CHAPTER 2

SOME FIXED-TIME STABILITY THEOREMS FOR IMPULSIVE SYSTEMS

2.1 Fixed-Time Stability of Dynamical Systems with Impulsive Effects

2.1.1 Introduction[1]

The objective of the present chapter of the thesis is to discuss some theoretical results on the stability of nonlinear impulsive dynamical systems within the fixed-time. Different from the Lyapunov stability theory and finite-time stability theory, fixed-time stability (FxTS) can ensure that the system's state trajectory converges to the equilibrium state after some fixed-time, which is called the settling-time function. The settling-time function is bounded and independent of the system's initial condition. Polyakov [78] first introduced the theory and derived a sufficient condition on FxTS of nonlinear systems as well as an estimation of settling-time. Subsequently, some interesting results have been obtained for FxTS of nonlinear systems [79, 80, 101]. The authors here have derived their results for the nonlinear systems without any impulsive effects. Recently, a significant number of articles [71, 94, 102, 103] have addressed the FTS/FxTS of nonlinear impulsive systems in which several results have been derived. It is worth

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mentioning the important studies on the stability of systems having stabilizing impulses as reported in the references [87, 89, 104, 105]. However, the effects of destabilizing impulses on FxTS are not explored in [87, 89]. Therefore, it is worth contributing to the analysis of the result of FxTS of impulsive systems with destabilizing impulses. The current chapter consideres two different problems in this respect that are presented in two subchapters. In the first subchapter, a new Lyapunov inequality has been designed to find several results for FxTS of nonlinear impulsive systems. The second subchapter addresses some results for the Cohen-Grossberg bidirectional associative memory neural network under destabilizing impulses. Based on the proposed Lyapunov inequality, we have designed the controller such that Cohen-Grossberg bidirectional associative memory achieve stability within a fixed-time. Moreover, we have verified the sufficient condition under which the system achieves FxTS under destabilizing impulses.

The present subchapter deals with the FxTS analysis of the nonlinear dynamical systems with impulsive effects. A novel criteria has been derived to achieve the FxTS of nonlinear systems under the effects of stabilizing and destabilizing impulses. The FxTS analysis due to the presence of destabilizing impulses in a dynamical system, that leads to behaviour of perturbing the systems' stability, has not been addressed much in the existing literature [87, 89]. Based on the concept of Lyapunov functional and average impulsive interval, two novel FxTS theorems are constructed here for stabilizing and destabilizing impulses separately to estimate the fixed-time convergence precisely. The theoretical derivation shows that the estimated fixed-time in this study is less conservative and more accurate as compared to the classical FxTS result presented in [87]. Further, the theoretical results are applied to the impulsive control of general neural network systems to guarantee the FxTS.

2.1.2 Problem Formulation and Preliminaries

Let us consider the dynamical system with impulsive effects described by

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \neq t_k, \\ \Delta x(t) = J_k\left(t_k, x(t_k)\right), & t = t_k,\ k \in \mathbb{N}, \\ x(t_0) = x_0, \end{cases} \tag{2.1.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state variable of the system (2.1.1) at instant t, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $J_k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions, and f meets the Lipschitz condition with respect to x. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+) = \lim\limits_{t \to t_k+0} x(t)$ and $x(t_k^-) = \lim\limits_{t \to t_k-0} x(t)$. Without any loss of generality, it is assumed that $x(t_k^-) = x(t_k)$, $k \in \mathbb{N}$, i.e., the solution of system (2.1.1) is left continuous at t_k. The impulsive time sequence $\zeta = \{t_k\}_{k\in\mathbb{N}}$ satisfies $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots,$ with $\lim\limits_{k\to\infty} t_k = +\infty$.

We recall following definitions to establish our main results:

Definition 2.1.2.1. (See [50]) The function $W : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to the class w_0 if

1. the function W is continuous on each of the sets $[t_k, t_{k+1}) \times \mathbb{R}^n$ and for $x \in \mathbb{R}^n$, $k \in \mathbb{N}$, $W(t_k^+, x) = \lim\limits_{(t,y)\to(t_k^+, x)} W(t, y)$ exists.
2. W is locally Lipschitz in x.

Definition 2.1.2.2. (See [87]) The origin of the system (2.1.1) is said to be globally finite-time stable (FTS) if it is globally Lyapunov stable and there exists a bounded function $T : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$, such that $x(t, x_0) = 0$ for all $t \geq T(x_0, \{t_k\})$. The function T is called the settling-time function.

Definition 2.1.2.3. (See [87]) The origin of system (2.1.1) is said to be globally FxTS if it is globally FTS, and the settling-time function T is independent of x_0, i.e., there exists $0 < T_{max} < \infty$ such that $T(x_0, \{t_k\}) \leq T_{max}$ for all $x_0 \in \mathbb{R}^n$.

In this subchapter, the FxTS of nonlinear impulsive system (2.1.1) is based on the class of impulsive sequences which satisfies the following definition.

Definition 2.1.2.4. (See [106]) Suppose $N_\zeta(s,t)$ is the number of impulsive time of the impulsive sequence $\zeta = \{t_1, t_2, t_3, \dots\}$ in the interval (t,s). If there exist a positive real number τ_a and a positive integer N_0, then the upper and lower bounds of $N_\zeta(s,t)$ can be estimated as

$$\frac{s-t}{\tau_a} - N_0 \leq N_\zeta(s,t) \leq \frac{s-t}{\tau_a} + N_0, \forall s \geq t \geq 0,$$

where the average impulsive interval of impulsive sequence $\zeta = \{t_l\}_{l \in \mathbb{N}}$ is equal to τ_a.

2.1.3 Main Results

The goal of this subsection is to derive certain sufficient conditions to guarantee the FxTS of system (2.1.1) under both the stabilizing and destabilizing impulsive effects.

Theorem 2.1.3.1. (Theorem on FxTS of systems with stabilizing impulses): For the system (2.1.1), if there exists a function $W(t, x(t)) \in w_0$ and some positive constants w_1, w_2, a, m_1, m_2, m_3, q_1, q_2, α satisfying $q_1 > 1$, $0 < q_2 < 1$ and impulsive strength $0 < \alpha \leq 1$ such that the following conditions hold:

(i) $w_1||x(t)||^a \leq W(t, x(t)) \leq w_2||x(t)||^a$, for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$,

(ii) when $t \neq t_k$,

$$\dot{W}(t,x(t)) \leq \begin{cases} -m_2 W^{q_2}(t,x(t)) - m_3 W(t,x(t)), & 0 < W(t,x(t)) < 1, \\ -m_3 W(t,x(t)) - m_1 W^{q_1}(t,x(t)), & W(t,x(t)) \geq 1, \\ 0, & W(t,x(t)) = 0, \end{cases}$$

(iii) when $t = t_k$, $W(t_k^+, x(t_k^+)) \leq \alpha W(t_k, x(t_k))$,

then the system (2.1.1) is FxTS, and the associated settling-time T is estimated as $T = \frac{\ln\left[1+\frac{m_1}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)\alpha^{N_0(1-q_1)}}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)} + \frac{\ln\left[\frac{m_2\alpha^{N_0(1-q_2)}}{\alpha^{-N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)}$, for $0 < \alpha < 1$ and $T = \frac{\ln\left[1+\frac{m_1}{m_3}\right]}{m_3(q_1-1)} - \frac{\ln\left[\frac{m_2}{m_3+m_2}\right]}{m_3(1-q_2)}$, for $\alpha = 1$.

Proof. Consider the following comparison system proposed as

$$\begin{cases} \dot{V}(t,x(t)) = \begin{cases} -m_2 V^{q_2}(t,x(t)) - m_3 V(t,x(t)), & 0 < V(t,x(t)) < 1,\ t \neq t_k, \\ -m_3 V(t,x(t)) - m_1 V^{q_1}(t,x(t)), & V(t,x(t)) \geq 1, \quad t \neq t_k, \\ 0, & V(t,x(t)) = 0, \quad t \neq t_k, \end{cases} \\ V(t_k^+, x(t_k^+)) = \alpha V(t_k, x(t_k)),\ t = t_k, \\ V(0,x(0)) = V_0. \end{cases} \tag{2.1.2}$$

Comparing the conditions (ii) and (iii) with Eq. (2.1.2), one can easily find that $0 \leq W(t,x(t)) \leq V(t,x(t))$. Therefore, $W(t,x(t)) = 0$ if $V(t,x(t)) = 0$. Now, it is therefore needed to prove the results corresponding to the problem of system (2.1.2) to establish the FxTS of system (2.1.1).

For convenience, denote $V(t) = V(t,x(t))$. When $V(t) \geq 1$, the term $-m_3 V(t) - m_1 V^{q_1}(t)$ will be a dominating one in the system (2.1.2). Let $S(t) = V^{1-q_1}(t)$, which implies that $S(t) \to 1$ as $V(t) \to 1$, and $S(t) \to 0$ as $V(t) \to \infty$. The equation satisfied by $S(t)$ can be obtained as

$$\begin{cases} \dot{S}(t) = m_3(q_1-1)S(t) + m_1(q_1-1), & 0 < S(t) \leq 1,\ t \neq t_k, \\ S(t_k^+) = \alpha_1 S(t_k), & t = t_k, \\ S(0) = S_0 = V_0^{1-q_1}, \end{cases} \tag{2.1.3}$$

where $\alpha_1 = \alpha^{1-q_1} \in [1, \infty)$. Note that $V(t) \to 1$ is similar to $S(t) \to 1$. Let us investigate the system (2.1.2) for the following two cases: $0 < \alpha < 1$ and $\alpha = 1$.

Firstly, for the case of $0 < \alpha < 1$, one has $\alpha_1 \in (1, \infty)$. Based on the formula of variation of parameters, one can obtain

$$S(t) = V(t,0)S(0) + m_1(q_1-1)\int_0^t V(t,s)ds,\ t \geq 0, \tag{2.1.4}$$

where

$$\begin{aligned} V(t,s) &= e^{m_3(q_1-1)(t-s)} \prod_{s \leq t_k \leq t} \alpha_1 \\ &= e^{m_3(q_1-1)(t-s)} \alpha_1^{N_\zeta(t,s)}. \end{aligned}$$

Substituting $V(t,s)$ in Eq. (2.1.4), we obtain

$$S(t) = e^{m_3(q_1-1)t}\alpha_1^{N_\zeta(t,0)} S(0) + m_1(q_1-1)\int_0^t e^{m_3(q_1-1)(t-s)}\alpha_1^{N_\zeta(t,s)}ds,\ t \geq 0. \tag{2.1.5}$$

Using the transformation $S(t) = V^{1-q_1}(t)$, the Eq. (2.1.5) can be written as

$$\begin{aligned} V^{q_1-1}(t) &\leq \left[e^{m_3(q_1-1)t}\alpha_1^{N_\zeta(t,0)} V^{1-q_1}(0) + m_1(q_1-1)\int_0^t e^{m_3(q_1-1)(t-s)}\alpha_1^{N_\zeta(t,s)}ds\right]^{-1} \\ &\leq e^{-m_3(q_1-1)t}\alpha_1^{-\frac{t}{\tau_a}+N_0} V^{q_1-1}(0), \end{aligned}$$

$$\begin{aligned} V(t) \leq & e^{-[m_3+\frac{\ln\alpha_1}{\tau_a(q_1-1)}]t}\alpha_1^{\frac{N_0}{(q_1-1)}}V(0) \\ \leq & \alpha_1^{\frac{N_0}{(q_1-1)}}V(0). \end{aligned} \tag{2.1.6}$$

From condition (i) and inequality (2.1.6), the solution of the system (2.1.1) is therefore Lyapunov stable. Now, we will proceed to show FxTS of the nonlinear impulsive system (2.1.1).

In view of Eq. (2.1.5), it is clear that $S(0) < 1$, therefore $\lim\limits_{t\to\infty} S(t) = \infty$ and $S(t)$ is monotonically increasing on $[0, \infty)$. This implies that there exists a $T_1 > 0$ such that $\lim\limits_{t\to T_1} S(t) = 1$ where $0 < S(t) < 1$ for $0 < t < T_1$. Now, for $t \geq T_1$, we have $S(t) = 1$. Therefore, it can be derived from Eq. (2.1.5) that

$$e^{m_3(q_1-1)t}\left[\alpha_1^{N_\zeta(t,0)}S(0) + m_1(q_1-1)\int_0^t e^{-m_3(q_1-1)s}\alpha_1^{N_\zeta(t,s)}ds\right] = 1, \tag{2.1.7}$$

which implies that

$$m_1(q_1-1)e^{m_3(q_1-1)t}\int_0^t e^{-m_3(q_1-1)s}\alpha_1^{N_\zeta(t,s)}ds \leq 1. \tag{2.1.8}$$

Considering the definition of $N_\zeta(t,s)$, one can derive from Eq. (2.1.8) that

$$\begin{aligned} m_1(q_1-1)e^{m_3(q_1-1)t}\int_0^t e^{-m_3(q_1-1)s}\alpha_1^{\frac{t-s}{\tau_a}-N_0}ds \leq & 1, \\ i.e.,\ \alpha_1^{-N_0}m_1(q_1-1)e^{m_3(q_1-1)t}e^{\frac{\ln\alpha_1}{\tau_a}t}\int_0^t e^{-\left[m_3(q_1-1)+\frac{\ln\alpha_1}{\tau_a}\right]s}ds \leq & 1, \end{aligned}$$

$$i.e.,\ \left[m_3(q_1-1)+\frac{\ln\alpha_1}{\tau_a}\right]t \leq \ln\left[1+\frac{m_3(q_1-1)+\frac{\ln\alpha_1}{\tau_a}}{\alpha_1^{-N_0}m_1(q_1-1)}\right],$$

$$i.e., \left[m_3(q_1-1)+\frac{\ln\alpha_1}{\tau_a}\right]t \leq \ln\left[1+\frac{m_3(q_1-1)+\frac{\ln\alpha_1}{\tau_a}}{\alpha_1^{-N_0}m_1(q_1-1)}\right],$$

$$i.e.,\ t \leq \frac{\ln\left[1+\frac{\alpha_1^{N_0}\left(m_3(q_1-1)+\frac{\ln\alpha_1}{\tau_a}\right)}{m_1(q_1-1)}\right]}{\left[m_3(q_1-1)+\frac{\ln\alpha_1}{\tau_a}\right]}.$$

Substituting $\alpha_1=\alpha^{1-q_1}$, the above inequality leads to

$$t \leq \frac{\ln\left[1+\frac{\alpha^{N_0(1-q_1)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)}. \tag{2.1.9}$$

Assume $T_1=\frac{\ln\left[1+\frac{\alpha^{N_0(1-q_1)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)}$.

In view of the system (2.1.3), it is easy to obtain

$$S(t;0,S_0)=\begin{cases} e^{m_3(q_1-1)t}\alpha_1^{N_\zeta(t,0)}S(0)+m_1(q_1-1)\int_0^t e^{m_3(q_1-1)(t-s)}\alpha_1^{N_\zeta(t,s)}ds, & 0\leq t<T_1,\\ 1, & t=T_1, \end{cases} \tag{2.1.10}$$

where $T_1=\frac{\ln\left[1+\frac{\alpha^{N_0(1-q_1)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)}$.

When $0<V(t)<1$, the dominating term will be replaced by a new term $-m_2V^{q_2}(t)-m_3V(t)$. Let $S(t)=V^{1-q_2}(t)$, when $0<V(t)<1$. Since $0<q_2<1$, it follows that $S(t)\to 1$ when $V(t)\to 1$ and $S(t)\to 0$ when $V(t)\to 0$. Therefore, the corresponding equation for $S(t)$ is as follows:

$$\begin{cases} \dot{S}(t) = -m_2(1-q_2) - m_3(1-q_2)S(t), & 0 < S(t) < 1,\, t \neq t_k,\, t \geq T_1, \\ S(t_k^+) = \alpha_2 S(t_k), & t = t_k,\, t \geq T_1, \\ S(T_1) = S(T_1; 0, S_0) = 1, \end{cases} \tag{2.1.11}$$

in which $\alpha_2 = \alpha^{1-q_2} \in (0,1]$. Note that $V(t) \to 0$ is similar to $S(t) \to 0$. In the case when $0 < \alpha < 1$, one has $\alpha_2 \in (0,1)$.

Now, from the formula of variation of parameters, we get

$$S(t) = V(t,T_1)S(T_1) - m_2(1-q_2)\int_0^t V(t,s)ds,$$

where

$$\begin{aligned} V(t,s) &= e^{-m_3(1-q_2)(t-s)}\alpha_2^{N_\zeta(t,s)}, \\ V(t,0) &= e^{-m_3(1-q_2)t}\alpha_2^{N_\zeta(t,0)}. \end{aligned}$$

When $0 < S(t) < 1$, it can be derived from system (2.1.11) that

$$S(t) = e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{N_\zeta(t,T_1)}S(T_1) - m_2(1-q_2)\int_{T_1}^t e^{-m_3(1-q_2)(t-s)}\alpha_2^{N_\zeta(t,s)}ds. \tag{2.1.12}$$

Since $S(T_1) = \alpha_2^{N_\zeta(T_1,T_1)}S(T_1) = 1$ and $S(t)$ is monotonically increasing on $[0,\infty)$, there exists a $T_2 > 0$ such that $\lim\limits_{t \to T_2} S(t) = 0$.

When $0 < \alpha < 1$, it follows from the Eq. (2.1.12) and the definition of $N_\zeta(t,s)$ that

$$\begin{aligned} S(t) &= e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{N_\zeta(t,T_1)}S(T_1) - m_2(1-q_2)\int_{T_1}^t e^{-m_3(1-q_2)(t-s)}\alpha_2^{N_\zeta(t,s)}ds, \\ &\leq e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{\frac{t-T_1}{\tau_a}-N_0} - m_2(1-q_2)\int_{T_1}^t e^{-m_3(1-q_2)(t-s)}\alpha_2^{\frac{t-s}{\tau_a}+N_0}ds \end{aligned}$$

$$
\begin{aligned}
i.e., S(t) &\leq e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{\frac{t-T_1}{\tau_a}-N_0} - m_2(1-q_2)e^{-m_3(1-q_2)t}\alpha_2^{\frac{t}{\tau_a}+N_0}\int_{T_1}^{t} e^{\left[m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right]s}ds \\
&= e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{\frac{t-T_1}{\tau_a}-N_0} - m_2(1-q_2)\alpha_2^{N_0}\left[\frac{1-e^{-\left[m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right](t-T_1)}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}\right] \\
&= \left[\alpha_2^{-N_0} + \frac{m_2(1-q_2)\alpha_2^{N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}\right]e^{-\left[m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right](t-T_1)} - \frac{m_2(1-q_2)\alpha_2^{N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}.
\end{aligned}
\tag{2.1.13}
$$

Considering $A = \alpha_2^{-N_0} + \frac{m_2(1-q_2)\alpha_2^{N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}$ and $B = \frac{m_2(1-q_2)\alpha_2^{N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}$, we can note that $A > B$ and $Ae^{-\left[m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right](t-T_1)}$ is continuously decreasing, i.e., the right-hand side of Eq. (2.1.13) must be zero after some time. We therefore have

$$
\begin{aligned}
e^{-\left[m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right](t-T_1)} &= \frac{m_2(1-q_2)\alpha_2^{N_0}}{\alpha_2^{-N_0}\left(m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right)+m_2(1-q_2)\alpha_2^{N_0}} \\
i.e., t &= \frac{\ln\left[\frac{m_2(1-q_2)\alpha_2^{N_0}}{\alpha_2^{-N_0}\left(m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right)+m_2(1-q_2)\alpha_2^{N_0}}\right]}{\frac{\ln\alpha_2}{\tau_a}-m_3(1-q_2)} + T_1.
\end{aligned}
$$

Substituting $\alpha_2 = \alpha^{1-q_2}$, the above inequality leads to

$$
t = \frac{\ln\left[\frac{m_2\alpha^{N_0(1-q_2)}}{\alpha^{-N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)} + T_1.
\tag{2.1.14}
$$

Assume $T_2 = \frac{\ln\left[\frac{m_2\alpha^{N_0(1-q_2)}}{\alpha^{-N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)}$.

