Chapter 1

Introduction

1.1 Fractional Calculus

Fractional Calculus (FC) is a branch of mathematics that studies and applies integrals and derivatives of arbitrary order. Thus, fractional Calculus may be considered as an extension of the classical calculus that deals with integrals and derivatives of any real or complex order [9, 10]. FC first appeared in the communication between G.A. de L'Hôpital and G.W. Leibnitz, at the end of the 17^{th} Century, in 1695. But, in 1819, Lacorix was the first to explain the fractional derivative by taking the n^{th} derivative of the power function $y = t^k$, and using the Gamma function, it was found

$$\frac{d^n y}{dt^n} = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} t^{k-n}, \qquad k \geq n.$$

Further, for $n = 1/2$, $k = 1$,

$$\frac{d^{1/2} y}{dt^{1/2}} = \frac{2\sqrt{t}}{\sqrt{\pi}}.$$

After that, over the years, several mathematicians have contributed to the development of FC, including Grünwald, Letnikov, Riemann, Liouville, Fourier, Abel, and many others.

In 1823, Abel [11] was the first to study the application of FC in solving Tautochrone problem. Since then, several books and papers on fractional calculus have been written by mathematicians [9, 10, 12, 13, 14, 15]. The field of fractional calculus was being developed over three centuries as a pure mathematical theory. But, in the last few decades, applications of FC were found not only in mathematics but also in other fields, such as bio-engineering [16], viscoelasticity [17, 18], dynamics of interfaces among nanoparticles and substrates [19], control theory [20], physics [21], economics [22] and many others can be seen in the literature [23, 24, 25, 26, 27, 28].

1.1.1 Definitions and Properties of Fractional Integrals and Derivatives

Here, we discuss only the essential definitions and properties of fractional integrals and derivatives. The detailed idea about FC can be seen in [9, 10].

Definition 1.1. [9] Let $[c, d]$ be a finite interval on $\mathbb{R}$. The Riemann-Liouville (R-L) fractional integrals $I^{\alpha}_{c^+}y$ (left) and $I^{\alpha}_{d^-}y$ (right) for function y of order $\alpha > 0$ are defined by

$$({}_cI^{\alpha}_t\, y)(t) = (I^{\alpha}_{c^+}y)(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-\tau)^{\alpha-1} y(\tau)\, d\tau, \tag{1.1}$$

$$({}_tI^{\alpha}_d\, y)(t) = (I^{\alpha}_{d^-}y)(t) = \frac{1}{\Gamma(\alpha)} \int_t^d (\tau-t)^{\alpha-1} y(\tau)\, d\tau. \tag{1.2}$$

Definition 1.2. [9] The Riemann-Liouville fractional derivatives (RLFDs) $D^{\alpha}_{c^+}y$ (left) and $D^{\alpha}_{d^-}y$ (right) of $y(t)$ of order $\alpha \in (0,1)$ are defined by

$$({}_cD^{\alpha}_t\, y)(t) = (D^{\alpha}_{c^+}y)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_c^t (t-\tau)^{-\alpha}y(\tau)\,d\tau, \tag{1.3}$$

$$({}_tD^{\alpha}_d\, y)(t) = (D^{\alpha}_{d^-}y)(t) = \frac{(-1)}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_t^d (\tau-t)^{-\alpha}y(\tau)\,d\tau. \tag{1.4}$$

Definition 1.3. [9] The Caputo fractional derivative (CFD) ${}^CD^{\alpha}_{c^+}y$ (left) and ${}^CD^{\alpha}_{d^-}y$ (right) of $y(t)$ of order $\alpha \in (0,1)$ are defined by

$$\left({}^C{}_cD^{\alpha}_t y\right)(t) = \left({}^CD^{\alpha}_{c^+}y\right)(t) = \ D^{\alpha}_{c^+}[y(t)-y(c)], \tag{1.5}$$

$$\left({}^C{}_tD^{\alpha}_d y\right)(t) = \left({}^CD^{\alpha}_{d^-}y\right)(t) = \ D^{\alpha}_{d^-}[y(t)-y(d)]. \tag{1.6}$$

