct projective modules. The class of semi-simple direct projective modules lies between the class of direct projective modules and simple direct projective modules. The simple direct projective modules was introduced in [18] which are defined as an R -module M is simple direct projective if for every simple direct summand K of M such that $M/T \cong K \leq^{\oplus} M$, then T will also be direct summand of M

Definition 2.2.1. *An R -module M is said to be semi-simple direct projective if for every semi-simple direct summand K of M such that $M/T \cong K \leq^{\oplus} M$, then T is a direct summand of M .*

Example 2.2.2. *The following are some examples of semi-simple direct projective modules:*

1. *Every module with $\text{rad}(M) = 0$ is a semi-simple direct projective module; in this case, every simple submodule is a direct summand.*
2. *An indecomposable R -module M which is not simple, then $M \oplus E(M)$ is a semi-simple direct projective module. Since $M \oplus E(M)$ has no simple summands, it is a semi-simple direct projective trivially.*

Remark 2.2.3. *The following implications hold and are irreversible:*

Direct Projective module \implies Semi-simple direct Projective module \implies Simple direct Projective module.

To prove the above implications are strict. Following are some examples.

1. Let us consider \mathbb{Z} -module $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ this module is semi-simple direct projective but not a direct projective module.
2. Consider $S = \prod_{p \in P} \mathbb{Z}_p$ and $S' = \bigoplus_{p \in P} \mathbb{Z}_p$ where P is set of primes. We claim that \mathbb{Z} -module $M = S \oplus S'$ is a simple direct projective module but not a semi-simple direct projective module. Since an epimorphism from S to S' does not split as S' is an essential submodule of S . However, M is a simple direct projective as every epimorphism from M to its simple summand splits. Moreover, we observe that the class of semi-simple direct projective modules and the class of simple direct projective modules are equivalent over an Artinian ring.
3. Every finitely generated module is a semi-simple direct projective module if it is a simple direct projective module.

Proposition 2.2.4. *Every direct summand of the semi-simple direct projective module is semi-simple direct projective.*

Proof:- Let M be a semi-simple direct projective module, and N be any direct summand of M . If N has no semi-simple summand, then N is semi-simple direct projective. In another case, let us consider P as a semi-simple summand of N . Since $P \leq^{\oplus} N$ implies $P \leq^{\oplus} M$. Therefore there exists X submodule of M such that, $M/X \cong P \leq^{\oplus} M$ and $X \leq^{\oplus} M$ (As M semi-simple direct projective). Since $P \oplus X = M$ implies $(P \oplus X) \cap N = M \cap N$ by Modular law $P \oplus (X \cap N) = N$ implies $(X \cap N) \leq^{\oplus} N$. Therefore we can write $N/(X \cap N) \cong P \leq^{\oplus} N$. Since P is an arbitrary semi-simple summand. Hence N is a semi-simple direct projective module.

Proposition 2.2.5. *Let M be a semi-simple direct projective module where $M = M_1 \oplus M_2$ and M_2 is a semi-simple summand of M . If $f : M_1 \rightarrow M_2$, be homomorphism, then $\text{Ker}(f)$ is a direct summand of M_1 .*

Proof:- Let $f : M_1 \rightarrow M_2$ be a module homomorphism. Consider canonical projection $\pi : M \rightarrow M_1$, then $f \circ \pi : M \rightarrow M_2$. Therefore $M/(\text{Ker}(f \circ \pi)) \cong \text{Img}(f) \leq^\oplus M_2$. By definition of semi-simple direct projective modules $\text{Ker}(f \circ \pi) \leq^\oplus M$. But $\text{Ker}(f \circ \pi) = M_2 \oplus \text{Ker}(f)$ therefore, $\text{Ker}(f) \leq^\oplus M$. Since $\text{Ker}(f)$ is a submodule of M_1 implies $\text{Ker}(f) \leq^\oplus M_1$.

Corollary 2.2.6. *The following statements are equivalent for semi-simple direct projective module M :*

1. *Every epimorphism from module M to its semi-simple summand splits;*
2. *Let K and T are direct summands of M , such that M/K and M/T are semi-simple summands, then $T \cap K$ is a direct summand of M ;*

Proof:- (1) \Rightarrow (2) Let f be an epimorphism from $f : M \rightarrow M/K$ and g be an epimorphism such that $g : M \rightarrow M/T$. Since M is semi-simple direct projective hence both g and f split. Let g' be the required splitting of g , then $f \circ g' : M/T \rightarrow M/K$. And $\text{Ker}(f \circ g') = T \cap K$. Again due to M semi-simple direct projective $\text{Ker}(f \circ g')$ direct summand of M . Hence $T \cap K$ is a direct summand of M .

