

# Chapter 3

## Artificial Delayed Output Twisting Algorithm

### 3.1 Introduction

For uncertain systems with relative degree two, *Twisting controller*, forces both, the position or its equivalent output variable as well as its derivative to zero simultaneously in finite time. However, it requires the derivative of the output [62]. For having the information of both the output and its derivative, either two sensors are used or an observer or differentiator is used in addition to one sensor. Having more number of sensors can increase the cost of the system [30]. Use of observer and differentiator also come with their own share of problems. The performance of observers are known to be inferior as compared to the physical sensors when the parameters change during the operation of the system. Differentiators are known to be highly sensitive to noise in the output channel [62].

Many systems with time delay which are represented by functional differential equation has application in different areas [70]. Thus, analysis of stability of systems with time delay becomes important [71]. In this chapter we propose an artificially delayed output feedback based controller which is structurally similar to the twisting algorithm, but instead of using the information of output and its derivative, uses the information of output and an artificially delayed version of it.

The stability analysis of systems with time delay mostly considers the linear time-delay models due to the complexities involved [3]. However, in [71] on the introduction

of delay performance of the system is boosted.

For the proposed controller which requires the information about output and its delayed values, we obtain the condition on controller gains by the application of standard Lyapunov technique in presence of time-varying disturbance. The condition for the delay has been obtained by using the Lyapunov-Razumikhin approach [72], using the same Lyapunov function that is used for obtaining the controller gains. Apart from time-varying disturbance rejection, the proposed controller can also be used as an observer. Moreover, it can also mitigate the unmatched disturbance for an uncertain second order system.

## 3.2 Notions

$\mathbb{R}$  denotes the set of real numbers,  $\mathbb{R}_+$  denotes the set of positive real numbers. The sign function is defined as  $\text{sign}(x) := 1$  for  $x > 0$ ,  $\text{sign}(x) := -1$  for  $x < 0$  and  $\text{sign}(x) := [-1, 1]$  for  $x = 0$ .  $C_{[-\tau, 0]}$  denotes the Banach space of continuous functions  $\phi : [-\tau, 0] \rightarrow \mathbb{R}^n$  with  $\|\phi\| = \sup_{-\tau \leq \zeta \leq 0} |\phi(\zeta)|$  where  $|\cdot|$  denotes the standard norm. Dilation matrix is given as  $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$  where  $r = [r_1, \dots, r_n]^T$ .  $B_\rho^\tau$  denotes a sphere of radius  $\rho > 0$  in  $C_{[-\tau, 0]}$  and is given by  $B_\rho^\tau = \{\phi \in C_{[-\tau, 0]} : \|\phi\|_r \leq \rho\}$ .

Weighted homogeneous functions and vector field have several elegant properties, we are going to recall a definition of weighted homogeneity for delayed functions as well as vector field, which will be used in the construction of the proposed artificial delayed output twisting algorithm of this note.

**Definition 1** [73]  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be weighted homogeneous of degree  $m$  if  $f(\Lambda_r(\lambda)x) = \lambda^m f(x)$ , for any  $x \in \mathbb{R}^n$  and for all  $\lambda > 0$ .

**Definition 2** [73]  $f : C_{[-\tau, 0]} \rightarrow \mathbb{R}$  is called  $r$ -homogeneous, where  $r_i > 0 \quad \forall i = 1 \dots n$ , of degree  $m$  if  $f(\Lambda_r(\lambda)\phi) = \lambda^m f(\phi)$  for any  $\phi \in C_{[-\tau, 0]}$  and for all  $\lambda > 0$ .

**Definition 3** [73]  $f : C_{[-\tau, 0]} \rightarrow \mathbb{R}^n$  is called  $r$ -homogeneous, where  $r_i > 0 \quad \forall i = 1 \dots n$ , of degree  $m$  if  $f(\Lambda_r(\lambda)\phi) = \lambda^m \Lambda_r(\lambda) f(\phi)$  for any  $\phi \in C_{[-\tau, 0]}$  and for all  $\lambda > 0$ .

Similarly, weighted homogeneous functions have several elegant properties, we are going to recall a result about continuous real-valued homogeneous functions ([74], Lemma 4.2), which will be used in the proof of the main Theorems of this chapter.

**Lemma 3** Suppose  $V_1$  and  $V_2$  are continuous real-valued functions  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , homogeneous with the same weights and degrees  $l_1 > 0$  and  $l_2 > 0$ , respectively, and  $V_1$  is positive-definite. Then for every  $x \in \mathbb{R}^n$ ,

$$\left[ \min_{\{z:V_1(z)=1\}} \right] [V_1(z)]^{\frac{l_2}{l_1}} \leq V_2(z) \leq \left[ \max_{\{z:V_1(z)=1\}} \right] [V_1(z)]^{\frac{l_2}{l_1}}.$$

**Lemma 4** [73] Let  $f : C_{[-\tau,0]} \rightarrow \mathbb{R}^n$  be locally bounded and  $r$ -homogeneous with degree  $d$ , then there exists  $k > 0$  such that  $\|f(x)\|_r < k \max_{1 \leq i \leq n} \|x\|_r^{1+d/r_i}$ ,  $\forall x \in C_{[-\tau,0]}$ .