If $y(t) \in AC[c,d]$, then CFD of $y(t)$ can be defined as

$$\left({}^CD^{\alpha}_{c^+}y\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int\limits_c^t (t-\tau)^{-\alpha}g'(\tau)\,d\tau, \tag{1.7}$$

$$\left({}^CD^{\alpha}_{d^-}y\right)(t) = \frac{(-1)}{\Gamma(1-\alpha)}\int\limits_t^d (\tau-t)^{-\alpha}g'(\tau)\,d\tau. \tag{1.8}$$

Lemma 1.4. [9] *For $0<\alpha\le 1$ and $y(t)\in AC[c,d]$, then*

$$\left(I^{\alpha}_{c^+}\ {}^CD^{\alpha}_{c^+}y\right)(t) = y(t)-y(c), \tag{1.9}$$

$$\left(I^{\alpha}_{d^-}\ {}^CD^{\alpha}_{d^-}y\right)(t) = y(t)-y(d). \tag{1.10}$$

Lemma 1.5. [9, 29] *For $\alpha>0$, $p\ge 1$, $q\ge 1$, and $1/p+1/q\le 1+\alpha$, then following holds.*

(i) Let $y(t) \in L^p(c,d)$ *and* $h(t) \in L^q(c,d)$*. Then*

$$\int_c^d y(t)(I_{c+}^{\alpha} h)(t)\,dt = \int_c^d h(t)(I_{d-}^{\alpha} y)(t)\,dt. \tag{1.11}$$

(ii) Let $y(t) \in I_{d-}^{\alpha}(L^p)$ *and* $h(t) \in I_{c+}^{\alpha}(L^q)$*. Then*

$$\int_c^d y(t)\left({}^{C}D_{c+}^{\alpha} h\right)(t)\,dt = \int_c^d h(t)(D_{d-}^{\alpha} y)(t)\,dt. \tag{1.12}$$

(iii) Let $y(t) \in I_{c+}^{\alpha}(L^p)$ *and* $h(t) \in I_{d-}^{\alpha}(L^q)$*. Then*

$$\int_c^d y(t)\left({}^{C}D_{d-}^{\alpha} h\right)(t)\,dt = \int_c^d h(t)(D_{a+}^{\alpha} y)(t)\,dt. \tag{1.13}$$

Lemma 1.6. [9] *For* $\alpha > 0$*, the fractional integral operators* $I_{c+}^{\alpha} y$ *(left) and* $I_{d-}^{\alpha} y$ *(right) are bounded in* $L^p(c,d)$ $(1 \leq p \leq \infty)$*.*

$$\|I_{c+}^{\alpha} y\|_p \leq K_{\alpha}\|y\|_p, \quad \|I_{d-}^{\alpha} y\|_p \leq K_{\alpha}\|y\|_p, \quad \left(K_{\alpha} = \frac{(d-c)^{\alpha}}{\Gamma(\alpha+1)}\right). \tag{1.14}$$

1.1.2 Generalized Fractional Integral and Derivatives

Here, we give some definitions of generalized fractional integral and derivatives [8].

Definition 1.7. For $\alpha > 0$, the left generalized fractional integral of function $y(t)$ is defined as

$$({}_{c}I_{t;[z;\omega]}^{\alpha} y)(t) = \frac{[\omega(t)]^{-1}}{\Gamma(\alpha)} \int_c^t \frac{\omega(\tau) z'(\tau) y(\tau))}{(z(t) - z(\tau))^{1-\alpha}}\,d\tau, \tag{1.15}$$

where $z(t) > 0$ and $\omega(t) \neq 0$ are scale and weight functions respectively.

Definition 1.8. For integer order $n \geq 1$, the left generalized derivative of a function $y(t)$ with respect to weight function $\omega(t)$ is defined as

$$(D^n_{[z;\omega;L]}y)(t) = [\omega(t)]^{-1}\left[\left(\frac{1}{z'(t)}D_t\right)^n(\omega(t)y(t))\right], \tag{1.16}$$

where D_t is the integer-order derivative with respect to t.

