Chapter 5

Quasi-projective synchronization of inertial complex-valued recurrent neural networks with mixed time-varying delay and mismatched parameters

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This chapter investigates the quasi-projective synchronization of inertial complex-valued recurrent neural networks in the presence of mixed time-varying delay and mismatched parameters. Firstly, inertial in neural networks have proposed by Babcock and Westervelt [116] in 1986. They have added inertia to an RLC circuit that connects a neuron's output to its input, and analyzed its dynamical behavior. In consequence of the analysis, they found that the addition of an inertial term to the rate equation of a simple electronic neural network consisting of one or two neurons may exhibit complex dynamic behaviors, including spontaneous oscillation, instability, ringing around the equilibrium point, and chaotic responses to periodic drives. These behaviors are similar to those observed in real biological neurons. The findings of this research highlight the importance of inertial effects in controlling nonlinear dynamical behaviors and suggest possible avenues for further studies on the stability of nonlinear electronic networks. It is necessary to have systems with chaotic natures or complex dynamic behaviors in engineering applications. Chaotic systems, for example, increase the security strength of signals passing from a transmitter to a receiver in secure communication. Therefore, the synchronization problem of inertial complex-valued recurrent neural networks has been paid much attention to researchers [117, 118, 119, 120, 121].

Using the concept of inertial neural networks, sufficient criteria have been derived for achieving synchronization between nonidentical complex-valued recurrent neural networks in the presence of mixed time-varying delay. In the end of this chapter, two numerical examples are considered, which have discussed the efficiency of obtained results.

5.1 Preliminaries and System Formulation

This chapter is concerned with the following inertial CVRNNs with mixed delay as drive system as

$$\begin{aligned}\ddot{\omega}_u(t) =& - d_u\dot{\omega}_u(t) - c_u\omega_u(t) + \sum_{v=1}^{n} a_{uv} f_v(\omega_v(t)) + \sum_{v=1}^{n} b_{uv} g_v(\omega_v(t-\sigma_v(t))) \\ &+ \sum_{v=1}^{n} p_{uv} \int_{t-\tau_v(t)}^{t} h_v(\omega_v(s))ds + I_u(t), \end{aligned} \tag{5.1.1}$$

under the initial conditions $\omega_u(s) = \psi_u(s), \dot{\omega}_u(s) = \phi_u(s),\ s \in [-\bar{\sigma}, 0]$,

which can be rewritten in compact form as

$$\ddot{\omega}(t) = - D\dot{\omega}(t) - C\omega(t) + Af(\omega(t)) + Bg(\omega(t-\sigma(t))) + P\int_{t-\tau(t)}^{t} h(\omega(s))ds + I(t), \tag{5.1.2}$$

where $u \in I \triangleq \{1, 2, ..., n\}$, $\omega(t) = (\omega_1(t), \omega_2(t), ..., \omega_n(t))^T$ represents the state vector of the neuron at time t. $D = \text{diag}(d_1, d_2, ..., d_n)$ and $C = \text{diag}(c_1, c_2, ..., c_n)$ are non-negative constant vectors, $A = (a_{uv})_{n\times n} \in C^{n\times n}$ is the connection weight matrix of the neuron at time t. $B = (b_{uv})_{n\times n}$ and $P = (p_{uv})_{n\times n} \in C^{n\times n}$ denote the neuron connection weight matrix at $t-\sigma(t)$ and $t-\tau(t)$, respectively. $f(\omega(t)) = (f_1(\omega_1(t)), f_2(\omega_2(t)), ..., f_n(\omega_n(t)))^T$, $g(\omega(t-\sigma(t))) = (g_1(\omega_1(t-\sigma_1(t))), g_2(\omega_2(t-\sigma_2(t))), ..., g_n(\omega_n(t-\sigma_n(t))))^T$ and $h(\omega(t)) = (h_1(\omega_1(t)), h_2(\omega_2(t)), ..., h_n(\omega_n(t)))^T$ are the complex-valued nonlinear activation functions of the neurons at time t, time-varying discrete delay $t-\sigma(t)$ and distributed delay $t-\tau(t)$. $\sigma_v(t)$ and $\tau_v(t)$ denote the bounded discrete and distributed time-varying delays, respectively satisfying $0 \leq \sigma(t) \leq \sigma$ and $0 \leq \tau(t) \leq \tau$ with $\bar{\sigma}= max\{\sigma, \tau\}$, $I = (I_1, I_2, ..., I_n)^T$ denotes external input vector of the neuron at time t satisfying $|I_u(t)| \leq I_u$ for all $u =$ 1,2,3,...,n. $\phi = (\phi_1, \phi_2, ..., \phi_n)^T$ and $\psi = (\psi_1, \psi_2, ..., \psi_n)^T$ are bounded and continuous functions in $C([-\bar{\sigma}, 0], C^n)$.

The following variable transformation is considered as

$$w(t) = \dot{\omega}(t) + \xi\omega(t).$$

Choosing $\xi \in R$, the equation (5.1.2) may be transformed into

$$\begin{aligned}\dot{\omega}(t) =& -\xi\omega(t) + w(t),\\ \dot{w}(t) =& -\gamma w(t) + \delta\omega(t) + Af(\omega(t)) + Bg(\omega(t-\sigma(t))) + P\int_{t-\tau(t)}^{t} h(\omega(s))ds + I(t),\end{aligned} \tag{5.1.3}$$

where $\gamma= D-\xi$, $\delta= \gamma\xi - C$.

The corresponding response system is described as

$$\begin{aligned}\ddot{\widetilde{\omega}}_u(t) =& -d'_u\dot{\widetilde{\omega}}_u(t) - c'_u\widetilde{\omega}_u(t) + \sum_{v=1}^{n} a'_{uv}f_v(\widetilde{\omega}_v(t)) + \sum_{v=1}^{n} b'_{uv}g_v(\widetilde{\omega}_v(t-\sigma_v(t)))\\ &+ \sum_{v=1}^{n} p'_{uv}\int_{t-\tau_v(t)}^{t} h_v(\widetilde{\omega}_v(s))ds + I'_u(t) + \Omega(t),\end{aligned} \tag{5.1.4}$$

under the initial conditions $\widetilde{\omega}_u(s) = \widetilde{\psi}_u(s), \dot{\widetilde{\omega}}_u(s) = \widetilde{\phi}_u(s)$, $s \in [-\bar{\sigma}, 0]$, which can be rewritten in compact form as

$$\begin{aligned}\ddot{\tilde{\omega}}(t) =& - D'\dot{\tilde{\omega}}(t) - C'\tilde{\omega}(t) + A'f(\tilde{\omega}(t)) + B'g(\tilde{\omega}(t-\sigma(t))) + P'\int_{t-\tau(t)}^{t} h(\tilde{\omega}(s))ds \\ &+ I'(t) + \Omega(t), \end{aligned} \tag{5.1.5}$$

$\tilde{\omega}(t) = (\tilde{\omega}_1(t), \tilde{\omega}_2(t), ..., \tilde{\omega}_n(t))^T$ represents the state vector of the neuron at time t. $D' = \text{diag}(d'_1, d'_2, ..., d'_n)$ and $C' = \text{diag}(c'_1, c'_2, ..., c'_n)$ are non-negative constant vectors, $A' = (a'_{uv})_{n\times n} \in C^{n\times n}$ is the connection weight matrix of the neuron at time t. $B' = (b'_{uv})_{n\times n}$ and $P' = (p'_{uv})_{n\times n} \in C^{n\times n}$ denote the neuron connection weight matrix at $t-\sigma(t)$ and $t-\tau(t)$, respectively. $\Omega(t)$ is control input to be designed.

Considering the variable transformation as

$$\tilde{w}(t) = \dot{\tilde{\omega}}(t) + \xi'\tilde{\omega}(t),$$

by choosing some scalar $\xi' \in R$, the equation (5.1.5) may be transformed into

$$\begin{aligned}\dot{\tilde{\omega}}(t) =& - \xi'\tilde{\omega}(t) + \tilde{w}(t), \\ \dot{\tilde{w}}(t) =& - \gamma'\tilde{w}(t) + \delta'\tilde{\omega}(t) + A'f(\tilde{\omega}(t)) + B'g(\tilde{\omega}(t-\sigma(t))) + P'\int_{t-\tau(t)}^{t} h(\tilde{\omega}(s))ds \\ &+ I'(t) + \Omega(t), \end{aligned} \tag{5.1.6}$$

where $\gamma' = D' - \xi'$, $\delta' = \gamma'\xi' - C'$.

Suppose $\omega(t) = \alpha(t) + i\beta(t)$, $w(t) = u(t) + iv(t)$, $\alpha(t), \beta(t), u(t)$ and $v(t) \in R$. Now separating equations (5.1.3) and (5.1.6) into real and imaginary parts, the drive system (5.1.3) can be written as

$$\begin{aligned}
\dot{\alpha}(t) =& -\xi\alpha(t) + u(t),\\
\dot{u}(t) =& -\gamma u(t) + \delta\alpha(t) + A^R f^R(\alpha(t), \beta(t)) - A^I f^I(\alpha(t), \beta(t))\\
&+ B^R g^R(\alpha(t-\sigma(t)), \beta(t-\sigma(t))) - B^I g^I(\alpha(t-\sigma(t)), \beta(t-\sigma(t)))\\
&+ P^R \int_{t-\tau(t)}^{t} h^R(\alpha(s), \beta(s))ds - P^I \int_{t-\tau(t)}^{t} h^I(\alpha(s), \beta(s))ds + I^R(t), \quad (5.1.7)
\end{aligned}$$

$$\begin{aligned}
\dot{\beta}(t) =& -\xi\beta(t) + v(t),\\
\dot{v}(t) =& -\gamma v(t) + \delta\beta(t) + A^R f^I(\alpha(t), \beta(t)) + A^I f^R(\alpha(t), \beta(t))\\
&+ B^R g^I(\alpha(t-\sigma(t)), \beta(t-\sigma(t))) + B^I g^R(\alpha(t-\sigma(t)), \beta(t-\sigma(t)))\\
&+ P^R \int_{t-\tau(t)}^{t} h^I(\alpha(s), \beta(s))ds + P^I \int_{t-\tau(t)}^{t} h^R(\alpha(s), \beta(s))ds + I^I(t). \quad (5.1.8)
\end{aligned}$$