Therefore, in view of Eq. (2.1.11), we can write

$$
S(t+T_1;0,S(T_1)) = \begin{cases} e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{N_\zeta(t,T_1)}S(T_1), & \\ -m_2(1-q_2)\int_{T_1}^{t} e^{-m_3(1-q_2)(t-s)}\alpha_2^{N_\zeta(t,s)}ds, & 0 \leq t < T_2, \\ 0, & t \geq T_2, \end{cases}
\tag{2.1.15}
$$

where $T_2 = \frac{\ln\left[\frac{m_2\alpha^{N_0(1-q_2)}}{\alpha^{-N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)}$.

On the basis of the above discussions, when $0<\alpha<1$, one may see that $S(t;0,S_0)=0$ for all $t\geq T_1+T_2$. It is therefore found that for any initial value $V_0>0$, a fixed time T_1+T_2 exists which is independent of $V_0>0$, such that $\lim_{t\to T_1+T_2} V(t)=0$ and $V(t)\equiv 0$ for $t\geq T_1+T_2$. Since $||x(t)||\leq \sqrt[a]{\frac{W(t,x(t))}{w_1}} \leq \sqrt[a]{\frac{V(t,x(t))}{w_1}}$, one has $\lim_{t\to T_1+T_2}||x(t)||=0$ and $||x(t)||\equiv 0$ for $t\geq T_1+T_2$. Hence, the system (2.1.1) is FxTS, and the associated settling-time $T=T_1+T_2$ is estimated as

$$T = \frac{\ln\left[1+\frac{\alpha^{N_0(1-q_1)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)} + \frac{\ln\left[\frac{m_2\alpha^{N_0(1-q_2)}}{\alpha^{-N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)}. \tag{2.1.16}$$

Secondly, when $\alpha=1$, it can be observed that $\alpha_1=1$ and $\alpha_2=1$, and fromEq. (2.1.5), we have

$$T_1 = \frac{\ln\left[1+\frac{m_3}{m_1}\right]}{m_3(q_1-1)}. \tag{2.1.17}$$

According to Eq. (2.1.12),

$$T_2 = \frac{\ln\left[\frac{m_2}{m_3+m_2}\right]}{-m_3(1-q_2)}. \tag{2.1.18}$$

Further, when $\alpha=1$, for any initial value $V_0>0$, $\lim_{t\to T_1+T_2} V(t)=0$ and $V(t)\equiv 0$ for $t\geq T_1+T_2$. Hence, the system (2.1.1) is FxTS, and the associated settling-time $T=T_1+T_2$ is estimated as

$$T = \frac{\ln\left[1+\frac{m_3}{m_1}\right]}{m_3(q_1-1)} - \frac{\ln\left[\frac{m_2}{m_3+m_2}\right]}{m_3(1-q_2)}. \tag{2.1.19}$$

For any initial value $V_0>0$, we know that based on the above analysis for two instances, there exists a $T>0$ such that $\lim_{t\to T} V(t)=0$ and $V(t)\equiv 0$ for $t\geq T$. It is followed by the condition $||x(t)|| \leq \sqrt[a]{\frac{W(t,x(t))}{w_1}} \leq \sqrt[a]{\frac{V(t,x(t))}{w_1}}$ that $\lim\limits_{t\to T} x(t) = 0$ and $x(t) \equiv 0$ for $t \geq T$. Hence, the system (2.1.1) is FxTS, with T as the settling time. The proof of Theorem 2.1.3.1 is therefore completed here.

Remark 2.1.3.1. Theorem 2.1.3.1 establishes a novel result on FxTS of impulsive system (2.1.1) which is different from the existing results as reported by Li et al. [87]. A less conservative and higher-precision estimate of settling time is provided in the proof of the Theorem 2.1.3.1. Since Lyapunov inequality has one more term than the inequality in [87], the convergence rate of FxTS of system (2.1.1) is faster than the convergence rate in [87]. The claim is verified in the numerical section of the subchapter.

Corollary 2.1.3.1. For the system (2.1.1), if there exists a function $W(t,x(t)) \in w_0$ and some positive constants w_1, w_2, a, m_1, m_2, m_3, q, q_1, q_2, α satisfying $qq_1 > 1$, $0 < qq_2 < 1$ and $0 < \alpha \leq 1$ such that the following conditions hold:

(i) $w_1||x(t)||^a \leq W(t,x(t)) \leq w_2||x(t)||^a$, for $t \in \mathbb{R}_+,\ \ x \in \mathbb{R}^n$,

(ii) when $t \neq t_k$,

$$\dot{W}(t,x(t)) \leq -\left(m_1 W^{q_1}(t,x(t)) + m_2 W^{q_2}(t,x(t)) + m_3 W(t,x(t))\right)^q,$$

(iii) when $t = t_k$, $W(t_k^+, x(t_k^+)) \leq \alpha W(t_k, x(t_k))$,

then the system (2.1.1) is FxTS, and the associated settling-time T is estimated by
$T = \frac{\ln\left[1+\frac{\alpha^{N_0(1-qq_1)}\left(m_3^q - \frac{\ln\alpha}{\tau_a}\right)}{m_1^q}\right]}{\left(m_3^q - \frac{\ln\alpha}{\tau_a}\right)(qq_1-1)} + \frac{\ln\left[\frac{m_2^q\alpha^{N_0(1-qq_2)}}{\alpha^{-N_0(1-qq_2)}\left(m_3^q - \frac{\ln\alpha}{\tau_a}\right)+m_2^q\alpha^{N_0(1-qq_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a} - m_3^q\right)(1-qq_2)}$, when $0 < \alpha < 1$, and
$T = \frac{\ln\left[1+\frac{m_3^q}{m_1^q}\right]}{m_3^q(q_1-1)} - \frac{\ln\left[\frac{m_2^q}{m_3^q+m_2^q}\right]}{m_3^q(1-q_2)}$, when $\alpha = 1$.

Corollary 2.1.3.2. Consider the following scalar system as

$$\begin{cases} \dot{z}(t) = -m_1\mathrm{sign}(z(t))|z(t)|^{q_1} - m_2\mathrm{sign}(z(t))|z(t)|^{q_2} - m_3\mathrm{sign}(z(t))|z(t)|, \quad t \neq t_k, \\ z(t_k^+) = \alpha z(t_k),\ \ k \in \mathbb{N}, \end{cases} \tag{2.1.20}$$

where $m_1 > 0$, $m_2 > 0$, $m_3 > 0$, $q_1 > 1$, $0 < q_2 < 1$, $0 < \alpha \leq 1$. Then the system (2.1.20) is FxTS, and the associated settling-time T is estimated by

$T = \frac{\ln\left[1+\frac{\alpha^{N_0(1-q_1)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)} + \frac{\ln\left[\frac{m_2\alpha^{N_0(1-q_2)}}{\alpha^{-N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)}$, when $0 < \alpha < 1$, and $T = \frac{\ln\left[1+\frac{m_3}{m_1}\right]}{m_3(q_1-1)} - \frac{\ln\left[\frac{m_2}{m_3+m_2}\right]}{m_3(1-q_2)}$, when $\alpha = 1$.

Remark 2.1.3.2. It should be noted that the impulsive effects are of two types viz., stabilizing impulsive effect and destabilizing impulsive effect. For $0 < \alpha \leq 1$, the impulses are beneficial in stabilizing the systems. It is clear from Theorem 2.1.3.1 that the impulse effects considered are stabilizing impulses. Can the impulsive systems maintain FxTS if the impulses are destabilizing? This problem will be solved by the following theorem.

Theorem 2.1.3.2. (Theorem on FxTS of systems with destabilizing impulses): If all the assumptions of Theorem 2.1.3.1 are satisfied except that the impulsive strength $0 < \alpha \leq 1$ is replaced by $\alpha > 1$, and if the following condition is satisfied:

$$\tau_a > \frac{\ln\alpha}{m_3}, \tag{2.1.21}$$

then the system (2.1.1) is FxTS, and the associated settling-time T is estimated by $\frac{\ln\left[1+\frac{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1\alpha^{N_0(1-q_1)}}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)} + \frac{\ln\left[\frac{m_2\alpha^{-N_0(1-q_2)}}{\alpha^{N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{-N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)}$.

Proof. Consider the following proposed comparison system:

$$\begin{cases} \dot{V}(t,x(t)) = \begin{cases} -m_2 V^{q_2}(t,x(t)) - m_3 V(t,x(t)), & 0 \leq V(t,x(t)) < 1,\ t \neq t_k, \\ -m_3 V(t,x(t)) - m_1 V^{q_1}(t,x(t)), & V(t,x(t)) \geq 1, \quad t \neq t_k, \\ 0, & V(t,x(t)) = 0, \quad t \neq t_k, \end{cases} \\ V(t_k^+, x(t_k^+)) = \alpha V(t_k, x(t_k)),\ t = t_k, \\ V(0,x(0)) = V_0. \end{cases} \tag{2.1.22}$$

Then by comparing conditions (ii) and (iii) with Eq. (2.1.22), we can easily obtain that $0 \leq W(t,x(t)) \leq V(t,x(t))$. Therefore, $W(t,x(t)) = 0$ if $V(t,x(t)) = 0$. Hence, in order to establish the FxTS of system (2.1.1), we have to prove the same corresponding to the problem of system (2.1.22).

For convenience, let us assume that $V(t) = V(t,x(t))$. When $V(t) \geq 1$, the term $-m_3V(t) - m_1V^{q_1}(t)$ will be the dominating term in the system (2.1.22). Let $S(t) = V^{1-q_1}(t)$, then we have $S(t) \to 1$ when $V(t) \to 1$ and $S(t) \to 0$ for $V(t) \to \infty$. The equation for $S(t)$ can be obtained as follows:

$$\begin{cases} \dot{S}(t) = m_3(q_1 - 1)S(t) + m_1(q_1 - 1), & t \neq t_k, \\ S(t_k^+) = \alpha_1 S(t_k), & t = t_k, \\ S(0) = S_0 = V_0^{1-q_1}, \end{cases} \tag{2.1.23}$$

where $\alpha_1 = \alpha^{1-q_1} \in (0,1)$. Note that $V(t) \to 1$ is similar to $S(t) \to 1$.

Now, from the formula of variation of parameters, when $0 < S(t) < 1$, one can derive from (2.1.23) that

$$S(t) = e^{m_3(q_1-1)t} \alpha_1^{N_\zeta(t,0)} S(0) + m_1(q_1 - 1) \int_0^t e^{m_3(q_1-1)(t-s)} \alpha_1^{N_\zeta(t,s)} ds. \tag{2.1.24}$$

Since $S(t) = V^{1-q_1}(t)$, therefore Eq. (2.1.24) can be written as

$$
\begin{aligned}
V^{q_1-1}(t) \leq & \left[e^{m_3(q_1-1)t} \alpha_1^{N_\zeta(t,0)} V^{1-q_1}(0) + m_1(q_1-1) \int_0^t e^{m_3(q_1-1)(t-s)} \alpha_1^{N_\zeta(t,s)} ds \right]^{-1} \\
\leq & e^{-m_3(q_1-1)t} \alpha_1^{-\frac{t}{\tau_a} - N_0} V^{q_1-1}(0), \\
i.e.,\ V(t) \leq & e^{-[m_3 - \frac{\ln \alpha}{\tau_a}]t} \alpha_1^{-\frac{N_0}{(q_1-1)}} V(0) \\
\leq & \alpha_1^{-\frac{N_0}{(q_1-1)}} V(0).
\end{aligned}
\tag{2.1.25}
$$

From condition (i), and inequalities (2.1.21) and (2.1.25), the solution of the system (2.1.1) is Lyapunov stable. Now, we will proceed to show the FxTS of the nonlinear impulsive system (2.1.1). It follows from Eq. (2.1.24), when $0 < \alpha_1 < 1$, there exists $T_1 > 0$ such that $\lim\limits_{t \to T_1} S(t) = 1$ and $0 < S(t) < 1$ for $0 < t < T_1$ and also considering the definition of $N_\zeta(t,s)$, we have

$$
\begin{aligned}
e^{m_3(q_1-1)t} m_1(q_1-1) \int_0^t e^{-m_3(q_1-1)s} \alpha_1^{N_\zeta(t,s)} ds \leq & 1, \\
i.e.,\ m_1(q_1-1) e^{m_3(q_1-1)t} \int_0^t e^{-m_3(q_1-1)s} \alpha_1^{\frac{t-s}{\tau_a} + N_0} ds \leq & 1, \\
i.e.,\ m_1(q_1-1) \alpha_1^{N_0} e^{m_3(q_1-1)t} e^{\frac{\ln \alpha_1}{\tau_a} t} \int_0^t e^{-\left[m_3(q_1-1) + \frac{\ln \alpha_1}{\tau_a}\right]s} ds \leq & 1,
\end{aligned}
$$

$$
\begin{aligned}
i.e.,\ \frac{e^{\left[m_3(q_1-1) + \frac{\ln \alpha_1}{\tau_a}\right]t}}{m_3(q_1-1) + \frac{\ln \alpha_1}{\tau_a}} \leq & \frac{1}{\alpha_1^{N_0} m_1(q_1-1)} + \frac{1}{m_3(q_1-1) + \frac{\ln \alpha_1}{\tau_a}}, \\
i.e.,\ t \leq & \frac{\ln \left[1 + \frac{m_3(q_1-1) + \frac{\ln \alpha_1}{\tau_a}}{\alpha_1^{N_0} m_1(q_1-1)} \right]}{\left[m_3(q_1-1) + \frac{\ln \alpha_1}{\tau_a} \right]}.
\end{aligned}
$$

Substituting $\alpha_1 = \alpha^{1-q_1}$ into the above inequality, we get

$$t \leq \frac{\ln\left[1 + \frac{\left(m_3 - \frac{\ln\alpha}{\tau_a}\right)}{m_1 \alpha^{N_0(1-q_1)}}\right]}{\left(m_3 - \frac{\ln\alpha}{\tau_a}\right)(q_1 - 1)}. \tag{2.1.26}$$

In view of system (2.1.23), we obtain

$$S(t;0,S_0) = \begin{cases} e^{m_3(q_1-1)t}\alpha_1^{N_\zeta(t,0)}S(0) + m_1(q_1-1)\int_0^t e^{m_3(q_1-1)(t-s)}\alpha_1^{N_\zeta(t,s)}ds, & 0 \leq t < T_1, \\ 1, & t = T_1, \end{cases} \tag{2.1.27}$$

where $T_1 = \frac{\ln\left[1+\frac{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1\alpha^{N_0(1-q_1)}}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)}$. When $0 < V(t) < 1$, the dominating term will be replaced by a new term $-m_2V^{q_2}(t) - m_3V(t)$. Now from the previous theorem, we have

$$\begin{cases} \dot{S}(t) = -m_2(1-q_2) - m_3(1-q_2)S(t), & 0 < S(t) < 1,\ t \neq t_k,\ t \geq T_1, \\ S(t_k^+) = \alpha_2 S(t_k), & t = t_k,\ t \geq T_1, \\ S(T_1) = S(T_1;0,S_0) = 1, \end{cases} \tag{2.1.28}$$

where $\alpha_2 = \alpha^{1-q_2} \in (1,\infty)$. Note that $V(t) \to 0$ is similar to $S(t) \to 0$. Now, from the formula of variation of parameters, when $0 < S(t) < 1$, it can be deduced from the system (2.1.28) that

$$S(t) = e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{N_\zeta(t,T_1)}S(T_1) - m_2(1-q_2)\int_{T_1}^t e^{-m_3(1-q_2)(t-s)}\alpha_2^{N_\zeta(t,s)}ds. \tag{2.1.29}$$

It follows from Eq. (2.1.29) that when $1 < \alpha_2 < \infty$, there exists a $T_2 > 0$ such that $\lim_{t\to T_2} S(t) = 0$ and also considering the definition of $N_\zeta(t,s)$, we have

$$\begin{aligned}
R(t) =& e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{N_\zeta(t,T_1)}S(T_1) - m_2(1-q_2)\int_{T_1}^{t} e^{-m_3(1-q_2)(t-s)}\alpha_2^{N_\zeta(t,s)}ds, \\
\leq & e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{\frac{t-T_1}{\tau_a}+N_0} - m_2(1-q_2)\int_{T_1}^{t} e^{-m_3(1-q_2)(t-s)}\alpha_2^{\frac{t-s}{\tau_a}-N_0}ds \\
=& \left[\alpha_2^{N_0} + \frac{m_2(1-q_2)\alpha_2^{-N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}\right] e^{-\left[m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right](t-T_1)} - \frac{m_2(1-q_2)\alpha_2^{-N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}.
\end{aligned} \tag{2.1.30}$$

Taking $A = \alpha_2^{N_0} + \frac{m_2(1-q_2)\alpha_2^{-N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}$ and $B = \frac{m_2(1-q_2)\alpha_2^{-N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}$, it is seen that $A > B$ and $Ae^{-\left[m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right](t-T_1)}$ is continuously decreasing, i.e., the right-hand side of Eq. (2.1.30) must be zero after some time. Therefore, we have

$$\left[\alpha_2^{N_0} + \frac{m_2(1-q_2)\alpha_2^{-N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}}\right] e^{-\left[m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right](t-T_1)} = \frac{m_2(1-q_2)\alpha_2^{-N_0}}{m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}},$$

$$i.e.,\ t = \frac{\ln\left[\frac{m_2(1-q_2)\alpha_2^{-N_0}}{\alpha_2^{N_0}\left(m_3(1-q_2)-\frac{\ln\alpha_2}{\tau_a}\right)+m_2(1-q_2)\alpha_2^{-N_0}}\right]}{\left(\frac{\ln\alpha_2}{\tau_a}-m_3\right)(1-q_2)} + T_1.$$

Substituting $\alpha_2 = \alpha^{1-q_2}$ into the above inequality, we obtain

$$t = \frac{\ln\left[\frac{m_2\alpha^{-N_0(1-q_2)}}{\alpha^{N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{-N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)} + T_1. \tag{2.1.31}$$

In view of Eq. (2.1.28), we obtain

$$S(t+T_1;0,S(T_1)) = \begin{cases} e^{-m_3(1-q_2)(t-T_1)}\alpha_2^{N_\zeta(t,T_1)}S(T_1), & \\ -m_2(1-q_2)\int_{T_1}^{t} e^{-m_3(1-q_2)(t-s)}\alpha_2^{N_\zeta(t,s)}ds, & 0 \leq t < T_2, \\ 0, & t \geq T_2, \end{cases} \tag{2.1.32}$$

where $T_2 = \frac{\ln\left[\frac{m_2\alpha^{-N_0(1-q_2)}}{\alpha^{N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{-N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)}$.

