Chapter 3

A numerical approach to solve two-dimension fractional reaction advection diffusion equation of variable-order with Vieta-Lucas polynomials

3.1 Introduction

It is well known that the non-integer calculus is used by many researchers in various fields like wave propagation, porous media, turbulence, signal processing, fluid-dynamic, colored noise, solid mechanics, viscoelasticity, etc. Non-integer calculus was initiated by Leibniz and L' Hopital, and was extraordinarily explored by many great mathematicians like, Abel, Euler, Laplace, Liouville, and Riemann [94]. It can be seen that the non-integer calculus has gained great advantages in many physical problems which display the dynamical behavior of a system where the succeeding state of the system depends not only on its present situation but also on all its preceding conditions ([95], [96]). This turns out the non-locality nature of the partial differential equations of fractional order. These partial differential equations with fractional order derivatives in space-time or space or in time describe certain phenomena and processes in the fields of viscoelasticity, electromagnetics, diffusion procedures, electrochemistry, etc. ([97], [98], [99]). The theories of fractional calculus and its applications are well covered in ([53], [100], [101]).

Despite the beneficial attributes of fractional partial differential equations, finding the analytical solutions of these equations are generally difficult ([102], [103], [104]). Therefore, several numerical or approximate approaches have been developed for solving such types of equations in the last few decades. Some of the approaches are variation iteration method [105], Galerkin method [106], finite difference method [107], collocation method ([108], [109]), homotopy perturbation technique [110], etc. An implicit method is discussed by Langlands and Henry [111] to solve the fractional diffusion equation, and they also discussed the stability and accuracy of the scheme. In 2014, Wang et al. [112] applied a Haar wavelet method to find the solutions of fractional partial differential equations. In 2016, a compact alternating direction implicit method has been proposed in [113] to solve two-dimension diffusion equations involving time-fractional derivative. In 2018, the Galerkin method is successfully applied by Li et al. [114], to solve the multi-dimensional non-linear reaction subdiffusion equations. Singh et al. [115] discussed a two-dimension fractional reaction advection diffusion equation in their study, and they applied a non-standard finite difference scheme and collocation approach to solve their problem. In 2022, Craciun and Singh [116] presented an algorithm based on operational matrices of Lucas polynomials and collocation scheme to solve the nonlinear fractional reaction advection diffusion equations.

In the last decade, the researchers ([117], [118]) who are working in the field of fractional calculus have generalized the fractional differential operators from the constant-order to the variable-order because variable-order fractional differential operators have the potential to explain various physical situations. In the variable-order fractional differential equation, the orders of fractional derivatives may be a function of time and/or space. The fundamental of variable-order operators can be seen in Lorenzo and Hartley [56]. The applications of variable-order operators can be found in ([119], [120]). Some applications of variable-order fractional derivatives in diffusion processes can be seen in (Shen et al. [121], Sun et al. [122], Wei et al. [123], Wei et al. [124]). Recently, several mathematicians have investigated fractional reaction advection-diffusion equations of variable-order due to its wide range of applications in many complex physical problems of energy and mass transfer, fluid transport in porous media, various chemical reactions, transport of water vapor in the atmosphere, etc. In 2020, Li and Wu [125] presented a novel meshless approach to solve the time-fractional reaction advection diffusion equations of variable-order. Dwivedi et al. [126] discussed a fractional reaction advection diffusion equations of variable-order in the heterogeneous medium, and presented the solution of the problem by finite difference method and Fibonacci collocation approach. Hosseininia et al. [127] presented a two-dimension model of reaction advection diffusion equations with a time-fractional derivative in the Heydari-Hosseininia sense, and a hybrid method is discussed to solve the problem. In 2022, Jaiswal et al. [128] presented a Legendre collocation scheme to solve two-dimension fractional differential equations with Dirichlet boundary conditions. Kheirkhah et al. [129] established a numerical approach to solve the RADE with Caputo fractional derivative. In 2023, Partohaghighi et al. [130] presented a model of one-dimension advection-dispersion equation with HH time derivative of variable-order and discussed a numerical solution to the problem which is based on the shifted Vieta-Lucas polynomials.

In order to analyze the diffusion of the solute in porous media in two-dimension, the author present two-dimensional physical model in this chapter with the help of the theory of fractional differential operators which paves the way for exploiting the classical models of integer orders to fractional orders. The physical interpretation of fractional orders describes the behavior memory feature of dynamical processing which may not be supported by actual the local limit such as anomalous transport processes and difusion. The non-local character simply means that the evolution of solution at the present time level stacks up its solution at all prior time levels which saved in storage at every time levels. The following model is showing the physical behavior of a two-dimensional advection–reaction–diffusion phenomenon for solute profile in porous media:

$$\begin{aligned}\frac{\partial^{\alpha(\mathcal{S},t)}u(\mathcal{S},t)}{\partial t^{\alpha(\mathcal{S},t)}} &= \upsilon\left(\frac{\partial^{\beta(\mathcal{S},t)}u(\mathcal{S},t)}{\partial x^{\beta(\mathcal{S},t)}} + \frac{\partial^{\gamma(\mathcal{S},t)}u(\mathcal{S},t)}{\partial y^{\gamma(\mathcal{S},t)}}\right)\\ &\quad - \kappa\left(u^{\eta}\frac{\partial^{\delta(\mathcal{S},t)}u(\mathcal{S},t)}{\partial x^{\delta(\mathcal{S},t)}} + u^{\eta}\frac{\partial^{\sigma(\mathcal{S},t)}u(\mathcal{S},t)}{\partial y^{\sigma(\mathcal{S},t)}}\right) + \lambda\psi(u,\mathcal{S},t), \qquad (3.1)\end{aligned}$$

with the following conditions:

$$\begin{aligned} u(x,y,0) &= f_1(x,y),\\ u(0,y,t) &= f_2(y,t),\\ u(1,y,t) &= f_3(y,t),\\ u(x,0,t) &= f_4(x,t),\\ u(x,1,t) &= f_5(x,t). \qquad (3.2)\end{aligned}$$

where $0 < \alpha(\mathcal{S},t),\ \delta(\mathcal{S},t),\ \sigma(\mathcal{S},t) \leq 1$, $1 < \beta(\mathcal{S},t),\ \gamma(\mathcal{S},t) \leq 2$, is an arbitrary fractional variable-order, $t > 0$, $\mathcal{S} = (x,y)$, $(x,y) \in$ [0,1]×[0,1] and $u(\mathcal{S},t)$ is the concentration of solute in fluid, $\alpha(\mathcal{S},t)$ is the time fractional derivative of variable-order, $\beta(\mathcal{S},t)$, $\gamma(\mathcal{S},t)$, $\delta(\mathcal{S},t)$ and $\sigma(\mathcal{S},t)$ are the space fractional derivatives of variable=order, $\psi(u)$ is the forced term, $f_1(x,y)$ is a known function which represents the initial solute concentration, $f_2(y,t)$, $f_3(y,t)$, $f_4(x,t)$ and $f_5(x,t)$ are the known functions which represent the solute concentration at the boundary points at any time t and υ, κ, λ are constants.