Definition 1.9. For $\alpha > 0$, the left generalized fractional derivative (GFD) of a function $y(t)$ is defined as

$$({}_cD^{\alpha}_{t;[z;\omega,1]}y)(t) = (D^n_{[z;\omega;L]}\ {}_aI^{n-\alpha}_{t;[z;\omega]}y)(t), \tag{1.17}$$

where $n \in \mathbb{Z}^+$, and $n-1 < \alpha < n$. The Eq. (1.17) is generalized from the R-L type fractional derivative and thus known as generalized RLFD of type 1. The weight and scale functions $\omega(t)$ and $z(t)$, respectively, are chosen suitably to make the integrals and derivatives finite. If we choose $z(t) = t$ and $\omega(t) = 1$, then Eq. (1.17) reduces to the left RLFD. For $n = 1$, the new GFD of order $\alpha \in (0,1)$ in Eq. (1.17) becomes

$$({}_cD^{\alpha}_{t;[z;\omega,1]}y)(t) = \frac{[\omega(t)]^{-1}}{\Gamma(1-\alpha)\,z'(t)}\frac{d}{dt}\int_c^t \frac{\omega(\tau)z'(\tau)y(\tau))}{(z(t)-z(\tau))^{\alpha}}\,d\tau. \tag{1.18}$$

1.2 Fractional Variational Problems

Among several fractional order systems, fractional calculus of variation (FCV) has achieved much attention in recent years. The field of calculus of variation (CV) has a long history and importance in various disciplines. Naturally, functional optimization problems occur in engineering and sciences where optimization of fuctionals, like, Lagrangian, strain, potential and total energy etc are studied. It is stated that the classical variational problem was considered by Leonhard Euler as follows.

$$J(y) = \int_c^d G(t, y(t), y'(t))\, dt, \tag{1.19}$$

with BCs

$$y(c) = y_c, \qquad\qquad y(d) = y_d, \tag{1.20}$$

where y_c, y_d are fixed real numbers, the Lagrangian G is continuously differentiable, and $y(t)$ is differentiable for all $t \in [c, d]$ such that $y'(t)$ exist and continuous. In 1750, for solving variational problems, Swiss mathematician Leonhard Euler and Italian mathematician Joseph-Louis Lagrange discovered an equation called the Euler–Lagrange equation, which must be satisfied by every functional optimizer. For a broad investigation of CV theory, we refer [30].

Although, the traditional calculus of variation involves integer order derivatives. However, recently, it has been seen that the fractional derivative term was considered in functionals to obtain nonconservative term in the desired differential equation. To deal with these types of problems, fractional calculus of variation has been investigated. In fractional calculus, the fractional variational theory has been found as an emerging area. A variational problem containing one or more fractional derivative term in either objective functional or constraints, is known as fractional variational problem (FVP). The FVP can be defined in the following form [31]

$$J(y) = \int_c^d G(t, y(t), {}_cD_t^{\alpha} y(t))\, dt, \tag{1.21}$$

with BCs

$$y(c) = y_c, \qquad\qquad y(d) = y_d, \tag{1.22}$$

where $\alpha \in (0,1)$ and the fractional derivative ${}_cD_t^\alpha$ is the R-L or Caputo fractional derivative. The Lagrangian G is continuously differentiable, and $y(t)$ is absolutely continuous for all $t \in [c,d]$ such that ${}_cD_t^\alpha y(t)$ exist and continuous. For solving the FVPs, in 2002, Agrawal [31] generalized the Euler–Lagrange equation for RLFDs.

1.3 Fractional Sturm-Liouville Problems

The well-known Sturm-Liouville problems evolved in mathematics almost two centuries ago. It is defined as the second order ordinary differential equation with some boundary conditions (BCs) as follows:

$$-\frac{d}{dt}\left(\psi(t)\frac{dy}{dt}\right) + \xi(t)y = \lambda\omega(t)y, \tag{1.23}$$

BCs:

$$a_1y(c) + a_2y'(c) = 0, \qquad b_1y(d) + b_2y'(d) = 0, \tag{1.24}$$

where $t \in \Theta = [c,d]$, $\psi(t) > 0, \psi'(t), \xi(t)$, and $\omega(t)$ are in $C(\Theta)$. Further, $a_1^2 + a_2^2 \neq 0$ and $b_1^2 + b_2^2 \neq 0$. For suitable BCs, and specific values of the parameter λ the non-trivial solutions of Eq. (1.23) exists. The parameter λ is known as the eigenvalue, and corresponding solutions are known as eigenfunctions. For detailed study of classical SLPs, we refer to [32, 33, 34, 35].