Now, on the basis of the above discussions, when $\alpha > 1$, one may see that $S(t;0,S_0) = 0$ for all $t \geq T_1 + T_2$. It is found that for any initial value $V_0 > 0$, a fixed time $T_1 + T_2$ exists which is independent of $V_0 > 0$ such that $\lim\limits_{t\to T_1+T_2} V(t) = 0$ and $V(t) \equiv 0$ for $t \geq T_1 + T_2$. Since $||x(t)|| \leq \sqrt[a]{\frac{W(t,x(t))}{w_1}} \leq \sqrt[a]{\frac{V(t,x(t))}{w_1}}$, one has $\lim\limits_{t\to T_1+T_2} ||x(t)|| = 0$ and $||x(t)|| \equiv 0$ for $t \geq T_1 + T_2$. Hence, the system (2.1.1) is FxTS, and the associated settling-time $T = T_1 + T_2$ is estimated as

$$T = \frac{\ln\left[1 + \frac{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1\alpha^{N_0(1-q_1)}}\right]}{\left(m_3 - \frac{\ln\alpha}{\tau_a}\right)(q_1 - 1)} + \frac{\ln\left[\frac{m_2\alpha^{-N_0(1-q_2)}}{\alpha^{N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{-N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a} - m_3\right)(1 - q_2)}. \tag{2.1.33}$$

For any initial value $V_0 > 0$, based on the above analysis it is known that for $\alpha > 1$, there exists a $T > 0$ such that $\lim\limits_{t\to T} V(t) = 0$ and $V(t) \equiv 0$ for $t \geq T$. It is followed by the condition $||x(t)|| \leq \sqrt[a]{\frac{W(t,x(t))}{w_1}} \leq \sqrt[a]{\frac{V(t,x(t))}{w_1}}$ that $\lim\limits_{t\to T} x(t) = 0$ and $x(t) \equiv 0$ for $t \geq T$. Hence, the system (2.1.1) is FxTS, with T as the settling time. The proof of Theorem 2.1.3.2 is thus completed.

Remark 2.1.3.3. It is to be mentioned that Theorem 2.1.3.2 deals with the problem towards achieving the FxTS in the presence of destabilizing impulsive effects, i.e. $\alpha > 1$. We have found the condition $\tau_a > \frac{\ln\alpha}{m_3}$, with the help of condition (ii) and definition of average impulsive interval, for which the system (2.1.1) will be FxTS even under the influence of destabilizing impulses. It is to be noted that it had been claimed in [87] that FxTS of system (2.1.1) is not possible for destabilizing impulses.

Corollary 2.1.3.3. For system (2.2.1), if there exist a function $W(t,x(t)) \in w_0$ and some positive constants w_1, w_2, a, m_1, m_2, m_3, q_1, q_2, α satisfying $qq_1 > 1$, $0 < qq_2 < 1$ and $\alpha > 1$ such that the following conditions hold:

(i) $w_1||x(t)||^a \leq W(t, x(t)) \leq w_2||x(t)||^a$, for $t \in \mathbb{R}_+,\ \ x \in \mathbb{R}^n$,

(ii) when $t \neq t_k$,

$$\dot{W}(t, x(t)) \leq -\left(m_1 W^{q_1}(t, x(t)) + m_2 W^{q_2}(t, x(t)) + m_3 W(t, x(t))\right)^q,$$

(iii) when $t = t_k$, $W(t_k^+, x(t_k^+)) \leq \alpha W(t_k, x(t_k))$,

then the system (2.2.1) is FxTS, and the associated settling-time T is estimated as

$$T = \frac{\ln\left[1+\frac{\left(m_3^q-\frac{\ln\alpha}{\tau_a}\right)}{m_1^q\alpha^{N_0(1-qq_1)}}\right]}{\left(m_3^q-\frac{\ln\alpha}{\tau_a}\right)(qq_1-1)} + \frac{\ln\left[\frac{m_2^q\alpha^{-N_0(1-qq_2)}}{\alpha^{N_0(1-qq_2)}\left(m_3^q-\frac{\ln\alpha}{\tau_a}\right)+m_2^q\alpha^{-N_0(1-qq_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3^q\right)(1-qq_2)}.$$

Corollary 2.1.3.4. Consider the following scalar system as

$$\begin{cases} \dot{z}(t) = -m_1\text{sign}(z(t))|z(t)|^{q_1} - m_2\text{sign}(z(t))|z(t)|^{q_2} - m_3\text{sign}(z(t))|z(t)|, \quad t \neq t_k, \\ z(t_k^+) = \alpha z(t_k),\ k \in \mathbb{N}, \end{cases} \tag{2.1.34}$$

where $m_1 > 0$, $m_2 > 0$, $m_3 > 0$, $q_1 > 1$, $0 < q_2 < 1$, $\alpha > 1$, then the system (2.1.34) is FxTS, and the associated settling-time T is estimated as

$$T = \frac{\ln\left[1+\frac{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)}{m_1\alpha^{N_0(1-q_1)}}\right]}{\left(m_3-\frac{\ln\alpha}{\tau_a}\right)(q_1-1)} + \frac{\ln\left[\frac{m_2\alpha^{-N_0(1-q_2)}}{\alpha^{N_0(1-q_2)}\left(m_3-\frac{\ln\alpha}{\tau_a}\right)+m_2\alpha^{-N_0(1-q_2)}}\right]}{\left(\frac{\ln\alpha}{\tau_a}-m_3\right)(1-q_2)}.$$

2.1.4 Numerical Simulation and Discussions

The objective of this subsection is to demonstrate the effectiveness of the proposed theoretical results. For this, following two numerical examples have been considered for presentation.

Example 2.1.4.1. Consider the system

$$\begin{cases} \dot{z}(t) = -m_1 \mathrm{sign}(z(t))|z(t)|^{q_1} - m_2 \mathrm{sign}(z(t))|z(t)|^{q_2} - m_3 \mathrm{sign}(z(t))|z(t)|, \quad t \neq t_k, \\ z(t_k^+) = \alpha z(t_k), \ k \in \mathbb{N}. \end{cases} \tag{2.1.35}$$

From Corollary 2.1.3.2, the system (2.1.35) is FxTS for stabilizing impulses. The impulsive sequence is defined as $\alpha = 0.5$, $N_0 = 1$, $\tau_a = 2$. The numerical results corresponding to this example is plotted in Fig. 2.1.1 (a) and 2.1.1 (b) under two sets of parameters, which are chosen as follows:

(a) $m_1 = 4$, $m_2 = 3$, $m_3 = 2$, $q_1 = 1.3$, $q_2 = 0.7$; (b) $m_1 = 0.1$, $m_2 = 2$, $m_3 = 3$, $q_1 = 2.5$, $q_2 = 0.3$. By carrying out computation on the basis of our results, we find the settling time $T \approx 1.88$ under the first set (a) and $T \approx 1.63$ is obtained under the second set (b). This verifies that the proposed estimation is less conservative as compared to the previous results as reported in Ref. [87].

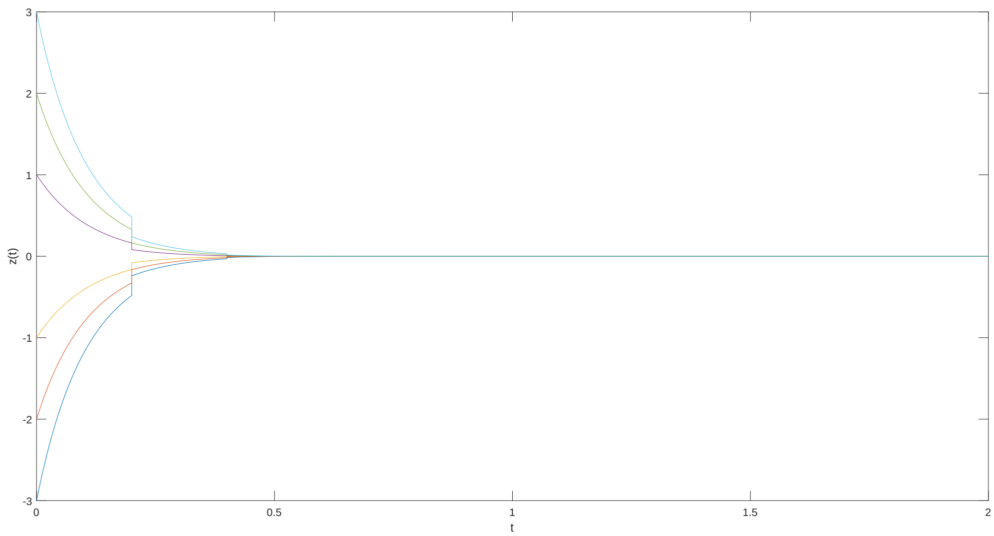
Similarly, we can obtain from Corollary 2.1.3.4 that the system (2.1.35) is FxTS for destabilizing impulses. The impulsive sequence is defined as $\alpha = 1.1$, $N_0 = 1$, $\tau_a = 2$. With the help of computational work, we find the settling time $T \approx 1.58$ under the first set (a) and $T \approx 1.28$ under the second set (b). The simulated results for state trajectories are shown in Fig. 2.1.2 (a) and 2.1.2 (b) under two sets of parameters.

Example 2.1.4.2. Consider a general neural network with three nodes described by

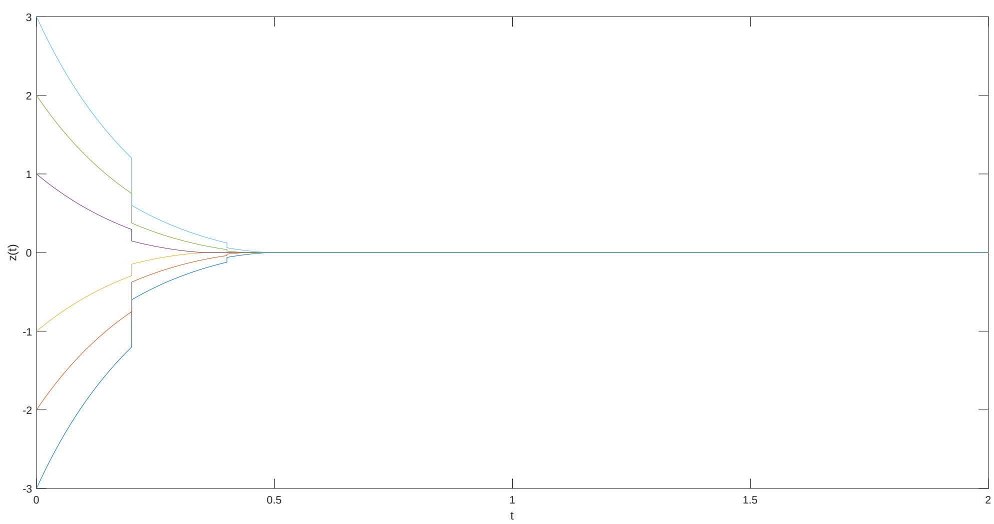
$$\begin{cases} \dot{x}_i(t) = -Dx_i(t) + Bf(x_i(t)) + u_i(t), \quad t \neq t_k, \\ \triangle x_i(t_k) = d_i x_i(t_k), \ k \in \mathbb{N}, \end{cases} \tag{2.1.36}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T$, and $D = \mathrm{diag}\{1, 1, 1\}$,

$f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), f_3(x_{i3}(t)))^T$ with $f_i(z) = \frac{1}{2}(|z+1| - |z-1|)$,

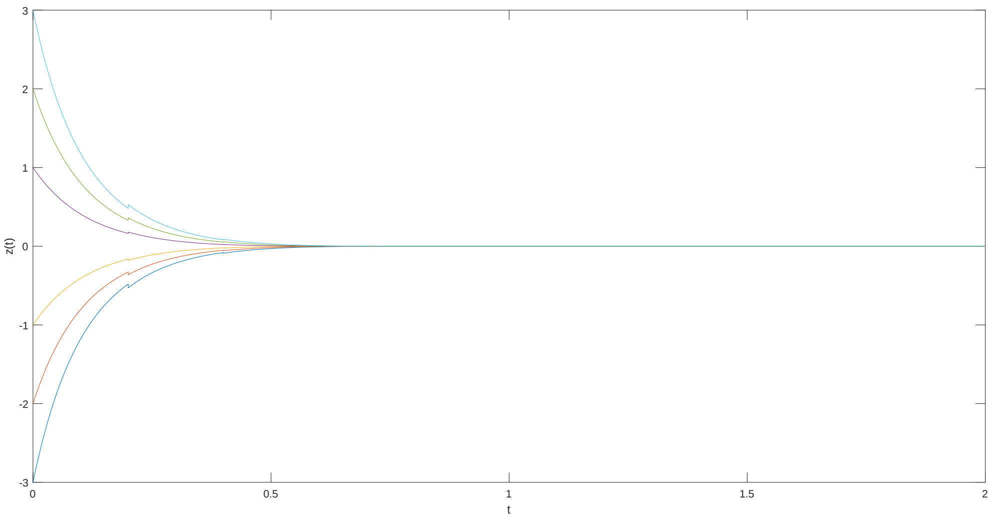


2.1.1 (a)

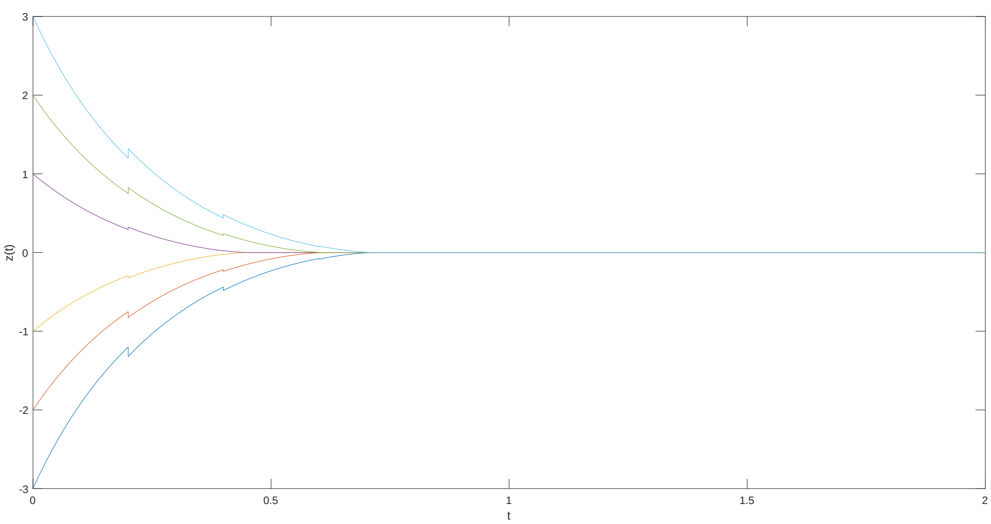


2.1.1 (b)

Figure 2.1.1: (a) State trajectories of system (2.1.35) with stabilizing impulses for the first set of parameters (b) State trajectories of system (2.1.35) with stabilizing impulses for the second set of parameters



2.1.2 (a)



2.1.2 (b)

Figure 2.1.2: (a) State trajectories of system (2.1.35) with destabilizing impulses for the first set of parameters (b) State trajectories of system (2.1.35) with destabilizing impulses for the second set of parameters

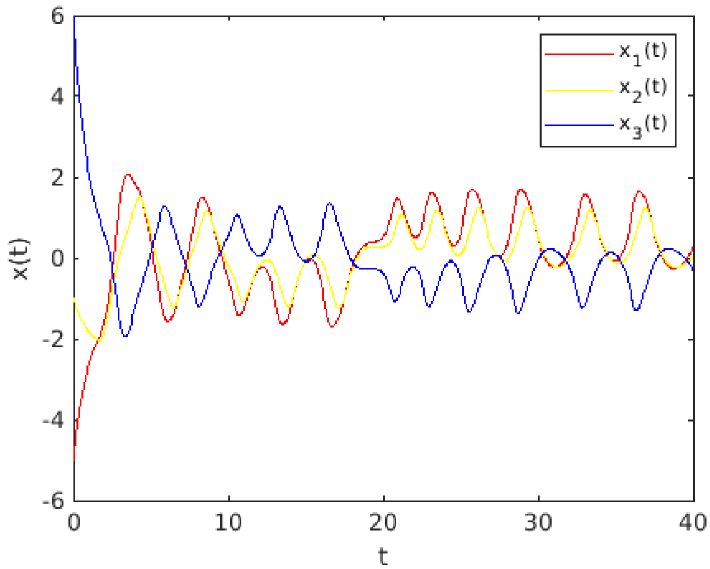


Figure 2.1.3: State trajectories of system (2.1.36) without control.

$$B = \begin{bmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{bmatrix}.$$

Here the controller is considered as

$$u_i(t) = -\Phi_1 x_i(t) - \Phi_2 x_i^{\frac{v}{u}}(t) - \Phi_3 x_i^{\frac{q}{p}}(t), \tag{2.1.37}$$

where Φ_1, Φ_2, Φ_3 are positive numbers and u, v, p, q, are odd positive numbers, satisfying $v < u$ and $q > p$.

Here, this novel controller has been used to handle the FxTS problem of the neural network with continuous dynamical function [107]. This controller is novel as compared to that applied in [108] and it is more simpler and can be applied to the systems more conveniently.

The simulated results for the state trajectories of the system (33) without using the control are plotted in Fig. 2.1.3, which indicates that the neural network (2.1.36) can not be stable without control protocol (2.1.37).

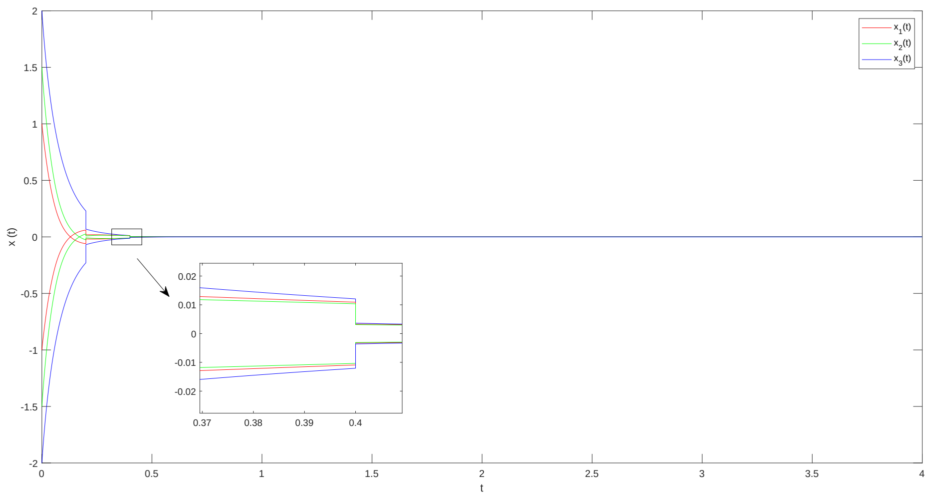


Figure 2.1.4: State trajectories of system (2.1.36) with stabilizing impulses.