This chapter aims to present a numerical algorithm that depends on the approximation of variable-order Caputo fractional derivative with the operational matrix of shifted Vieta-Lucas polynomials, and collocation approach. First, the operational matrices of shifted Vieta-Lucas polynomials are constructed for the fractional derivative of variable-order, and then the proposed scheme is applied to solve the fractional time-space reaction advection diffusion equations of variable-order which is the generalisation of fractional reaction advection diffusion equations of constant order. The convergence and error bound of the scheme are also theatrically analysed, and it is found that the approximate solution converges rapidly to the exact solution as the number of approximating polynomials (Vieta-Lucas polynomials) increases. To show the accuracy and efficiency of the proposed algorithms, some examples of reaction advection diffusion equations of variable-order are considered and the obtained results are discussed through graphs and tables.

The following is a breakdown of the chapter's structure. In section 3.2, the features of the shifted Vieta-Lucas polynomials are described. The approximation of an arbitrary function is mentioned in section 3.3. The operational matrices for differentiation of the shifted Vieta-Lucas polynomials are constructed in section 3.4. A brief description of the scheme for the general case is discussed in section 3.5. Section 3.6 will wrap up the conversation on convergence and error analysis. The numerical computation of this study is described in section 3.7, and finally, the conclusion is presented in section 3.8.

3.2 Properties of Shifted Vieta-Lucas Polynomials

The Vieta-Lucas polynomials lies in the interval of [-2,2] and is characterized by the following recurrence relation:

$$VL_m(x) = xVL_{m-1}(x) - VL_{m-2}(x), \quad m = 2, 3, ...,$$

where

$$VL_0(x) = 2, \quad VL_1(x) = x.$$

The Vieta-Lucas polynomials ($VL_m(x)$) are orthogonal on [-2,2] w.r.t. the weight function $\frac{1}{\sqrt{4-x^2}}$ as given below

$$\langle VL_{m1}(x), VL_{m2}(x)\rangle = \int_{-2}^{2} \frac{1}{\sqrt{4-x^2}} VL_{m1}(x) VL_{m2}(x) dx = \begin{cases} 0, & m1 \neq m2 \neq 0, \\ 4\pi, & m1 = m2 = 0, \\ 2\pi, & m1 = m2 \neq 0, \end{cases}$$

where $m1$, $m2 \in \mathbb{N} \cup \{0\}$.

In order to use a new class of Vieta-Lucas polynomials on the closed interval [0,1], we can define the tranformation x=$4x-2$ for [0,1]. Let us denote the shifted Vieta-Lucas polynomials $VL_i(4x-2)$ as ($VL_i^*(x)$). Then $VL_i^*(x)$ can be calculated as

$$VL_m^*(x) = (4x-2)VL_m^*(x) - VL_{m-1}^*(x), \quad m = 2, 3, ...,$$

where $VL_0^*(x)$=2 and $VL_1^*(x)$=$4x-2$. The series form of the $VL_i^*(x)$ is given below:

$$VL_m^*(x) = 2m\sum_{i=0}^{m}(-1)^i\frac{4^{m-i}\Gamma(2m-i)}{\Gamma(i+1)\Gamma(2m-2i+1)}x^{m-i}, \quad m=2,3,..., \tag{3.3}$$

or

$$VL_m^*(x) = 2n\sum_{i=0}^{m}(-1)^{m-i}\frac{4^{i}\Gamma(m+i)}{\Gamma(m-i+1)\Gamma(2i+1)}x^{i}, \quad m=2,3,... \tag{3.4}$$

The orthogonality condition for the polynomials $VL_m^*(x)$ w.r.t. weight function $\chi(x)=\frac{1}{\sqrt{x-x^2}}$ is given by

$$\langle VL_{m1}^*(x), VL_{m2}^*(x)\rangle = \int_0^1 \chi(x)VL_{m1}^*(x)VL_{m2}^*(x)dx = \begin{cases} 0, & m1 \neq m2 \neq 0, \\ 4\pi, & m1 = m2 = 0, \\ 2\pi, & m1 = m2 \neq 0. \end{cases} \tag{3.5}$$

3.3 Approximation of an Arbitrary Function

Let us suppose $\varphi(t) = [VL_0^*(t), VL_1^*(t), ..., VL_m^*(t)]^T \in L^2[0,1]$ is the set of shifted Vieta-Lucas polynomials. Then a function $u(t) \in L^2[0,1]$ can be represented in terms of shifted Vieta-Lucas polynomials as

$$u(t) = \sum_{i=0}^{\infty} c_i VL_i^*(t), \tag{3.6}$$

where c_i are given as

$$c_i = \begin{cases} \frac{1}{4\pi}\int_0^1 \frac{u(t)VL_i^*(t)}{\sqrt{t-t^2}}dt, & i=0, \\ \frac{1}{2\pi}\int_0^1 \frac{u(t)VL_i^*(t)}{\sqrt{t-t^2}}dt, & i=1,2,...,m. \end{cases} \tag{3.7}$$

Let us truncate the series (3.6) as

$$u(t) \simeq \sum_{i=0}^{m} c_i VL_i^*(t) = C^T \varphi(t), \tag{3.8}$$

where T represent the transpose, and

$$C = [c_0, c_1, ..., c_m]^T,$$

$$\varphi(t) = [VL_0^*(t), VL_1^*(t), ..., VL_m^*(t)]^T. \tag{3.9}$$

The polynomials $\varphi(t)$ can be written as

$$\varphi(t) = \mathcal{V}P(t), \tag{3.10}$$

where $P(t) = [1, t, t^2, ..., t^m]$, and the matrix $\mathcal{V}$ is $(m+1) \times (m+1)$ has the following form:

$$\mathcal{V} = \begin{pmatrix} v_{0,0} & 0 & 0 & 0 & \cdots & 0 \\ v_{1,0} & v_{1,1} & 0 & 0 & \cdots & 0 \\ v_{2,0} & v_{2,1} & v_{2,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ v_{m,0} & v_{m,1} & v_{m,2} & v_{m,3} & \cdots & v_{m,m} \end{pmatrix}, \tag{3.11}$$

where entries of the matrix $\mathcal{V}$ are given by

$$(v_{i,j})_{0\le i,j\le m} = \begin{cases} 2, & i = j = 0, \\ \frac{2m(-1)^{m-i}4^i\Gamma(m+i)}{\Gamma(m-i+1)\Gamma(2i+1)}, & i \ge j, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

Also, Eq. (3.10) can be rewritten as

$$P(t) = \mathcal{V}^{-1}\varphi(t). \tag{3.13}$$

Similiarly, an arbitrary function $u(x, y, t) \in L^2[0, 1] \times L^2[0, 1] \times L^2[0, 1]$ can be expressed in terms of shifted Vieta-Lucas polynomials as

$$\begin{aligned} u(x, y, t) &\simeq \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} c_{ijk} VL_i^*(x) VL_j^*(y) VL_k^*(t) \\ &= \varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y)), \end{aligned} \tag{3.14}$$

where $\mathbb{A}$=$[c_{ijk}]$ is the $(m + 1) \times (m + 1)^2$ matrix whose entries are

$$c_{ijk} = \begin{cases} \frac{1}{64\pi^3}\int_0^1\int_0^1\int_0^1 \frac{u(x,y,t)VL_i^*(x)VL_j^*(y)VL_k^*(t)}{\sqrt{x-x^2}\sqrt{y-y^2}\sqrt{t-t^2}}dxdydt, & i = j = k = 0, \\ \mathbb{Q}\int_0^1\int_0^1\int_0^1 \frac{u(x,y,t)VL_i^*(x)VL_j^*(y)VL_k^*(t)}{\sqrt{x-x^2}\sqrt{y-y^2}\sqrt{t-t^2}}dxdydt, & \text{otherwise.} \end{cases} \tag{3.15}$$

where $\mathbb{Q}$ has the following form :

$$\mathbb{Q} = \begin{cases} \frac{1}{8\pi^3}, & \text{when } i \neq 0,\ j \neq 0,\ k \neq 0, \\ \frac{1}{16\pi^3}, & \text{when } i \neq 0,\ j \neq 0,\ or\ j \neq 0,\ k \neq 0,\ or\ k \neq 0,\ i \neq 0, \\ \frac{1}{32\pi^3}, & \text{when } i \neq 0,\ or\ j \neq 0,\ or\ k \neq 0. \end{cases}$$