The corresponding response system of equation (5.1.6) is

$$\begin{aligned}
\dot{\tilde{\alpha}}(t) =& -\xi'\tilde{\alpha}(t) + \tilde{u}(t),\\
\dot{\tilde{u}}(t) =& -\gamma'\tilde{u}(t) + \delta'\tilde{\alpha}(t) + A'^R f^R(\tilde{\alpha}(t), \tilde{\beta}(t)) - A'^I f^I(\tilde{\alpha}(t), \tilde{\beta}(t))\\
&+ B'^R g^R(\tilde{\alpha}(t-\sigma(t)), \tilde{\beta}(t-\sigma(t))) - B'^I g^I(\tilde{\alpha}(t-\sigma(t)), \tilde{\beta}(t-\sigma(t)))\\
&+ P'^R \int_{t-\tau(t)}^{t} h^R(\tilde{\alpha}(s), \tilde{\beta}(s))ds - P'^I \int_{t-\tau(t)}^{t} h^I(\tilde{\alpha}(s), \tilde{\beta}(s))ds + I'^R(t) + \Omega^R(t),
\end{aligned}$$

(5.1.9)

$$
\begin{aligned}
\dot{\widetilde{\beta}}(t) =& -\xi'\widetilde{\beta}(t) + \widetilde{v}(t),\\
\dot{\widetilde{v}}(t) =& -\gamma'\widetilde{v}(t) + \delta'\widetilde{\beta}(t) + A'^R f^I(\widetilde{\alpha}(t),\widetilde{\beta}(t)) + A'^I f^R(\widetilde{\alpha}(t),\widetilde{\beta}(t))\\
&+ B'^R g^I(\widetilde{\alpha}(t-\sigma(t)),\widetilde{\beta}(t-\sigma(t))) + B'^I g^R(\widetilde{\alpha}(t-\sigma(t)),\widetilde{\beta}(t-\sigma(t)))\\
&+ P'^R\int_{t-\tau(t)}^{t} h^I(\widetilde{\alpha}(s),\widetilde{\beta}(s))ds + P'^I\int_{t-\tau(t)}^{t} h^R(\widetilde{\alpha}(s),\widetilde{\beta}(s))ds + I'^I(t) + \Omega^I(t).
\end{aligned}
\tag{5.1.10}
$$

Let us consider projective error as $\epsilon^R(t) = \widetilde{\alpha}(t) - r\alpha(t)$, $\epsilon^I(t) = \widetilde{\beta}(t) - r\beta(t)$, $\hat{\epsilon}^R(t)$ $= \widetilde{u}(t) - ru(t)$, $\hat{\epsilon}^I(t) = \widetilde{v}(t) - rv(t)$ and the control input vectors $\Omega^R(t)$ and $\Omega^I(t)$ are defined by

$$
\begin{aligned}
\Omega^R(t) =& -\kappa\epsilon^R(t),\\
\Omega^I(t) =& -\kappa\epsilon^I(t),
\end{aligned}
\tag{5.1.11}
$$

where $\kappa = \text{diag}(\kappa_1, \kappa_2, ..., \kappa_n)$.

Therefore from equations (5.1.7) and (5.1.9), we get

$$
\begin{aligned}
\dot{\epsilon}^R(t) =& -\xi'\epsilon^R(t) + \hat{\epsilon}^R(t) + N_1(t),\\
\dot{\hat{\epsilon}}^R(t) =& -\gamma'\hat{\epsilon}^R(t) + (\delta' - \kappa)\epsilon^R(t) + A'^R f^R(\epsilon^R(t),\epsilon^I(t)) - A'^I f^I(\epsilon^R(t),\epsilon^I(t))\\
&+ B'^R g^R(\epsilon^R(t-\sigma(t)),\epsilon^I(t-\sigma(t))) - B'^I g^I(\epsilon^R(t-\sigma(t)),\epsilon^I(t-\sigma(t)))\\
&+ P'^R\int_{t-\tau(t)}^{t} h^R(\epsilon^R(s),\epsilon^I(s))ds - P'^I\int_{t-\tau(t)}^{t} h^I(\epsilon^R(s),\epsilon^I(s))ds + M_1(t).
\end{aligned}
\tag{5.1.12}
$$

Also from equations (5.1.8) and (5.1.10), we get

$$
\begin{aligned}
\dot{\epsilon}^I(t) =& -\xi'\epsilon^I(t) + \hat{\epsilon}^I(t) + N_2(t),\\
\dot{\hat{\epsilon}}^I(t) =& -\gamma'\hat{\epsilon}^I(t) + (\delta' - \kappa)\epsilon^I(t) + A'^R f^I(\epsilon^R(t),\epsilon^I(t)) + A'^I f^R(\epsilon^R(t),\epsilon^I(t))
\end{aligned}
$$

$$+ B'^R g^I(\epsilon^R(t-\sigma(t)), \epsilon^I(t-\sigma(t))) + B'^I g^R(\epsilon^R(t-\sigma(t)), \epsilon^I(t-\sigma(t)))$$
$$+ P'^R \int_{t-\tau(t)}^{t} h^I(\epsilon^R(s), \epsilon^I(s))ds + P'^I \int_{t-\tau(t)}^{t} h^R(\epsilon^R(s), \epsilon^I(s))ds + M_2(t), \quad (5.1.13)$$

where

$$\begin{aligned}
N_1(t) =& r(\xi - \xi')\alpha(t), \\
M_1(t) =& r(\gamma - \gamma')u(t) + r(\delta' - \delta)\alpha(t) + r(A'^R - A^R)f^R(\alpha(t), \beta(t)) \\
&+ r(A'^I - A^I)f^I(\alpha(t), \beta(t)) + r(B'^R - B^R)g^R(\alpha(t-\sigma(t)), \beta(t-\sigma(t))) \\
&+ r(B'^I - B^I)g^I(\alpha(t-\sigma(t)), \beta(t-\sigma(t))) + r(P'^R - P^R) \\
&\times \int_{t-\tau(t)}^{t} h^R(\alpha(s), \beta(s))ds + r(P'^I - P^I) \int_{t-\tau(t)}^{t} h^I(\alpha(s), \beta(s))ds \\
&+ I'^R(t) - rI^R(t).
\end{aligned}$$

Using the norm on both sides, we have

$$\begin{aligned}
\|N_1(t)\| \leq& |r|\|(\xi - \xi')\|\|\alpha(t)\|, \\
\|M_1(t)\| \leq& |r|\|(\gamma - \gamma')\|\|u(t)\| + |r|\|(\delta' - \delta)\|\|\alpha(t)\| \\
&+ |r|\|(A'^R - A^R)\|\|f^R(\alpha(t), \beta(t))\| + |r|\|(A'^I - A^I)\|\|f^I(\alpha(t), \beta(t))\| \\
&+ |r|\|(B'^R - B^R)\|\|g^R(\alpha(t-\sigma(t)), \beta(t-\sigma(t)))\| \\
&+ |r|\|(B'^I - B^I)\|\|g^I(\alpha(t-\sigma(t)), \beta(t-\sigma(t)))\| \\
&+ |r|\|(P'^R - P^R)\| \int_{t-\tau(t)}^{t} \|h^R(\alpha(s), \beta(s))\|ds \\
&+ |r|\|(P'^I - P^I)\| \int_{t-\tau(t)}^{t} \|h^I(\alpha(s), \beta(s))\|ds + (\|I'^R\| + |r|\|I^R\|). \\
\|N_1(t)\| \leq& |r|(\|\xi\| + \|\xi'\|)\vartheta, \\
\|M_1(t)\| \leq& |r|(\|\gamma\| + \|\gamma'\|)\widetilde{\vartheta} + |r|(\|\delta'\| + \|\delta\|)\vartheta + |r|(\|A'^R\| + \|A^R\|)\vartheta L_f \\
&+ |r|(\|A'^I\| + \|A^I\|)\vartheta L_{f'} + |r|(\|B'^R\| + \|B^R\|)\vartheta L_g
\end{aligned}$$

$$+ |r|(\|B'^I\| + \|B^I\|)\vartheta L_{g'} + |r|(\|P'^R\| + \|P^R\|)\tau\vartheta L_h$$
$$+ |r|(\|P'^I\| + \|P^I\|)\tau\vartheta L_{h'} + \eta_1.$$

Therefore

$$\|N_1(t)\| \leq K_1, \quad \|M_1(t)\| \leq k_2. \tag{5.1.14}$$

Now

$$\begin{aligned}
N_2(t) =& r(\xi - \xi')\beta(t),\\
M_2(t) =& r(\gamma - \gamma')v(t) + r(\delta' - \delta)\beta(t) + r(A'^R - A^R)f^I(\alpha(t), \beta(t))\\
&+ r(A'^I - A^I)f^R(\alpha(t), \beta(t)) + r(B'^R - B^R)g^I(\alpha(t - \sigma(t)), \beta(t - \sigma(t)))\\
&+ r(B'^I - B^I)g^R(\alpha(t - \sigma(t)), \beta(t - \sigma(t))) + r(P'^R - P^R)\int_{t-\tau(t)}^{t} h^I(\alpha(s), \beta(s))ds\\
&+ r(P'^I - P^I)\int_{t-\tau(t)}^{t} h^R(\alpha(s), \beta(s))ds + I'^I(t) - rI^I(t),
\end{aligned}$$

$$\begin{aligned}
\|N_2(t)\| \leq& |r|\|(\xi - \xi')\|\|\beta(t)\|,\\
\|M_2(t)\| \leq& |r|\|(\gamma - \gamma')\|\|v(t)\| + |r|\|(\delta' - \delta)\|\|\beta(t)\| + |r|\|(A'^R - A^R)\|\|f^I(\alpha(t), \beta(t))\|\\
&+ |r|\|(A'^I - A^I)\|\|f^R(\alpha(t), \beta(t))\|\\
&+ |r|\|(B'^R - B^R)\|\|g^I(\alpha(t - \sigma(t)), \beta(t - \sigma(t)))\|\\
&+ |r|\|(B'^I - B^I)\|\|g^R(\alpha(t - \sigma(t)), \beta(t - \sigma(t)))\|\\
&+ |r|\|(P'^R - P^R)\|\int_{t-\tau(t)}^{t}\|h^I(\alpha(s), \beta(s))\|ds\\
&+ |r|\|(P'^I - P^I)\|\int_{t-\tau(t)}^{t}\|h^R(\alpha(s), \beta(s))\|ds + (\|I'^I\| + |r|\|I^I\|).\\
\|N_2(t)\| \leq& |r|(\|\xi\| + \|\xi'\|)\vartheta,\\
\|M_2(t)\| \leq& |r|(\|\gamma\| + \|\gamma'\|)\widetilde{\vartheta} + |r|(\|\delta'\| + \|\delta\|)\vartheta + |r|(\|A'^R\| + \|A^R\|)\vartheta L_{f'}
\end{aligned}$$

$$+|r|(\|A'^I\|+\|A^I\|)\vartheta L_f+|r|(\|B'^R\|+\|B^R\|)\vartheta L_g,$$
$$+|r|(\|B'^I\|+\|B^I\|)\vartheta L_g+|r|(\|P'^R\|+\|P^R\|)\tau\vartheta L_{h'}$$
$$+|r|(\|P'^I\|+\|P^I\|)\tau\vartheta L_h+\eta_2.$$