Now, considering $d_i = -0.7$, it is followed as in [87] that $\alpha = 0.8$. Hence the impulses are now stabilizing. In view of the Theorem 2.1.3.1, taking $N_0 = 1$, $\tau_a = 2$, and setting the parameters as $m_1 = 3$, $m_2 = 2$, $m_3 = 1$, $q_1 = 2$, $q_2 = 0.5$, the system (2.1.36) with stabilizing impulses can be FxTS under the fixed-time control protocol given by (2.1.37) with $\Phi_1 = 2.5$, $\Phi_2 = 10$, $p = 7$, $q = v = 9$, $u = 11$. It is derived from (2.1.16) that the system (2.1.36) is FxTS within settling time $T \approx 1.29$. The corresponding state trajectories are shown in Fig. 2.1.4 and verifies the results as obtained in Theorem 2.1.3.1.

On the other hand, when $d_i = 0.3$, one can find $\alpha = 2.8$, and hence the corresponding impulses are destabilizing. By Theorem 2.1.3.2, it is easy to show from (2.1.33) that the system (2.1.36) is FxTS under the control protocol (2.1.37) and the settling-time $T \approx 2.91$. The results in this case are further shown in Fig. 2.1.5.

2.1.5 Conclusion

In this subchapter, the FxTS for nonlinear dynamical systems with impulsive effects has been investigated. Two theorems for FxTS of the system (2.1.1) with stabilizing and destabilizing impulses have been constructed using the Lyapunov functional and

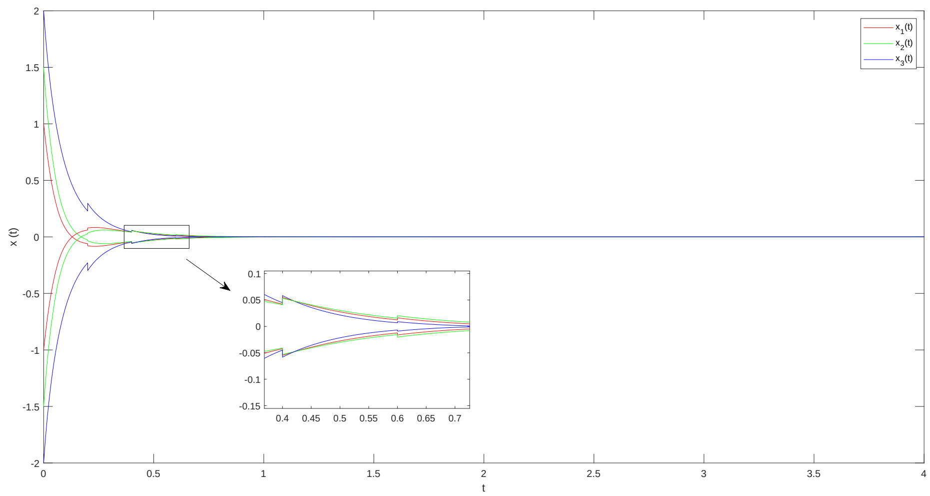


Figure 2.1.5: State trajectories of system (2.1.36) with destabilizing impulses

average impulsive interval. Contrary to the claim given in [87, 89] that FxTS of system (2.1.1) cannot be possible for destabilizing impulses, here the condition $\tau_a > \frac{\ln \alpha}{m_3}$ is found to achieve the FxTS of the system (2.1.1). It is also verified in the numerical section of this subchapter that the estimated settling-time functions are more effective than that reported previously in literature. Finally, two numerical examples including one as general neural network, are considered and the numerical simulation results show the effectiveness of the proposed approach. One can be optimistic that the present results will be beneficial for the researchers to understand FxTS of impulsive dynamical systems in different situations. The future works will be focused on the FxTS of delayed systems or hybrid systems via feedback control situations ([109, 110]).

2.2 Fixed-Time Stability of Cohen-Grossberg BAM Neural Networks with Impulsive Perturbations

2.2.1 Introduction[2]

In this present subchapter, the problem of FxTS of a Cohen-Grossberg bidirectional associative memory neural network (CGBAMNN) with destabilizing impulsive effects is derived. A novel sufficient condition for impulsive dynamical systems to be FxTS for destabilizing impulses is obtained. Different from the usual Lyapunov inequality, we obtain the results under perturbation. Based on the average impulsive interval and the comparison principle, a novel sufficient condition $\tau_a > \frac{\ln \xi}{c}$ for the impulsive dynamical systems to be FxTS for destabilizing impulses is obtained. Contrary to the claim in the previous articles [87, 88, 89, 96, 111], that the results of FxTS are not possible to obtain for destabilizing impulses, we have extended their results for destabilizing impulses by using a new type of Lyapunov inequality that contains three terms to obtain the results under impulsive perturbation. Based on the new type of Lyapunov inequality, we constructed two different continuous controllers by following [89]: one with signum term and another without signum term and applied them to FxTS of CGBAMNNs with destabilizing impulsive effects. The settling-time functions obtained in this subchapter depend on the class of impulsive sequences which relax the drawback of the settling-time obtained in [89]. Finally, two numerical examples, one as a cyber-physical system with deception attacks and another one as a neural network, are given to validate the efficiency of the obtained theoretical results.

2.2.2 Model Formulation and Mathematical Preliminaries

Let us consider a CGBAMNN with impulsive effects as

[2]The content of this subchapter is published in ***Neurocomputing***, 550, 126501 (2023)

$$\begin{cases} \dot{x}_r(t) = -a_r(x_r(t))\left\{b_r(x_r(t)) - \sum_{s=1}^{i_2} h_{rs} f_s(y_s(t)) - I_r\right\}, & t \neq t_l,\, t \in \mathbb{R}_+, \\ \dot{y}_s(t) = -c_s(y_s(t))\left\{d_s(y_s(t)) - \sum_{r=1}^{i_1} w_{sr} g_r(x_r(t)) - J_s\right\}, & t \neq t_l,\, t \in \mathbb{R}_+, \\ \Delta x_r(t_l) = \varrho x_r(t_l), & l \in \mathbb{N}, \\ \Delta y_s(t_l) = \sigma y_s(t_l), & l \in \mathbb{N}, \end{cases} \tag{2.2.1}$$

where $r = 1, 2, \ldots, i_1$ and $s = 1, 2, \ldots, i_2$ be the number of neurons in the X-layer and the Y-layer, respectively; $x_r(t)$ and $y_s(t)$ are the state variables; $a_r(x_r(t))$ and $c_s(y_s(t))$ represent the amplification functions; $b_r(x_r(t))$ and $d_s(y_s(t))$ denote appropriately behaved functions; h_{rs} and w_{sr} represent the connection weights; $f_s(y_s(t))$ and $g_r(x_r(t))$ represent the activation functions of the s-th neuron in X-layer and the r-th neuron in Y-layer, respectively; I_r and J_s denote an external inputs to the r-th neuron in X-layer and the s-th neuron in the Y-layer, respectively. Here for all r and s, $\Delta x_r(t_l) = x_r(t_l^+) - x_r(t_l)$ and $\Delta y_s(t_l) = y_s(t_l^+) - y_s(t_l)$, where $x_r(t_l^+) = \lim\limits_{t \to t_l+0} x_r(t_l)$ and $y_s(t_l^+) = \lim\limits_{t \to t_l+0} y_s(t_l)$. Without loss of generality, we assume that $x_r(t_l^-) = \lim\limits_{t \to t_l-0} x_r(t_l) = x_r(t_l)$ and $y_s(t_l^-) = \lim\limits_{t \to t_l-0} y_s(t_l) = y_s(t_l)$, i.e., the state of system (2.2.1) is left continuous at every impulsive point t_l; the impulsive strengths ϱ and σ are taken as positive constants. The impulsive sequence $\zeta = \{t_l\}_{l \in \mathbb{N}}$ satisfies $0 \leq t_0 < t_1 < t_2 < \cdots < t_l < \cdots$, with $\lim\limits_{l \to +\infty} t_l = +\infty$. Let $x(t) = (x_1(t), x_2(t), \ldots, x_{i_1}(t))^T$, $y(t) = (y_1(t), y_2(t), \ldots, y_{i_2}(t))^T$, $w(t) = (x^T(t), y^T(t))^T = (w_1(t), w_2(t), \ldots, w_{i_1+i_2}(t))^T$.

For convenience, let us take the following fundamental assumptions.

Assumption 2.2.2.1. The amplification functions $a_r(x)$ and $c_s(y)$ are continuous and bounded, i.e, they satisfy $\hat{a}_r \leq a_r(x) \leq \check{a}_r$ and $\hat{c}_s \leq c_s(y) \leq \check{c}_s$, for all x,$y \in \mathbb{R}$, where $\hat{a}_r$, $\check{a}_r$, $\hat{c}_s$ and $\check{c}_s$ are some known positive constants.

Assumption 2.2.2.2. The appropriately behaved functions $b_r(x)$ and $d_s(y)$ are monotonically differentiable, i.e., this satisfy $\hat{b}_r \leq \dot{b}_r(x) \leq \check{b}_r$ and $\hat{d}_s \leq \dot{d}_s(y) \leq \check{d}_s$, for all $x,y \in \mathbb{R}$, where $\hat{b}_r$, $\check{b}_r$, $\hat{d}_s$ and $\check{d}_s$ are some known positive constants.

Assumption 2.2.2.3. The activation functions $f_s(.)$ and $g_r(.)$ satisfy the Lipschitz criterion, i.e., there exist positive Lipschitz constants A_s and B_r such that $|f_s(x)-f_s(y)| \leq A_s|x-y|, \quad |g_r(x)-g_r(y)| \leq B_r|x-y|$, for all $x,y \in \mathbb{R}$.

We also recall the following definitions and lemma:

Definition 2.2.2.1. (See [89]) A function W: $\mathbb{R}^{i_1+i_2} \to \mathbb{R}_+$ is said to belong to class w_0, if it satisfies

(1) W is continuous on each of the sets $[t_l, t_{l+1}) \times \mathbb{R}^{i_1+i_2}$ and for $u \in \mathbb{R}^{i_1+i_2}$, $l \in \mathbb{N}$ and $W(t_l^+, u) = \lim\limits_{(t,y)\to(t_l^+,u)} W(t,y)$ exists.

(2) W is locally Lipschitz in u.

Definition 2.2.2.2. (See [89]) The origin of system (2.2.1) is said to be globally FTS if

(i) the origin of system (2.2.1) is Lyapunov stable;

(ii) the origin of system (2.2.1) is finite-time convergent that means for any initial state $u_0 \in \mathbb{R}^{i_1+i_2}$, there exists $T : \mathbb{R}^{i_1+i_2}\backslash\{0\} \to (0,+\infty)$, such that $\lim\limits_{t\to T(u_0)} u(t,u_0) = 0$ and $u(t,u_0) = 0$ for all $t \geq T(u_0)$. The function T is said to be settling-time function.

Definition 2.2.2.3. (See [89]) The origin of system (2.2.1) is called to be FxTS if it is FTS and settling-time function $T(u_0)$ is bounded, i.e., $\exists\ T_{max} > 0$ such that $T(u_0) \leq T_{max}$, $\forall\ u_0 \in \mathbb{R}^{i_1+i_2}$.

Lemma 2.2.2.1. (See [112]) Let v_1, v_2, $\ldots$, $v_{i_1+i_2} \geq 0$, $0 < m \leq 1$, $n > 1$, then the following inequalities hold

$$\sum_{k=1}^{i_1+i_2} v_k^m \geq \left(\sum_{k=1}^{i_1+i_2} v_k\right)^m, \sum_{k=1}^{i_1+i_2} v_k^n \geq (i_1+i_2)^{1-n}\left(\sum_{k=1}^{i_1+i_2} v_k\right)^n.$$

2.2.3 Main Results

2.2.3.1 Fixed-Time Stability for Nonlinear Impulsive Dynamical Systems

In this subsection, the sufficient conditions of FxTS are provided for nonlinear impulsive dynamical systems (2.2.2) with destabilizing impulsive effects. The obtained results will be used to design continuous controllers for FxTS of CGBAMNN under the effects of destabilizing impulses.

The following nonlinear dynamical system is considered with impulsive effects:

$$\begin{cases} \dot{u}(t) = \mathrm{F}(u(t)), & t \neq t_l,\, t \in \mathbb{R}_+, \\ \mathit{\Delta} u(t_l) = \mathrm{J}(u(t_l)), & l \in \mathbb{N}, \\ u(t_0) = u_0, \end{cases} \tag{2.2.2}$$

where $u(t) \in \mathbb{R}^{i_1+i_2}$ represent the system's state of (2.2.2) at time t, $\mathrm{F}, \mathrm{J} : \mathbb{R}^{i_1+i_2} \to \mathbb{R}^{i_1+i_2}$ are locally Lipschitz with $\mathrm{F}(0) = 0$ and $\mathrm{J}(0) = 0$. For all $l \in \mathbb{N}$, $\mathit{\Delta} u(t_l) = u(t_l^+) - u(t_l)$, where $u(t_l^+) = \lim\limits_{t \to t_l + 0} u(t)$. Without any loss of generalization, we will assume that the solution $u(t)$ is left continuous at the impulsive point, i.e., $u(t_l^-) = \lim\limits_{t \to t_l - 0} u(t) = u(t_l)$.

Theorem 2.2.3.1. Suppose the function $W(u(t)) \in w_0$ be positive definite and radially unbounded which satisfies the following conditions:

$$\begin{cases} \dot{W}(u(t)) \leq -aW^{\alpha}(u(t)) - bW^{\beta}(u(t)) - cW(u(t)), & t \neq t_l, \\ W(u(t_l^+)) \leq \xi W(u(t_l)), & t = t_l, \end{cases} \tag{2.2.3}$$

where a, b, $c > 0$, $0 < \alpha < 1$, $\beta > 1$, $\xi > 1$, then the system (2.2.2) is FxTS if

$$\tau_a > \frac{\ln \xi}{c}. \tag{2.2.4}$$

The estimated settling-time function is given by $T = \frac{\ln\left[1+\frac{\left(c-\frac{\ln\xi}{\tau_a}\right)}{b\xi^{N_0(1-\beta)}}\right]}{\left(c-\frac{\ln\xi}{\tau_a}\right)(\beta-1)} + \frac{\ln\left[\frac{a\xi^{-N_0(1-\alpha)}}{\xi^{N_0(1-\alpha)}\left(c-\frac{\ln\xi}{\tau_a}\right)+a\xi^{-N_0(1-\alpha)}}\right]}{\left(\frac{\ln\xi}{\tau_a}-c\right)(1-\alpha)}$,

where N_0 is a positive integer.

Proof. For the purpose of comparison principle, we consider the auxiliary differential equations

$$\begin{cases} \dot{M}(t) = \begin{cases} -aM^{\alpha}(t) - cM(t), & 0 \leq M(t) < 1, t \neq t_l, \\ -bM^{\beta}(t) - cM(t), & M(t) \geq 1, t \neq t_l, \\ 0, & M(t) = 0, t \neq t_l, \end{cases} \\ M(t_l^+) = \xi M(t_l), t = t_l, \\ M(0) = W(u(0)) = M_0. \end{cases} \tag{2.2.5}$$

Comparing system (2.2.5) with system (2.2.3), we get $0 \leq W(t) \leq M(t)$. Consequently, if there exists a $T > 0$ such that $\lim\limits_{t\to T} M(t) = 0$ and $M(t) \equiv 0\ \forall t \geq T$, then $\lim\limits_{t\to T} W(t) = 0$ and $W(t) \equiv 0\ \forall t \geq T$. In order to establish the FxTS of system (2.2.2), it is now necessary to consider the corresponding problem of systems (2.2.3).

Define $P(t) = M^{1-\beta}(t)$ when $M(t) \geq 1$. Thus we have $P(t) \to 1$ as $M(t) \to 1$ and $P(t) \to 0$ as $M(t) \to \infty$, we have

$$\begin{cases} \dot{P}(t) = c(\beta-1)P(t) + b(\beta-1), \quad 0 < P(t) \leq 1,\, t \neq t_l, \\ P(t_l^+) = \xi_1 P(t_l),\, t = t_l, \\ P(0) = P_0 = M_0^{1-\beta}, \end{cases} \tag{2.2.6}$$

where $\xi_1 = \xi^{1-\beta} \in (0,1)$. It is simple to see that $M(t) \to 1$ is similar to $P(t) \to 1$. Using the transformation $P(t) = M^{1-\alpha}(t)$, where $0 < M(t) < 1$, it can observed that $P(t) \to 1$ as $M(t) \to 1$ and $P(t) \to 0$ as $M(t) \to 0$. Therefore, we have the following equations as

$$\begin{cases} \dot{P}(t) = -a(1-\alpha) - c(1-\alpha)P(t), \quad 0 < P(t) < 1, t \neq t_l, \\ P(t_l^+) = \xi_2 P(t_l),\, t = t_l, \\ P(0) = 1, \end{cases} \tag{2.2.7}$$

where $\xi_2 = \xi^{1-\alpha} \in (1,\infty)$. It follows that $M(t) \to 0$ is similar to $P(t) \to 0$.

Now from the above discussion, the FxTS of system (2.2.5) have been transformed into the following two problems (i) the solution of system (2.2.6) tends to 1. in the fixed-time T_1 and (ii) the solution of system (2.2.7) converge to 0 from 1 in a fixed-time T_2. When the initial value of system (2.2.5) is less than 1, then the FxTS of system (2.2.5) is similar to the corresponding problem of system (2.2.7). In other words, the FxTS of system (2.2.5) can be attained within the settling-time $T = T_1 + T_2$. The above (i) and (ii) will be considered in the case of impulsive strength $\xi > 1$.

