

CERTAIN FUNCTIONAL SPACE AND PSEUDO-
DIFFERENTIAL OPERATORS INVOLVING THE
WEINSTEIN TRANSFORM



Thesis submitted in partial fulfillment
for the Award of Degree
Doctor of Philosophy

by

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It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

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I, **Sitaram Yadav**, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of **Prof. Santosh Kumar Upadhyay** from **July, 2019** to **July, 2024** at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, thesis, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.


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Abbreviations

- $\mathbb{N} = \{1, 2, 3, \dots\}$
- $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$
- $\mathbb{R} = (-\infty, \infty)$
- $\mathbb{R}^+ = \mathbb{I} = (0, \infty)$
- $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$ (*n-times*)
- $\|x\|^2 = \sum x_i^2$
- $D_x \equiv \frac{\partial}{\partial x}$
- $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$
- $\mathbb{N}_0^{n+1} = \mathbb{N}_0 \times \mathbb{N}_0 \times \dots \times \mathbb{N}_0$ (*n+1- times*)
- $|x| = \sum_i x_i$
- $x = (x', x_{n+1}) = (x_1, x_2, x_3, \dots, x_n, x_{n+1}) \in \mathbb{R}_0^{n+1}$
- $-x = (-x', x_{n+1}) = (-x_1, -x_2, -x_3, \dots, -x_n, x_{n+1}) \in \mathbb{R}_0^{n+1}$
- $\xi = (\xi', \xi_{n+1}) = (\xi_1, \xi_2, \dots, \xi_n, \xi_{n+1}) \in \mathbb{C}^{n+1}$
- $-\xi = (-\xi', \xi_{n+1}) = (-\xi_1, -\xi_2, \dots, -\xi_n, \xi_{n+1}) \in \mathbb{C}^{n+1}$
- $\langle x, \xi \rangle = \sum_{k=1}^{n+1} x_k \xi_k$
- $|\xi|^2 = \sum_{k=1}^{n+1} \xi_k^2$
- $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} D_{n+1}^{\alpha_{n+1}}$, $\alpha \in \mathbb{N}_+^{n+1}$
- $x^m = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} x_{n+1}^{m_{n+1}}$, $m \in \mathbb{N}_0^{n+1}$
- $\alpha! = \alpha_1! \alpha_2! \dots \alpha_{n+1}!$ for $\alpha \in \mathbb{N}_0^{n+1}$
- $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \dots \binom{\alpha_n}{\beta_n} \binom{\alpha_{n+1}}{\beta_{n+1}}$ for $\alpha, \beta \in \mathbb{N}_0^{n+1}$
- $C_*(\mathbb{R}_+^{n+1})$, The space of continuous functions on \mathbb{R}_+^{n+1} , even with respect to last variable
- $C_*^\infty(\mathbb{R}_+^{n+1})$, The space of infinitely differentiable function on \mathbb{R}_+^{n+1} , even with respect to

last variable

- $S_*(\mathbb{R}_+^{n+1})$, The Schwartz space of rapidly decreasing functions on \mathbb{R}_+^{n+1} , even with respect to last variable.
- $S'_*(\mathbb{R}_+^{n+1})$, the dual of the Schwartz space $S_*(\mathbb{R}_+^{n+1})$.