3.4 Construction of Operational Matrix

From Eq. (3.10), we have

$$D_t^{\mu(\mathcal{S},t)}\varphi(t) = D_t^{\mu(\mathcal{S},t)}(\mathcal{V}P(t)) = \mathcal{V}D_t^{\mu(\mathcal{S},t)}[1, t, t^2, ..., t^m]^T. \tag{3.16}$$

According to Eq. (3.3), Eq. (3.16) becomes

$$D_t^{\mu(\mathcal{S},t)}\varphi(t) = \mathcal{V}\left[0, \frac{\Gamma(2)}{\Gamma(2-\mu(\mathcal{S},t))}t^{1-\mu(\mathcal{S},t)}, \ldots, \frac{\Gamma(m+1)}{\Gamma(m+1-\mu(\mathcal{S},t))}t^{m-\mu(\mathcal{S},t)}\right]^T,$$

$$D_t^{\mu(\mathcal{S},t)}\varphi(t) = \mathcal{V}R^{\mu(\mathcal{S},t)}P(t), \tag{3.17}$$

where

$$R^{\mu(\mathcal{S},t)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(-\mu(t(\mathcal{S},t))}t^{-\mu(\mathcal{S},t)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(-\mu(\mathcal{S},t))}t^{-\mu(\mathcal{S},t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(m+1)}{\Gamma(m+1-\mu(\mathcal{S},t))}t^{-\mu(\mathcal{S},t)} \end{pmatrix}.$$

Substituting Eq. (3.13) in Eq. (3.17), we get

$$D_t^{\mu(\mathcal{S},t)}\varphi(t) = \mathcal{V}R^{\mu(\mathcal{S},t)}\mathcal{V}^{-1}\varphi(t), \tag{3.18}$$

where, $\mathcal{V}R^{\mu(\mathcal{S},t)}\mathcal{V}^{-1}$ is the required operational matrix.

Consequently, for the Eq. (3.14), we have

$$\begin{aligned} D_t^{\mu(\mathcal{S},t)}u(\mathcal{S},t) &\simeq D_t^{\mu(\mathcal{S},t)}(\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))) \\ &= D_t^{\mu(\mathcal{S},t)}(\varphi^T(t))\mathbb{A}(\varphi(x)\otimes\varphi(y)) \\ &= (\mathcal{V}R^{\mu(\mathcal{S},t)}\mathcal{V}^{-1}\varphi(t))^T\mathbb{A}(\varphi(x)\otimes\varphi(y)). \end{aligned} \tag{3.19}$$

3.5 Discription of the Present Method

Here, we apply operational matrix scheme associated with the shifted Vieta-Lucas polynomials to find the approximate solution of a nonlinear variable-order fractional reaction advection diffusion equations (3.1)-(3.2):

Firstly, we shall approximate $u(\mathcal{S},t)$ by shifted Vieta-Lucas polynomials as

$$u(\mathcal{S},t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} c_{ijk} VL_i^*(x)VL_j^*(y)VL_k^*(t), \tag{3.20}$$

where c_{ijk} are the unknown coefficients for $i = 0,1,...,m,\ j = 0,1,...,m$ and $k = 0,1,...,m$.

Now, we write

$$u(\mathcal{S},t) = \varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y)), \tag{3.21}$$

where $\mathbb{A} = [c_{ijk}]$ is the $(m+1)\times(m+1)^2$ matrix whose entries are unknowns and $\varphi(t) = [VL_0^*(t), VL_1^*(t), ..., VL_m^*(t)]^T$ is a column vector.

Now, substituting the following approximations

$$\begin{aligned}
\frac{\partial^{\alpha(\mathcal{S},t)}u}{\partial t^{\alpha(\mathcal{S},t)}} &= \left(D_t^{\alpha(\mathcal{S},t)}\varphi^T(t)\right)\mathbb{A}\left(\varphi(x)\otimes\varphi(y)\right),\\
\frac{\partial^{\beta(\mathcal{S},t)}u(\mathcal{S},t)}{\partial x^{\beta(\mathcal{S},t)}} &= \varphi^T(t)\mathbb{A}\left(D_x^{\beta(\mathcal{S},t)}\varphi(x)\otimes\varphi(y)\right),\\
\frac{\partial^{\gamma(\mathcal{S},t)}u(\mathcal{S},t)}{\partial y^{\gamma(\mathcal{S},t)}} &= \varphi^T(t)\mathbb{A}\left(\varphi(x)\otimes D_y^{\gamma(\mathcal{S},t)}\varphi(y)\right),\\
\frac{\partial^{\delta(\mathcal{S},t)}u(\mathcal{S},t)}{\partial x^{\delta(\mathcal{S},t)}} &= \varphi^T(t)\mathbb{A}\left(D_x^{\delta(\mathcal{S},t)}\varphi(x)\otimes\varphi(y)\right),\\
\frac{\partial^{\sigma(\mathcal{S},t)}u(\mathcal{S},t)}{\partial y^{\sigma(\mathcal{S},t)}} &= \varphi^T(t)\mathbb{A}\left(\varphi(x)\otimes D_y^{\sigma(\mathcal{S},t)}\varphi(y)\right),
\end{aligned}$$

in the Eq.(3.1), we get

$$\begin{aligned}
&\left(D_t^{\alpha(\mathcal{S},t)}\varphi^T(t)\right)\mathbb{A}\left(\varphi(x)\otimes\varphi(y)\right)\\
&\qquad= \upsilon\varphi^T(t)\mathbb{A}\left((D_x^{\beta(\mathcal{S},t)}\varphi(x)\otimes\varphi(y))\right) + \upsilon\varphi^T(t)\mathbb{A}\left((\varphi(x)\otimes D_y^{\gamma(\mathcal{S},t)}\varphi(y))\right)\\
&\qquad- \kappa\varphi^T(t)\mathbb{A}\varphi(x)\otimes\varphi(y)\varphi^T(t)\mathbb{A}\left(D_x^{\delta(\mathcal{S},t)}\varphi(x)\otimes\varphi(y)\right)\\
&\qquad- \kappa\varphi^T(t)\mathbb{A}\varphi(x)\otimes\varphi(y)\varphi^T(t)\mathbb{A}\left(\varphi(x)\otimes D_y^{\sigma(\mathcal{S},t)}\varphi(y)\right)\\
&\qquad+ \lambda\psi\left(\varphi^T(t)\mathbb{A}\varphi(x)\otimes\varphi(y)\right). \qquad (3.22)
\end{aligned}$$

From the conditions Eq. (3.21) and the Eq. (3.2), we get

$$\begin{aligned}
\varphi^T(0)\mathbb{A}(\varphi(x)\otimes\varphi(y)) =& f_1(x,y),\\
\varphi^T(t)\mathbb{A}(\varphi(0)\otimes\varphi(y)) =& f_2(y,t),\\
\varphi^T(t)\mathbb{A}(\varphi(1)\otimes\varphi(y)) =& f_3(y,t),\\
\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(0)) =& f_4(x,t),\\
\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(1)) =& f_5(x,t). \qquad (3.23)
\end{aligned}$$

Now, we collocate Eq. (3.22) and Eq. (3.23) at collocation points $x_i = y_i = t_i = \frac{2i-1}{2m+1}$ that produce $(m-1)\times(m-1)\times m$ equations, and $(m+1)^2+4m^2$ equations, respectively. This step produces a system of $(m+1)^3$ nonlinear algebraic equations. The solution of that non-linear system produces matrix $\mathbb{A}$. In this way, we can get a numerical solution of our consider fractional reaction advection diffusion equations of variable-order (3.1)-(3.2).