From the formula for variation of parameters, when $0 < P(t) < 1$, the solution of system (2.2.6) can be written as

$$P(t) = e^{c(\beta-1)t}\xi_1^{N_\zeta(t,0)}P(0) + b(\beta-1)\int_0^t e^{c(\beta-1)(t-s)}\xi_1^{N_\zeta(t,s)}ds. \tag{2.2.8}$$

From Definition 2.1.2.4, Eq. (2.2.8) and $P(0) < 1$, one can observe that $P(t)$ is increasing and $\lim_{t\to\infty} P(t) = \infty$, when $0 < \xi_1 < 1$. There exists $T_1 > 0$ such that $\lim_{t\to T_1} P(t) = 1$ and $0 < P(t) < 1\ \forall\ 0 < t < T_1$ that is

$$e^{c(\beta-1)t}\left[\xi_1^{N_\varsigma(t,0)}P(0) + b(\beta-1)\int_0^t e^{-c(\beta-1)s}\xi_1^{N_\varsigma(t,s)}ds\right] = 1.$$

Note that

$$b(\beta-1)e^{c(\beta-1)t}\int_0^t e^{-c(\beta-1)s}\xi_1^{N_\varsigma(t,s)}ds \leq 1,$$

$$i.e.,\ b(\beta-1)e^{c(\beta-1)t}\int_0^t e^{-c(\beta-1)s}\xi_1^{\frac{t-s}{\tau_a}+N_0}ds \leq 1,$$

$$i.e.,\ \frac{e^{\left[c(\beta-1)+\frac{\ln\xi_1}{\tau_a}\right]t}}{\left(c(\beta-1)+\frac{\ln\xi_1}{\tau_a}\right)} \leq \frac{1}{\xi_1^{-N_0}b(\beta-1)} + \frac{1}{\left(c(\beta-1)+\frac{\ln\xi_1}{\tau_a}\right)},$$

$$i.e.,\ \left[c(\beta-1)+\frac{\ln\xi_1}{\tau_a}\right]t \leq \ln\left[1+\frac{\left(c(\beta-1)+\frac{\ln\xi_1}{\tau_a}\right)}{\xi_1^{-N_0}b(\beta-1)}\right],$$

$$i.e.,\ t \leq \frac{\ln\left[1+\frac{\left(c(\beta-1)+\frac{\ln\xi_1}{\tau_a}\right)}{\xi_1^{N_0}b(\beta-1)}\right]}{\left[c(\beta-1)+\frac{\ln\xi_1}{\tau_a}\right]} = T_1.$$

Substituting $\xi_1 = \xi^{1-\beta}$, we get

$$T_1 = \frac{\ln\left[1+\frac{\left(c-\frac{\ln\xi}{\tau_a}\right)}{b\xi^{N_0(1-\beta)}}\right]}{\left(c-\frac{\ln\xi}{\tau_a}\right)(\beta-1)}. \tag{2.2.9}$$

Similar to Eq. (2.2.8), one has the following from Eq. (2.2.7)

$$P(t) = e^{-c(1-\alpha)t}\xi_2^{N_\zeta(t,0)}P(0) - a(1-\alpha)\int_0^t e^{-c(1-\alpha)(t-s)}\xi_2^{N_\zeta(t,s)}ds. \tag{2.2.10}$$

From Definition 2.1.2.4, $1 < \xi_2 < \infty$ and $P(0) = 1$, we get

$$\begin{aligned} P(t) =& \xi_2^{N_\zeta(t,0)}e^{-c(1-\alpha)t} - a(1-\alpha)\int_0^t e^{-c(1-\alpha)(t-s)}\xi_2^{N_\zeta(t,s)}ds \\ \leq& \xi_2^{\frac{t}{\tau_a}+N_0}e^{-c(1-\alpha)t} - a(1-\alpha)\int_0^t e^{-c(1-\alpha)(t-s)}\xi_2^{\frac{t-s}{\tau_a}-N_0}ds \\ =& \left[\xi_2^{N_0} + \frac{a(1-\alpha)\xi_2^{-N_0}}{c(1-\alpha)-\frac{\ln\xi_2}{\tau_a}}\right]e^{-\left[c(1-\alpha)-\frac{\ln\xi_2}{\tau_a}\right]t} - \frac{a(1-\alpha)\xi_2^{-N_0}}{c(1-\alpha)-\frac{\ln\xi_2}{\tau_a}}. \end{aligned} \tag{2.2.11}$$

Let $h(t) = \left[\xi_2^{N_0} + \frac{a(1-\alpha)\xi_2^{-N_0}}{c(1-\alpha)-\frac{\ln\xi_2}{\tau_a}}\right]e^{-\left[c(1-\alpha)-\frac{\ln\xi_2}{\tau_a}\right]t} - \frac{a(1-\alpha)\xi_2^{-N_0}}{c(1-\alpha)-\frac{\ln\xi_2}{\tau_a}}$, then we have $h(0) = \xi_2^{N_0} > 0$, $h(+\infty) = -\frac{a(1-\alpha)\xi_2^{-N_0}}{c(1-\alpha)-\frac{\ln\xi_2}{\tau_a}} < 0$, and $\dot{h}(t) < 0$. From the fundamental concept of calculus, there exists a unique $T_2 > 0$ such that $h(T_2) = 0$. This implies $h(t)$ is decreasing. Since $P(t) \leq h(t)$ and $P(t) > 0$ then $P(t) \to 0$ as $t \to T_2$. From inequality (2.2.11), we obtain

$$T_2 = \frac{\ln\left[\frac{a(1-\alpha)\xi_2^{-N_0}}{\xi_2^{N_0}\left(c(1-\alpha)-\frac{\ln\xi_2}{\tau_a}\right)+a(1-\alpha)\xi_2^{-N_0}}\right]}{\left(\frac{\ln\xi_2}{\tau_a}-c\right)(1-\alpha)}.$$

Substituting $\xi_2 = \xi^{1-\alpha}$ in the above equation, we get

$$T_2 = \frac{\ln\left[\frac{a\xi^{-N_0(1-\alpha)}}{\xi^{N_0(1-\alpha)}\left(c-\frac{\ln\xi}{\tau_a}\right)+a\xi^{-N_0(1-\alpha)}}\right]}{\left(\frac{\ln\xi}{\tau_a}-c\right)(1-\alpha)}. \tag{2.2.12}$$

From Eq. (2.2.9) and Eq. (2.2.12), $M(t)$ is approaching to 1 in fixed-time T_1 and approaching to 0 in fixed-time T_2. Therefore, the system (2.2.2) is FxTS with the settling-time given by $T = T_1 + T_2 = \frac{\ln\left[1+\frac{\left(c-\frac{\ln\xi}{\tau_a}\right)}{b\xi^{N_0(1-\beta)}}\right]}{\left(c-\frac{\ln\xi}{\tau_a}\right)(\beta-1)} + \frac{\ln\left[\frac{a\xi^{-N_0(1-\alpha)}}{\xi^{N_0(1-\alpha)}\left(c-\frac{\ln\xi}{\tau_a}\right)+a\xi^{-N_0(1-\alpha)}}\right]}{\left(\frac{\ln\xi}{\tau_a}-c\right)(1-\alpha)}$. Therefore the proof is completed.

Remark 2.2.3.1. Different from Theorem 2.2.3.1 in [89], a new Lyapunov inequality has been taken in Theorem 2.2.3.1 of this subchapter. Based on the inequality, a novel sufficient condition for system (2.2.2) to be FxTS has been derived under the influence of destabilizing impulsive effects. If we consider $c = 0$, $\xi = 1$ in Eq. (2.2.3) then we get the same results as obtained in [89]. Therefore, the obtained extended results have been obtained in comparison with [89]. In [87], the authors have claimed that FxTS results of system (2.2.2) are not possible for destabilizing impulses when we consider the Lyapunov inequality $\dot{W}(u(t)) \leq -aW^{\alpha}(u(t)) - bW^{\beta}(u(t))$, $0 < \alpha < 1$, $\beta > 1$. This motivates us to consider a linear term in the inequality due to which here a sufficient condition $\tau_a > \frac{\ln\xi}{c}$ is derived for system (2.2.2) to be FxTS with destabilizing impulses.

Remark 2.2.3.2. As we know FxTS is a special case of FTS. The term $-aW^{\alpha}(u(t))$, $0 < \alpha < 1$ in Theorem 2.2.3.1 is sufficient to get the FTS of the impulsive dynamical systems. To realize FxTS, the second nonlinear term $-bW^{\beta}(u(t))$, $\beta > 1$ should be added which pulls the trajectory of the system with stabilizing impulses into a ball of radius one in a fixed-time, (see [87, 88, 89, 96, 111]). However, in the case of impulsive perturbation to the system, a linear term $-cW(u(t))$ should be added to the Lyapunov inequality. For sufficient value of control gain $c > 0$ such that $\tau_a > \frac{\ln\xi}{c}$, the controller compels the system's trajectory to converge into a ball of radius one in a fixed time. Otherwise, impulsive perturbation destroys the system's stability which is verified numerically in the numerical and simulations section of this subchapter.

Remark 2.2.3.3. In [71, 72, 73, 74, 98], the impulsive strength is dependent on the parameters of the continuous Lyapunov inequality, i.e., $W(u(t_l^+)) \leq \xi^{\frac{1}{(1-\alpha)(\beta-1)}} W(u(t_l))$, where $0 < \alpha < 1$ and $\beta > 1$, which is conservative discrete time behaviour of Lyapunov inequality. However, here an impulsive strength is chosen, which is independent of the parameters of the continuous Lyapunov condition α and β, i.e., $W(u(t_l^+)) \leq \xi W(u(t_l))$. Therefore, the obtained results in this article are less conservative.

Corollary 2.2.3.1. Consider a scalar system with impulsive effect

$$\begin{cases} \dot{u}(t) = -au^{\frac{\alpha_1}{\beta_1}}(t) - bu^{\frac{\alpha_2}{\beta_2}}(t) - cu(t), \quad t \neq t_l, t \in \mathbb{R}_+, \\ u(t_l^+) = \xi u(t_l), l \in \mathbb{N}, \end{cases} \tag{2.2.13}$$

where a, b, $c > 0$ and $\xi > 1; \alpha_1, \beta_1, \alpha_2, \beta_2$ are odd positive integers such that $\alpha_1 < \beta_1$, $\alpha_2 > \beta_2$. If $\tau_a > \frac{\ln \xi}{c}$ then system (2.2.13) is FxTS and the estimated settling-time function is given by $\mathrm{T} = \frac{\ln\left[1 + \frac{\left(c - \frac{\ln \xi}{\tau_a}\right)}{b\xi^{N_0\left(\frac{\beta_2-\alpha_2}{\beta_2}\right)}}\right]}{\left(c - \frac{\ln \xi}{\tau_a}\right)\left(\frac{\alpha_2-\beta_2}{\beta_2}\right)} + \frac{\ln\left[\frac{a\xi^{-N_0\left(\frac{\beta_1-\alpha_1}{\beta_1}\right)}}{\xi^{N_0\left(\frac{\beta_1-\alpha_1}{\beta_1}\right)}\left(c - \frac{\ln \xi}{\tau_a}\right) + a\xi^{-N_0\left(\frac{\beta_1-\alpha_1}{\beta_1}\right)}}\right]}{\left(\frac{\ln \xi}{\tau_a} - c\right)\left(\frac{\beta_1-\alpha_1}{\beta_1}\right)}$.

Remark 2.2.3.4. In Corollary 2.2.3.1, a simple scalar impulsive system is considered with destabilizing impulses and shown that for the appropriate parameters $a, b, c, \alpha_1, \beta_1, \alpha_2, \beta_2$ we can estimate the settling-time for the system to be FxTS if sufficient condition $\tau_a > \frac{\ln \xi}{c}$ holds. On the other hand, in [89], the settling-time can not be estimated for $\xi > 1$. This demonstrates the superiority of our results.

2.2.3.2 Fixed-Time Stability for Impulsive CGBAMNNs with the First Continuous Controller

In this subsection, a drive is taken to control CGBAMNNs with destabilizing impulsive effects to the equilibrium point $(x^\star, y^\star)$ similar to [113]. To control the system, we design a continuous controller based on the Lyapunov inequality (2.2.3). Thus, the equilibrium point $(x^\star, y^\star)$ can be shifted to the origin using the translation $p_r(t) = x_r(t) - x_r^\star$,

$q_s(t) = y_s(t) - y_s^\star$ which satisfy the following system:

$$\begin{cases} \dot{p_r}(t) = -\widetilde{a}_r(p_r(t)) \left\{ \widetilde{b}_r(p_r(t)) - \sum_{s=1}^{i_2} h_{rs} \widetilde{f}_s(q_s(t)) \right\}, & t \neq t_l, t \in \mathbb{R}_+, \\ \dot{q}_s(t) = -\widetilde{c}_s(q_s(t)) \left\{ \widetilde{d}_s(q_s(t)) - \sum_{r=1}^{i_1} w_{sr} \widetilde{g}_r(p_r(t)) \right\}, & t \neq t_l, t \in \mathbb{R}_+, \\ \Delta p_r(t_l) = \varrho p_r(t_l), & l \in \mathbb{N}, \\ \Delta q_s(t_l) = \sigma q_s(t_l), & l \in \mathbb{N}, \end{cases} \tag{2.2.14}$$

where $r = 1, 2, \ldots, i_1,\ s = 1, 2, \ldots, i_2; \widetilde{a}_r(p_r(t)) = a_r(p_r(t) + x_r^\star)$, $\widetilde{b}_r(p_r(t)) = b_r(p_r(t) + x_r^\star) - b_r(x_r^\star)$, $\widetilde{f}_s(q_s(t)) = f_s(q_s(t) + y_s^\star) - f_s(y_s^\star)$, $\widetilde{c}_s(q_s(t)) = c_s(q_s(t) + y_s^\star)$, $\widetilde{d}_s(q_s(t)) = d_s(q_s(t) + y_s^\star) - d_s(y_s^\star)$, $\widetilde{g}_r(p_r(t)) = g_r(p_r(t) + x_r^\star) - g_r(x_r^\star)$. For convenience, we can write $p(t) = (p_1(t), p_2(t), \ldots, p_{i_1}(t))^T$, $q(t) = (q_1(t), q_2(t), \ldots, q_{i_2}(t))^T$, $\widetilde{w}(t) = (p^T(t), q^T(t))^T = (\widetilde{w}_1(t), \widetilde{w}_2(t), \ldots, \widetilde{w}_{i_1+i_2}(t))^T$.

From Assumption (2.2.2.2), function $b_r(x)$ and $d_s(y)$ are differentiable, i.e., they satisfy $0 < \hat{b}_r \leq \dot{b}_r(x) \leq \check{b}_r$ and $0 < \hat{d}_s \leq \dot{d}_s(y) \leq \check{d}_s$, for all $x \in \mathbb{R}^{i_1}$ and $y \in \mathbb{R}^{i_2}$. From mean value theorem, we get

$$\begin{aligned} b_r(x_r) - b_r(x_r^\star) &= \dot{b}_r(x_r^\star + \ell(x_r - x_r^\star))(x_r - x_r^\star), 0 < \ell < 1, \\ \widetilde{b}_r(p_r(t)) &= \dot{b}_r(x_r^\star + \ell p_r(t))p_r(t), 0 < \ell < 1. \end{aligned} \tag{2.2.15}$$

Similarly, we obtain

$$\begin{aligned} d_s(y_s) - d_s(y_s^\star) &= \dot{d}_s(y_s^\star + \ell(y_s - y_s^\star))(y_s - y_s^\star),\ 0 < \ell < 1, \\ \widetilde{d}_s(q_s(t)) &= \dot{d}_s(y_s^\star + \ell q_s(t))q_s(t), 0 < \ell < 1. \end{aligned} \tag{2.2.16}$$

In order to control the system (2.2.14), the first kind of continuous controller is designed as

$$\begin{cases} u_r(t) = & -k_r^1 p_r(t) - k_1 \mathrm{sign}(p_r(t))|p_r(t)|^{\alpha} - k_2 \mathrm{sign}(p_r(t))|p_r(t)|^{\beta}, \\ v_s(t) = & -m_s^1 q_s(t) - m_1 \mathrm{sign}(q_s(t))|q_s(t)|^{\alpha} - m_2 \mathrm{sign}(q_s(t))|q_s(t)|^{\beta}, \end{cases} \tag{2.2.17}$$

where $0 < \alpha < 1$, $\beta > 1$ and k_1, k_2,m_1,m_2 are positive constants and k_r^1, m_s^1 denote control gains for all r and s.

Using the controllers (2.2.17) in the system (2.2.14), we get

$$\begin{cases} \dot{p_r}(t) = -\widetilde{a}_r(p_r(t)) \left\{ \widetilde{b}_r(p_r(t)) - \sum_{s=1}^{i_2} h_{rs} \widetilde{f}_s(q_s(t)) \right\} + u_r(t), & t \neq t_l, t \in \mathbb{R}_+, \\ \dot{q}_s(t) = -\widetilde{c}_s(q_s(t)) \left\{ \widetilde{d}_s(q_s(t)) - \sum_{r=1}^{i_1} w_{sr} \widetilde{g}_r(p_r(t)) \right\} + v_s(t), & t \neq t_l, t \in \mathbb{R}_+, \\ \Delta p_r(t_l) = \varrho p_r(t_l), & l \in \mathbb{N}, \\ \Delta q_s(t_l) = \sigma q_s(t_l), & l \in \mathbb{N}. \end{cases} \tag{2.2.18}$$

Theorem 2.2.3.2 Suppose that Assumptions (2.2.2.1)-(2.2.2.3) hold and the control gains k_r^1 and m_s^1 satisfy

$$\begin{cases} k_r^1 > \sum_{s=1}^{i_2} \check{c}_s B_r |w_{sr}| - \hat{a}_r \hat{b}_r, \\ m_s^1 > \sum_{r=1}^{i_1} \check{a}_r A_s |h_{rs}| - \hat{c}_s \hat{d}_s, \end{cases} \tag{2.2.19}$$

then the system (2.2.1) can achieve FxTS under controller (2.2.17) if

$$\tau_a > \frac{\ln \widetilde{\xi}}{\delta}. \tag{2.2.20}$$

The estimated settling-time function is given by $\mathrm{T} = \frac{\ln\left[1 + \frac{\left(\delta - \frac{\ln \widetilde{\xi}}{\tau_a}\right)}{\widetilde{\psi}\widetilde{\xi}^{N_0(1-\beta)}}\right]}{\left(\delta - \frac{\ln \widetilde{\xi}}{\tau_a}\right)(\beta - 1)} + \frac{\ln\left[\frac{\phi\widetilde{\xi}^{-N_0(1-\alpha)}}{\widetilde{\xi}^{N_0(1-\alpha)}\left(\delta - \frac{\ln \widetilde{\xi}}{\tau_a}\right) + \phi\widetilde{\xi}^{-N_0(1-\alpha)}}\right]}{\left(\frac{\ln \widetilde{\xi}}{\tau_a} - \delta\right)(1-\alpha)}$, where $\widetilde{\xi} = \max\{|1+\varrho|, |1+\sigma|\} > 1$, $\eta_r = k_r^1 - \sum_{s=1}^{i_2} \check{c}_s B_r |w_{sr}| + \hat{a}_r \hat{b}_r$, $\mu_s = m_s^1 - \sum_{r=1}^{i_1} \check{a}_r A_s |h_{rs}| + \hat{c}_s \hat{d}_s$ and $\delta = \min\{\min_r \eta_r, \min_s \mu_s\}$.