Thus

$$\|N_2(t)\|\leq\widetilde{K}_1, \|M_2(t)\|\leq\widetilde{K}_2. \tag{5.1.15}$$

From equations (5.1.12)-(5.1.15), we get

$$\begin{aligned}\dot{\theta}(t)=&-\bar{\xi}\theta(t)+\hat{\theta}(t)+N(t),\\ \dot{\hat{\theta}}(t)=&-\bar{\gamma}\hat{\theta}+\bar{\delta}\theta(t)+\bar{A}_1\bar{f}_1(\theta(t))+\bar{A}_2\bar{f}_2(\theta(t))+\bar{B}_1\bar{g}_1(\theta(t-\sigma(t)))+\bar{B}_2\bar{g}_2(\theta(t-\sigma(t)))\\ &+\bar{P}_1\int_{t-\tau(t)}^{t}\bar{h}_1(\theta(s))ds+\bar{P}_2\int_{t-\tau(t)}^{t}\bar{h}_2(\theta(s))ds+M(t),\end{aligned} \tag{5.1.16}$$

where $\theta(t)=\begin{pmatrix}\epsilon^R(t)\\ \\ \epsilon^I(t)\end{pmatrix}$, $\hat{\theta}(t)=\begin{pmatrix}\hat{\epsilon}^R(t)\\ \\ \hat{\epsilon}^I(t)\end{pmatrix}$, $\bar{f}_1(\theta(t))=\begin{pmatrix}f^R(\epsilon^R(t),\epsilon^I(t))\\ \\ f^R(\epsilon^R(t),\epsilon^I(t))\end{pmatrix}$,

$\bar{f}_2(\theta(t))=\begin{pmatrix}f^I(\epsilon^R(t),\epsilon^I(t))\\ \\ f^I(\epsilon^R(t),\epsilon^I(t))\end{pmatrix}$, $\bar{g}_1(\theta(t-\sigma(t)))=\begin{pmatrix}g^R(\epsilon^R(t-\sigma(t)),\epsilon^I(t-\sigma(t)))\\ \\ g^R(\epsilon^R(t-\sigma(t)),\epsilon^I(t-\sigma(t)))\end{pmatrix}$,

$\bar{g}_2(\theta(t-\sigma(t)))=\begin{pmatrix}g^I(\epsilon^R(t-\sigma(t)),\epsilon^I(t-\sigma(t)))\\ \\ g^I(\epsilon^R(t-\sigma(t)),\epsilon^I(t-\sigma(t)))\end{pmatrix}$, $\bar{h}_1(\theta(t))=\begin{pmatrix}h^R(\epsilon^R(t),\epsilon^I(t))\\ \\ h^R(\epsilon^R(t),\epsilon^I(t))\end{pmatrix}$,

$$\bar{h_2}(\theta(t)) = \begin{pmatrix} h^I(\epsilon^R(t),\epsilon^I(t)) \\ \\ h^I(\epsilon^R(t),\epsilon^I(t)) \end{pmatrix}, \ N(t) = \begin{pmatrix} N_1(t) \\ \\ N_2(t) \end{pmatrix}, \ \bar{\xi}= \begin{pmatrix} \xi' & 0 \\ \\ 0 & \xi' \end{pmatrix}, \quad \bar{\gamma}= \begin{pmatrix} \gamma' & 0 \\ \\ 0 & \gamma' \end{pmatrix},$$

$$\bar{\delta}= \begin{pmatrix} \delta'-\kappa & 0 \\ \\ 0 & \delta'-\kappa \end{pmatrix}, \quad \bar{A}_1= \begin{pmatrix} A'^R & 0 \\ \\ 0 & A'^I \end{pmatrix}, \quad \bar{A}_2= \begin{pmatrix} -A'^I & 0 \\ \\ 0 & A'^R \end{pmatrix}, \quad \bar{B}_1= \begin{pmatrix} B'^R & 0 \\ \\ 0 & B'^I \end{pmatrix},$$

$$\bar{B}_2= \begin{pmatrix} -B'^I & 0 \\ \\ 0 & B'^R \end{pmatrix}, \quad \bar{P}_1= \begin{pmatrix} P'^R & 0 \\ \\ 0 & P'^I \end{pmatrix}, \quad \bar{P}_2= \begin{pmatrix} -P'^I & 0 \\ \\ 0 & P'^R \end{pmatrix}, \quad M(t) = \begin{pmatrix} M_1(t) \\ \\ M_2(t) \end{pmatrix}.$$

Assumption 7. *[122] Let us consider $z(t) = \nu(t) + i\eta(t)$, and $\tilde{z}(t) = \tilde{\nu}(t) + i\tilde{\eta}(t)$, then the state activation functions of ith neuron can be expressed in the following forms:*

$f_i(z) = f_i^R(\nu,\eta) + if_i^I(\nu,\eta)$, $g_i(z) = g_i^R(\nu,\eta) + ig_i^I(\nu,\eta)$, $h_i(z) = h_i^R(\nu,\eta) + ih_i^I(\nu,\eta)$, *where $f_i^R(\nu,\eta), f_i^I(\nu,\eta), g_i^R(\nu,\eta), g_i^I(\nu,\eta) \in R$, and for any $\nu,\tilde{\nu},\eta,\tilde{\eta}$, there exist positive constants $P^{RR}, P^{RI}, P^{IR}, P^{II}, Q^{RR}, Q^{RI}, Q^{IR}, Q^{II}, H^{RR}, H^{RI}, H^{IR}, H^{II}$, such that the following inequalities hold:*

$$\begin{aligned} |f_i^R(\nu,\eta) - f_i^R(\tilde{\nu},\tilde{\eta})| &\le P_i^{RR}|\nu-\tilde{\nu}| + P_i^{RI}|\eta-\tilde{\eta}|, \\ |f_i^I(\nu,\eta) - f_i^I(\tilde{\nu},\tilde{\eta})| &\le P_i^{IR}|\nu-\tilde{\nu}| + P_i^{II}|\eta-\tilde{\eta}|, \\ |g_i^R(\nu,\eta) - g_i^R(\tilde{\nu},\tilde{\eta})| &\le Q_i^{RR}|\nu-\tilde{\nu}| + Q_i^{RI}|\eta-\tilde{\eta}|, \\ |g_i^I(\nu,\eta) - g_i^I(\tilde{\nu},\tilde{\eta})| &\le Q_i^{IR}|\nu-\tilde{\nu}| + Q_i^{II}|\eta-\tilde{\eta}|, \\ |h_i^R(\nu,\eta) - h_i^R(\tilde{\nu},\tilde{\eta})| &\le R_i^{RR}|\nu-\tilde{\nu}| + R_i^{RI}|\eta-\tilde{\eta}|, \\ |h_i^I(\nu,\eta) - h_i^I(\tilde{\nu},\tilde{\eta})| &\le R_i^{IR}|\nu-\tilde{\nu}| + R_i^{II}|\eta-\tilde{\eta}|. \end{aligned}$$

For the convenience, we have

$P^{RR} = diag\{P_1^{RR}, P_2^{RR}, ..., P_n^{RR}\}, P^{RI} = diag\{P_1^{RI}, P_2^{RI}, ..., P_n^{RI}\},$

$P^{IR} = diag\{P_1^{IR}, P_2^{IR}, ..., P_n^{IR}\}, P^{II} = diag\{P_1^{II}, P_2^{II}, ..., P_n^{II}\},$

$Q^{RR} = diag\{Q_1^{RR}, Q_2^{RR}, ..., Q_n^{RR}\}, Q^{RI} = diag\{Q_1^{RI}, Q_2^{RI}, ..., Q_n^{RI}\},$

$Q^{IR} = diag\{Q_1^{IR}, Q_2^{IR}, ..., Q_n^{IR}\}, Q^{II} = diag\{Q_1^{II}, Q_2^{II}, ..., Q_n^{II}\},$

$R^{RR} = diag\{R_1^{RR}, R_2^{RR}, ..., R_n^{RR}\}, R^{RI} = diag\{R_1^{RI}, R_2^{RI}, ..., R_n^{RI}\},$

$R^{IR} = diag\{R_1^{IR}, R_2^{IR}, ..., R_n^{IR}\}, R^{II} = diag\{R_1^{II}, R_2^{II}, ..., R_n^{II}\}.$

Assumption 8. *For any $u, v \in R$ and $g_i(.) : R \to R$, then there exist constants $l_i > 0, i = 1, 2, ..., n$ such that*

$$0 \leq \frac{g_i(u) - g_i(v)}{u - v} \leq l_i, \forall i = 1, 2, ..., n.$$

Lemma 5.1.1. [117] (c_q inequality). For all $\nu, \eta \in R$, $q > 0$, the inequality $(|\nu| + |\eta|)^q \leq c_q(|\nu|^q + |\eta|^q)$ holds, where

$$c_q = \begin{cases} 1, 0 < q \leq 1, \\ 2^{q-1}, q > 1. \end{cases}$$

Lemma 5.1.2. [113] Based on Assumption 8, suppose $\|.\|_q$ be an induced norm on $R^{n\times n}$ and $\mu_q(.)$ be the corresponding matrix measure with q = 1,∞. Then $\mu_q(AG_1(\theta(t))) \leq \mu_q(A^*L)$ for $L = diag(l_1^+, l_2^+, ..., l_n^+)$, $G_1(\theta(t)) = diag\{\frac{g_1^R(x_1(t))}{x_1(t)}, ..., \frac{g_n^R(x_n(t))}{x_n(t)}\}$ with $g_u^R(0) = 0$ for u, $\theta(t) = diag(x_1(t), x_2(t), ..., x_n(t)), A = (a_{uv}) \in R^{n\times n}, A^* = (a_{uv}^*) \in R^{n\times n}$ with

$$a_{uv}^* = \begin{cases} max\{0, a_{uu}^*\}, u = v, \\ a_{uv}^*, u \neq v. \end{cases}$$