3.6 Convergence and Error Analysis

Theorem 3.1. Suppose that $\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))$ is the approximation of $u(x,y,t)$ by shifted Vieta-Lucas polynomials. If the function $u(x,y,t)$ has sixth-order continuous derivatives, then

$$\begin{aligned}
&\|c_{000}\| \leq L_{0,0,0},\\
&\|c_{i00}\| \leq \frac{L_{2,0,0}}{(i-1)^2},\ \|c_{0j0}\| \leq \frac{L_{0,2,0}}{(j-1)^2},\ \|c_{00k}\| \leq \frac{L_{0,0,2}}{(k-1)^2},\\
&\|c_{ij0}\| \leq \frac{L_{2,2,0}}{(i-1)^2(j-1)^2},\ \|c_{0jk}\| \leq \frac{L_{0,2,2}}{(j-1)^2(k-1)^2},\ \|c_{i0k}\| \leq \frac{L_{2,0,2}}{(i-1)^2(k-1)^2},\\
&\|c_{ijk}\| \leq \frac{L_{2,2,2}}{(i-1)^2(j-1)^2(k-1)^2},\quad for\ \ i,j,k>1,
\end{aligned}$$

where

$$\begin{aligned}
L_{0,0,0} &= max\left\{\|u(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,0,0} &= max\left\{\|u_{xx}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,2,0} &= max\left\{\|u_{yy}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,0,2} &= max\left\{\|u_{tt}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,2,0} &= max\left\{\|u_{xxyy}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,2,2} &= max\left\{\|u_{yytt}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,0,2} &= max\left\{\|u_{xxtt}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,2,2} &= max\left\{\|u_{xxyytt}(x,y,t)\| : (x,y,t)\in[0,1]\times[0,1]\times[0,1]\right\}.
\end{aligned}$$

Proof. From Eq. (3.15), the coefficient c_{000} is given by

$$c_{000} = \frac{1}{64\pi^3}\int_0^1\int_0^1\int_0^1 \frac{u(x,y,t)VL_0^*(x)VL_0^*(y)VL_0^*(t)}{\sqrt{x-x^2}\sqrt{y-y^2}\sqrt{t-t^2}}dxdydt. \tag{3.24}$$

Using the following variables

$$2x-1=\cos(\xi_1), 2y-1=\cos(\xi_2), 2t-1=\cos(\xi_3), \tag{3.25}$$

in Eq. (3.24), we obtain

$$c_{000}=\frac{8}{64\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}u\left(\frac{cos(\xi_1)+1}{2},\frac{\cos(\xi_2)+1}{2},\frac{\cos(\xi_3)+1}{2}\right)d\xi_1 d\xi_2 d\xi_3. \tag{3.26}$$

From Eq.(3.26), we have $\|c_{000}\|\leq L_{0,0,0}$.

From Eq. (3.15), the coefficients c_{00k} for $k>0$ are computed as

$$c_{00k}=\frac{1}{32\pi^3}\int_0^1\int_0^1\int_0^1\frac{u(x,y,t)VL_0^*(x)VL_0^*(y)VL_k^*(t)}{\sqrt{x-x^2}\sqrt{y-y^2}\sqrt{t-t^2}}dxdydt. \tag{3.27}$$

By using the Eq. (3.25) and the following property

$$VL_{r+1}^*(s)=\cos((r+1)s), \quad s=\frac{\cos(t)+1}{2}, \tag{3.28}$$

in Eq. (3.27), we get

$$c_{00k}=\frac{8}{32\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}u\left(\frac{cos(\xi_1)+1}{2},\frac{\cos(\xi_2)+1}{2},\frac{\cos(\xi_3)+1}{2}\right)\cos(k\xi_3)d\xi_1 d\xi_2 d\xi_3. \tag{3.29}$$

Now, integrating Eq. (3.29) twice with respect to ξ_3, we get

$$c_{00k}=\frac{1}{32\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}u_{\xi_3\xi_3}\left(\frac{cos(\xi_1)+1}{2},\frac{\cos(\xi_2)+1}{2},\frac{\cos(\xi_3)+1}{2}\right)\Omega_k(\xi_3)d\xi_1 d\xi_2 d\xi_3, \tag{3.30}$$

where

$$\Omega_k(\xi_3) = \left(\frac{\sin(k-1)\xi_3}{k(k-1)} - \frac{\sin(k+1)\xi_3}{k(k+1)}\right)\sin(\xi_3), \tag{3.31}$$

which implies $\|c_{00k}\| \leq \frac{L_{0,0,2}}{(k-1)^2}$.

Similarly one can conclude $\|c_{i00}\| \leq \frac{L_{2,0,0}}{(i-1)^2}$ and $\|c_{0j0}\| \leq \frac{L_{0,2,0}}{(j-1)^2}$.

Again considering Eq. (3.15) for the coefficient c_{ij0}, which gives

$$c_{ij0} = \frac{1}{16\pi^3}\int_0^1\int_0^1\int_0^1 \frac{u(x,y,t)VL_i^*(x)VL_j^*(y)VL_0^*(t)}{\sqrt{x-x^2}\sqrt{y-y^2}\sqrt{t-t^2}}dxdydt. \tag{3.32}$$

From Eq. (3.25), (3.28), (3.32), we have

$$c_{ij0} = \frac{8}{16\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u\left(\frac{cos(\xi_1)+1}{2},\frac{\cos(\xi_2)+1}{2},\frac{\cos(\xi_3)+1}{2}\right) \times \cos(i\xi_1)\cos(j\xi_2)d\xi_1 d\xi_2 d\xi_3. \tag{3.33}$$

After integrating (3.33) twice with respect to ξ_1 and ξ_2, we get

$$c_{ij0} = \frac{1}{128\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u_{\xi_1\xi_1\xi_2\xi_2}\left(\frac{cos(\xi_1)+1}{2},\frac{\cos(\xi_2)+1}{2},\frac{\cos(\xi_3)+1}{2}\right) \times \Omega_i(\xi_1)\Omega_j(\xi_2)d\xi_1 d\xi_2 d\xi_3. \tag{3.34}$$