In recent years, many researchers have worked on certain generalizations of the SLPs. Mainly they are interested in the integer order SLPs. However, SLPs of fractional order have become an interesting topic for researchers in recent years. By replacing the integer-order derivatives in Eq. (1.23) with fractional derivatives, the FSLP is obtained [36, 37, 38]. Now, we define a kind of regular FSLP [39]

$$ {}_tD_1^{\alpha}\left(\psi(t){}_0^CD_t^{\alpha}y(t)\right) + \xi(t)y(t) = \lambda\omega(t)y(t), \quad t \in [c,d] \tag{1.25} $$

with BCs

$$ y(c) = 0, \qquad\qquad y(d) = 0, \tag{1.26} $$

where $\psi(t) > 0, \psi'(t), \xi(t)$, and $\omega(t)$ are in $C[c,d]$.

1.4 Motivation

Fractional calculus has shown its importance in different areas of science, engineering, and mathematics. The concept of fractional calculus is under development until recently. In the last few years, significant progress are observed in this field. In FC, fractional derivatives and integrals play a vital role in formulating a mathematical model for studying the model's non-local theory and memory effects. Such models [40, 41] present valuable understandings of phenomena like anomalous diffusion and non-Markovian processes. Consequently, FC contributes a more comprehensive understanding of real-world dynamics across engineering, physics, and mathematics. There are kinds of literature on the application of FDs and integrals; here, we cite some of them [21, 42, 43, 44].

Recent studies show that numerous physical systems can be modeled more accurately by utilizing formulations involving fractional derivatives. In recent years, FCVs has gained significant attention among various fractional order systems. The concept of the integer order CVs are old and well established. By incorporating fractional derivatives into traditional CVs, this framework facilitates a more precise depiction of complex systems governed by fractional dynamics. Consequently, it offers a more profound comprehension of their behaviour, especially in non-conservative systems. Riewe [1] began the area of FCV and presented the concept of Lagrangian, Hamiltonian for the non-conservative system by using a fractional derivative. Later, in 2001, Agrawal [45] presented a fractional derivative model for a damped system. In [46], Klimek introduced the symmetric fractional derivative and derived the Euler-Lagrange equation for models depending on the symmetric fractional derivative. Later, in 2002, Agrawal [31] proposed an Euler-Lagrange equation for FVPs. The solution to the classical variational problems can be found analytically using Euler-Lagrange equation, but the FVPs are difficult to solve using analytical methods. Therefore, numerical methods play an essential role in solving FVPs. In [47], author presented a finite element method for FVPs. Further, Agrawal obtained the solution in terms of power series for the Isoperimetric constraint FVP, and FDEs in [48]. Till now, an enormous amount of literature has been published on FVPs, and such work can be found in [49].

The principles of the fractional calculus of variations are used as a key point for developing the theories of FSLPs. In [50], authors established the fundamental theory of FSLPs by applying the methods of fractional variational analysis and investigated the properties of FSLPs. Also, fractional variational methods have been used to solve FSLPs [50, 51, 52].

In this thesis, we develop a numerical method for solving the FVPs with several dependent variables and discuss three numerical schemes for generalized isoperimetric constraint fractional variational problems (GICFVPs) described using the generalized fractional derivative (GFD). Also, using the fractional variational theory, we establish the essential properties of FSLPs for higher order fractional derivatives and in higher dimensions.

1.5 Literature Review

A review of past and current work on FVPs, FSLPs and their applications is presented in this section.