5.2 Main results

Theorem 5.2.1. Suppose the Assumption 7 holds. The trajectories of the systems (5.1.16) will be quasi-projective synchronized within a small region Ω, if there exists a matrix measure $\mu_q(.)$ for $q = 1, 2, \infty$ and the controller given in (5.1.11) , so that it satisfies the condition $\zeta_1 > \zeta_2 > 0$, where

$\zeta_1 = -(\mu_q(H) + 2l\|\bar{A}_1\|_q + 2l'\|\bar{A}_2\|_q)$, $\zeta_2 = (2m\|\bar{B}_1\|_q + 2m'\|\bar{B}_2\|_q + 2n\|\bar{P}_1\|_q + 2n'\|\bar{P}_2\|_q)$,

and H $= \begin{pmatrix} -\bar{\xi} & I \\ \bar{\delta} & -\bar{\gamma} \end{pmatrix}$, $l = max\{P^{RR}, P^{RI}\}$, $l' = max\{P^{IR}, P^{II}\}$, $m = max\{Q^{RR}, Q^{RI}\}$, $m' = max\{Q^{IR}, Q^{II}\}$, $n = max\{R^{RR}, R^{RI}\}$,$n' = max\{R^{IR}, R^{II}\}$.

The attractive region is estimated as

$$\Omega = \left\{ Z(t) \in R^n s.t \ \ \|Z(t)\|_q \leq \omega = \frac{\zeta_3}{k} \right\}.$$

Proof. Define $Z(t) = (\theta(t), \hat{\theta}(t))^T$. Now taking the upper right-hand Dini derivative of $\|Z(t)\|_q$ along the system (5.1.16), we obtain

$$\begin{aligned} D^+(\|Z(t)\|_q) =& \overline{\lim_{h\to 0^+}} \frac{\|Z(t+h)\|_q - \|Z(t)\|_q}{h} \\ =& \overline{\lim_{h\to 0^+}} \frac{\|Z(t) + h\dot{Z}(t) + \mathcal{O}(h^2)\|_q - \|Z(t)\|_q}{h}. \end{aligned} \tag{5.2.1}$$

Now,

$$\begin{aligned}
\|Z(t)+h\dot{Z}(t)+\mathcal{O}(h^2)\|_q =& \|\begin{pmatrix}\theta(t)\\ \\ \hat{\theta}(t)\end{pmatrix}+h\begin{pmatrix}\dot{\theta}(t)\\ \\ \dot{\hat{\theta}}(t)\end{pmatrix}+\mathcal{O}(h^2)\|_q \\
\leq& \|Z(t)+h\begin{pmatrix}-\bar{\xi} & I\\ \\ \bar{\delta} & -\bar{\gamma}\end{pmatrix}Z(t)\|_q+h\|\bar{A_1}\|_q\|\bar{f_1}(\theta(t))\|_q \\
&+h\|\bar{A_2}\|_q\|\bar{f_2}(\theta(t))\|_q+h\|\bar{B_1}\|_q\|\bar{g_1}(\theta(t-\sigma(t)))\|_q \\
&+h\|\bar{B_2}\|_q\|\bar{g_2}(\theta(t-\sigma(t)))\|_q+h\|\bar{P_1}\|_q\int_{t-\tau(t)}^{t}\|\bar{h_1}(\theta(s))\|_q ds \\
&+h\|\bar{P_2}\|_q\int_{t-\tau(t)}^{t}\|\bar{h_2}(\theta(s))\|_q ds+h\|\begin{pmatrix}N(t)\\ \\ M(t)\end{pmatrix}\|_q+\|\mathcal{O}(h^2)\|_q.
\end{aligned} \tag{5.2.2}$$

From Assumption 7 and Lemma 5.1.1, we have

$$\begin{aligned}
\|\bar{f_1}(\theta(t))\|_q^q \leq 2\|f_1^R(\theta(t))\|_q^q \leq& 2\bigg(P_q^{RR}\|e^R(t)\|_q+P_q^{RI}\|e^I(t)\|_q\bigg)^q \\
\leq& 2l^q\bigg(\|e^R(t)\|_q+\|e^I(t)\|_q\bigg)^q \\
\leq& 2^q l^q\bigg(\|e^R(t)\|_q^q+\|e^I(t)\|_q^q\bigg).
\end{aligned}$$

Thus

$$\|\bar{f}_1(\theta(t))\|_q \leq 2l\|\theta(t)\|_q \leq 2l\|Z(t)\|_q.$$

Similarly,

$$\begin{aligned}
&\|\bar{f}_2(\theta(t))\|_q \leq 2l'\|\theta(t)\|_q \leq 2l'\|Z(t)\|_q, \\
&\|\bar{g}_1(\theta(t-\sigma(t)))\|_q \leq 2m\|\theta(t-\sigma(t))\|_q \leq 2m\|Z(t-\sigma(t))\|_q, \\
&\|\bar{g}_2(\theta(t-\sigma(t)))\|_q \leq 2m'\|\theta(t-\sigma(t))\|_q \leq 2m'\|Z(t-\sigma(t))\|_q, \\
&\|\bar{h}_1(\theta(t))\|_q \leq 2n\|\theta(t)\|_q \leq 2n\|Z(t)\|_q, \\
&\|\bar{h}_2(\theta(t))\|_q \leq 2n'\|\theta(t)\|_q \leq 2n'\|Z(t)\|_q.
\end{aligned} \tag{5.2.3}$$

Using equation (5.2.3) in equation (5.2.2), we get

$$\begin{aligned}
\|Z(t) + h\dot{Z}(t) + \mathcal{O}(h^2)\|_q \leq &\|I + hH\|_q\|Z(t)\|_q + 2hl\|\bar{A}_1\|_q\|Z(t)\|_q + 2hl'\|\bar{A}_2\|_q\|Z(t)\|_q \\
&+ 2mh\|\bar{B}_1\|_q\|Z(t-\tau(t)))\|_q + 2m'h\|\bar{B}_2\|_q\|Z(t-\tau(t))\|_q \\
&+ 2hn\|\bar{P}_1\|_q \int_{t-\tau(t)}^{t} \|Z(s)\|_q ds + 2hn'\|\bar{P}_2\|_q \int_{t-\tau(t)}^{t} \|Z(s)\|_q ds \\
&+ h\|S(t)\|_q + \|\mathcal{O}(h^2)\|_q \\
\leq &\|I + hH\|_q\|Z(t)\|_q + 2h(l\|\bar{A}_1\|_q + l'\|\bar{A}_2\|_q)\|Z(t)\|_q \\
&+ 2h(m\|\bar{B}_1\|_q + m'\|\bar{B}_2\|_q)\|Z(t-\sigma(t))\|_q \\
&+ 2h(n\|\bar{P}_1\|_q + n'\|\bar{P}_2\|_q) \int_{t-\tau(t)}^{t} \|Z(s)\|_q ds + h\|S(t)\|_q \\
&+ \|\mathcal{O}(h^2)\|_q.
\end{aligned} \tag{5.2.4}$$

Substituting the equality (5.2.4) in equation (5.2.1), we get

$$\begin{aligned}
D^{+}(\|Z(t)\|_q) =& \overline{\lim_{h\to 0^+}} \frac{\|I+hH\|_q - 1}{h}\|Z(t)\|_q + 2(l\|\bar{A}_1\|_q + l'\|\bar{A}_2\|_q)\|Z(t)\|_q \\
&+ 2(m\|\bar{B}_1\|_q + m'\|\bar{B}_2\|_q)\|Z(t-\sigma(t))\|_q \\
&+ 2(n\|\bar{P}_1\|_q + n'\|\bar{P}_2\|_q)\int_{t-\tau(t)}^{t}\|Z(s)\|_q ds + \zeta_3 \\
\le&(\mu_q(H) + 2l\|\bar{A}_1\|_q + 2l'\|\bar{A}_2\|_q)\|Z(t)\|_q + (2m\|\bar{B}_1\|_q + 2m'\|\bar{B}_2\|_q) \\
&\times sup_{t-\sigma\le s\le t}\|Z(s)\|_q + (2n\|\bar{P}_1\|_q + 2n'\|\bar{P}_2\|_q)sup_{t-\tau\le s\le t}\|Z(s)\|_q + \zeta_3 \\
\le&(\mu_q(H) + 2l\|\bar{A}_1\|_q + 2l'\|\bar{A}_2\|_q)\|Z(t)\|_q + (2m\|\bar{B}_1\|_q + 2m'\|\bar{B}_2\|_q \\
&+ 2n\|\bar{P}_1\|_q + 2n'\|\bar{P}_2\|_q)sup_{t-\tau\le s\le t}\|Z(s)\|_q + \zeta_3 \\
\le& -\zeta_1\|Z(t)\|_q + \zeta_2 sup_{t-\tau\le s\le t}\|Z(s)\|_q + \zeta_3, \qquad (5.2.5)
\end{aligned}$$

where $\zeta_1 = -(\mu_q(H) + 2l\|\bar{A}_1\|_q + 2l'\|\bar{A}_2\|_q)$, $\zeta_2 = (2m\|\bar{B}_1\|_q + 2m'\|\bar{B}_2\|_q + 2n\|\bar{P}_1\|_q + 2n'\|\bar{P}_2\|_q)$.