Now from Eqs. (3.34) and (3.31), we have $\|c_{ij0}\| \leq \frac{L_{2,2,0}}{(i-1)^2(j-1)^2}$. Similiarly, it can be computed that $\|c_{0jk}\| \leq \frac{L_{0,2,2}}{(j-1)^2(k-1)^2}$ and $\|c_{i0k}\| \leq \frac{L_{2,0,2}}{(i-1)^2(k-1)^2}$. Moreover, for $i,j,k>1$, we have

$$c_{ijk} = \frac{8}{8\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u\left(\frac{cos(\xi_1)+1}{2},\frac{\cos(\xi_2)+1}{2},\frac{\cos(\xi_3)+1}{2}\right) \times \cos(i\xi_1)\cos(j\xi_2)\cos(k\xi_3)d\xi_1 d\xi_2 d\xi_3, \tag{3.35}$$

and applying the above process, we get

$$
\begin{aligned}
c_{ijk} = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u_{\xi_1\xi_2\xi_3}\left(\frac{cos(\xi_1)+1}{2}, \frac{\cos(\xi_2)+1}{2}, \frac{\cos(\xi_3)+1}{2}\right) \\
\times\, \Omega_i(\xi_1)\Omega_j(\xi_2)\Omega_k(\xi_3)d\xi_1 d\xi_2 d\xi_3,
\end{aligned} \tag{3.36}
$$

and consequently

$$
\|c_{ijk}\| \leq \frac{L_{2,2,2}}{(i-1)^2(j-1)^2(k-1)^2}. \tag{3.37}
$$

Theorem 2.2. Let $\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))$ be the approximation of $u(x,y,t)$ in the terms of shifted Vieta-Lucas polynomials. If the function $u(x,y,t)$ has continuous derivatives of sixth order, then

$$
\left\|u(x,y,t)-\varphi^T(t)\mathbb{A}(\varphi(x)\otimes\varphi(y))\right\|_{L^2} \leq \sqrt{G(i,j,k)},
$$

where

$$
\begin{aligned}
G(i,j,k) =& 32\pi^3\sum_{k=m+1}^{\infty}\frac{L^2_{0,0,2}}{(k-1)^4} + 32\pi^3\sum_{j=m+1}^{\infty}\frac{L^2_{0,2,0}}{(j-1)^4} + 32\pi^3\sum_{i=m+1}^{\infty}\frac{L^2_{2,0,0}}{(i-1)^4} \\
&+ 16\pi^3\sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty}\frac{L^2_{0,2,2}}{(j-1)^4(k-1)^4} + 16\pi^3\sum_{j=1}^{m}\sum_{k=m+1}^{\infty}\frac{L^2_{0,2,2}}{(j-1)^4(k-1)^4} \\
&+ 16\pi^3\sum_{i=m+1}^{\infty}\sum_{k=1}^{\infty}\frac{L^2_{2,0,2}}{(i-1)^4(k-1)^4} + 16\pi^3\sum_{i=1}^{m}\sum_{k=m+1}^{\infty}\frac{L^2_{2,0,2}}{(i-1)^4(k-1)^4} \\
&+ 16\pi^3\sum_{i=m+1}^{\infty}\sum_{j=1}^{\infty}\frac{L^2_{2,2,0}}{(i-1)^4(j-1)^4} + 16\pi^3\sum_{i=1}^{m}\sum_{j=m+1}^{\infty}\frac{L^2_{2,2,0}}{(i-1)^4(j-1)^4} \\
&+ 8\pi^3\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=m+1}^{\infty}\frac{L^2_{2,2,2}}{(i-1)^4(j-1)^4(k-1)^4} \\
&+ 8\pi^3\sum_{i=m+1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{L^2_{2,2,2}}{(i-1)^4(j-1)^4(k-1)^4}.
\end{aligned} \tag{3.38}
$$

Proof. Let us assume that

$$\mathcal{Q}(i,j,k) = \left\| u(x,y,t) - \varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y)) \right\|_{L^2}^2 . \tag{3.39}$$

So, from Eqs. (3.14) and (3.39), we get

$$\begin{aligned}\mathcal{Q}(i,j,k) = \int_0^1 \int_0^1 \int_0^1 (&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{ijk} VL_i^*(x)VL_j^*(y)VL_k^*(t) \\ &- \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} c_{ijk} VL_i^*(x)VL_j^*(y)VL_k^*(t))^2 \chi(x)\chi(y)\chi(t)dxdydt,\end{aligned}$$

$$\begin{aligned}= \int_0^1 \int_0^1 \int_0^1 (&\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=m+1}^{\infty} c_{ijk} VL_i^*(x)VL_j^*(y)VL_k^*(t) \\ &+ \sum_{i=0}^{m}\sum_{j=m+1}^{\infty}\sum_{k=0}^{\infty} c_{ijk} VL_i^*(x)VL_j^*(y)VL_k^*(t) \\ &+ \sum_{i=m+1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{ijk} VL_i^*(x)VL_j^*(y)VL_k^*(t))^2 \chi(x)\chi(y)\chi(t)dxdydt,\end{aligned}$$

$$\begin{aligned}&= \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=m+1}^{\infty} c_{ijk} \int_0^1 VL_i^{*^2}(x)\chi(x)dx \int_0^1 VL_j^{*^2}(y)\chi(y)dy \int_0^1 VL_k^{*^2}(t)\chi(t)dt \\ &+ \sum_{i=0}^{m}\sum_{j=m+1}^{\infty}\sum_{k=0}^{\infty} c_{ijk} \int_0^1 VL_i^{*^2}(x)\chi(x)dx \int_0^1 VL_j^{*^2}(y)\chi(y)dy \int_0^1 VL_k^{*^2}(t)\chi(t)dt \\ &+ \sum_{i=m+1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{ijk} \int_0^1 VL_i^{*^2}(x)\chi(x)dx \int_0^1 VL_j^{*^2}(y)\chi(y)dy \int_0^1 VL_k^{*^2}(t)\chi(t)dt.\end{aligned} \tag{3.40}$$

By using orthogonality condition of shifted Vieta-Lucas polynomials in Eq. (3.40), we get

$$\begin{aligned}\mathcal{Q}(i,j,k) =& 32\pi^3 \sum_{i=m+1}^{\infty} c_{i00} + 32\pi^3 \sum_{j=m+1}^{\infty} c_{0j0} + 32\pi^3 \sum_{k=m+1}^{\infty} c_{00k} \\ &+ 8\pi^3 \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=m+1}^{\infty} c_{ijk} + 8\pi^3 \sum_{i=m+1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{ijk} \\ &+ 16\pi^3 \sum_{i=m+1}^{\infty}\sum_{k=1}^{\infty} c_{i0k} + 16\pi^3 \sum_{i=1}^{m}\sum_{k=m+1}^{\infty} c_{i0k} \\ &+ 16\pi^3 \sum_{i=m+1}^{\infty}\sum_{j=1}^{\infty} c_{ij0} + 16\pi^3 \sum_{i=1}^{m}\sum_{j=m+1}^{\infty} c_{ij0} \\ &+ 16\pi^3 \sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty} c_{0jk} + 16\pi^3 \sum_{j=1}^{m}\sum_{k=m+1}^{\infty} c_{0jk}. \end{aligned} \tag{3.41}$$

Theorem 3.1 and Eq. (3.41) give rise to

$$\begin{aligned}\mathcal{Q}(i,j,k) \leq& 32\pi^3 \sum_{k=m+1}^{\infty} \frac{L_{0,0,2}^2}{(k-1)^4} + 32\pi^3 \sum_{j=m+1}^{\infty} \frac{L_{0,2,0}^2}{(j-1)^4} + 32\pi^3 \sum_{i=m+1}^{\infty} \frac{L_{2,0,0}^2}{(i-1)^4} \\ &+ 16\pi^3 \sum_{i=m+1}^{\infty}\sum_{k=1}^{\infty} \frac{L_{2,0,2}^2}{(i-1)^4(k-1)^4} + 16\pi^3 \sum_{i=1}^{m}\sum_{k=m+1}^{\infty} \frac{L_{2,0,2}^2}{(i-1)^4(k-1)^4} \\ &+ 16\pi^3 \sum_{i=m+1}^{\infty}\sum_{j=1}^{\infty} \frac{L_{2,2,0}^2}{(i-1)^4(j-1)^4} + 16\pi^3 \sum_{i=1}^{m}\sum_{j=m+1}^{\infty} \frac{L_{2,2,0}^2}{(i-1)^4(j-1)^4} \\ &+ 16\pi^3 \sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty} \frac{L_{0,2,2}^2}{(j-1)^4(k-1)^4} + 16\pi^3 \sum_{j=1}^{m}\sum_{k=m+1}^{\infty} \frac{L_{0,2,2}^2}{(j-1)^4(k-1)^4} \\ &+ 8\pi^3 \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=m+1}^{\infty} \frac{L_{2,2,2}^2}{(i-1)^4(j-1)^4(k-1)^4} \\ &+ 8\pi^3 \sum_{i=m+1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{L_{2,2,2}^2}{(i-1)^4(j-1)^4(k-1)^4}. \end{aligned} \tag{3.42}$$

Finally, square root of Eq. (3.42) gives the required result.