1.5.1 Literature Review on Fractional Variational Problem

In the 19th century, the initial link between fractional calculus and the calculus of variations emerged through the work of Niels Abel. In 1823, Abel utilized fractional calculus to solve an integral equation related to an extension of the tautochrone problem [11]. However, the calculus of variations with fractional derivatives was born at the end of 20th century with the contribution of Fred Riewe. He derived a generalized version of the Euler–Lagrange equations for problems of the calculus of variations with fractional derivatives, when investigating non-conservative Lagrangian and Hamiltonian mechanics [1, 2]. In 2001, Agrawal [45] presented a fractional derivative model for a damped system. In [31], Agrawal presented the necessary conditions of optimization for FVPs with respect to Riemann-Liouville derivatives. In [46], Klimek introduced the symmetric fractional derivative and derived the Euler-Lagrange equation for models depending on the symmetric fractional derivative. Further in 2002, Klimek [53] derived the stationarity-conservation laws by using linear fractional differential equations(FDEs) with variable coefficients. In [54], the author discussed some complex FVPs containing infinite integrals, isoperimetric constraints, or holonomic constraints. Agrawal [55] discussed three new fractional operators and showed that many fractional derivatives like Riemann-Liouville, Caputo, etc., can be written in form of these operators, and they are obtained by considering a special kernel. In [56], Yousefi et al. presented a necessary condition for the FVPs with completely free boundary conditions having an extremum. We can see the brief history of fractional calculus of variations in [3, 6, 7].

FVP with integral constraint is called an isoperimetric constraint fractional variational problem (ICFVP). In recent years, several researchers have broadly discussed ICVFPs due to its importance in control theory. In [4], author derived the transversality conditions for FVPs and presented an application of the formulations. Almeida et al. [57] studied the isoperimetric problem of FCV with left and right RLFD and derived the optimal conditions for FVPs and ICFVPs.

Generally, it's difficult to find the exact solution of the fractional variational problem and to move out this difficulty, investigations of the numerical methods become necessary. In 2008, Agrawal [47] proposed a finite element method for FVPs. He obtained the solution in terms of power series for the ICFVP, and FDEs in [48]. In [58], author studied the fractional oscillator problem and derived the solution on finite time interval using Mellin transform and fixed point theorem. Agrawal et al. [59] discussed the numerical method for the class of parametric problem of FCV using finite element approximation and derived the fractional orthogonal functions. In 2013, Wang et al. [60] developed a numerical approach for FVPs with indefinite integrals defined in terms of the Caputo fractional derivative, and discussed the fractional variational error. In [61], author discussed an approximate method based on Legendre polynomial for solving FVPs along with several dependent variables, several-order fractional derivatives, and a set of boundary conditions. In 2013, Zayernouri et al. [62] investigated and established the Jacobi poly-fractonomials. Later, in 2014, Zayernouri et al. [63] proposed a spectral element method for fractional ODEs using Jacobi poly-fractonomials which leads to exponentially fast decay of the error. Further, Mao et al. [64] developed a highly accurate Rayleigh-Ritz method for solving FVPs using Jacobi poly-fractonomials as a basis, and also demonstrated that the approximation error decreases exponentially. In 2015, Pandey et al. [65] developed a numerical scheme for generalized isoperimetric constraint fractional variational problems defined in terms of A-operator. Later, in 2015, Pandey et al. [66] presented a comparative study of four different approximation methods for solving isoperimetric constraint FVPs in terms of the A-operator. In 2018, Zaky et al. [67] developed a family of orthogonal systems of fractional functions based on Jacobi polynomials through a fractional coordinate transform, and introduced new types of fractional Jacobi-Gauss-type interpolation for FVPs. Ordokhani et al. [68] presented a Rayleigh-Ritz method for solving FVPs using Müntz–Legendre orthonormal polynomials as a basis function, and also estimated the error in the sense of Sobolev norms. In 2018, Ezz-Eldien et al. [69] proposed a new numerical method using shifted Chebyshev polynomials to approximate the solutions of FVPs containing indefinite integrals and Riemann-Liouville fractional integral. Further, Ezz-Eldien et al. [70] developed a direct method based on shifted Legendre orthonormal polynomials to approximate the isoperimetric FVPs. Dehghan et al. [71] investigated the Rayleigh-Ritz method based on modified Jacobi polynomials to solve the FVPs, and fractional optimal control problems. In 2019, Singh et al. [72] presented a Ritz method for solving the non-linear FVPs by using Jacobi polynomials. In 2020, Dehestani et. al [73] discussed a new modified Bessel wavelet method for solving the FVPs, and also investigated the error analysis of the discussed method. To better understand FVPs, we can find a list of references in [49, 5, 74, 75, 76].