**Lemma 5** [73] Let  $f : C_{[-\tau,0]} \rightarrow \mathbb{R}^n$  be  $r$ -homogeneous with degree  $d$  and uniformly continuous in  $B_\rho^\tau$  for some  $\rho > 0$ , then for any  $\eta > 0$  there exists  $k > 0$  such that  $\|F(x) - F(z)\|_r \leq \max\{k \max_{1 \leq i \leq n} \|x - z\|_r^{1+d/r_i}, \eta\}$ ,  $\forall x, z \in B_\rho^\tau$ .

### 3.3 Main Results

Let us consider a system  $\dot{x}_1 = u + d(t)$ ,  $x_1 \in \mathbb{R}$ ,  $d : \mathbb{R}_+ \rightarrow \mathbb{R}$ , where  $d(t)$  is perturbations/disturbances, satisfying  $|\dot{d}(t)| \leq d_0$ . Let the proposed controller  $u$  be taken as  $u := -k_1 \int_0^t \text{sign}(x_1(s))ds - k_2 \int_0^t \text{sign}(\alpha(x_1(s), x_1(s-\tau)))ds$  with  $x(s) = \phi(s)$ ,  $s \in [-\tau, 0]$  where  $k_1, k_2$  are the positive gains of the controllers, designed later in the manuscript,  $\tau$  denotes the specified artificial delay and function  $\alpha$  is selected such that the artificial delayed closed loop system is weighted homogeneous [73]. In this paper we have taken,  $\alpha(x_1(t), x_1(t-\tau))$  as  $|x_1|^{0.5} \text{sign}(x_1(t)) - |x_1(t-\tau)|^{0.5} \text{sign}(x_1(t-\tau))$ . After substitution of control  $u$ , the closed loop system can be written as

$$\begin{aligned} \dot{x}_1 = x_2, \quad \dot{x}_2 = & -k_1 \text{sign}(x_1(t)) \\ & - k_2 \text{sign}(\alpha(x_1(t), x_1(t-\tau))) + \dot{d}(t). \end{aligned} \tag{3.1}$$

Considering the representation where  $x_t(s) = x(t+s)$  and  $-\tau \leq s \leq 0$ , the autonomous functional differential inclusion (3.1) has state  $x = [x_1 \ x_2]^T \in \mathbb{R}^2$  and the state function  $x_t \in C_{[-\tau,0]}$

**Comment 1** where  $C_{[-\tau,0]}$  denotes the Banach space of continuous functions  $\phi : [-\tau, 0] \rightarrow \mathbb{R}^2$  with  $\|\phi\| = \sup_{-\tau \leq \zeta \leq 0} |\phi(\zeta)|$ , where  $|\cdot|$  denotes the standard Euclidean norm

Equation (3.1) can also be written as

$$\begin{aligned} \dot{x}_{t_1}(0) = x_{t_2}(0), \quad \dot{x}_{t_2}(0) \in & -k_1 \text{sign}(x_{t_1}(0)(t)) \\ & - [k_2 - d_0, k_2 + d_0] \text{sign}(\alpha(x_{t_1}(0), x_{t_1}(-\tau))), \end{aligned} \tag{3.2}$$

Let the right hand side of (3.2) be written as  $F[x_1(t), x_1(t - \tau)]$  with  $x_{t_0}(\tau) = \phi(-\tau)$ ,  $\tau \in [0, \tau_0]$  ( $\tau_0 > 0$ ). The functional  $F : C_{[-\tau, 0]} \rightarrow \mathbb{R}^2$  is locally bounded and is discontinuous in  $x_1 \cup \alpha(x_1, x_1(t - \tau))$ . Moreover, the functional  $F$  is continuous except for the several points defined by

$$\begin{aligned} R_i &= \{x_{t_1}(0) | x_{t_1}(0) = 0 \cup \alpha(x_{t_1}(0)(t), x_{t_1}(-\tau)) = 0\} \\ &\in C_{[-\tau, 0]}, \end{aligned} \tag{3.3}$$

where  $i = 1, \dots, m$  and  $m$  denotes number of conditions for which  $F$  is discontinuous. Let  $R_f = \cup_{i=1}^m R_i$  denotes all the points where the functional  $F$  is discontinuous. A functional following this property is said to be piecewise continuous [75]. The solution of such functionals can be understood in terms of generalized set-valued mapping with respect to the functional  $F$ . With the set-valued mapping, define  $\mathbf{K}[F](x_t) = \text{co}\{\lim_{i \rightarrow \infty} F(x_t^i) | x_t^i \rightarrow x_t, x_t^i \notin R_f\}$ , where ‘‘co’’ denotes the convex hull. Further, an equilibrium point of (3.2) is a point  $0 \in C_{[-\tau, 0]}$  such that  $0 \in \mathbf{K}[F](0)$ .

**Remark 3** *Suppose that  $t_0$  is the initial time when the trajectories of (3.2) hit discontinuous set (3.3). It is important to mention here that no trajectory of (3.2) is going to stay on the discontinuous set (3.3) for the following cases:  $(x_1(t) = 0 \cap x_2(t) \neq 0) \cup (x_1(t - \tau) = 0 \cap x_2(t) \neq 0) \cup (x_1(t) \cap x_1(t - \tau) = 0 \cap x_2(t) \neq 0) \forall t \geq t_0$ , except when  $x_1(t) = x_2(t) = 0$  if  $k_1 > k_2 + d_0$ . Therefore, condition on the delay for the asymptotic stability at the origin remains the same as the Lemma 4 [73].*