3.7 Examples and Discussion

In this section, we take the following examples to show the accuracy, efficiency, and applicability of the proposed approach for solving the fractional reaction advection diffusion equations of variable-order:

Example 3.7.1 Considering the following reaction advection diffusion equations [131] on a finite rectangular domain $[0,1] \times [0,1]$:

$$\frac{\partial^{\alpha(x,y,t)}u(x,y,t)}{\partial t^{\alpha(x,y,t)}} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} - \frac{\partial u(x,y,t)}{\partial x} - \frac{\partial u(x,y,t)}{\partial y} + \frac{2t^{2-\alpha(x,y,t)}}{\Gamma(3-\alpha(x,y,t))} + 2x + 2y - 4, \tag{3.43}$$

with the initial condition

$$u(x,y,0) = x^2 + y^2,$$

and Dirichlet boundary conditions in the form $u(0,y,t) = y^2+t^2, u(x,0,t) = x^2+t^2,$ $u(1,y,t) = y^2 + t^2 + 1,\ u(x,1,t) = x^2 + t^2 + 1, \quad t > 0.$

The analytical solution to the Example 3.7.1 is

$$u(x,y,t) = x^2 + y^2 + t^2. \tag{3.44}$$

For this problem the L_2 and L_∞ errors for $u(x,y,t)$ are tabulated in Table (3.1) and these are defined below

$$L_2 = \sqrt{\int_0^1 \int_0^1 \left\| u(x,y,t) - u'(x,y,t) \right\|^2 dxdy}, \tag{3.45}$$

TABLE 3.1: Obtained L_2 and L_∞ errors for example 3.7.1.

$\alpha(x,y,t)=0.5$				
t	L_∞ [131]	L_∞ (our method)	L_2 [131]	L_2(our method)
0.2	2.9999×10^{-5}	5.1742×10^{-6}	1.6997×10^{-4}	1.1596×10^{-6}
0.4	3.1223×10^{-5}	1.5776×10^{-5}	1.7686×10^{-4}	1.4466×10^{-6}
0.6	3.1766×10^{-5}	3.1310×10^{-5}	1.7975×10^{-4}	1.5938×10^{-6}
0.8	3.2083×10^{-5}	6.7867×10^{-5}	1.8144×10^{-4}	1.0583×10^{-6}
1.0	3.2296×10^{-5}	3.1815×10^{-5}	1.8256×10^{-4}	2.2192×10^{-5}
$\alpha(x,y,t)=0.8$				
t	L_∞ [131]	L_∞ (our method)	L_2 [131]	L_2 (our method)
0.2	2.0416×10^{-4}	5.2112×10^{-5}	1.1577×10^{-3}	2.1149×10^{-6}
0.4	2.1276×10^{-4}	3.5035×10^{-5}	1.2020×10^{-3}	9.9906×10^{-6}
0.6	2.1522×10^{-4}	5.8978×10^{-5}	1.2147×10^{-3}	1.1635×10^{-6}
0.8	2.1641×10^{-4}	3.7668×10^{-5}	1.2210×10^{-3}	7.8907×10^{-7}
1.0	2.1712×10^{-4}	4.7359×10^{-5}	1.2248×10^{-3}	1.0357×10^{-5}
$\alpha(x,y,t)$				
t	L_∞ [131]	L_∞ (our method)	L_2 [131]	L_2 (our method)
0.2	1.1668×10^{-4}	3.2174×10^{-5}	6.6238×10^{-4}	1.8182×10^{-6}
0.4	1.2258×10^{-4}	1.2396×10^{-5}	6.9398×10^{-4}	3.6736×10^{-6}
0.6	1.2612×10^{-4}	1.1349×10^{-5}	7.1434×10^{-4}	7.2942×10^{-6}
0.8	1.2974×10^{-4}	4.9678×10^{-5}	7.3611×10^{-4}	1.8193×10^{-6}
1.0	1.3394×10^{-4}	1.7013×10^{-5}	7.6200×10^{-4}	1.0468×10^{-5}

TABLE 3.2: Obtained L_2 and L_∞ errors for example 3.7.2.

$\alpha(x,y,t)$	L_2 error	L_∞ error	$\alpha(x,y,t)$	L_2 error	L_∞ error
$\frac{1-(xyt)^2}{10}$	4.3862E-16	1.1102E-16	$\frac{2^{xyt}-\cos(xyt)}{10}$	3.2125E-16	1.3322E-15
$\frac{2-\cos(xyt)}{20}$	3.4961E-16	1.1102E-15	$\frac{5-3^{xyt}+\sin(xyt)}{40}$	6.0721E-16	2.6645E-15
$\frac{3-\sin^2(xyt)}{10}$	4.1372E-16	2.2205E-16	$\frac{1-xyt+\sin(xyt)}{10}$	3.4059E-16	3.3306E-16
$\frac{4-e^x yt+xyt}{30}$	5.9485E-16	9.9921E-16	$\frac{1-(xyt)^3+\cos^2(xyt)}{10}$	6.5061E-16	1.2213E-15
$\frac{5-x+y^2-t^2}{50}$	2.7663E-16	5.5512E-15	$\frac{1+(xyt)^2-\sin^3(xyt)}{20}$	2.9433E-16	1.3323E-15

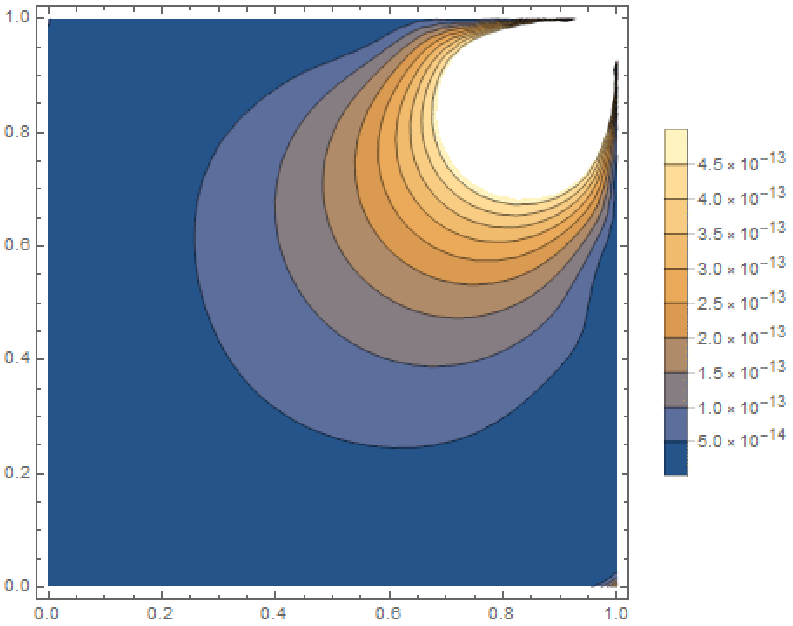

FIGURE 3.1: Absolute error at $\alpha(x, y, t)$=0.1 for example 3.7.2.