1.5.2 Literature Review on Fractional Sturm-Liouville Problem

Two mathematicians, Sturm and Liouville studied the second-order linear homogeneous differential equation (1.23) with specific boundary conditions over two centuries ago. Such problems are known as Sturm–Liouville problems (SLPs). The field of SLP has demonstrated its significant importance across mathematics, science, and engineering, with extensive applications [32, 34]. There have been numerous literatures on SLP [77, 78, 79], as well as a few books on the application of Sturm-Liouville theory [35, 80, 81, 82].

Here, our focus is the FSLPs, *i.e.* SLPs defined in terms of the fractional derivatives (FDs). The FSLPs have become an interesting topic for researchers in recent years. By replacing the integer-order derivatives in Eq. (1.23) with fractional derivatives, different type of FSLPs [36, 37, 38] are obtained. The emerging interest of several mathematicians in FSLPs is because of the orthogonal eigenfunctions of FSLP, which are used to solve fractional partial differential equations [83]. In 2012, Klimek et al. [84] defined a regular FSLP and studied the properties of the eigenvalues and eigenfunctions of the problem. Further, in [85], Klimek and Agrawal introduced two classes of FSLPs, *i.e.* regular and singular FSLP, and defined the fractional Legendre equation (FLE) a type of singular FSLP and discussed its solution. In [29], the authors investigated the fundamental properties of the Sturm-Liouville theory for the FSLP defined using Riemann–Liouville, Caputo, or Liouville fractional operators. Zayernouri et al. [62] discussed two kinds of regular and singular FSLPs and derived analytical eigensolutions as non-polynomial functions called Jacobi polyfractonomials. In 2014, Klimek et al. [50] proposed a variational approach for solving FSLPs of order $\alpha \in (1/2, 1)$ and proved that there exist a denumerable set of eigenvalues and demonstrated that the eigenfunctions corresponding distinct eigenvalues are orthogonal to each other. In [52], authors presented a numerical algorithm to solve the higher-order FSLPs. Klimek et al. [86] solved the space-time fractional diffusion equations on a finite domain using the theorem of the existence of the solutions to the FSLP and the method of separating variables. In 2020, Pandey et al. [39] extended the theory of Klimek et al. [50] and proved that the countable eigenvalues and corresponding eigenfunctions exist for $0 < \alpha < 1$. In [87], the authors showed that FSLPs have countable real eigenvalues with finite multiplicity and obtained the lower eigenvalue bound. In [88, 89], Khosravian et al. investigated theory and applications of FSLPs for unbounded domains. In [90, 91], authors studied a discontinuous FSLP with transmission conditions and proved that the main problem's eigenvalues and corresponding eigenfunctions are same as those of the constructed operator in Hilbert spaces. The authors in [92] studied conformable FSLP, constructed a Green function for the problem, and gave the eigenfunction expansions. In 2020, Dehghan et al. [93] considered the Dirichlet type of FSLP and showed the existence of finite number of real eigenvalues and infinitely many complex eigenvalues for $1/2 < \alpha < 1$. In [94] authors studied the tempered fractional Sturm-Liouville problems (TFSLPs) and developed a Petrov—Galerkin spectral method using tempered Jacobi poly-fractonomials for solving tempered fractional differential equations. In 2021, Yadav et al. [95] presented a finite difference method to solve a regular TFSLPs and obtained the weak solution of tempered fractional diffusion equation. In [96], authors showed the application of the fractional Sturm-Liouville theory to a fractional Sturm—Liouville telegraph equation. Later in 2023, Pandey et al. [97] proved the Sturm's theorems for generalized FSLPs.