In general, finding gain conditions based on the Lyapunov function for any class of functional differential inclusion is not a straight forward problem. Therefore, we introduce the following time-varying change of state variables of the system (3.1) to simplify the problem  $z_1(t) := \frac{x_1(t)}{L(t)}$ ,  $z_2(t) := \frac{x_2(t)}{L(t)}$ ,  $L(t) > 0$ ,  $\forall t \geq 0$ , in the new co-ordinates, then system (3.1) is given by

$$\begin{aligned} \dot{z}_1 &= - \left( \frac{\dot{L}}{L} \right) z_1 + z_2; \quad \dot{z}_2 = - \left( \frac{\dot{L}}{L} \right) z_2 - \frac{k_1}{L} \text{sign}(z_1) \\ &\quad - \frac{k_2}{L} \text{sign}(\alpha(L(t) z_1(t), L(t - \tau) z_1(t - \tau))) + \frac{\dot{d}}{L}. \end{aligned} \tag{3.4}$$

In general, an algebraic equivalence of systems (3.1) and (3.4) does not preserve the stability properties of a dynamical system. For this, it is necessary and sufficient to have topological equivalence: algebraic equivalence plus the condition  $|L(t)| \leq p_1$  and

$|1/L(t)| \leq p_2$  for all  $t \geq 0$  where  $p_1$  and  $p_2$  are fixed constants [61]. We are also assuming that derivative of function  $L(t)$  is bounded by  $p_3$  for all  $t \geq 0$ . It is easy to find  $L(t)$  which satisfies the above mentioned properties. For example, a logistic function  $L(t) = \frac{L_0}{2 - \exp(-l_0 t)}$  where  $L_0$  and  $l_0$  are the curve's maximum value and the logistic growth rate of the function, respectively.

The following Theorem gives the practical asymptotic stability of the closed loop system (3.1) with respect to  $B_\epsilon^r$ .

**Theorem 3** Consider a  $L(t)$  with given properties as mentioned above and assume that the gains are selected such that the following inequality is fulfilled for  $k_1 \geq k_2$ ,

$\max \left\{ L(t) \left( \frac{2^{\frac{3}{2}}}{3} \pi_2 - \pi_1 \right), |\dot{d}(t)| \right\} \leq (k_1 + k_2) \leq L(t) \left( \frac{2^{\frac{3}{2}}}{3} \pi_2 + \pi_1 \right)$  for all  $t \geq 0$  with  $\pi_1 \geq \frac{2^2 2^{\frac{5}{6}}}{3^2} \pi_2$  where  $\pi_i; i = 1, 2$  are positive constants. Then, the origin of the system (3.1) is practical asymptotically stable with respect to  $B_\epsilon^r$  in spite of disturbance  $\sup_t |\dot{d}(t)| \leq d_0$  for sufficiently small artificial delay  $0 < \tau < \tau_0$  where  $\tau_0$  is a positive constant.

**Proof :** Consider the following Lyapunov function in the new coordinates

$$V(z) = \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{3}{2}} + \pi_2 z_1 z_2. \quad (3.5)$$

Applying Young's inequality to the term  $\pi_2 |z_1| |z_2|$ , it can be shown that proposed Lyapunov function (3.5) is bounded from below by zero.

$$\begin{aligned} V(z) &\geq \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{3}{2}} - \pi_2 |z_1| |z_2| \\ &\geq \left( \pi_1 |z_1| \right)^{\frac{3}{2}} + \left( \frac{1}{2} z_2^2 \right)^{\frac{3}{2}} - \pi_2 \left( \frac{2}{3} g^{\frac{3}{2}} |z_1|^{\frac{3}{2}} + \frac{1}{3} g^{-3} |z_2|^3 \right) \\ &= \left( \pi_1^{\frac{3}{2}} - \frac{2}{3} \pi_2 g^{\frac{3}{2}} \right) |z_1|^{\frac{3}{2}} + \left( \left( \frac{1}{2} \right)^{\frac{3}{2}} - \frac{1}{3} \pi_2 g^{-3} \right) |z_2|^3. \end{aligned} \quad (3.6)$$

where  $g > 0$ .  $V \geq 0$  for all  $z$  if each of  $\left( \pi_1^{\frac{3}{2}} - \frac{2}{3} \pi_2 g^{\frac{3}{2}} \right)$  and  $\left( \frac{1}{2} \right)^{\frac{3}{2}} - \frac{1}{3} \pi_2 g^{-3}$  are greater than 0. Suppose  $2^{\frac{1}{2}} \left( \frac{\pi_2}{3} \right)^{\frac{1}{3}} < g < \left( \frac{3}{2\pi_2} \right)^{\frac{2}{3}} \pi_1$ , which implies  $\left( \frac{3}{2\pi_2} \right)^{\frac{2}{3}} \pi_1 \geq 2^{\frac{1}{2}} \left( \frac{\pi_2}{3} \right)^{\frac{1}{3}}$ . Thus,  $\pi_1 \geq \frac{2^{\frac{1}{2}} 2^{\frac{2}{3}}}{3} \pi_2$ . Further, selecting  $g$  to be the linear combination of  $2^{\frac{1}{2}} \left( \frac{\pi_2}{3} \right)^{\frac{1}{3}}$  and  $\left( \frac{3}{2\pi_2} \right)^{\frac{2}{3}} \pi_1$  will lead to  $V \geq 0$  as well  $V$  as a convex function. Therefore, if we are able to establish  $\dot{V} < 0$  in all argument then  $V = 0$  is the global minima. Thus  $g = a 2^{\frac{1}{2}} \left( \frac{\pi_2}{3} \right)^{\frac{1}{3}} + (1-a) \left( \frac{3}{2\pi_2} \right)^{\frac{2}{3}} \pi_1$ ,  $0 \leq a \leq 1$ . Now our next aim is to show  $\dot{V} < 0$ ,

$$\begin{aligned} \dot{V} &= \left\{ \frac{3}{2} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} \pi_1 \text{sign}(z_1) + \pi_2 z_2 \right\} \dot{z}_1 \\ &\quad + \left\{ \frac{3}{2} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} z_2 + \pi_2 z_1 \right\} \dot{z}_2 \end{aligned}$$