Then from the equality (5.2.5), we have $0 < \zeta_2 < \zeta_1$ and employing Lemma 3.1, we have

$$\|Z(t)\|_q \le \; sup_{-\tau\le s\le 0}\|Z(s)\|_q e^{-kt} + \frac{\zeta_3}{k}.$$

As *e(t)* converges exponentially towards a desirable region $\Omega = \{Z(t) : \|Z(t)\|_q \le \frac{\zeta_3}{k}\}$, therefore from Definition 1.5.2, we can conclude that the desired quasi-projective synchronization between ICVRNNs (5.1.1) and (5.1.4) is achieved. □

Theorem 5.2.2. Let us consider that the Assumption 8 holds. The trajectories of systems (5.1.16) will be projective quasi-synchronized under the controller (5.1.11) if the following conditions are satisfied such that $\zeta_1 > \zeta_2 > 0$,

where $\zeta_1 = -\left\{\mu_q(H) + \mu_q(\hat{A}G^*) + \|\hat{A}\|_q(l^+ + 3l^-)\right\}$,
$\zeta_2 = (\|\bar{B}_1\|_q 2^{\frac{1}{q}} l^+ + \|\bar{B}_2\|_q 2^{\frac{1}{q}} l^- + \|\bar{P}_1\|_q 2^{\frac{1}{q}} n^+ + \|\bar{P}_2\|_q 2^{\frac{1}{q}} n^-)$,

and $H = \begin{pmatrix} -\bar{\xi} & I \\ \bar{\delta} & -\bar{\gamma} \end{pmatrix}$, $\hat{A} = \begin{pmatrix} 0 & 0 \\ A_1 & A_2 \end{pmatrix}$,

$G^* = diag\{l_1^+, l_2^+, ..., l_n^+, l_1^-, l_2^-, ..., l_n^-, 0, 0, ..., 0\}$,$l^+ = max_{1\leq v\leq n}\{l_v^+\}$, $l^- = max_{1\leq v\leq n}\{l_v^-\}$, q = 1,$\infty$.

Proof. Similar to the proof of Theorem 5.2.1, we obtain

$$\begin{aligned}
D^+(\|Z(t)\|_q) =& \overline{\lim_{h\to 0^+}} \frac{\|Z(t+h)\|_q - \|Z(t)\|_q}{h} \\
=& \overline{\lim_{h\to 0^+}} \frac{\|Z(t) + h\dot{Z}(t) + \mathcal{O}(h^2)\|_q - \|Z(t)\|_q}{h} \\
\leq& \overline{\lim_{h\to 0^+}} \frac{1}{h}\Bigg[\|Z(t) + h\begin{pmatrix} -\bar{\xi} & I \\ \bar{\delta} & -\bar{\gamma} \end{pmatrix} Z(t) + h\begin{pmatrix} 0 \\ A_1 f_1(\theta(t)) + A_2 f_2(\theta(t)) \end{pmatrix}\|_q \\
&+ \|\bar{B}_1\|_q \|\bar{g}_1(\theta(t-\sigma(t)))\|_q + h\|\bar{B}_2\|_q \|\bar{g}_2(\theta(t-\sigma(t)))\|_q \\
&+ h\|\bar{P}_1\|_q \int_{t-\tau(t)}^{t} \|\bar{h}_1(\theta(s))\|_q ds + h\|\bar{P}_2\|_q \int_{t-\tau(t)}^{t} \|\bar{h}_2(\theta(s))\|_q ds \\
&+ \|\begin{pmatrix} N(t) \\ M(t) \end{pmatrix}\|_q - \|Z(t)\|_q\Bigg]. \qquad (5.2.6)
\end{aligned}$$

Suppose $\mu(t) = (\mu_1(t), \mu_2, ..., \mu_n(t))^T$, $f(\mu(t))= (f_1(\mu(t)), f_2(\mu(t)), ..., f_n(\mu(t)))^T$, then we can define

$G_1(\mu(t)) = diag\Big\{\frac{f_1(\mu_1(t))}{\mu_1(t)}, \frac{f_2(\mu_2(t))}{\mu_2(t)}, ..., \frac{f_n(\mu_n(t))}{\mu_n(t)}\Big\}$.

Let $G(\theta(t)) = diag\{G_1(\mu(t)), G_2(\mu(t)), 0, ..., 0\}$. Also under the Assumption 8, it follows that

$$\begin{pmatrix} 0 \\ A_1 f_1(\theta(t)) + A_2 f_2(\theta(t)) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} f_1(\theta(t)) \\ f_2(\theta(t)) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} f^R(\widetilde{\alpha}(t)) - f^R(\alpha(t)) \\ f^R(\widetilde{\alpha}(t)) - f^R(\alpha(t)) \\ f^I(\widetilde{\beta}(t)) - f^I(\beta(t)) \\ f^I(\widetilde{\beta}(t)) - f^I(\beta(t)) \end{pmatrix}$$

$$= \hat{A} \begin{pmatrix} G_1(\epsilon^R(t))\epsilon^R(t) \\ G_2(\epsilon^I(t))\epsilon^I(t) \\ 0 \\ 0 \end{pmatrix} + \hat{A}(X_1(t) - X_2(t) + X_3(t) + X_4(t))$$

$$\leq \hat{A}G(\theta(t))E(t) + \hat{A}(X_1(t) - X_2(t) + X_3(t) + X_4(t)),$$

where

$$X_1(t)=\begin{pmatrix} 0 \\ f^R(\widetilde{\alpha}(t))-f^R(\alpha(t)) \\ 0 \\ 0 \end{pmatrix}, X_2(t)=\begin{pmatrix} 0 \\ f^I(\widetilde{\alpha}(t))-f^I(\alpha(t)) \\ 0 \\ 0 \end{pmatrix},$$

$$X_3(t)=\begin{pmatrix} 0 \\ 0 \\ f^I(\widetilde{\alpha}(t))-f^I(\alpha(t)) \\ 0 \end{pmatrix}, X_4(t)=\begin{pmatrix} 0 \\ 0 \\ 0 \\ f^I(\widetilde{\alpha}(t))-f^I(\alpha(t)) \end{pmatrix}.$$

Now, from Assumption 8, we get

$$\begin{aligned}
\|X_1(t)\|_q =&\|f^R(\widetilde{\alpha}(t))-f^R(\alpha(t))\|_q \\
=&(\sum_{v=1}^{m}|f^R(\widetilde{\alpha}(t))-f^R(\alpha(t))|^q)^{\frac{1}{q}} \\
\leq&(\sum_{v=1}^{m}(l_v^+|\widetilde{\alpha}(t)-\alpha(t)|)^q)^{\frac{1}{q}} \\
\leq&l^+(\sum_{v=1}^{m}(|\widetilde{\alpha}(t)-\alpha(t)|)^q)^{\frac{1}{q}} \\
=&l^+\|e^R(t)\|_q\leq l^+\|Z(t)\|_q. \qquad (5.2.7)
\end{aligned}$$

Similarly, we get

$\|X_2(t)\|_q \leq l^-\|Z(t)\|_q$, $\|X_3(t)\|_q \leq l^-\|Z(t)\|_q$, $\|X_4(t)\|_q \leq l^-\|Z(t)\|_q$,

and also we get

$$\begin{aligned}
\|\bar{g_1}(\theta(t-\tau(t)))\|_q =&\| \begin{pmatrix} g_1^R(\widetilde{\alpha}(t-\sigma(t))) - g_1^R(\alpha(t-\sigma(t))) \\ \\ g_1^R(\widetilde{\alpha}(t-\sigma(t))) - g_1^R(\alpha(t-\sigma(t))) \end{pmatrix} \|_q \\
=&\left(2\sum_{v=1}^{m} |g_1^R(\widetilde{\alpha}(t-\sigma(t))) - g_1^R(\alpha(t-\sigma(t)))|^q\right)^{\frac{1}{q}} \\
\leq&\left(2\sum_{v=1}^{m} (l_v^+|\widetilde{\alpha}(t-\sigma(t)) - \alpha(t-\sigma(t)))|)^q\right)^{\frac{1}{q}} \\
\leq&2^{\frac{1}{q}} l^+\left(2\sum_{v=1}^{m} |\widetilde{\alpha}(t-\sigma(t)) - \alpha(t-\sigma(t))|^p\right)^{\frac{1}{q}} \\
=&2^{\frac{1}{q}} l^+\|e^R(t-\sigma(t))\|_q \leq 2^{\frac{1}{q}} l^+\|Z(t-\sigma(t))\|_q. \qquad (5.2.8)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\|\bar{g_2}(\theta(t-\tau(t)))\|_q \leq 2^{\frac{1}{q}} l^-\|Z(t-\tau(t))\|_q, \\
&\|\bar{h_1}(\theta(t))\|_q \leq 2^{\frac{1}{q}} n^+\|Z(t)\|_q, \\
&\|\bar{h_2}(\theta(t))\|_q \leq 2^{\frac{1}{q}} n^-\|Z(t)\|_q. \qquad (5.2.9)
\end{aligned}$$