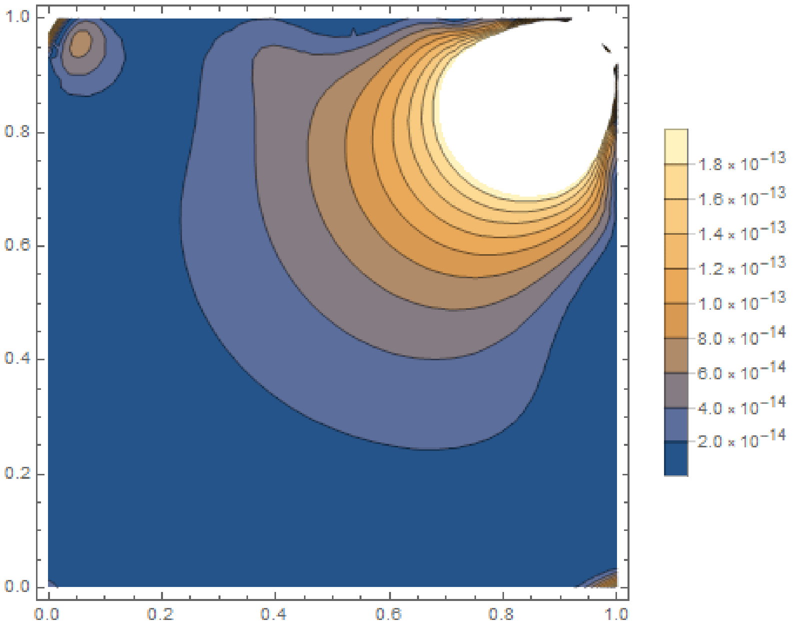

FIGURE 3.2: Absolute error at $\alpha(x, y, t)$=0.3 for example 3.7.2.

$$L_\infty = \max_{\{0<x<1\}} \max_{\{0<y<1\}} \left\| u(x, y, t) - u^{'}(x, y, t) \right\|. \tag{3.46}$$

where $u(x, y, t)$ and $u^{'}(x, y, t)$ are the approximate and exact solutions.

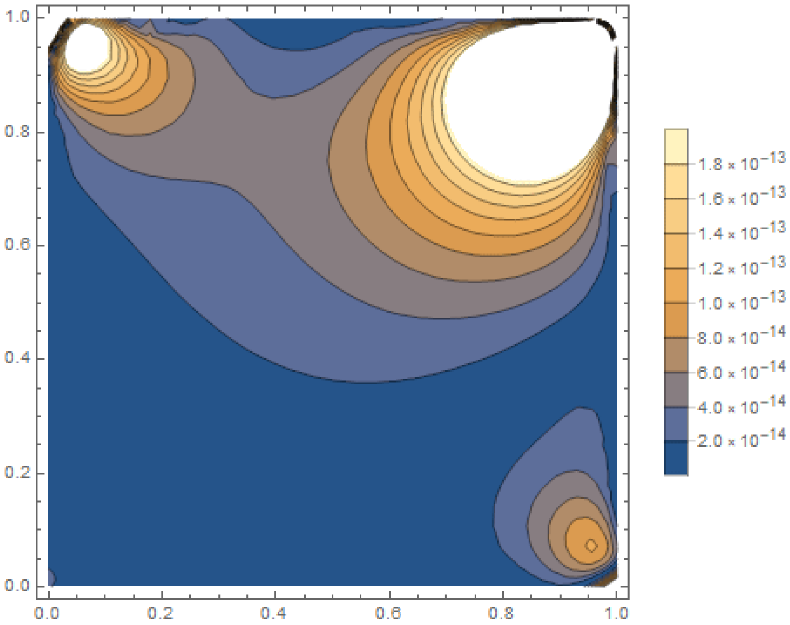

FIGURE 3.3: Absolute error at $\alpha(x, y, t)$=0.5 for example 3.7.2.

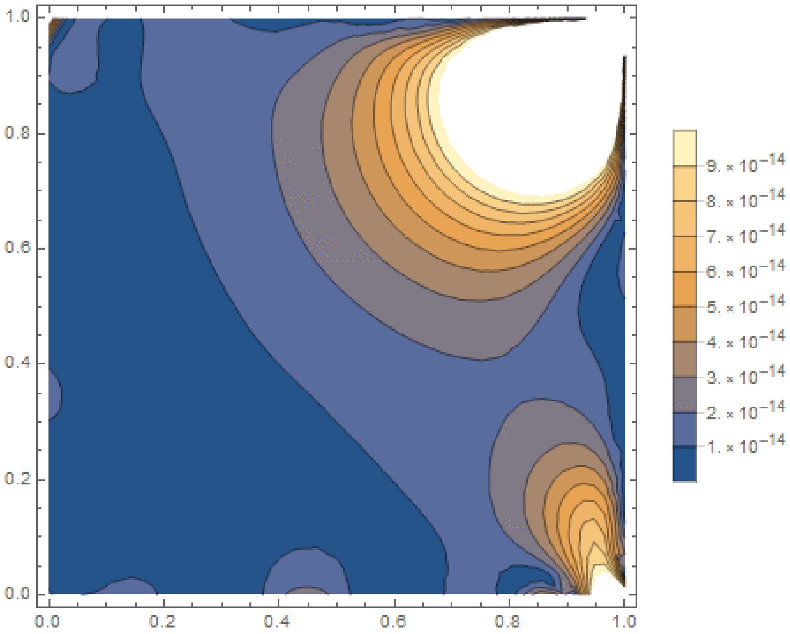

FIGURE 3.4: Absolute error at $\alpha(x, y, t)$=0.7 for example 3.7.2.

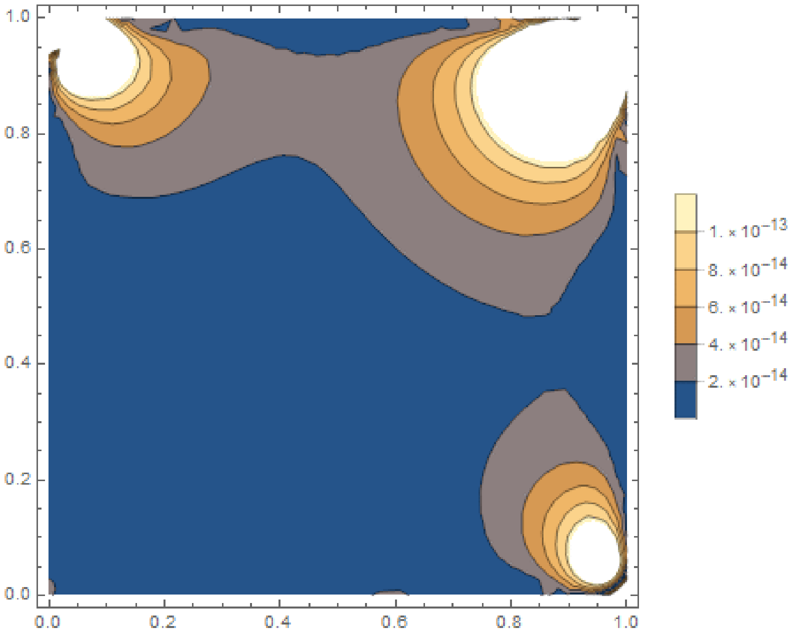

FIGURE 3.5: Absolute error at $\alpha(x,y,t)$=0.9 for example 3.7.2.