$\dot{V}$  can be rewritten as,

$$\dot{V} = -W_1(z) \left( \frac{\dot{L}}{L} \right) + W_2(z) \left( \frac{\dot{d}}{L} \right) - W_3^*(z), \quad (3.7)$$

where,

$$\begin{aligned} W_1(z) &= \frac{3}{2} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} \left( \pi_1 |z_1| + z_2^2 \right) + 2\pi_2 z_1 z_2 \\ W_2(z) &= \frac{3}{2} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} z_2 + \pi_2 z_1 \\ W_3^*(z) &= \left( k_1 \frac{\pi_2}{L} + k_2 \frac{\pi_2}{L} \text{sign}(z_1 \chi) \right) |z_1| \\ &+ \frac{3}{2} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} \left( \left( \frac{k_1}{L} - \pi_1 \right) \text{sign}(z_1 z_2) \right) |z_2| \\ &+ \frac{3}{2} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} \left( \frac{k_2}{L} \text{sign}(z_2 \chi) \right) |z_2| - \pi_2 z_2^2, \end{aligned} \quad (3.8)$$

and  $\chi = \alpha(L(t)z_1(t), L(t-\tau)z_1(t-\tau))$ . We are going to show that  $W_3^*(z)$  would dominate over  $W_2(z)$ . Since,  $\frac{3}{2} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} |z_2| \geq \frac{3}{2} \left( \frac{1}{2} \right)^{\frac{1}{2}} z_2^2$ , therefore,

$$\pi_2 (2)^{\frac{1}{2}} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} |z_2| \geq -\pi_2 z_2^2 \text{ and } W_3^* \leq W_3',$$

where,

$$\begin{aligned} W_3'(z) &= \left( k_1 \frac{\pi_2}{L} + k_2 \frac{\pi_2}{L} \text{sign}(z_1 \chi) \right) |z_1| + \frac{3}{2} \left( \pi_1 |z_1| + \frac{1}{2} z_2^2 \right)^{\frac{1}{2}} \\ &\left( \frac{2^{\frac{3}{2}}}{3} \pi_2 + \left( \frac{k_1}{L} - \pi_1 \right) \text{sign}(z_1 z_2) + \frac{k_2}{L} \text{sign}(z_2 \chi) \right) |z_2|. \end{aligned} \quad (3.9)$$

For  $W_3'$  to be greater than zero  $\forall z$ , both the coefficients of equation (3.9) should be independently greater than zero, that is, if

$$k_1 \frac{\pi_2}{L} + k_2 \frac{\pi_2}{L} \text{sign}(z_1 \chi) \geq 0, \quad (3.10)$$

$$\frac{2^{\frac{3}{2}}}{3} \pi_2 + \left( \frac{k_1}{L} - \pi_1 \right) \text{sign}(z_1 z_2) + \frac{k_2}{L} \text{sign}(z_2 \chi) \geq 0. \quad (3.11)$$

These two inequalities are satisfied if

$$\begin{aligned} k_1 + k_2 &\geq 0; \quad k_1 - k_2 \geq 0, \\ \frac{2^{\frac{3}{2}}}{3} \pi_2 + \frac{k_1}{L} - \pi_1 + \frac{k_2}{L} &\geq 0, \quad \frac{2^{\frac{3}{2}}}{3} \pi_2 - \frac{k_1}{L} + \pi_1 - \frac{k_2}{L} \geq 0, \end{aligned} \quad (3.12)$$

which can be rewritten as

$$\begin{aligned} k_1 &\geq k_2 \\ L \left( \frac{2^{\frac{3}{2}}}{3} \pi_2 - \pi_1 \right) &\leq (k_1 + k_2) \leq L \left( \frac{2^{\frac{3}{2}}}{3} \pi_2 + \pi_1 \right). \end{aligned} \quad (3.13)$$

Since

$$W_3'(z) \geq W_3^f(z) \triangleq A|z_1| + B \left( \pi_1|z_1| + \frac{1}{2}z_2^2 \right)^{\frac{1}{2}} |z_2|, \quad (3.14)$$

with

$$\begin{aligned} A &= \min_z \left[ k_1 \frac{\pi_2}{L} + k_2 \frac{\pi_2}{L} \text{sign}(z_1 \chi) \right] \geq 0 \\ B &= \min_z \left[ \frac{2\frac{3}{2}}{3} \pi_2 + \left( \frac{k_1}{L} - \pi_1 \right) \text{sign}(z_1 z_2) + \frac{k_2}{L} \text{sign}(z_2 \chi) \right] \geq 0, \end{aligned} \quad (3.15)$$