Substituting the inequalities (5.2.7)-(5.2.9) in equation (5.2.6), we get

$$
\begin{aligned}
D^{+}(\|Z(t)\|_q) \leq & \overline{\lim_{h\to 0^+}} \frac{1}{h}\Big[\|Z(t)+h(H+\hat{A}G(\theta(t)))Z(t)+\hat{A}(X_1(t)-X_2(t)+X_3(t) \\
& +X_4(t))\|_q + \|\bar{B}_1\|_q\|\bar{g}_1(\theta(t-\sigma(t)))\|_q + h\|\bar{B}_2\|_q\|\bar{g}_2(\theta(t-\sigma(t)))\|_q \\
& + h\|\bar{P}_1\|_q\int_{t-\tau(t)}^{t}\|\bar{h}_1(\theta(s))\|_q ds + h\|\bar{P}_2\|_q\int_{t-\tau(t)}^{t}\|\bar{h}_2(\theta(s))\|_q ds \\
& + h\|S(t)\| - \|Z(t)\|_q\Big] \\
\leq & \overline{\lim_{h\to 0^+}} \frac{\|I+h(H+\hat{A}G(\theta(t)))\|_q - 1}{h}\|Z(t)\|_q + \|\hat{A}\|_q\Big(\|X_1(t)\|_q + \|X_2(t)\|_q \\
& + \|X_3(t)\|_q + \|X_4(t)\|_q\Big) + \|\bar{B}_1\|_q\|\bar{g}_1(\theta(t-\sigma(t)))\|_q + \|\bar{B}_2\|_q\|\bar{g}_2(\theta(t-\sigma(t)))\|_q \\
& + \|\bar{P}_1\|_q\int_{t-\tau(t)}^{t}\|\bar{h}_1(\theta(s))\|_q ds + \|\bar{P}_2\|_q\int_{t-\tau(t)}^{t}\|\bar{h}_2(\theta(s))\|_q ds + \|S(t)\| \\
\leq & \overline{\lim_{h\to 0^+}} \frac{\|I+h(H+\hat{A}G(\theta(t)))\|_q - 1}{h}\|Z(t)\|_q + \|\hat{A}\|_q\Big(l^+ + 3l^-\Big)\|Z(t)\|_q \\
& + \|\bar{B}_1\|_q 2^{\frac{1}{q}} l^+\|Z(t-\sigma(t))\|_q + \|\bar{B}_2\|_q 2^{\frac{1}{q}} l^-\|Z(t-\sigma(t))\|_q \\
& + \|\bar{P}_1\|_q\int_{t-\tau(t)}^{t} 2^{\frac{1}{q}} n^+\|Z(s)\|_q ds + \|\bar{P}_2\|_q\int_{t-\tau(t)}^{t} 2^{\frac{1}{q}} n^-\|Z(s)\|_q ds + \zeta_3 \\
\leq & \mu_q(H+\hat{A}G(\theta(t)))\|Z(t)\|_q + \|\hat{A}\|_q\Big(l^+ + 3l^-\Big)\|Z(t)\|_q + (\|\bar{B}_1\|_q 2^{\frac{1}{q}} l^+ + \|\bar{B}_2\|_q 2^{\frac{1}{q}} l^-) \\
& \times \|Z(t-\sigma(t))\|_q + (\|\bar{P}_1\|_q 2^{\frac{1}{q}} n^+ + h\|\bar{P}_2\|_q 2^{\frac{1}{q}} n^-)\int_{t-\tau(t)}^{t}\|Z(s)\|_q ds + \zeta_3 \\
\leq & \Big\{\mu_q(H+\hat{A}G(\theta(t))) + \|\hat{A}\|_q(l^+ + 3l^-)\Big\}\|Z(t)\|_q + (\|\bar{B}_1\|_q 2^{\frac{1}{q}} l^+ + \|\bar{B}_2\|_q 2^{\frac{1}{q}} l^-) \\
& \times \|Z(t-\sigma(t))\|_q + (\|\bar{P}_1\|_q 2^{\frac{1}{q}} n^+ + h\|\bar{P}_2\|_q 2^{\frac{1}{q}} n^-)\int_{t-\tau(t)}^{t}\|Z(s)\|_q ds + \zeta_3 \\
\leq & \Big\{\mu_q(H+\hat{A}G(\theta(t))) + \|\hat{A}\|_q(l^+ + 3l^-)\Big\}\|Z(t)\|_q + (\|\bar{B}_1\|_q 2^{\frac{1}{q}} l^+ + \|\bar{B}_2\|_q 2^{\frac{1}{q}} l^-) \\
& \times sup_{t-\sigma\leq s\leq t}\|Z(s)\|_q + (\|\bar{P}_1\|_q 2^{\frac{1}{q}} n^+ + h\|\bar{P}_2\|_q 2^{\frac{1}{q}} n^-)sup_{t-\tau\leq s\leq t}\|Z(s)\|_q + \zeta_3 \\
\leq & \Big\{\mu_q(H+\hat{A}G(\theta(t))) + \|\hat{A}\|_q(l^+ + 3l^-)\Big\}\|Z(t)\|_q + (\|\bar{B}_1\|_q 2^{\frac{1}{q}} l^+ + \|\bar{B}_2\|_q 2^{\frac{1}{q}} l^- \\
& + \|\bar{P}_1\|_q 2^{\frac{1}{q}} n^+ + h\|\bar{P}_2\|_q 2^{\frac{1}{q}} n^-)sup_{t-\tau\leq s\leq t}\|Z(s)\|_q + \zeta_3. \qquad (5.2.10)
\end{aligned}
$$

According to Lemma 1.3 and Lemma 5.1.2, the following inequality is obtained

$$\mu_q(H + \hat{A}G(\theta(t))) \leq \mu_q(H) + \mu_q(\hat{A}G(\theta(t))) \leq \mu_q(H) + \mu_q(\hat{A}G^*). \tag{5.2.11}$$

Thus,

$$\begin{aligned} D^+(\|Z(t)\|_q) \leq & \left\{ \mu_q(H) + \mu_q(\hat{A}G^*) + \|\hat{A}\|_q(l^+ + 3l^-) \right\} \|Z(t)\|_q \\ & + (\|\bar{B}_1\|_q 2^{\frac{1}{q}} l^+ + \|\bar{B}_2\|_q 2^{\frac{1}{q}} l^- + \|\bar{P}_1\|_q 2^{\frac{1}{q}} n^+ \\ & + \|\bar{P}_2\|_q 2^{\frac{1}{q}} n^-) sup_{t-\tau \leq s \leq t} \|Z(s)\|_q + \zeta \\ \leq & - \zeta_1 \|Z(t)\|_q + \zeta_2 \|Z(s)\|_q + \zeta_3, \end{aligned} \tag{5.2.12}$$

where $\zeta_1 = -\left\{ \mu_q(H) + \mu_q(\hat{A}G^*) + \|\hat{A}\|_q(l^+ + 3l^-) \right\}$,
$\zeta_2 = (\|\bar{B}_1\|_q 2^{\frac{1}{q}} l^+ + \|\bar{B}_2\|_q 2^{\frac{1}{q}} l^- + \|\bar{P}_1\|_q 2^{\frac{1}{q}} n^+ + \|\bar{P}_2\|_q 2^{\frac{1}{q}} n^-)$.

Then from the condition (5.2.12), we have $0 < \zeta_2 < \zeta_1$ and using Lemma 3.1, we get

$$\|Z(t)\|_q \leq \ sup_{-\tau \leq s \leq 0} \|Z(s)\|_q e^{-kt} + \frac{\zeta_3}{k}.$$

Accordingly, the error system *e(t)* converges exponentially towards a desirable region $\Omega = \{Z(t) : \|Z(t)\|_q \leq \frac{\zeta_3}{k}\}$. This means that from Definition 1.5.2, it is proven that ICVRNNs (5.1.1) and (5.1.4) exhibit quasi-projective synchronization. □

Remark 5.2.1. The different values of ξ_v can be selected in the matrices $\bar{\xi}$, $\bar{\gamma}$ and $\bar{\delta}$ by using different variable transformations and existence of matrices $\bar{\xi}$, $\bar{\gamma}$ and $\bar{\delta}$ determine whether the conditions of Theorems 5.2.1 and 5.2.2 will be hold. Theorems 5.2.1 and 5.2.2 introduce the free-weight matrix $\bar{\xi}$ and make the results here less conservative than those of a certain variable substitution.

Remark 5.2.2. Throughout this chapter, Theorems 5.2.1 and 5.2.2 is proved by using matrix measure method to ensure the quasi-projective synchronization between the systems (5.1.1) and (5.1.4). Based on Lemma 1.3, 5.1.1 and 3.1, Assumptions 3 and 6, Definitions 1.4.4 and 1.5.2, the Theorem 5.2.1 is derived. But Lemma 5.1.2 and Assumption 8 are utilized in Theorem 5.2.2. Lemma 5.1.2 is invalid for q=2. Therefore, Theorem 5.2.2 is invalid for q=2. So, Theorems 5.2.1 and 5.2.2 will complement and enrich both.

5.3 Numerical Simulation

Here, two numerical simulations are given to demonstrate the effectiveness and viability of the proposed quasi-projective synchronization scheme.

Example 5.3.1. Let us consider the following two-dimensional ICVRNN system with mixed time-varying delay as the drive system as

$$\ddot{\omega}(t) = - D\dot{\omega}(t) - C\omega(t) + Af(\omega(t)) + Bg(\omega(t-\sigma(t))) + P\int_{t-\tau(t)}^{t} h(\omega(s))ds + I(t), \tag{5.3.1}$$

where $\omega_u(t) = \alpha_u(t) + i\beta_u(t)$, $\alpha_u(t), \beta_u(t) \in R^n$, $u = 1, 2$.

Here the parameters are taken as

$$D = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad C = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad A = \begin{pmatrix} 0.2-0.4i & 0.3+0.1i \\ 0.1+0.2i & -0.2+0.1i \end{pmatrix},$$

$$B = \begin{pmatrix} 0.3 - 0.1i & 0.1 - 0.2i \\ 0.2 + 0.4i & -0.3 + 0.2i \end{pmatrix}, P = \begin{pmatrix} -0.4 - 0.1i & 0.2 + 0.2i \\ 0.4 - 0.2i & 0.3 + 0.5i \end{pmatrix}, \xi = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

$$I(t) = \begin{pmatrix} 0.4sin(t) + i0.5cos(t) \\ 0.5cos(t) + i0.4sin(t) \end{pmatrix}, \sigma(t) = 0.1 + 0.2sin^2(t), \tau(t) = cos^2(t).$$

The activation functions are taken as

$$f_u(\omega_u(t)) = g_u(\omega_u(t)) = h_u(\omega_u(t)) = \frac{1}{10(1 + exp(-\alpha(t) + 2\beta(t)))} + i\frac{1 - exp(-2\alpha(t) - \beta(t))}{10(1 + exp(-2\alpha(t) - \beta(t)))}.$$

The three-dimensional plots of state trajectories $\omega_1(t)$ and $\omega_2(t)$ of the system (5.3.1) are shown in Figure 5.1.

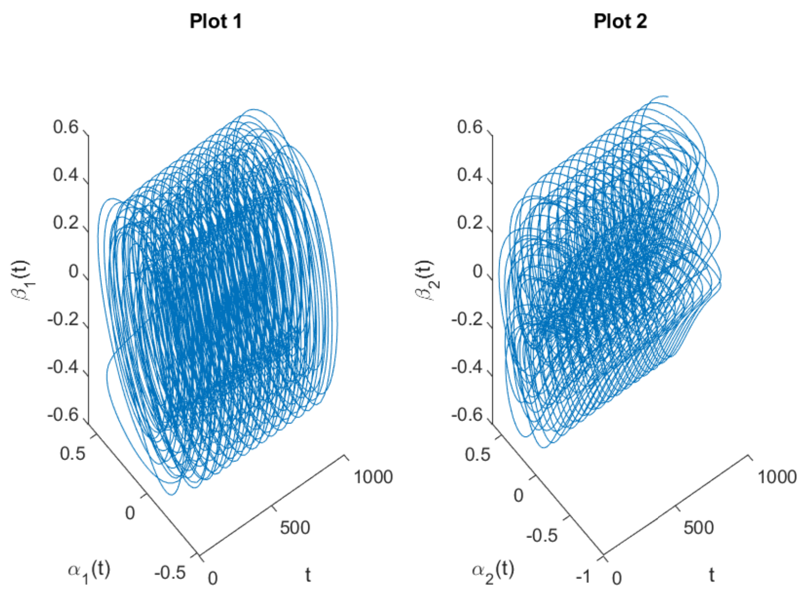

Figure 5.1: The plots of state trajectories $\omega_1(t)$ and $\omega_2(t)$ of the system (5.3.1) for Example 5.3.1.