Proof. Choose the following Lyapunov function as

$$W(t) = \sum_{k=1}^{i_1+i_2} |\widetilde{w}_k(t)| = \sum_{r=1}^{i_1} |p_r(t)| + \sum_{s=1}^{i_2} |q_s(t)|. \tag{2.2.21}$$

Differentiation of $W(t)$ along the solution of system (2.2.18) for $t \neq t_l$ is as follows:

$$\begin{aligned}\dot{W}(t) =& \sum_{r=1}^{i_1} \operatorname{sign}(p_r(t)) \left[-\widetilde{a}_r(p_r(t))\widetilde{b}_r(p_r(t)) + \widetilde{a}_r(p_r(t)) \sum_{s=1}^{i_2} h_{rs}\widetilde{f}_s(q_s(t)) + u_r(t) \right] \\ &+ \sum_{s=1}^{i_2} \operatorname{sign}(q_s(t)) \left[-\widetilde{c}_s(q_s(t))\widetilde{d}_s(q_s(t)) + \widetilde{c}_s(q_s(t)) \sum_{r=1}^{i_1} w_{sr}\widetilde{g}_r(p_r(t)) + v_s(t) \right].\end{aligned} \tag{2.2.22}$$

From inequalities (2.2.15), (2.2.16) and the Assumptions (2.2.2.1)-(2.2.2.3), we get

$$\begin{aligned}\dot{W}(t) =& -\sum_{r=1}^{i_1} \widetilde{a}_r(p_r(t))\dot{b}_r(x_r^\star + \ell p_r(t))|p_r(t)| - \sum_{r=1}^{i_1} k_r^1 |p_r(t)| \\ &- \sum_{r=1}^{i_1} k_1 |p_r(t)|^\alpha - \sum_{r=1}^{i_1} k_2 |p_r(t)|^\beta \\ &+ \sum_{r=1}^{i_1}\sum_{s=1}^{i_2} \widetilde{a}_r(p_r(t))h_{rs}\left(f_s(q_s(t)+y_s^\star) - f_s(y_s^\star)\operatorname{sign}(p_r(t))\right. \\ &- \sum_{s=1}^{i_2} \widetilde{c}_s(q_s(t))\dot{d}_s(y_s^\star + \ell q_s(t))|q_s(t)| - \sum_{s=1}^{i_2} m_s^1|q_s(t)| - \sum_{s=1}^{i_2} m_1|q_s(t)|^\alpha - \sum_{s=1}^{i_2} m_2|q_s(t)|^\beta \\ &+ \sum_{s=1}^{i_2}\sum_{r=1}^{i_1} w_{sr}\widetilde{c}_s(q_s(t))\left(g_r(p_r(t)+x_r^\star) - g_r(x_r^\star)\right)\operatorname{sign}(q_s(t)),\end{aligned}$$

$$\begin{aligned}\textit{i.e.}, \ \dot{W}(t) \leq& -\sum_{r=1}^{i_1} \hat{a}_r\hat{b}_r|p_r(t)| + \sum_{r=1}^{i_1}\sum_{s=1}^{i_2} \check{a}_r A_s |h_{rs}||q_s(t)| - \sum_{r=1}^{i_1} k_r^1|p_r(t)| - \sum_{r=1}^{i_1} k_1|p_r(t)|^\alpha \\ &- \sum_{r=1}^{i_1} k_2|p_r(t)|^\beta - \sum_{s=1}^{i_2} \hat{c}_s\hat{d}_s|q_s(t)| + \sum_{s=1}^{i_2}\sum_{r=1}^{i_1} \check{c}_s B_r |w_{sr}||p_r(t)| - \sum_{s=1}^{i_2} m_s^1|q_s(t)| \\ &- \sum_{s=1}^{i_2} m_1|q_s(t)|^\alpha - \sum_{s=1}^{i_2} m_2|q_s(t)|^\beta,\end{aligned}$$

$$i.e., \dot{W}(t) = \sum_{r=1}^{i_1}\left(-\hat{a}_r\hat{b}_r + \sum_{s=1}^{i_2}\check{c}_s B_r|w_{sr}| - k_r^1\right)|p_r(t)| - \sum_{r=1}^{i_1}k_1|p_r(t)|^{\alpha} - \sum_{r=1}^{i_1}k_2|p_r(t)|^{\beta}$$
$$+ \sum_{s=1}^{i_2}\left(-\hat{c}_s\hat{d}_s + \sum_{r=1}^{i_1}\breve{a}_r A_s|h_{rs}| - m_s^1\right)|q_s(t)| - \sum_{s=1}^{i_2}m_1|q_s(t)|^{\alpha} - \sum_{s=1}^{i_2}m_2|q_s(t)|^{\beta}. \tag{2.2.23}$$

If $k_r^1 > \sum_{s=1}^{i_2}\check{c}_s B_r|w_{sr}| - \hat{a}_r\hat{b}_r$, $m_s^1 > \sum_{r=1}^{i_1}\breve{a}_r A_s|h_{rs}| - \hat{c}_s\hat{d}_s$, $r = 1,2,\ldots,i_1$, $s = 1,2,\ldots,i_2$, then we obtain

$$\begin{aligned}\dot{W}(t) \leq &-\sum_{r=1}^{i_1}\eta_r|p_r(t)| - \sum_{r=1}^{i_1}k_1|p_r(t)|^{\alpha} - \sum_{r=1}^{i_1}k_2|p_r(t)|^{\beta} - \sum_{s=1}^{i_2}\mu_s|q_s(t)| \\ &-\sum_{s=1}^{i_2}m_1|q_s(t)|^{\alpha} - \sum_{s=1}^{i_2}m_2|q_s(t)|^{\beta},\end{aligned} \tag{2.2.24}$$

where $\eta_r = k_r^1 - \sum_{s=1}^{i_2}\check{c}_s B_r|w_{sr}| + \hat{a}_r\hat{b}_r > 0$ and $\mu_s = m_s^1 - \sum_{r=1}^{i_1}\breve{a}_r A_s|h_{sr}| + \hat{c}_s\hat{d}_s > 0$. Let $\eta = \min_r\{\eta_r\}$, $\mu_s = \min_s\{\mu_s\}$, $\delta = \min\{\eta,\mu\}$, $\phi = \min\{k_1, m_1\}$, $\psi = \min\{k_2, m_2\}$, then we have

$$\dot{W}(t) \leq -\delta\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)| - \phi\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)|^{\alpha} - \psi\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)|^{\beta}. \tag{2.2.25}$$

According to Lemma 2.2.2.1, we can derive the following inequalities:

$$\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)|^{\alpha} \geq \left(\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)|\right)^{\alpha}, \quad \sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)|^{\beta} \geq (i_1+i_2)^{1-\beta}\left(\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)|\right)^{\beta}. \tag{2.2.26}$$

From Eq. (2.2.25) and Eq. (2.2.26), one can get

$$\begin{aligned}\dot{W}(t) &\leq -\delta\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)| - \phi\left(\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)|\right)^{\alpha} - \psi(i_1+i_2)^{1-\beta}\left(\sum_{k=1}^{i_1+i_2}|\widetilde{w}_k(t)|\right)^{\beta} \\ &\leq -\delta W(t) - \phi W^{\alpha}(t) - \widetilde{\psi}W^{\beta}(t),\end{aligned} \tag{2.2.27}$$

where $\widetilde{\psi} = \psi(i_1 + i_2)^{1-\beta}$.

When $t = t_l$, it follows from Eq. (2.2.21) that

$$\begin{aligned} W(t_l^+) &= \sum_{r=1}^{i_1} |p_r(t_l^+)| + \sum_{s=1}^{i_2} |q_s(t_l^+)| \\ &= \sum_{r=1}^{i_1} |1+\varrho||p_r(t_l)| + \sum_{s=1}^{i_2} |1+\sigma||q_s(t_l)| \\ &\leq \sum_{k=1}^{i_1+i_2} \widetilde{\xi}|\widetilde{w}_k(t_l)| = \widetilde{\xi}W(t_l), \end{aligned} \tag{2.2.28}$$

where $\widetilde{\xi} = \max\{|1+\varrho|,\, |1+\sigma|\}$.

According to Theorem 2.2.3.1, the origin of system (2.2.1) can achieve FxTS under the controller (2.2.17) with the estimated settling-time as given by

$$T = \frac{\ln\left[1 + \frac{\left(\delta - \frac{\ln\widetilde{\xi}}{\tau_a}\right)}{\widetilde{\psi}\widetilde{\xi}^{N_0(1-\beta)}}\right]}{\left(\delta - \frac{\ln\widetilde{\xi}}{\tau_a}\right)(\beta-1)} + \frac{\ln\left[\frac{\phi\widetilde{\xi}^{-N_0(1-\alpha)}}{\widetilde{\xi}^{N_0(1-\alpha)}\left(\delta - \frac{\ln\widetilde{\xi}}{\tau_a}\right) + \phi\widetilde{\xi}^{-N_0(1-\alpha)}}\right]}{\left(\frac{\ln\widetilde{\xi}}{\tau_a} - \delta\right)(1-\alpha)}. \tag{2.2.29}$$

Therefore the proof is completed.

2.2.3.3 Fixed-Time Stability for Impulsive CGBAMNNs with the Second Continuous Controller

The second type of continuous controller based on the Lyapunov inequality (2.2.3) is designed as follows:

$$\begin{cases} u_r(t) = & -\Phi_r p_r(t) - k_3 p_r^{\frac{\alpha_1}{\beta_1}}(t) - k_4 p_r^{\frac{\alpha_2}{\beta_2}}(t), \\ v_s(t) = & -\Psi_s q_s(t) - m_3 q_s^{\frac{\alpha_1}{\beta_1}}(t) - m_4 q_s^{\frac{\alpha_2}{\beta_2}}(t), \end{cases} \tag{2.2.30}$$

where $k_3,\ k_4,\ m_3,\ m_4$ are positive constants; $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ are odd positive integers satisfying $\alpha_2 > \beta_2$, $\alpha_1 < \beta_1$ and Φ_r, Ψ_s denote control gains for all r and s.

Theorem 2.2.3.3. Suppose that Assumptions (2.2.2.1)-(2.2.2.3) hold and the control gains Φ_r and Ψ_s satisfy

$$\begin{cases} \Phi_r > \sum_{s=1}^{i_2} \frac{A_s \check{b}_r |h_{rs}|}{2\epsilon} + \frac{B_r \check{d}_r |w_{sr}| \varepsilon}{2}) - \hat{a}_r \hat{b}_r, \\ \Psi_s > \sum_{r=1}^{i_1} (\frac{A_s \check{b}_r |h_{rs}| \epsilon}{2} + \frac{B_r \check{d}_r |w_{sr}|}{2\varepsilon}) - \hat{c}_s \hat{d}_s, \end{cases} \tag{2.2.31}$$

where $0 < \epsilon, \varepsilon < 1$, then the system (2.2.1) is FxTS under controller (2.2.30) if

$$\tau_a > \frac{\ln \widetilde{\xi}}{\widetilde{\Theta}}. \tag{2.2.32}$$

The estimated settling-time is given by $\mathrm{T} = \frac{\ln\left[1 + \frac{\left(\widetilde{\Theta} - \frac{\ln \widetilde{\xi}}{\tau_a}\right)}{\widetilde{\rho} \widetilde{\xi}^{N_0(\frac{\beta_2 - \alpha_2}{2\beta_2})}}\right]}{\left(\widetilde{\Theta} - \frac{\ln \widetilde{\xi}}{\tau_a}\right)(\frac{\alpha_2 - \beta_2}{2\beta_2})} + \frac{\ln\left[\frac{\widetilde{\chi} \widetilde{\xi}^{-N_0(\frac{\beta_1 - \alpha_1}{2\beta_1})}}{\widetilde{\xi}^{N_0(\frac{\beta_1 - \alpha_1}{2\beta_1})}\left(\widetilde{\Theta} - \frac{\ln \widetilde{\xi}}{\tau_a}\right) + \widetilde{\chi} \widetilde{\xi}^{-N_0(\frac{\beta_1 - \alpha_1}{2\beta_1})}}\right]}{\left(\frac{\ln \widetilde{\xi}}{\tau_a} - \widetilde{\Theta}\right)(\frac{\beta_1 - \alpha_1}{2\beta_1})}$, where $\widetilde{\xi} = \max\{(1+\varrho)^2, (1+\sigma)^2\} > 1$, $\gamma_r = \Phi_r - \sum_{s=1}^{i_2} (\frac{A_s \check{a}_r |h_{rs}|}{2\epsilon} + \frac{B_r \check{c}_r |w_{sr}| \varepsilon}{2}) + \hat{a}_r \hat{b}_r$, $\lambda_s = \Psi_s - \sum_{r=1}^{i_1} (\frac{A_s \check{a}_r |h_{rs}| \epsilon}{2} + \frac{B_r \check{c}_r |w_{sr}|}{2\varepsilon}) + \hat{c}_s \hat{d}_s$ and $\widetilde{\Theta} = 2\min\{\min_r \gamma_r, \min_s \lambda_s\}$.

Proof. Choose the following Lyapunov function as

$$W(t) = \frac{1}{2} \sum_{k=1}^{i_1+i_2} \widetilde{w}_k^2(t) = \frac{1}{2} \sum_{r=1}^{i_1} p_r^2(t) + \frac{1}{2} \sum_{s=1}^{i_2} q_s^2(t). \tag{2.2.33}$$

The derivative of $W(t)$ along the along the system (2.2.18) for $t \neq t_l$ is given by

$$\begin{aligned} \dot{W}(t) &= \sum_{r=1}^{i_1} p_r(t) \dot{p}_r(t) + \sum_{s=1}^{i_2} q_s(t) \dot{q}_s(t) \\ &= \sum_{r=1}^{i_1} p_r(t) \left[-\widetilde{a}_r(p_r(t)) \left(\widetilde{b}_r(p_r(t)) - \sum_{s=1}^{i_2} h_{rs} \widetilde{f}_s(q_s(t)) \right) \right] \\ &+ \sum_{r=1}^{i_1} p_r(t) \left[-\Phi_r p_r(t) - k_3 p_r^{\frac{\alpha_1}{\beta_1}}(t) - k_4 p_r^{\frac{\alpha_2}{\beta_2}}(t) \right] \\ &+ \sum_{s=1}^{i_2} q_s(t) \left[-\widetilde{c}_s(q_s(t)) \left(\widetilde{d}_s(q_s(t)) - \sum_{r=1}^{i_1} w_{sr} \widetilde{g}_r(p_r(t)) \right) \right] \\ &+ \sum_{s=1}^{i_2} q_s(t) \left[-\Psi_s q_s(t) - m_3 q_s^{\frac{\alpha_1}{\beta_1}}(t) - m_4 q_s^{\frac{\alpha_2}{\beta_2}}(t) \right], \end{aligned}$$

$$\begin{aligned}\dot{W}(t) \leq & -\sum_{r=1}^{i_1} \hat{a}_r \hat{b}_r p_r^2(t) + \sum_{r=1}^{i_1}\sum_{s=1}^{i_2} A_s \breve{a}_r |h_{rs}||p_r(t)||q_s(t)| - \sum_{r=1}^{i_1} \Phi_r p_r^2(t) \\ & - k_3 \sum_{r=1}^{i_1} p_r^{\frac{\alpha_1+\beta_1}{\beta_1}}(t) - k_4 \sum_{r=1}^{i_1} p_r^{\frac{\alpha_2+\beta_2}{\beta_2}}(t) - \sum_{s=1}^{i_2} \hat{c}_s \hat{d}_s q_s^2(t) \\ & + \sum_{s=1}^{i_2}\sum_{r=1}^{i_1} B_r \check{c}_s |w_{sr}||q_s(t)||p_r(t)| - \sum_{s=1}^{i_2} \Psi_s q_s^2(t) \\ & - m_3 \sum_{s=1}^{i_2} q_s^{\frac{\alpha_1+\beta_1}{\beta_1}}(t) - m_4 \sum_{s=1}^{i_2} q_s^{\frac{\alpha_2+\beta_2}{\beta_2}}(t),\end{aligned}$$

$$\begin{aligned}\dot{W}(t) \leq & \sum_{r=1}^{i_1}\left[\sum_{s=1}^{i_2}(\frac{A_s \breve{a}_r |h_{rs}|}{2\epsilon} + \frac{B_r \check{c}_r |w_{sr}|\varepsilon}{2}) - \hat{a}_r \hat{b}_r - \Phi_r\right] p_r^2(t) - k_3 \sum_{r=1}^{i_1} p_r^{\frac{\alpha_1+\beta_1}{\beta_1}}(t) \\ & - k_4 \sum_{r=1}^{i_1} p_r^{\frac{\alpha_2+\beta_2}{\beta_2}}(t) + \sum_{s=1}^{i_2}\left[\sum_{r=1}^{i_1}(\frac{A_s \breve{a}_r |h_{rs}|\epsilon}{2} + \frac{B_r \check{c}_r |w_{sr}|}{2\varepsilon}) - \hat{c}_s \hat{d}_s - \Psi_s\right] q_s^2(t) \\ & - m_3 \sum_{s=1}^{i_2} q_s^{\frac{\alpha_1+\beta_1}{\beta_1}}(t) - m_4 \sum_{s=1}^{i_2} q_s^{\frac{\alpha_2+\beta_2}{\beta_2}}(t). \qquad (2.2.34)\end{aligned}$$

If $\Phi_r > \sum_{s=1}^{i_2} \frac{A_s \breve{a}_r |h_{rs}|}{2\epsilon} + \frac{B_r \check{c}_r |w_{sr}|\varepsilon}{2}) - \hat{a}_r \hat{b}_r$, $\Psi_s > \sum_{r=1}^{i_1}(\frac{A_s \breve{a}_r |h_{rs}|\epsilon}{2} + \frac{B_r \check{c}_r |w_{sr}|}{2\varepsilon}) - \hat{c}_s \hat{d}_s$, $r = 1, 2, \ldots, i_1$, $s = 1, 2, \ldots, i_2$, one can obtain

$$\begin{aligned}\dot{W}(t) \leq & -\sum_{r=1}^{i_1} \gamma_r p_r^2(t) - k_3 \sum_{r=1}^{i_1} p_r^{\frac{\alpha_1+\beta_1}{\beta_1}}(t) - k_4 \sum_{r=1}^{i_1} p_r^{\frac{\alpha_2+\beta_2}{\beta_2}}(t) \\ & - \sum_{s=1}^{i_2} \lambda_s q_s^2(t) - m_3 \sum_{s=1}^{i_2} q_s^{\frac{\alpha_1+\beta_1}{\beta_1}}(t) - m_4 \sum_{s=1}^{i_2} q_s^{\frac{\alpha_2+\beta_2}{\beta_2}}(t), \qquad (2.2.35)\end{aligned}$$

where $\gamma_r = \Phi_r - \sum_{s=1}^{i_2}(\frac{A_s \breve{a}_r |h_{rs}|}{2\epsilon} + \frac{B_r \check{c}_r |w_{sr}|\varepsilon}{2}) + \hat{a}_r \hat{b}_r > 0$ and $\lambda_s = \Psi_s - \sum_{r=1}^{i_1}(\frac{A_s \breve{a}_r |h_{rs}|\epsilon}{2} + \frac{B_r \check{c}_r |w_{sr}|}{2\varepsilon}) + \hat{c}_s \hat{d}_s > 0$.