$W_3^f(z)$  is a continuous and homogeneous positive definite function. According to Lemma 3, it follows that  $\forall z \in \mathbb{R}^2$ ,  $W_2(z) \leq \gamma W_3^f(z)$  is satisfied, with  $\gamma = \max_{\{z: W_3^f(z)=1\}} > 0$ , because both  $W_2(z)$  and  $W_3^f(z)$  are continuous and homogeneous with same weights and degree. Finally,

$$\begin{aligned} W_1(z) &= \frac{3}{2} \left( \pi_1|z_1| + \frac{1}{2}z_2^2 \right)^{\frac{1}{2}} \left( \pi_1|z_1| + z_2^2 \right) + 2\pi_2 z_1 z_2 \\ &\geq \frac{3}{2} (\pi_1|z_1|)^{\frac{1}{2}} \pi_1|z_1| + \frac{3}{2} \left( \frac{1}{2}z_2^2 \right)^{\frac{1}{2}} z_2^2 - 2\pi_2 \left( \frac{2}{3}g^{\frac{3}{2}}|z_1|^{\frac{3}{2}} + \frac{1}{3}g^{-3}|z_2|^3 \right) \\ &= \left( \frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\pi_2 g^{\frac{3}{2}} \right) |z_1|^{\frac{3}{2}} + \left( \frac{3}{2} - \frac{2\pi_2}{3}g^{-3} \right) |z_2|^3 \end{aligned} \quad (3.16)$$

$W_1(z)$  is positive-definite if  $\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\pi_2 g^{\frac{3}{2}} > 0$  and  $\frac{3}{2} - \frac{2\pi_2}{3}g^{-3} > 0$  or equivalently  $\frac{2\frac{5}{6}}{3\frac{3}{2}}\pi_2^{\frac{1}{3}} < g < \frac{3\frac{4}{3}}{2^2} \frac{\pi_1}{\pi_2^{\frac{2}{3}}}$  and such a value of  $g$  exists if  $\pi_1 > \frac{2\frac{5}{6}2^2}{3^2}\pi_2$ . Thus  $\pi_1$  should be selected such that  $\pi_1 > \frac{2\frac{5}{6}2^2}{3^2}\pi_2$ . It can be noted that it also fulfills  $\pi_1 \geq \frac{2\frac{1}{2}2\frac{2}{3}}{3}\pi_2$  required for  $V \geq 0$ . Thus, we have obtained the condition for gains.

Next we proceed for getting the condition on delay. It is based on the fundamental result develop in [73]. Let  $x_0 \in B_\rho^\tau$  for some  $\rho > 0$  and  $\tau > 0$ , then

$$x_1(t, x_0) = x_0(0) + \int_0^t F[x_1(s, x_0), x_1(s - \tau, x_0)] ds \quad (3.17)$$

denotes the solution of dynamics represented by (3.1) on the interval  $[0, t] \forall t \geq 0$ ,  $\|x_0\| \leq 2\bar{\rho}_r(\rho)$ . For all  $\|\phi\| \leq 2\bar{\rho}_r(\rho)$  according to Lemma 3, there exists  $k > 0$  for almost everywhere such that  $|F[\phi(0), \phi(-\tau)]|_r \leq k \max_{1 \leq i \leq n} \|\phi\|_r^{1+q/r_i}$ , then using Lemma 4  $|F[\phi(0), \phi(-\tau)]| \leq \bar{\sigma}_r(|F[\phi(0), \phi(-\tau)]|_r) \leq \bar{\sigma}_r(k \max_{1 \leq j \leq n} \|\phi\|_r^{1+q/r_i}) \leq \bar{\sigma}_r(k \max_{1 \leq j \leq n} [\rho_r^{-1}(\|\phi\|)]^{1+q/r_i}) = \iota_k(\|\phi\|)$  where  $\iota_k(s) = \bar{\sigma}_r(k \max_{1 \leq j \leq n} [\rho_r^{-1}(s)]^{1+q/r_i})$  is a function of class  $\kappa_\infty$  for  $q > -\min_{1 \leq i \leq n} r_i$ . Let's select  $0 < \tau < \frac{\bar{\rho}_r(\rho)}{\iota_k(2\bar{\rho}_r(\rho))}$ , then under this restriction the  $\|x(t, x_0)\| \leq 2\bar{\rho}_r(\rho)$  for  $0 < t \leq \tau$ . Now lets suppose  $\|x_s\| < 2\bar{\rho}_r(\rho)$  for all  $s \in [0, t']$  for some  $t' \leq \tau$  and  $\|x_s\| \geq 2\bar{\rho}_r(\rho)$  for  $s \in [t', \tau]$ , then From (3.17),

$$\begin{aligned} |x(t', x_0)| &\leq |x_0(0)| + \int_0^{t'} |F[x(s, x_0), x(s - \tau, x_0)]| ds \\ &\leq |x_0(0)| + \int_0^{t'} \iota_k(\|x_s\|) ds \leq |x_0(0)| + t' \sup_{0 \leq s < t'} \iota_k(\|x_s\|) \\ &< \bar{\rho}_r(\rho) + t' \iota_k(2\bar{\rho}_r(\rho)) < 2\bar{\rho}_r(\rho) \end{aligned} \quad (3.18)$$

This is a contradiction, thus  $\|x_s\| \leq 2\bar{\rho}_r(\rho)$  for  $s \in [0, \tau]$ . Hence for any  $\rho > 0$  there exists  $\tau > 0$  such that when the initial point lies within  $B_\rho^\tau$  then the trajectory  $x_t \in B_{\rho'}^\tau$ , for  $t \in [0, \tau]$ , where  $\rho' = \underline{\rho}_r^{-1} [2\bar{\rho}_r(\rho)]$ .