The corresponding two dimensional response system is considered as

$$\ddot{\tilde{\omega}}(t) = -D'\dot{\tilde{\omega}}(t) - C'\tilde{\omega}(t) + A'f(\tilde{\omega}(t)) + B'g(\tilde{\omega}(t-\sigma(t))) + P'\int_{t-\tau(t)}^{t} h(\tilde{\omega}(s))ds + I'(t) + \Omega(t), \tag{5.3.2}$$

with the following parameters

$$D' = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}, \quad C' = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.6 \end{pmatrix}, \quad A' = \begin{pmatrix} 0.3-0.5i & 0.4+0.2i \\ 0.1-0.3i & 0.2-0.3i \end{pmatrix},$$

$$B' = \begin{pmatrix} -0.2+0.2i & 0.2-0.3i \\ 0.3+0.5i & 0.2-0.3i \end{pmatrix}, P' = \begin{pmatrix} 0.3+0.2i & 0.3-0.2i \\ 0.3+0.1i & 0.4+0.4i \end{pmatrix}, \xi' = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},$$

$$I'(t) = \begin{pmatrix} 0.2cos(t) + i0.3sin(t) \\ 0.3sin(t) + i0.2cos(t) \end{pmatrix}, \kappa = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.7 \end{pmatrix},$$

where $\Omega(t) = -\kappa(e(t))$ is denoted as controller.

Let $P^{RR} = Q^{RR} = R^{RR} = 0.1$, $P^{RI} = Q^{RI} = R^{RI} = 0.2$, $P^{IR} = Q^{IR} = R^{IR} = 0.4$, $P^{II} = Q^{II} = R^{II} = 0.2$.

Let us also consider the initial conditions as

$\omega_1(s) = -0.15 + 0.45i, \omega_2(s) = -0.25 + 0.15i,$

$\widetilde{\omega}_1(s) = -0.30 + 0.30i$, $\widetilde{\omega}_2(s) = 0.30 + 0.25i$. Choosing $r = 0.2$, we can find that $k_1 = 8.1227$, $k_2 = 0.2334$, i.e., $k_1 > k_2$, and $\zeta = 5.8957$.

Thus all the conditions of Theorem 5.2.1 hold. Hence the system (5.3.1) will be quasi-projective synchronized with the system (5.3.2) having estimated error level $= 0.9294$.

Figures 5.2 and 5.3 represent the state evaluation curves of the systems (5.3.1) and (5.3.2) under the controllers with mismatched parameters. In Figure 5.4, Plot 1 shows the quasi-projective synchronization of the error system (5.1.16) without controller and Plot 2 shows that of system (5.1.16) under controller with upper bound $= 0.9294$.

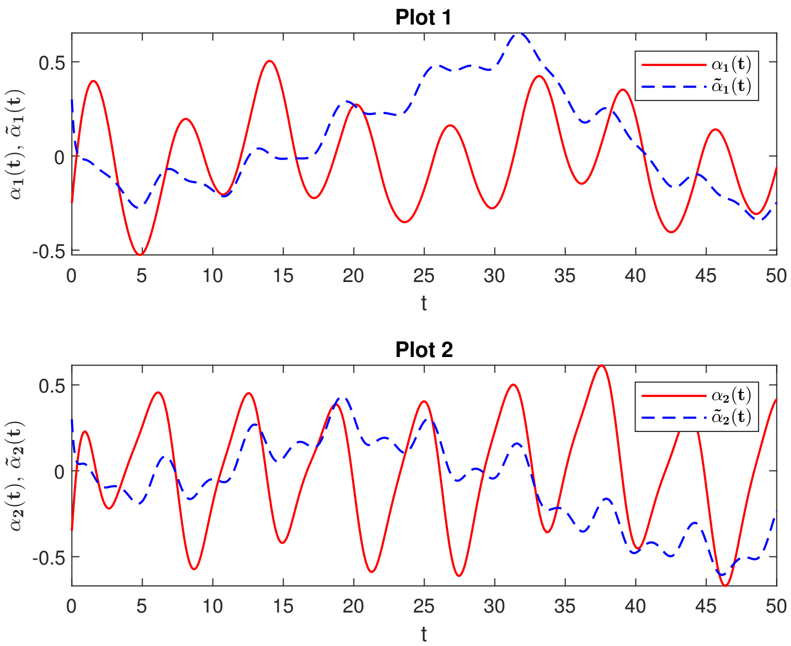

Figure 5.2: The plots of real state trajectories $\alpha_1(t)$, $\alpha_2(t)$ and $\widetilde{\alpha}_1(t)$, $\widetilde{\alpha}_2(t)$ of drive system (5.3.1) and response system (5.3.2) for Example 5.3.1.

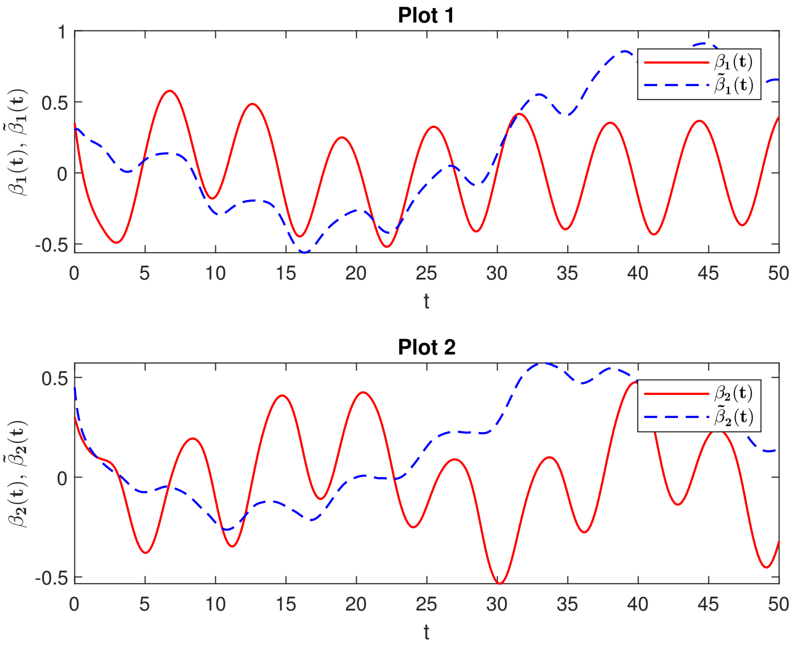

Figure 5.3: The plots of imaginary state trajectories $\beta_1(t)$, $\beta_2(t)$ and $\widetilde{\beta}_1(t)$, $\widetilde{\beta}_2(t)$ of drive system (5.3.1) and response system (5.3.2) for Example 5.3.1.

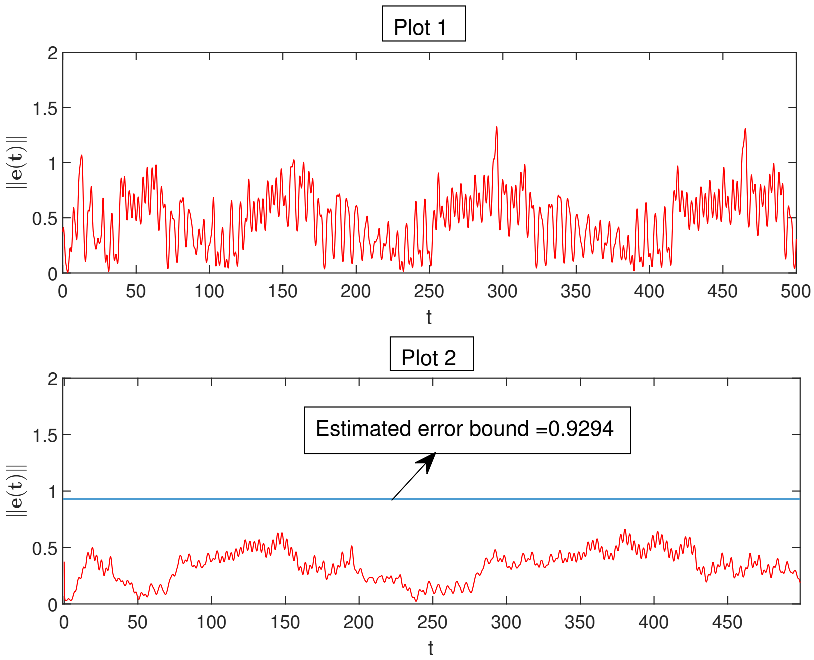

Figure 5.4: Plots of the error system (5.3.1) without controllers and with controllers having error bound = 0.9294 for Example 5.3.1.

Example 5.3.2. Let us consider the following parameters of the two-dimensional ICVRNN system (5.3.1) with mixed time-varying delay as the drive system as

$$D = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}, \quad C = \begin{pmatrix} 1.3 & 0 \\ 0 & 1.2 \end{pmatrix}, \quad A = \begin{pmatrix} 0.3 - 0.4i & 0.2 + 0.1i \\ 0.2 + 0.3i & -0.2 + 0.3i \end{pmatrix},$$

$$B = \begin{pmatrix} 0.3 - 0.2i & 0.2 - 0.3i \\ 0.2 + 0.3i & -0.4 + 0.2i \end{pmatrix}, P = \begin{pmatrix} -0.5 - 0.2i & 0.2 + 0.3i \\ 0.3 - 0.2i & 0.3 - 0.5i \end{pmatrix}, \xi = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$I(t) = \begin{pmatrix} 0.6 + i0.4 \\ 0.7 + i0.5 \end{pmatrix}, \sigma(t) = 0.1 + 0.2sin^2(t), \tau(t) = cos^2(t).$$

The activation functions are taken as

$$f_u(\omega_u(t)) = g_u(\omega_u(t)) = h_u(\omega_u(t)) = \frac{1 - exp(-0.6\alpha_u)}{2(1 + exp(-0.6\alpha_u))} + i\frac{1}{2(1 + exp(-0.8\beta_u))}.$$

The following parameters of the corresponding response system (5.3.2) are

$$D' = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.7 \end{pmatrix}, \quad C' = \begin{pmatrix} 1.4 & 0 \\ 0 & 1.9 \end{pmatrix}, \quad A' = \begin{pmatrix} 0.3 + 0.4i & -0.5 - 0.2i \\ 0.5 + 0.3i & 0.2 - 0.4i \end{pmatrix},$$

$$B' = \begin{pmatrix} -0.2+0.2i & 0.2+0.1i \\ 0.2-0.3i & -0.3+0.2i \end{pmatrix}, P' = \begin{pmatrix} 0.3-0.2i & 0.3+0.2i \\ 0.2+0.1i & -0.2+0.3i \end{pmatrix}, \xi' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

$$I'(t) = \begin{pmatrix} 0.2+i0.3 \\ 0.3+i0.2 \end{pmatrix}, \kappa = \begin{pmatrix} 4.5 & 0 \\ 0 & 3.7 \end{pmatrix},$$

where $\Omega(t) = -\kappa e(t)$ is denoted as controller.