In [50], authors established the fundamental theory of FSLPs by applying the methods of fractional variational analysis and investigated the applications of FCVs for FSLPs. Further, the variational approach is found useful for solving FSLPs [50, 39, 98, 51, 99, 100]. Later, in 2021, Ferreira et al. [101], presented the fractional variational approach for solving the higher dimension FSLP and proved the main result of the existence of eigenvalues (EVs) and corresponding eigenfunctions (EFs) for fractional order $\alpha \in (1/2, 1]$. Moreover, they also showed that the eigenfunctions are orthogonal and the eigenvalues are real and simple.

1.6 Thesis Objectives

In this thesis, we study the approximation method for solving FVPs and FSLPs defined in terms of Caputo, Riemann-Liouville and generalized fractional derivatives. Many research works have been carried out on FVPs and FSLPs described using numerous fractional derivatives, but there is still room for contribution to these problems. This thesis aims to develop the numerical method for solving FVPs with several dependent variables and three numerical schemes are formulated for generalized isoperimetric constraint fractional variational problems (GICFVPs) described using the generalized fractional derivative (GFD). Furthermore, using a fractional variational approach, we extend the theories of FSLPs for higher-order fractional derivatives and in higher dimensions.

Here are the thesis's pointwise objectives:

(i) To develop a numerical method for solving FVPs with several dependent variables.

(ii) To develop numerical schemes for solving quadratic type ICFVPs defined using generalized RLFD.

(iii) To study a variational and numerical approximation methods for higher-order FSLPs.

(iv) To investigate the essential properties of higher dimensions of FSLPs using variational principles.

1.7 Outline of the Thesis

The structure of the thesis is outlined as follows:

Chapter 1 briefly introduces the thesis, including the definitions and important properties of fractional integrals and derivatives. Further, it contains literature reviews on FC, FVP, and FSLP, where both historical and contemporary research in these areas are examined and discussed.

Chapter 2 discusses a numerical method for solving the FVP with several dependent variables in terms of CFD and a set of homogeneous initial conditions. Using Jacobi poly-fractonomial, we approximate the solution of FVPs. After that, by Ritz method, the considered FVP is reduced to a system of algebraic equations. Furthermore, by resolving this set of algebraic equations, an approximate solution for the FVP is achieved. Later, we extend the method for non homogeneous initial conditions. The convergence and error analysis are also discussed. Finally, we conduct illustrative examples to demonstrate the validity and relevance of the proposed method.

In Chapter 3, we consider the isoperimetric constrained FVP. Here, we develop three numerical schemes, *i.e.* linear, quadratic and quadratic linear, for solving the generalized isoperimetric constrained FVP (GICFVP) defined using the GFD wih fixed BCs. In numerical methods, the solution is estimated on discretized subdomains of the domain by using linear and quadratic polynomial approximations in the first and second schemes, respectively. In the last scheme, the linear and quadratic functions are used together to approximate the solution. After approximating the solution and using the Lagrange multiplier technique, the minimization problem reduces to the eigenvalues problem. The eigenfunction corresponding to the minimum eigenvalue minimizes the considered cost functional. Further, we also discuss the convergence and error analysis of the schemes in detail. At the last, we present some numerical results varying the scale and weight functions. Finally, the obtained result using different schemes are compared .

In chapter 4, we study the higher-order FSLP defined in terms of right R-L and left Caputo fractional derivatives for $\alpha \in (0, 1)$. Using variational technique, we demonstrate that the FSLP has a countable set of eigenvalues and corresponding unique eigenfunctions. Moreover, We present two findings indicating that the eigenfunctions associated with different eigenvalues exhibit orthogonality, and the smallest (first) eigenvalue serves as the minimizer of the functional. Also, we present a numerical method for FSLP and validate the theoretical results. Further, the analytical convergence and the absolute error of the method are analyzed.

Chapter 5 is the generalization of chapter 4 in higher dimension. Here, we consider the N-dimensional FSLP of fractional order $(0, 1]$. The FSLP is defined using a fractional version of the gradient operator in terms of left and right CFDs. It is also demonstrated that FSLP possesses a countable set of eigenvalues, each corresponding to unique orthogonal solutions of the problem. Eventually, we use the fractional Laplace equation as an illustration to validate our analytical results.

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