Let $\gamma = \min_r\{\gamma_r\}$, $\lambda = \min_s\{\lambda_s\}$, $\varTheta = \min\{\gamma, \lambda\}$, $\chi = \min\{k_3, m_3\}$, $\rho = \{k_4, m_4\}$. Therefore, we get

$$\dot{W}(t) \leq -\Theta \sum_{k=1}^{i_1+i_2} \widetilde{w}_k^2(t) - \chi \sum_{k=1}^{i_1+i_2} \widetilde{w}_k^{\frac{\alpha_1+\beta_1}{\beta_1}}(t) - \rho \sum_{k=1}^{i_1+i_2} \widetilde{w}_k^{\frac{\alpha_2+\beta_2}{\beta_2}}(t). \tag{2.2.36}$$

In view of Lemma 2.2.2.1, we obtain

$$\sum_{k=1}^{i_1+i_2} \widetilde{w}_k^{\frac{\alpha_1+\beta_1}{\beta_1}}(t) = \sum_{k=1}^{i_1+i_2} \left(\widetilde{w}_k^2(t)\right)^{\frac{\alpha_1+\beta_1}{2\beta_1}} \geq \left(\sum_{k=1}^{i_1+i_2} \widetilde{w}_k^2(t)\right)^{\frac{\alpha_1+\beta_1}{2\beta_1}}, \tag{2.2.37}$$

$$\sum_{k=1}^{i_1+i_2} \widetilde{w}_k^{\frac{\alpha_2+\beta_2}{\beta_2}}(t) = \sum_{k=1}^{i_1+i_2} \left(\widetilde{w}_k^2(t)\right)^{\frac{\alpha_2+\beta_2}{2\beta_2}} \geq (i_1+i_2)^{\frac{\beta_2-\alpha_2}{2\beta_2}} \left(\sum_{k=1}^{i_1+i_2} \widetilde{w}_k^2(t)\right)^{\frac{\alpha_2+\beta_2}{2\beta_2}}. \tag{2.2.38}$$

Using inequalities (2.2.36), (2.2.37) and (2.2.38), we get

$$\begin{aligned}
\dot{W}(t) &\leq -\Theta \sum_{k=1}^{i_1+i_2} \widetilde{w}_k^2(t) - \chi \left(\sum_{k=1}^{i_1+i_2} \widetilde{w}_k^2(t)\right)^{\frac{\alpha_1+\beta_1}{2\beta_1}} - \rho(i_1+i_2)^{\frac{\beta_2-\alpha_2}{2\beta_2}} \left(\sum_{k=1}^{i_1+i_2} \widetilde{w}_k^2(t)\right)^{\frac{\alpha_2+\beta_2}{2\beta_2}} \\
&\leq -2\Theta W(t) - \chi 2^{\frac{\alpha_1+\beta_1}{2\beta_1}} W^{\frac{\alpha_1+\beta_1}{2\beta_1}}(t) - \rho 2^{\frac{\alpha_2+\beta_2}{2\beta_2}} (i_1+i_2)^{\frac{\beta_2-\alpha_2}{2\beta_2}} W^{\frac{\alpha_2+\beta_2}{2\beta_2}}(t) \\
&= -\widetilde{\Theta} W(t) - \widetilde{\chi} W^{\frac{\alpha_1+\beta_1}{2\beta_1}}(t) - \widetilde{\rho} W^{\frac{\alpha_2+\beta_2}{2\beta_2}}(t),
\end{aligned} \tag{2.2.39}$$

where $\widetilde{\Theta} = 2\Theta$, $\widetilde{\chi} = \chi 2^{\frac{\alpha_1+\beta_1}{2\beta_1}}$, $\widetilde{\rho} = \rho 2^{\frac{\alpha_2+\beta_2}{2\beta_2}} (i_1+i_2)^{\frac{\beta_2-\alpha_2}{2\beta_2}}$.

When $t = t_l$, it follows from Eq. (2.2.33) that

$$\begin{aligned}
W(t_l^+) &= \frac{1}{2} \sum_{r=1}^{i_1} p_r^2(t_l^+) + \frac{1}{2} \sum_{s=1}^{i_2} q_s^2(t_l^+) \\
&= \frac{1}{2} \sum_{r=1}^{i_1} (1+\varrho)^2 p_r^2(t_l) + \frac{1}{2} \sum_{s=1}^{i_2} (1+\sigma)^2 q_s^2(t_l) \\
&= \frac{\widetilde{\xi}}{2} \sum_{k=1}^{i_1+i_2} \widetilde{w}_k^2(t_l) = \widetilde{\xi} W(t_l),
\end{aligned} \tag{2.2.40}$$

where $\widetilde{\xi} = \max\{(1+\varrho)^2, (1+\sigma)^2\}$.

From Theorem 2.2.3.1, the origin of system (2.2.1) can achieve FxTS under controller (2.2.30) with the estimated settling-time given by

$$T = \frac{\ln\left[1 + \frac{\left(\widetilde{\Theta} - \frac{\ln \widetilde{\xi}}{\tau_a}\right)}{\widetilde{\rho}\widetilde{\xi}^{N_0(\frac{\beta_2-\alpha_2}{2\beta_2})}}\right]}{\left(\widetilde{\Theta} - \frac{\ln \widetilde{\xi}}{\tau_a}\right)(\frac{\alpha_2-\beta_2}{2\beta_2})} + \frac{\ln\left[\frac{\widetilde{\chi}\widetilde{\xi}^{-N_0(\frac{\beta_1-\alpha_1}{2\beta_1})}}{\widetilde{\xi}^{N_0(\frac{\beta_1-\alpha_1}{2\beta_1})}\left(\widetilde{\Theta} - \frac{\ln \widetilde{\xi}}{\tau_a}\right) + \widetilde{\chi}\widetilde{\xi}^{-N_0(\frac{\beta_1-\alpha_1}{2\beta_1})}}\right]}{\left(\frac{\ln \widetilde{\xi}}{\tau_a} - \widetilde{\Theta}\right)(\frac{\beta_1-\alpha_1}{2\beta_1})}. \tag{2.2.41}$$

Therefore the proof is completed.

Remark 2.2.3.5. As we know that the stability of the impulsive system is a subject to the class of impulsive sequence. In Theorem 2.2.3.2. and Theorem 2.2.3.3, the estimated settling-time is dependent on the impulsive sequence with the conditions $\tau_a > \frac{\ln \widetilde{\xi}}{\delta}$ and $\tau_a > \frac{\ln \widetilde{\xi}}{\widetilde{\Theta}}$ respectively. Intuitively, the conditions state that the occurrence of impulsive perturbation to the system must be separated by the length of lower bounds of τ_a for the system (2.2.18) to be FxTS. In [89], the lower bounds of control gains like the inequalities (2.2.19) and (2.2.31) of this subchapter have been derived corresponding to both the continuous controllers. However, the settling-time functions in [89] are independent of impulsive sequence which is not realistic for the applications of impulsive systems.

2.2.4 Numerical Simulations and Discussion

In this subsection, two numerical examples are provided to verify the stability results and to discuss the superiority of the obtained results in comparison to the existing results.

Example 2.2.4.1 Consider a model of the cyber-physical system with impulsive deception attacks that can be described as

$$\begin{cases} \dot{x}_1(t) = 0.1\sin(x_2(t)) + u_1(t), & t \neq t_l, \\ \dot{x}_2(t) = 0.1\sin(x_1(t) + u_2(t), & t \neq t_l, \\ x_1(t_l^+) = \xi x_1(t_l), & t = t_l, \\ x_2(t_l^+) = \xi x_2(t_l), & t = t_l, \end{cases} \tag{2.2.42}$$

where $u_1(t)$,$u_2(t)$ are the controllers given by

$$\begin{cases} u_1(t) = 0.4\mathrm{sign}(x_1(t))|x_1(t)|^{0.5} - 1.2\mathrm{sign}(x_1(t))|x_1(t)|^{1.5} - 0.2x_1(t), \\ u_2(t) = 0.4\mathrm{sign}(x_2(t))|x_2(t)|^{0.5} - 1.2\mathrm{sign}(x_2(t))|x_2(t)|^{1.5} - 0.2x_2(t), \end{cases} . \tag{2.2.43}$$

The nonlinear impulsive model (2.2.42) is a particular case of the model investigated in [114], where random cyber attacks on the system at the impulsive times have been considered. In our case, we considered that perturbation with magnitude $\xi > 1$ occurs to the signals $x_1(t), x_2(t)$ at deterministic moments of the impulsive sequence. The signals without deception attacks and the controller for the initial states $x_1(0) = 0.2, x_2(0) = 0.1$ are shown in Fig. 2.2.1 (a). Consider the impulsive sequence $\{t_l\}_{l\in\mathbb{N}}$ from [106] that can be described as

$$t_l - t_{l-1} = \begin{cases} \theta, & \text{if } \mathrm{mod}(l, N_0) \neq 0, \\ N_0(\tau_a - \theta) + \theta, & \text{if } \mathrm{mod}(l, N_0) = 0, \end{cases} \tag{2.2.44}$$

where θ and N_0 are positive numbers satisfying $\theta \leq \tau_a$ and $N_0 \in \mathbb{N}$, then τ_a and N_0 are the average impulsive interval and elasticity number of $\{t_l\}_{l\in\mathbb{N}}$ respectively. When perturbations occur at the impulsive times of the impulsive sequence (2.2.44), then the signals become unbounded and unreachable to the receptor that can be seen in Fig. 2.2.1 (b). In order to make the signals to reach the receptor, the continuous controller

(2.2.43) is applied to the states. In Fig. 2.2.1 (b), it can be seen that the signals achieved steady states in a fixed-time.

Choose the Lyapunov function $W(t) = |x_1(t)| + |x_2(t)|$, then the derivative of $W(t)$ along the solution of the system (2.2.42) for $t \neq t_l$ is

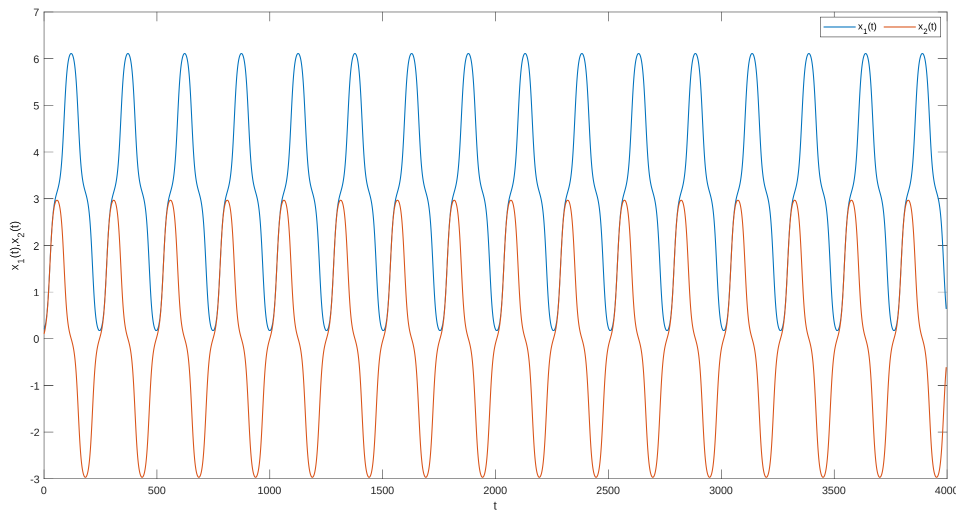
$$\begin{aligned}
\dot{W}(t) &\leq -0.2(|x_1(t)| - |x_2(t)|) - 0.4(|x_1(t)|^{0.5} + |x_2(t)|^{0.5}) - 1.2(|x_1(t)|^{1.5} - |x_2(t)|^{1.5}) \\
&\quad + 0.1(\sin(x_1(t) + \sin(x_2(t))), \\
i.e., \ \dot{W}(t) &\leq -0.1(|x_1(t)| - |x_2(t)|) - 0.4(|x_1(t)|^{0.5} + |x_2(t)|^{0.5}) - 1.2(|x_1(t)|^{1.5} - |x_2(t)|^{1.5}).
\end{aligned}$$

From Lemma 2.2.2.1, we get

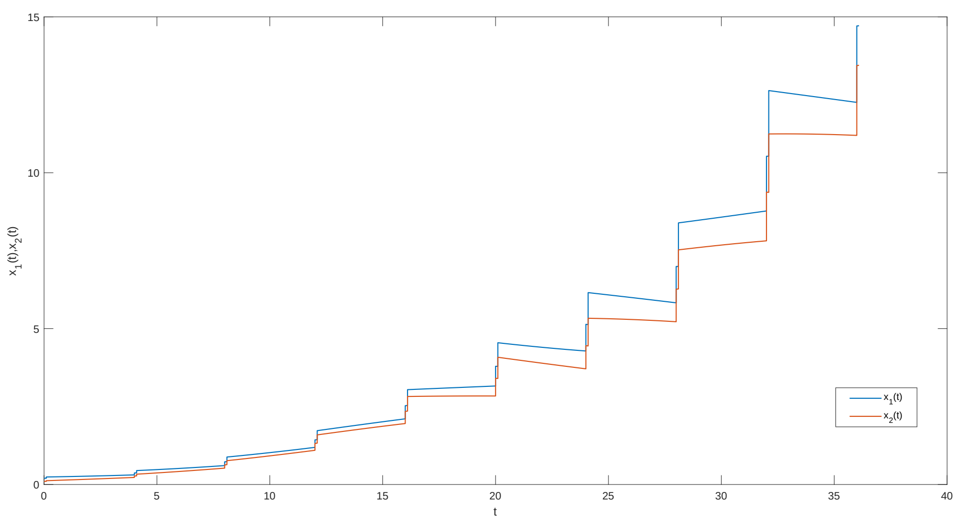
$$\dot{W}(t) \leq -0.1W(t) - 0.4W^{0.5}(t) - 0.8W^{1.5}(t),$$

and $W(t_l^+) = \xi W(t_l)$ when $t = t_l$. Hence, all the conditions of Theorem 2.2.3.1 are satisfied with $a = 0.4$, $b = 0.8$ and $c = 0.1$. The impulsive sequence is taken as $N_0 = 2$, $\theta = 0.1$ and the impulsive strength as $\xi = 1.2$ so that we get $\frac{\ln \xi}{c} = 1.82$ and choose $\tau_a = 2$ such that $\tau_a > \frac{\ln \xi}{c}$, thus, sufficient condition of Theorem 2.2.3.1 is satisfied. Hence, FxTS of the system (2.2.42) is achieved with the settling-time estimated by $T = 10.07$.

Remark 2.2.4.1. One of the main scientific contributions in this subchapter is the derivation of a sufficient condition $\tau_a > \frac{\ln \xi}{c}$ which assures that if the impulsive systems experience perturbations at the impulse moments separated by at least $\frac{\ln \xi}{c}$, then the solutions to the impulsive systems must be FxTS. Otherwise, the impulsive system may loss the stability due to having perturbations at the impulse times. This statement can be numerically verified from Fig.2.2.2 (a) and 2.2.2 (b).

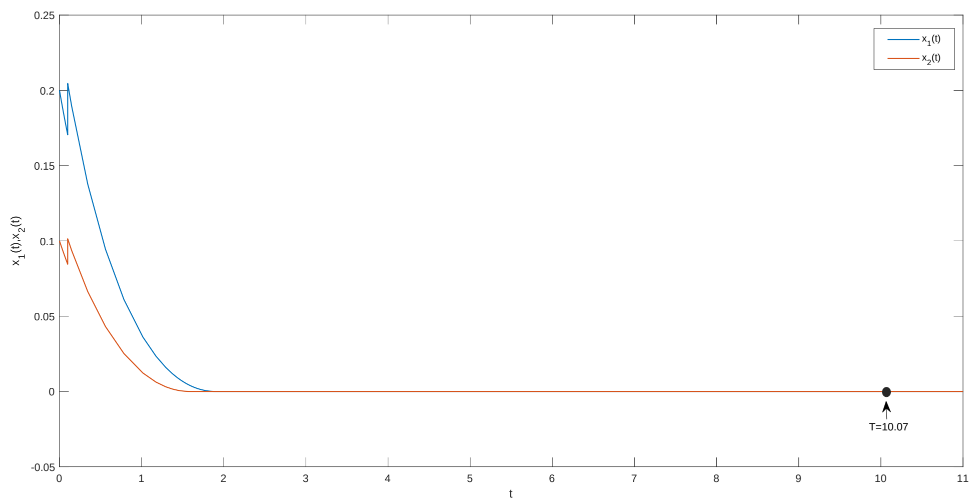


2.2.1 (a)

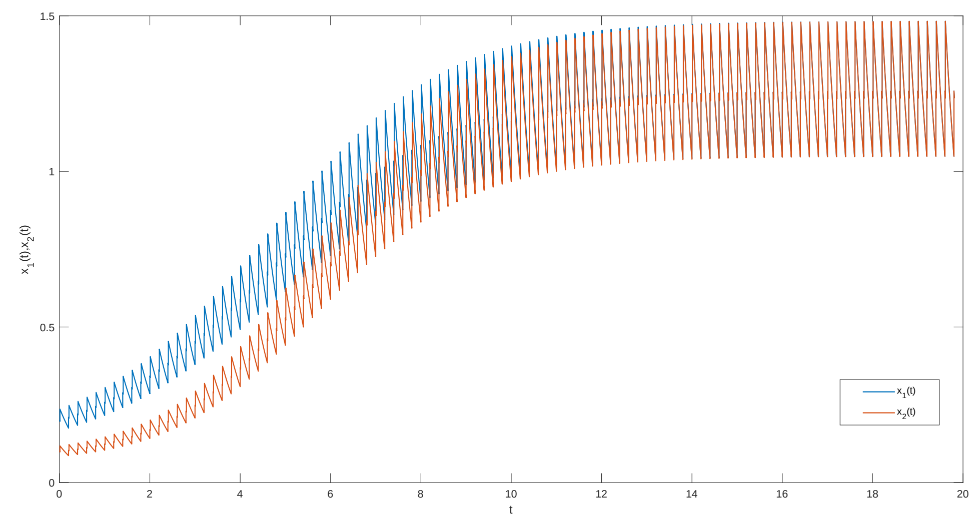


2.2.1 (b)

Figure 2.2.1: (a) Oscillatory states of the system (2.2.42) without controller (2.2.43) and the impulsive perturbation (b) States of the system (2.2.42) without controller (2.2.43) but having impusive perturbations $\xi = 1.2$ at the moments of impulsive sequence $\tau_a = 2, N_0 = 2,$ and $\theta = 0.1$



2.2.2 (a)



2.2.2 (b)

Figure 2.2.2: (a) A plot to demonstrate fixed-time stability of system (2.2.42) with sufficient condition $\tau_a = 2 > 1.82$ and the controller (2.2.43) (b) The states of system (2.2.42) for $\tau_a = 0.1 < 1.82$,,i.e., without sufficient condition and with controller (2.2.43).

Example 2.2.4.2. Consider a numerical model of CGBAMNN with impulsive effects as

$$\begin{cases} \dot{x}_r(t) = -a_r(x_r(t))\left\{b_r(x_r(t)) - \sum_{s=1}^{2} h_{rs} f_s(y_s(t)) - I_r\right\}, & t \neq t_l,\, t \in \mathbb{R}_+, \\ \dot{y}_s(t) = -c_s(y_s(t))\left\{d_s(y_s(t)) - \sum_{r=1}^{2} w_{sr} g_r(x_r(t)) - J_s\right\}, & t \neq t_l,\, t \in \mathbb{R}_+, \\ \varDelta x_r(t_l) = \varrho x_r(t_l), & l \in \mathbb{N}, \\ \varDelta y_s(t_l) = \sigma y_s(t_l), & l \in \mathbb{N}, \end{cases} \tag{2.2.45}$$

where $r, s = 1, 2$, $a_1(x_1(t)) = 0.8 + \frac{0.2}{1+x_1^2(t)}$, $a_2(x_2(t)) = 0.6 + \frac{0.3}{1+x_2^2(t)}$, $c_1(y_1(t)) = 1.4 + \frac{0.6}{1+y_1^2(t)}$, $c_2(y_2(t)) = 0.74 + \frac{0.6}{1+y_2^2(t)}$, $b_1(x_1(t)) = 0.5x_1(t)$, $b_2(x_2(t)) = 0.74x_2(t)$, $d_1(y_1(t)) = 0.2y_1(t)$, $d_2(y_2(t)) = 0.3y_2(t)$, $M = \{h_{rs}\}_{2\times 2} = \begin{bmatrix} -2.5 & 0.5 \\ 0.3 & -1.5 \end{bmatrix}$, $D = \{w_{sr}\}_{2\times 2} = \begin{bmatrix} -0.4 & -1.3 \\ 1.5 & -0.4 \end{bmatrix}$, $\varrho = 0.05$, $\sigma = 0.1$, $I_r = J_s = 0$.