The three-dimensional plots of state trajectories $\omega_1(t)$ and $\omega_2(t)$ of the system (5.3.2) are shown in Figure 5.5

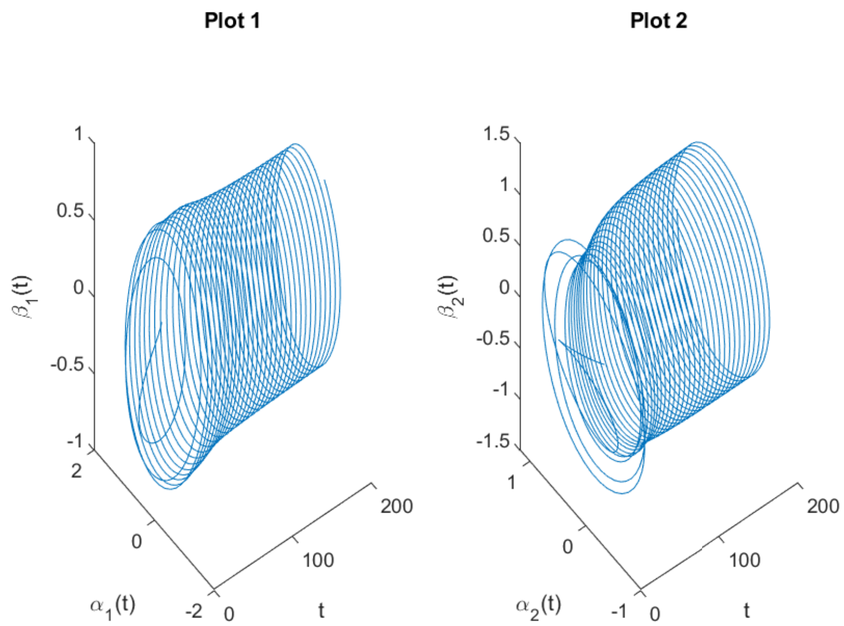

Figure 5.5: The state trajectories $\omega_1(t)$ and $\omega_2(t)$ of the system (5.3.1) for Example 5.3.2.

For $l_k^+ = n_k^+ = 0.15$, $l_k^- = n_k^- = 0.1$, let us consider the initial conditions as

$\omega_1(s) = -0.15 + 0.45i, \omega_2(s) = -0.25 + 0.15i,$

$\widetilde{\omega}_1(s) = -0.30 + 0.30i, \widetilde{\omega}_2(s) = 0.30 + 0.25i.$

Suppose $r = 0.2$, we can find that $k_1 = 15.8427$, $k_2 = 1.4531$, i.e., $k_1 > k_2$, $\zeta = 7.3453$. Thus all the conditions of Theorem 5.2.2 hold. Hence the system (5.3.1) will be projective synchronized with the system (5.3.2) having estimated error level $= 0.8892$.

Figures 5.6 and 5.7 represent the state evaluation curves of the systems (5.3.1) and (5.3.2) under the controllers with mismatched parameters. In Figure 5.8, Plot 1 shows the quasi- projective synchronization to the error systems (5.1.16) without controller and Plot 2 shows the similar thing under controller with error bound $= 0.8892$.

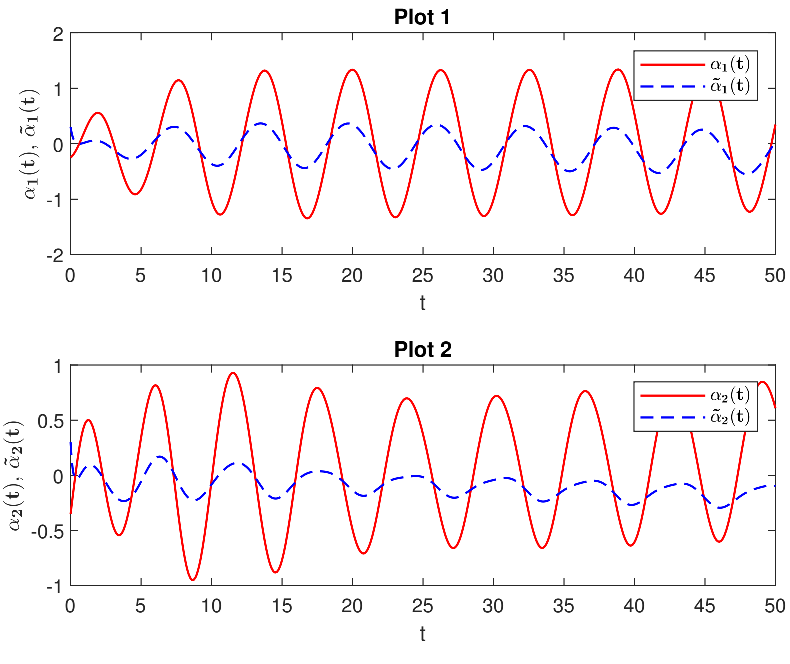

Figure 5.6: The plots of real state trajectories $\alpha_1(t)$, $\alpha_2(t)$ and $\widetilde{\alpha}_1(t)$, $\widetilde{\alpha}_2(t)$ of drive system (5.3.1) and response system (5.3.2) for Example 5.3.2.

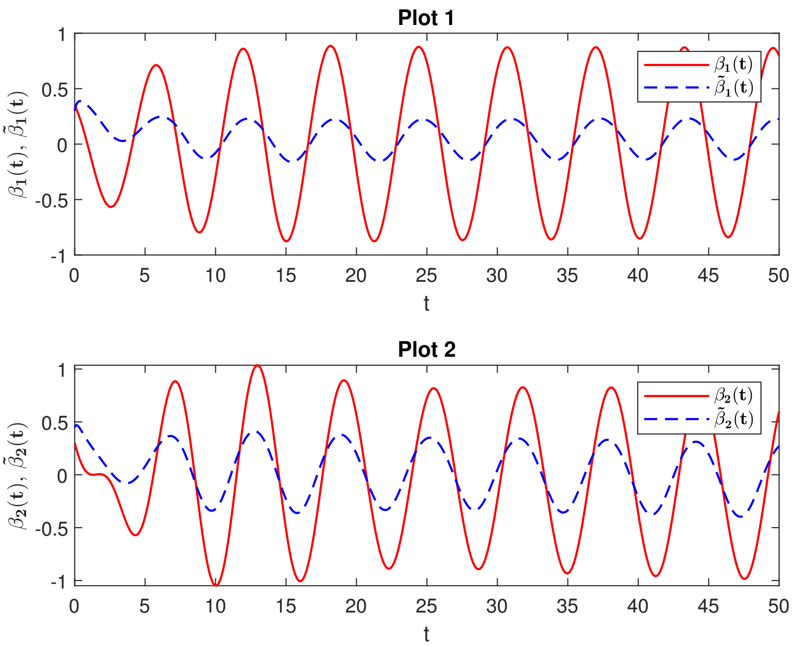

Figure 5.7: The plots of imaginary state trajectories $\beta_1(t)$, $\beta_2(t)$ and $\widetilde{\beta}_1(t)$, $\widetilde{\beta}_2(t)$ of drive system (5.3.1) and response system (5.3.2) for Example 5.3.2.

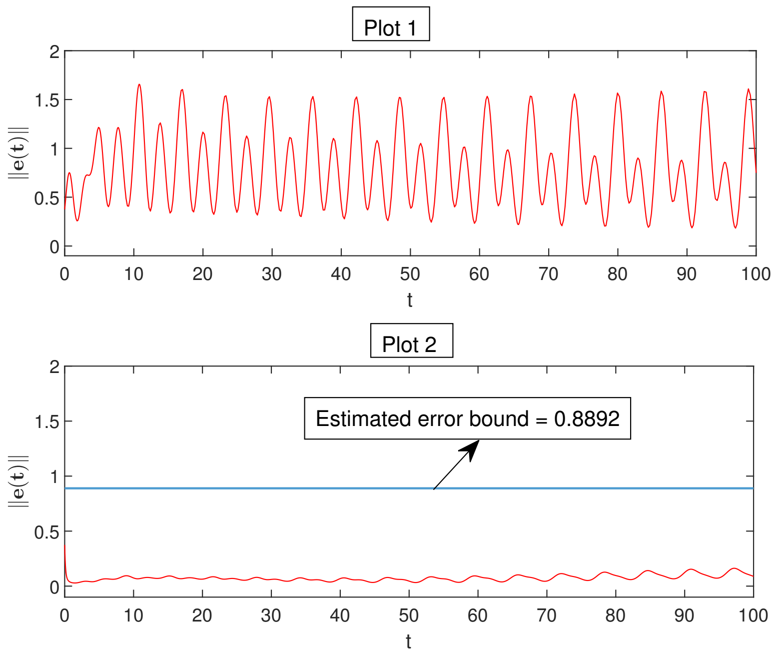

Figure 5.8: Plots of the error system (5.3.1) without controllers and with controllers having error bound = 0.8892 for Example 5.3.2.

5.4 Conclusion

In this chapter, the quasi-projective synchronization problem on the ICVRNNs with mixed time-varying delay and parameters' mismatched has been studied. The considered system has been transformed into the system of first order differential equations by implementing a suitable variable transformation. Applying matrix measure approach with nonlinear Lipschitz activation functions, the quasi-projective synchronization criterion of ICVRNNs has been derived by constructing a suitable controller and also the upper bound of synchronization error is estimated. Some sufficient conditions are also furnished for some special cases. Two numerical simulation results are demonstrated to show the unwavering and accurateness of the theoretical results of the present chapter.

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