For a weighted Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , of degree  $v > -q$  the following quantities are defined

$$a = -\sup_{\zeta \in \mathbb{S}_r} \frac{\partial V}{\partial \zeta} F(\zeta, \zeta) > 0, \quad 0 < b = \sup_{|\zeta|_r \leq 1} \left| \frac{\partial V(\zeta)}{\partial \zeta} \right| < +\infty \quad (3.19)$$

$$c_1 |x|_r^v \leq V(x) \leq c_2 |x|_r^v \quad \forall x \in \mathbb{R}^n.$$

Let us also define  $\lambda_1 = |\phi(0)|_r$  and  $\lambda_2 = \|\phi\|_r$ , where  $\phi(0) = \Lambda_r(\lambda_1)\zeta$  for some  $\zeta \in \mathbb{S}_r$  and  $\phi = \Lambda_r(\lambda_2)\psi$  for some  $\psi \in S_r \subset C_{[-\tau, 0]}$  and  $|\psi(0)|_r \leq |\zeta|_r = 1$ . Again  $\frac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)] = \frac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(0)] + \frac{\partial V(\phi(0))}{\partial \phi(0)} \{F[\phi(0), \phi(-\tau)] - F[\phi(0), \phi(0)]\}$   
 $= \lambda_1^{q+v} \frac{\partial V}{\partial \zeta} F[\zeta, \zeta] + \lambda_2^{q+v} \frac{\partial V(\psi(0))}{\partial \psi(0)} \{F[\psi(0), \psi(-\tau)] - F[\psi(0), \psi(0)]\}$ . By Lemma 5 for all  $\psi \in S_r$  and for some  $\eta > 0$  (defined later), there exists  $G > 0$  such that,

$|F[\psi(0), \psi(-\tau)] - F[\psi(0), \psi(0)]|_r \leq \max\{G \max_{1 \leq i \leq n} |\psi(0) - \psi(-\tau)|_r^{1+q/r_i}, \eta\}$ . For the trajectory  $F$ , for  $t \geq \tau$ , there exists  $M > 0$  such that

$$|\phi(0) - \phi(-\tau)| \leq M\tau,$$

where,  $M = \sup_{|z|_r \leq \rho, |y|_r \leq \rho} |F[z, y]|$ ;  $\|\phi\|_r |\psi(0) - \psi(-\tau)|_r = |\phi(0) - \phi(-\tau)|_r \leq \underline{\sigma}_r^{-1}(G\tau)$  and  $|F[\psi(0), \psi(-\tau)] - F[\psi(0), \psi(0)]| \leq \bar{\sigma}_r \circ \max\{G \max_{1 \leq i \leq n} \|\phi\|_r^{-1+q/r_i} \underline{\sigma}_r^{-1}(G\tau)^{1+q/r_i}, \eta\}$ . Finally, on solution for  $t \leq \tau$   $|F[\psi(0), \psi(-\tau)] - F[\psi(0), \psi(0)]| \leq \bar{\sigma}_r \circ \max\{\mu_1(\tau)\mu_2(\lambda_2), \eta\}$  where

$$\mu_1(\tau) = G \max_{1 \leq i \leq n} \underline{\sigma}_r^{-1}(G\tau)^{1+q/r_i}$$

$$\mu_2(\lambda_2) = \begin{cases} \lambda_2^{-1-q/\min_{1 \leq i \leq n} r_i} & \text{if } \lambda_2 \geq 1 \\ \lambda_2^{-1-q/\max_{1 \leq i \leq n} r_i} & \text{if } \lambda_2 < 1 \end{cases}$$

For applying the Lyapunov-Razumikhin method to prove asymptotic stability, we consider some  $\gamma > 1$ , then it can be written that  $c_1 \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|_r^v \leq \sup_{\theta \in [-\tau, 0]} V[\phi(\theta)] < \gamma V[\phi(0)] \leq \gamma c_2 |\phi(0)|_r^v$ , which implies,

$$\sup_{\theta \in [-\tau, 0]} |\phi(\theta)|_r \leq (\gamma c_1^{-1} c_2)^{\frac{1}{v}} |\phi(0)|_r = (\gamma c_1^{-1} c_2)^{\frac{1}{v}} \lambda_1, \text{ so, } \lambda_2 = \|\phi\|_r = \left( \sum_{i=1}^n \sup_{\theta \in [-\tau, 0]} |\phi_i(\theta)|_r^{\frac{\rho}{r_i}} \right)^{\frac{1}{\rho}}$$

$$\leq \left( \sum_{i=1}^n \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|_r^{\rho} \right)^{\frac{1}{\rho}} = n^{\frac{1}{\rho}} \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|_r \leq R \lambda_1 \text{ where } R = n^{\frac{1}{\rho}} (\gamma c_1^{-1} c_2)^{\frac{1}{v}}.$$

Therefore,  $\frac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)] \leq b \bar{\sigma}_r \circ \max\{\mu_1(\tau)\mu_2(\lambda_2), \eta\} \lambda_2^{q+v} - a R^{-q-v} \lambda_2^{q+v}$