TABLE 3.3: The variation of L_2 and L_∞ errors with increasing value of degree of approximation for Example 3.7.3.

m	L_2 error	order of convergence	L_∞ error	order of convergence
4	3.58359E-5	-	4.85726E-5	-
8	2.85745E-6	2.15131	3.96945E-6	2.13039
13	3.85924E-7	2.26561	5.37584E-7	2.26251
20	4.83692E-8	2.56098	6.84736E-8	2.54108
28	7.86356E-9	2.81409	9.97537E-9	2.98403

Recently, Tayebi et al. [131] proposed a messless mothod to solve this problem. In Table (3.1), we present the numerical comparison of our result with the result of Tayebi et al. [131] for three choices of $\alpha(x,y,t)$=0.5, 0.8, and $0.8-0.1\cos(xt)\sin(x)-0.1\cos(yt)\sin(y)$. This comparison show that our results are more accurate than [131].

Example 3.7.2 Let us take the following diffusion equation with variable-order time fractional derivative:

$$\frac{\partial^{\alpha(x,y,t)}u(x,y,t)}{\partial t^{\alpha(x,y,t)}} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \frac{2x^2y^2t^{2-\alpha(x,y,t)}}{\Gamma(3-\alpha(x,y,t))} - 2x^2t^2 - 2y^2t^2, \tag{3.47}$$

with following conditions

$$u(x,y,0) = 0,\ u(0,y,t) = u(x,0,t) = 0,\ u(1,y,t) = y^2t^2,\ u(x,1,t) = x^2t^2. \tag{3.48}$$

for all $x \in [0,1], \quad y \in [0,1], \quad t \geq 0.$

It is found that the exact solution to the problem (3.47)-(3.48) is $u(x,y,t) = x^2y^2t^2$. This problem is solved at time $t = 0.5$ by utilizing the proposed algorithm for different value of $\alpha(x,y,t)$. Table (3.2) shows the L_2 and L_∞ errors for different types of functions $\alpha(x,y,t)$ which demonstrates the accuracy of our method. Figures (3.1)-(3.5) illustrates the characterization of absolute errors for $\alpha(x,y,t) = 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9.$ These figures also depict the accuracy of our approach.

Example 3.7.3 Consider the following non-linear fractional-order advection diffusion equations [132]:

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{x^3y^{1.4}}{\Gamma(3.9)}\frac{\partial^{1.9}u(x,y,t)}{\partial x^{1.9}} + \frac{x^{1.1}y^3}{\Gamma(3.6)}\frac{\partial^{1.6}u(x,y,t)}{\partial y^{1.6}} - (1+2x^{1.1}y^{1.4})e^{-t}x^{2.9}y^{2.6}, \tag{3.49}$$

subject to the initial condition

$$u(x,y,0) = x^{2.9}y^{2.6}, \tag{3.50}$$

and boundary conditions

$$u(0, y, t) = u(x, 0, t) = 0, \tag{3.51}$$

$$u(1, y, t) = e^{-t} y^{2.6}, \quad u(x, 1, t) = e^{-t} x^{2.9}. \tag{3.52}$$

It can be easily verify that the analytical solution of above problem (3.49)-(3.52) is given by,

$$u(x, y, t) = e^{-t} x^{2.9} y^{2.6}. \tag{3.53}$$

In the Table (3.2), we present calculated error and rate of convergence of the proposed method for Example 3.7.3 at $t = 0.5$. These obtained results show that the proposed operational matrix method is sufficiently near with the analytical solution.

Example 3.7.4 Consider the following space fractional order diffusion equation as

$$\frac{\partial u(x, y, t)}{\partial t} = \frac{\partial^{\beta} u(x, y, t)}{\partial x^{\beta}} + \frac{\partial^{\beta} u(x, y, t)}{\partial y^{\beta}} - q(x, y, t), \tag{3.54}$$

under the condition

$$\begin{aligned} &u(x, y, 0) = x^2 y^2, \\ &u(0, y, t) = 0, \\ &u(1, y, t) = e^{-t} y^2, \\ &u(x, 0, t) = 0, \\ &u(x, 1, t) = e^{-t} x^2. \end{aligned} \tag{3.55}$$

Table 3.4: The computational error and convergence order (CO) for example 3.7.4

$\alpha(x,y,t)$	m	L_∞ error	CO	L_2 error	CO
$0.55+0.2\sin(xyt)$	3	1.31797×10^{-2}	-	6.85639×10^{-3}	-
	6	2.33286×10^{-3}	1.54712	1.12119×10^{-3}	1.61789
	9	4.73607×10^{-4}	2.23519	2.51221×10^{-4}	2.09689
	12	9.88480×10^{-5}	2.96673	4.50153×10^{-5}	3.27661
	15	2.26724×10^{-5}	3.56989	9.99515×10^{-6}	3.62384
$\alpha(x,y,t)$	m	L_∞ error	CO	L_2 error	CO
$0.45+0.2\sin(xyt)$	3	1.32565×10^{-2}	-	6.90407×10^{-3}	-
	6	2.49937×10^{-3}	1.49071	1.24364×10^{-3}	1.53147
	9	4.62574×10^{-4}	2.36488	2.40196×10^{-4}	2.30510
	12	7.92367×10^{-5}	3.36244	3.03149×10^{-5}	3.94455
	15	1.26462×10^{-5}	4.41892	5.60496×10^{-6}	4.06471
$\alpha(x,y,t)$	m	L_∞ error	CO	L_2 error	CO
$0.35+0.2\sin(xyt)$	3	1.32990×10^{-2}	-	6.93080×10^{-3}	-
	6	2.56670×10^{-3}	1.46982	1.29467×10^{-3}	1.49899
	9	4.59114×10^{-4}	2.41267	2.36583×10^{-4}	2.38272
	12	6.79059×10^{-6}	3.64222	5.10434×10^{-5}	2.92270
	15	7.17396×10^{-6}	5.41243	4.83372×10^{-6}	5.67585

The exact solution of this problem is $u(x,y,t)=e^{-t}x^2y^2$ and the force function is calculated from (3.54)-(3.55).

The computational error and convergence rated ofr this example is presented in table 3.4. Table 3.4 is analysing the L_2 and L_∞ errors with increasing order of approximate polynomials for different space fractional which confirm the high accuracy of the scheme.

3.8 Conclusion

In our work, an operational matrix of variable-order fractional derivative is derived with the aid of Vieta-Lucas polynomials for the two-dimensions problems. Based on this operational matrix of Vieta-Lucas polynomials and collocation approach, a numerical scheme is discussed to solve the two-dimension nonlinear problem involving space-time fractional reaction advection diffusion equations of variable-order in the Caputo sense. Convergence and error analysis of the proposed algorithm are also discussed analytically and it is found that the error associated with the obtained approximate solution rapidly tends to zero as number of polynomials or degree of polynomials (Vieta-Lucas polynomials) enhances. The validity and applicability of the proposed algorithms are demonstrated through three examples, and it is found that the proposed algorithm is efficient and sufficiently accurate.

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