(2) \Rightarrow (1) Using the second theorem of homomorphism, we know that $M/K \cong T/(T \cap K)$. Let f be an epimorphism such that $f : M \rightarrow T/(T \cap K)$, we must prove that f splits. Since $\text{Ker}(f) = (T \cap K)$, which is a direct summand from (3), hence f splits.

Proposition 2.2.7. *The following statements are equivalent for an R -module M :*

1. R is a semi-simple Artinian ring;
2. Every factor module of $R \oplus R$ is semi-simple direct projective;
3. Every 2-generated R -module is a semi-simple direct projective.

Proof:- (1) \Rightarrow (2) and (1) \Rightarrow (3) are obvious.

(2) \Rightarrow (3) If every factor module of $R \oplus R$ is semi-simple direct projective, then 2-generated R -module is semi-simple direct projective. Since it is a homomorphic image of $R \oplus R$

(3) \Rightarrow (1) Let $R \oplus S$ be 2-generated R -module and it is semi-simple direct projective where S is semi-simple in R . Then epimorphism $f : R \rightarrow S$ splits. Hence S is projective in R .

In the following proposition, we discuss under what condition *semi-simple direct projective modules* are equivalent to quasi-projective modules.

Proposition 2.2.8. *Let R be an Artinian serial ring and Jacobson radical $J^2(R) = 0$. Then every semi-simple direct projective module over R is a quasi-projective module.*

Proof:- In Remark 2.2.3 we concluded that a simple direct projective module is equivalent to a semi-simple direct projective module over an Artinian ring. Here proof is followed from [18, Proposition 4.4].

The definition of a semi-simple direct projective envelope module is given below.

Definition 2.2.9. *An R -homomorphism $f : M \rightarrow P$ is called a semi-simple direct projective envelope of M if P is a semi-simple direct projective and diagram*

commutes

$$\begin{array}{ccc}
 & & P' \\
 & \nearrow f' & \downarrow \exists \Phi' \\
 M & \xrightarrow{f} & P
 \end{array}$$

where P' semi-simple direct projective module and Φ' is an isomorphism,

$$\begin{array}{ccc}
 & & P \\
 & \nearrow f & \downarrow \exists \Phi \\
 M & \xrightarrow{f} & P
 \end{array}$$

and the diagram above can be completed by an automorphism Φ .

Proposition 2.2.10. *The following statements are equivalent:*

1. R is a semi-simple Artinian ring;
2. Every finitely generated R -module has a semi-simple direct projective envelope;
3. Every 2-generated R -module has a semi-simple direct projective envelope.

Proof:- (1) \Rightarrow (2) and (2) \Rightarrow (3) are obvious.

(3) \Rightarrow (1) We must prove that every semi-simple submodule S of R -module M is projective. Let $h : R \rightarrow S$ is an epimorphism. Due to our assumption, we know that $M = R \oplus S$ is 2-generated and has a semi-simple direct projective envelope. By definition of semi-simple direct projective envelope let $g : M \rightarrow P$ is a homomorphism and P is a semi-simple direct projective module. Since S and R both are semi-simple direct projective modules, so from the diagram below,

$$\begin{array}{ccc}
 & & P \\
 & \nearrow g & \downarrow \exists \beta \\
 M & \xrightarrow{ps} & S
 \end{array}$$

$$\begin{array}{ccc}
 & & P \\
 & \nearrow g & \downarrow \exists \alpha \\
 M & \xrightarrow{p_R} & R
 \end{array}$$

where p_R and p_S are projection map from M to R and M to S respectively. This implies $\beta \circ g = p_S$ and similarly $\alpha \circ g = p_R$. So combining these two we get $p_S \oplus p_R = (\beta \oplus \alpha) \circ g$, hence g splits. Therefore M is the direct summand of a semi-simple direct projective module. From Proposition 2.2.4 we get M is a semi-simple direct projective. Therefore epimorphism h splits from Proposition 2.1.2, hence S is projective in R .