$= \|\phi\|_r^{q+v} (b\bar{\sigma}_r \circ \max\{\mu_1(\tau)\mu_2(\|\phi\|_r), \eta\} - aR^{-q-v})$ . Now select  $\eta = \frac{\bar{\sigma}_r^{-1}(aR^{-q-v} - \epsilon)}{b}$  and  $\rho > \epsilon > 0$  then for all  $\tau \in (0, \tau_0]$  where,

$$\tau_0 = \min \left\{ \frac{\bar{\rho}_r(\rho)}{\iota_k 2\bar{\rho}_r(\rho)}, \frac{1}{\mu_2(\epsilon)} \mu_1^{-1} \left( \frac{\bar{\sigma}_r^{-1}(aR^{-q-v} - \epsilon)}{b} \right) \right\}, \quad (3.20)$$

for some  $\epsilon \in (0, aR^{-q-v})$ , thus it can be written that  $\max_{\theta \in [-\tau, 0]} V[\phi(\theta)] < \gamma V[\phi(0)]$ , which implies,  $\frac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)] \leq -\epsilon |\phi(0)|_r^{q+v}$  for all  $\phi \in B_{\rho'}^\tau \setminus B_\epsilon^\tau$  that are solution of (3.1) for  $t \geq \tau$ . Hence, we have proved the asymptotic stability for  $t \geq \tau$ , while stability for  $t \leq \tau$  can be seen from (3.18). It gives asymptotic stability with respect to  $B_\epsilon^\tau$  in  $B_\rho^\tau$  for any delay  $0 < \tau \leq \tau_0$ .  $\square$

### Calculation of $\tau_0$ :

From the Lyapunov function (3.5) and given condition (3.19), one can calculate the value of parameters  $a = 58.5, b = 3.535, c_1 = 0.744$  and  $c_2 = 2.834$  by choosing  $\pi_1 = \pi_2 = 1, L(t) = \frac{2}{2 - \exp(-t)}, k_1=4, k_2=3, d_0 = 1$  and the initial condition of states as  $0.5 \leq |x(0)|_r \leq 5$ . In the closed loop system (3.1), for  $\alpha(x_1(t), x_1(t - \tau)) = |x_1|^{0.5} \text{sign}(x_1(t)) - |x_1(t - \tau)|^{0.5} \text{sign}(x_1(t - \tau)), q = -1$  and for the chosen Lyapunov function,  $v = 3$ . The other parameters for the calculation of maximum value of delay  $\tau_0$  are obtained as  $\underline{\sigma}_r(s) = 2^{-1/2}s, \bar{\sigma}_r(s) = s, \underline{\rho}_r^{-1}(s) = \left(\sum_{i=1}^n s^{r_i}\right)^{\frac{1}{4}}, \bar{\rho}_r(s) = \left(\sum_{i=1}^n s^{2r_i}\right)^{\frac{1}{2}}, \rho' = \underline{\rho}_r^{-1}[2\bar{\rho}_r(\rho)] = 9.431$  for  $r = [2 \ 1], M = 12.367, R = 2.784$  (for  $\gamma = 2$  and  $\rho = 2$ ). Therefore, for  $\epsilon = 5$ , the delay condition can be obtained from equation (3.20) as  $\tau_0 = \min\{1.456, 2.612\} = 1.456$ .

## 3.4 Applications

In this section, we demonstrate that the proposed artificial delayed output twisting algorithm works as a controller as well as an observer for uncertain second order systems. Full state feedback based twisting algorithm [62] is only able to mitigate the matched uncertainty/disturbance. However, we are going to show that artificial delayed output feedback twisting controller is also able to mitigate the unmatched disturbance which is an extra benefit of the proposed algorithm over the full state information based twisting controller. Another benefit comes into picture in terms of observer design, whereas the classical twisting [62] works only like a controller.

### 3.4.1 Controller Design for Second Order System

Consider the controller

$$u := -k_1 \text{sign}(x(t)) - k_2 \text{sign}(\alpha(x(t), x(t - \tau))),$$

for the system,  $\ddot{x} = u + d(t)$ , with relative degree 2 with respect to output  $x$ , where  $d(t)$  is the bounded matched disturbance. The system under consideration can be transformed into

$$\begin{aligned} \dot{x}_1 = x_2; \quad \dot{x}_2 = & -k_1 \text{sign}(x_1(t)) \\ & - k_2 \text{sign}(\alpha(x_1(t), x_1(t - \tau))) + d(t) \end{aligned} \quad (3.21)$$

taking  $x_1 = x$  and  $x_2 = \dot{x}$ . It can be noted that the control signal would be discontinuous in this case.

**Remark 4** *System (3.21) is structurally similar to the twisting controller given in [62]. Since the controller used in system (3.21) requires the information of present state and its delayed version, the proposed controller is named as "Artificial Delayed Output Twisting algorithm".*

### 3.4.2 Mitigation of unmatched disturbance

Consider the following system,  $\dot{x}_1 = x_2 + d_1(t)$ ;  $\dot{x}_2 = u + d_2(t)$ , where  $u$ ,  $d_1(t)$  and  $d_2(t)$  are control, unmatched and matched perturbations respectively. Assume that  $d_1(t)$  is Lipschitz continuous and  $d_2(t)$  is bounded. Suppose that first; we want to design full state feedback based twisting controller by converting unmatched disturbance into matched disturbance as follows:  $\dot{x}_1 = p(t)$ ;  $\dot{p}(t) = u + \dot{d}_1(t) + d_2(t)$ , where  $p(t) := x_2(t) + d_1(t)$ . Now, one can design classical twisting [62] as  $u := -k_1 \text{sign}(x_1) - k_2 \text{sign}(p)$  with  $k_1 > k_2 + |\dot{d}_1(t) + d_2(t)|$ . However, this controller is not practically feasible because it requires the explicit information of disturbance  $d_1(t)$ . Next, we are going to suggest a solution which is based on the proposed delayed output twisting algorithm.