Let the activation functions are defined as $g_r(x_r(t)) = \frac{1}{2}\tanh(x_r(t))$,$f_s(y_s(t)) = \frac{1}{2}(|y_s(t)+1|-|y_s(t)-1|)$, which satisfy Assumption (2.2.2.3) then we have $A_1 = A_2 = B_1 = B_2 = 1$ and obviously $0.8 \leq a_1(x_1) \leq 1$, $0.6 \leq a_2(x_2) \leq 0.9$, $1.4 \leq c_1(y_1) \leq 2$, $0.74 \leq c_2(y_2) \leq 1.34$, $\dot{b}_1(x_1) = 0.5$, $\dot{b}_2(x_2) = 0.74$, $\dot{d}_1(y_1) = 0.2$, $\dot{d}(y_2) = 0.3$, which satisfy the Assumptions (2.2.2.1)-(2.2.2.2). Without any controller, the system's trajectories are shown in Fig.2.2.3. According to Theorem 2.2.3.2, the control gains k_r^1, m_s^1, $r, s = 1, 2$ should satisfy

$$k_1^1 > \sum_{s=1}^{2} \check{c}_s B_1 |w_{s1}| - \hat{a}_1 \hat{b}_1 = 2.41,\ \ k_2^1 > \sum_{s=1}^{2} \check{c}_s B_2 |w_{s2}| - \hat{a}_2 \hat{b}_2 = 2.83,$$

$$m_1^1 > \sum_{r=1}^{2} \check{a}_r A_1 |h_{r1}| - \hat{c}_1 \hat{d}_1 = 2.49,\ \ m_2^1 > \sum_{r=1}^{2} \check{a}_r A_2 |h_{r2}| - \hat{c}_2 \hat{d}_2 = 1.63.$$

Now, we can choose $k_1^1 = k_2^1 = 4$, $m_1^1 = m_2^1 = 3$, $k_1 = m_1 = 1.25$, $k_2 = m_2 = 4$, $\alpha = 0.8$

and $\beta = 1.8$, so that we have $\phi = 1.25$, $\widetilde{\psi} = 1.3$, $\delta = 0.5$. The impulsive strength is taken as $\widetilde{\xi} = 1.3$, so that we get $\frac{\ln\widetilde{\xi}}{\delta} = 0.524 < \tau_a$. Choose $\tau_a = 0.8, N_0 = 2$, and $\theta = 0.4$. Then sufficient condition $\tau_a > \frac{\ln\widetilde{\xi}}{\delta}$ is satisfied and the system (2.2.45) with controller (2.2.17) achieves FxTS that can be seen in Fig.2.2.4 (a). Now the settling-time is calculated as $T = 5.89$ based on the given values of the parameters. In Fig.2.2.4 (a), it is observable that the system (2.2.45) is FxTS before $T = 5.89$. On the other hand, if the sufficient condition $\tau_a > \frac{\ln\widetilde{\xi}}{\delta}$ is not satisfied then there exists τ_a so that the system (2.2.45) will not be FxTS. Further, let $\tau_a = 0.03 < 0.524, N_0 = 2$, and $\theta = 0.02$ then it is clear from Fig. 2.2.4 (a) that the system's trajectories are not stable. Hence, the effectiveness of our sufficient condition to achieve FxTS of the impulsive systems with destabilizing impulses is numerically verified.

For the second type of the continuous controller (2.2.30), let us choose $k_3 = m_3 = 0.5$, $k_4 = m_4 = 1.5$, $\alpha_1 = 3$, $\beta_1 = 7$, $\alpha_2 = 9$, $\beta_2 = 5$, $\epsilon = \varepsilon = 0.5$. According to Theorem 2.2.3.3, the control gains Φ_r and Ψ_s, $r, s = 1, 2$ should satisfy

$$\Phi_1 > \sum_{s=1}^{2}(\frac{A_s\check{a}_1|h_{1s}|}{2\epsilon} + \frac{B_1\check{c}_1|w_{j1}|\varepsilon}{2}) - \hat{a}_1\hat{b}_1 = 3.55,$$

$$\Phi_2 > \sum_{s=1}^{2}(\frac{A_s\check{a}_2|h_{2s}|}{2\epsilon} + \frac{B_2\check{c}_2|w_{s2}|\varepsilon}{2}) - \hat{a}_2\hat{b}_2 = 1.79,$$

$$\Psi_1 > \sum_{r=1}^{2}(\frac{A_1\check{b}_r|h_{r1}|\epsilon}{2} + \frac{B_r\check{d}_r|w_{1r}|}{2\varepsilon}) - \hat{c}_1\hat{d}_1 = 0.47,$$

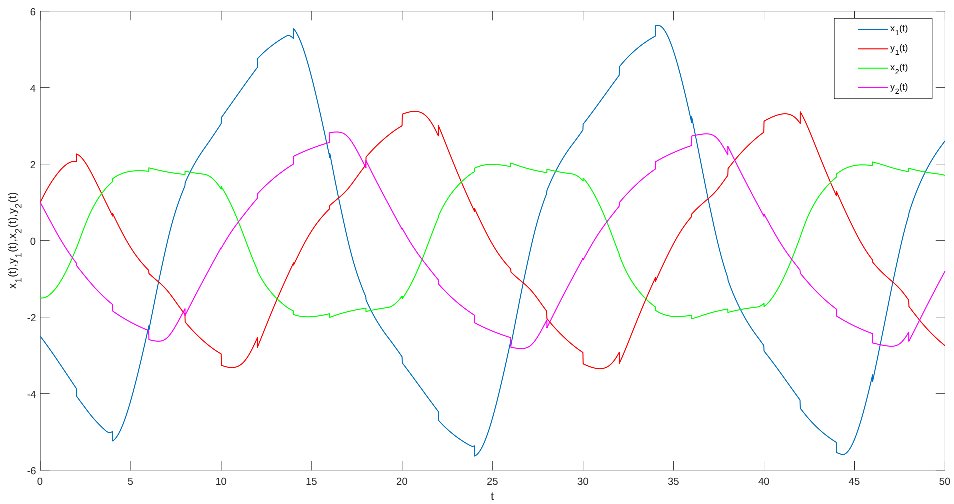
$$\Psi_2 > \sum_{r=1}^{2}(\frac{A_2\check{b}_r|h_{r2}|\epsilon}{2} + \frac{B_r\check{d}_r|w_{2r}|}{2\varepsilon}) - \hat{c}_2\hat{d}_2 = 0.85.$$

To satisfy the inequalities, the control gains are chosen as $\Phi_1 = \Phi_2 = 4$, $\Psi_1 = \Psi_2 = 2$, and we obtain by a simple calculation $\widetilde{\Theta} = 0.47$, $\widetilde{\chi} = 0.8$, $\widetilde{\rho} = 2.25$. The impulsive sequence is taken as $N_0 = 1$, $\theta = 0.2$ and the impulsive strength is assumed to be $\widetilde{\xi} = 1.2$. We calculate $\frac{\ln\widetilde{\xi}}{\widetilde{\Theta}} = 0.388$ and choose $\tau_a = 0.5$ such that $\tau_a > \frac{\ln\widetilde{\xi}}{\widetilde{\Theta}}$. Hence, sufficient condition of Theorem 2.2.3.3 is satisfied. The settling-time is estimated by $T = 2.62$. From Fig. 2.2.5 (b), it can be observed that the system (2.2.45) is FxTS. If sufficient condition $\tau_a > \frac{\ln\widetilde{\xi}}{\widetilde{\Theta}}$ is not satisfied then there exists τ_a such that the system (2.2.45) can not be FxTS, which is demonstrated in Fig.2.2.5 (a) for $\tau_a = 0.08, N_0 = 1,$ and $\theta = 0.05$. Hence, the sufficient condition is also effective with the second controller under the destabilizing impulsive effects to the system.

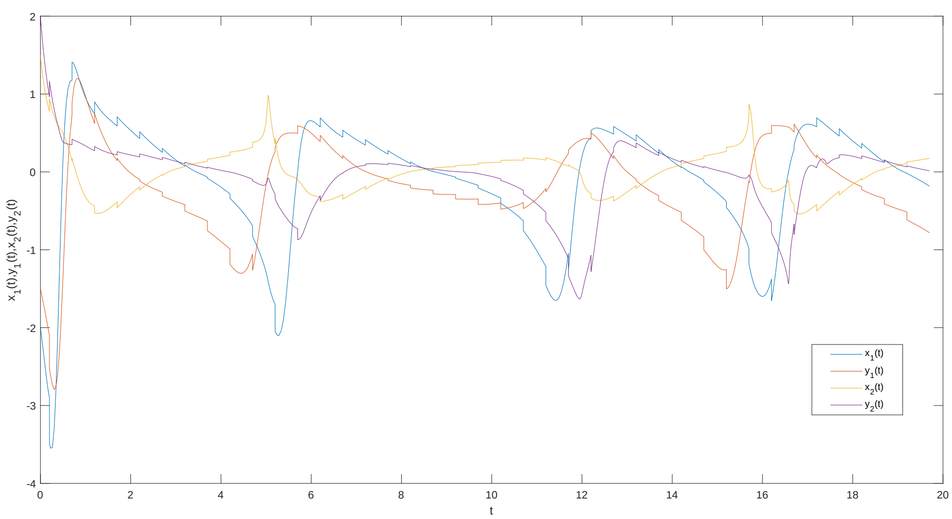
Remark 2.2.4.2. As discussed in the Remark 2.2.3.1 that two terms of Lyapunov inequality $\dot{W}(u(t)) \leq -aW^{\alpha}(u(t)) - bW^{\beta}(u(t))$, $0 < \alpha < 1$, $\beta > 1$ is not sufficient to achieve FxTS for the impulsive systems with destabilizing impulses. This can be verified numerically. For example, let the second continuous controller (2.2.30) is written without the control gain term as follows:

$$\begin{cases} u_r(t) = & -k_3 p_r^{\frac{\alpha_1}{\beta_1}}(t) - k_4 p_r^{\frac{\alpha_2}{\beta_2}}(t), \\ v_s(t) = & -m_3 q_s^{\frac{\alpha_1}{\beta_1}}(t) - m_4 q_s^{\frac{\alpha_2}{\beta_2}}(t). \end{cases} \tag{2.2.46}$$

Using the same parameters values as given above, it can be verified from Fig.2.2.3, that the system (2.2.45) is not FxTS. Hence, the obtained results in Theorem 2.2.3.1 of this subchapter are an extension of the previous articles [87, 88, 89] in which only stabilizing impulses have been considered.

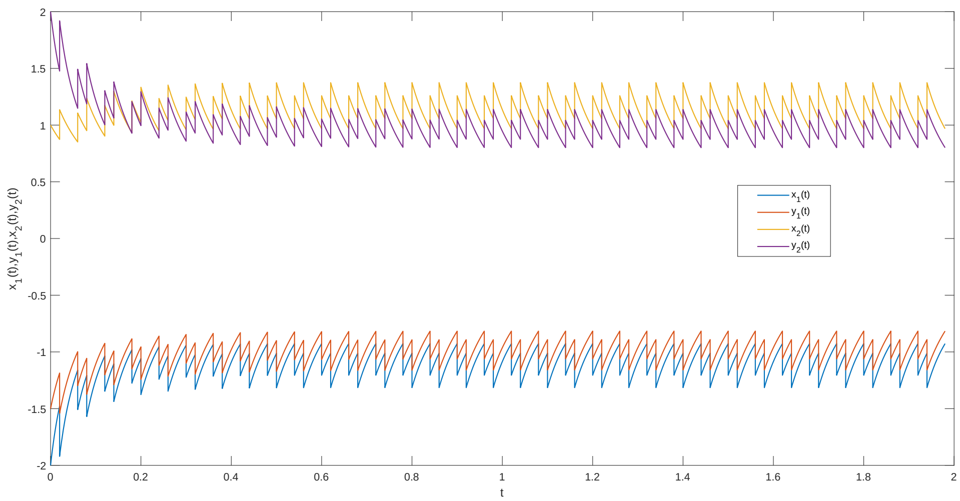


2.2.3 (a)

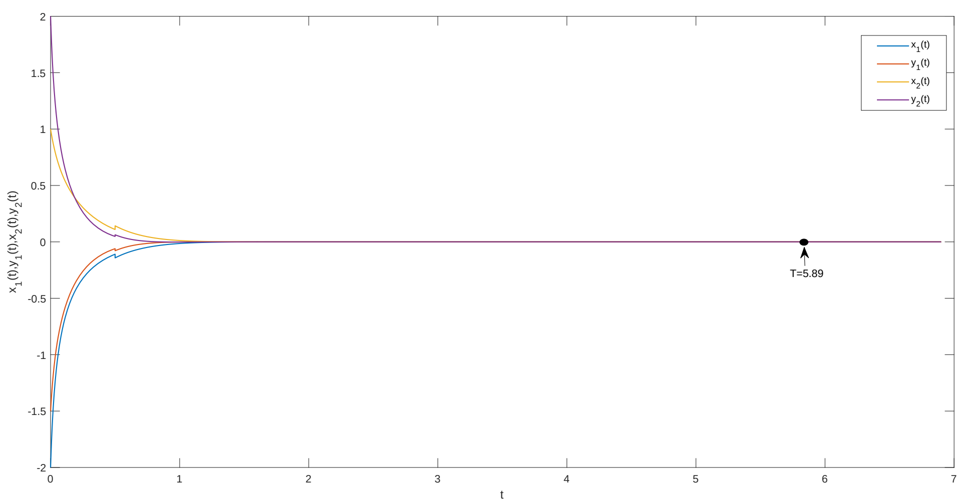


2.2.3 (b)

Figure 2.2.3: (a) Time evolution of state trajectories of system (2.2.42) without any controllers (b) Trajectories of the system (2.2.45) with controller (2.2.46) and $\tau_a = 0.5, N_0 = 1,$ and $\theta = 0.2$.

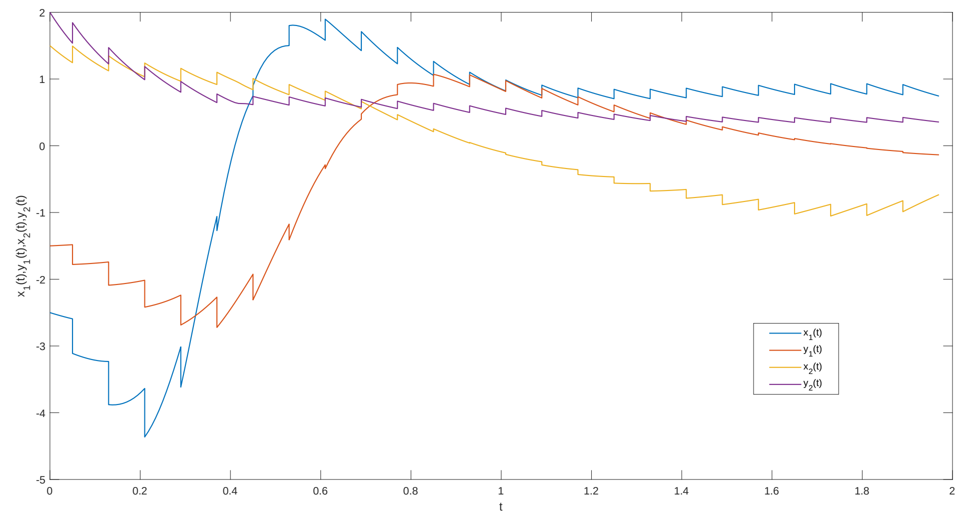


2.2.4 (a)

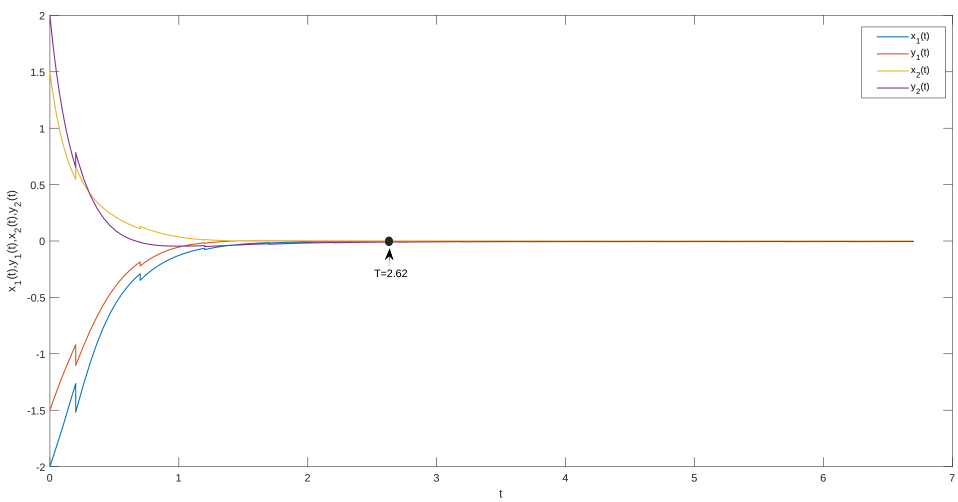


2.2.4 (b)

Figure 2.2.4: (a) Time evolution of system (2.2.45) with controller (2.2.17) and $\tau_a = 0.03 < 0.524$ (b) Time evolution of system (2.2.45) with controller (2.2.17) and $\tau_a = 0.8 > 0.524$.



2.2.5 (a)



2.2.5 (b)

Figure 2.2.5: (a) Time evolution of system (2.2.45) with controller (2.2.30) and $\tau_a = 0.08 < 0.388$ (b) Time evolution of system (2.2.45) with controller (2.2.30) and $\tau_a = 0.5 > 0.388$.

2.2.5 Conclusion

This subchapter investigates the effects of destabilizing impulses on FxTS of impulsive dynamical systems. A novel sufficient condition to achieve FxTS under the influence of impulsive perturbations to the systems has been obtained by using a new Lyapunov inequality. The comparison method and the concept of average impulsive interval have been applied to derive the results in this scientific contribution. From the sufficient condition, it is concluded that if the impulsive perturbations to the systems activate at the impulsive times which are separated by a certain length then FxTS of impulsive systems is possible to obtain. Two continuous controllers in which one contains signum terms and another does not, have been constructed based on the Lyapunov inequality to investigate FxTS of CGBAMNNs. The settling-time functions obtained in this subchapter depend on the parameters of impulsive sequences. This implies that the desired settling-time value can be set for FxTS by taking different classes of impulsive sequences. Finally, two numerical examples are given. First one is a cyber-physical model and the second one is a neural network to show the effectiveness of proposed theoretical results. In future work, it will be interesting to investigate FxTS of time-delayed impulsive systems with hybrid impulses.