Let us take a sliding surface,

$$\begin{aligned} S := & x_2 + k_1 \int_0^t \text{sign}(x_1(s)) ds + \\ & k_2 \int_0^t \text{sign}(\alpha(x_1(s), x_1(s - \tau))) ds. \end{aligned} \quad (3.22)$$

Therefore

$$\dot{S} = u + d_2 + k_1 \text{sign}(x_1(t)) + k_2 \text{sign}(\alpha(x_1(t), x_1(t - \tau))). \quad (3.23)$$

Let us take the controller  $u := -k_1 \text{sign}(x_1(t)) - k_2 \text{sign}(\alpha(x_1(t), x_1(t - \tau))) - K \text{sign}(S)$ , so (3.23) becomes  $\dot{S} = -K \text{sign}(S) + d_2$ , which is finite time stable when  $K > |d_2|$  and hence,  $S$  converges to 0. Thus from (1), we get,  $x_2 = -k_1 \int_0^t \text{sign}(x_1(s)) ds - k_2 \int_0^t \text{sign}(\alpha(x_1(s), x_1(s - \tau))) ds$ , which can be re-written as  $\dot{x}_1 = z$ ,  $\dot{z} = -k_1 \text{sign}(x_1(t)) - k_2 \text{sign}(\alpha(x_1(t), x_1(t - \tau))) + \dot{d}_1(t)$ , which is similar to that of (3.1). By theorem 3, the above system is asymptotically stable which in turn proves that the proposed controller mitigates unmatched disturbance.

### 3.4.3 Observer Design

Let us consider a system  $\dot{x}_1 = x_2$ ,  $\dot{x}_2 = u + d(t)$ , which is mimicked as  $\dot{\hat{x}}_1 := \hat{x}_2$   $\dot{\hat{x}}_2 := u + L_2$ , where,  $L_2$ ,  $u$  and  $d(t)$  are correction term, control input and bounded perturbation respectively. The estimation errors are  $e_1 := \hat{x}_1 - x_1$ ,  $e_2 := \hat{x}_2 - x_2$ . The closed loop system in terms of observer's estimation errors  $e^T = [e_1 \ e_2]$  is

$$\dot{e}_1 = e_2, \quad \dot{e}_2 = L_2 - d(t). \quad (3.24)$$

On choosing  $L_2 := -k_1 \text{sign}(e_1(t)) - k_2 \text{sign}(\alpha(e_1(t), e_1(t - \tau)))$ , the system (3.24) becomes structurally similar to system (3.1) and hence, asymptotically converges to zero by theorem 3. Thus, the states are estimated.

## 3.5 Simulation

The proposed artificial delayed controller is illustrated and validated using simulations. The initial condition  $x_1(0)$  and  $x_2(0)$  is taken as 5 and 2, respectively. The controller gains  $k_1$  and  $k_2$  are selected as 4 and 3, respectively, which satisfies the gain condition outlined in (3.12). The delay  $\tau$  has been taken as 0.1 for the purpose of simulation. Fig. 5.1 depicts the states in presence of time varying disturbance  $\dot{d} = \sin\left(\frac{t^2}{2}\right)$ . As evident from simulation the time varying disturbance is rejected on the application of the proposed control strategy.

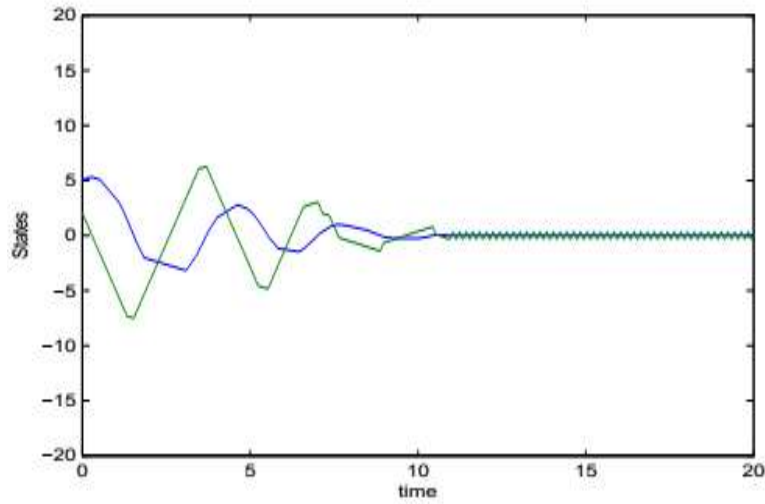


Figure 3.1: Evolution of states in presence of time-varying disturbance

### 3.6 Summary

In this chapter, a controller structurally similar to the twisting controller has been proposed which, for a system with relative degree two, needs the information about one of the states and its artificially delayed version. The condition on the controller gains are obtained for stable operation. The maximum delay upto which the controller remains stable has been obtained by the application of Lyapunov-Razumikhin approach. The simulation results confirms the convergence of states to equilibrium point when the controller gains and artificial delay are selected according to the results obtained.

In the next chapter, we study the application of Lyapunov-krasovskii functional method on linear system with time-varying delay to obtain maximum upper bound delay value by ensuring asymptotic stability of the system.