Chapter 4

$L^p_\alpha(\mathbb{R}^{n+1}_+)$ - Boundedness of Pseudo-Differential Operators involving the Weinstein transform

4.1 Introduction

The L^p- boundedness of pseudo differential operators is an important result and suitably used in various problems of mathematical analysis and its applications. Exploiting this theory, varieties of work have been done by many important mathematicians. Calderón and Vaillancourt [4] discussed the L^2- boundedness of M-order pseudo differential operators associated with symbol $p(x_1, x_2, \xi) \in S^M_{\rho,\delta_1,\delta_1}$ for $0 \leq \rho \leq \delta_1, \delta_2 < 1$ and $\frac{M}{n} \geq \frac{(\delta_1+\delta_2)}{2} - \rho$, Fefferman [17] investigated sharp L^p- boundedness results for pseudo-differential operators in the class $S^m_{\rho,\delta}$, Illner [30] found the L^p- boundedness of pseudo-differential operators with symbol $p(x, \xi, y) \in S^\mu_{\rho,\delta,\epsilon}$ for $\mu \leq (\rho-1)(n+1)$, Cato [31] constructed the L^2- boundedness for pseudo-differential operators with symbol $a(x,\xi)$ lies in $S^0_{\rho,\rho}$ for $0<\rho<1$, Nagase [43] investigated the L^p- boundedness of pseudo-differential operators with non-regular symbols, Hwang and Lee [29] found the L^p - boundedness of pseudo-differential operator with symbol class $S^m_{0,0}$ for $m=-n|1/p-1/2|$, Wong [81, p. 77] proved that the $L^p(\mathbb{R}^n)$ - boundedness of pseudo-diffrential operators T_σ for $\sigma\in S^0$. These L^p - boundedness results made strong foundations of pseudo-differential operators involving Fourier transform with different types of symbol classes. Exploiting the Hankel transform theory, Pathak and Upadhyay [57] established the L^p_μ - boundedness results of pseudo-differential operators with symbol class H^m. Using L^p_μ - boundedness properties, authors showed that $h_{\mu,a}$ is bounded linear operator from $W^{m,p}_\mu\to W^{0,p}_\mu$ and $W^{s,p}_\mu\to W^{s-m,p}_\mu$ respectively. These spaces are defined in [53]. Motivated from the aforesaid important literature, our main objective of this chapter is to investigate the $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators associated with certain class of symbol S^0 involving the Weinstein transform.

Contents of this chapter are organized in the following way:

Section 4.1 is introductory which describes the brief motivation of the $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators. In Section 4.2, an integral representation of pseudo-differential operators and other properties are obtained. The $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of the pseudo-differential operators T_σ associated with symbol $\sigma(x,\xi)$ in the symbol class S^0 involving the Weinstein transform techniques is obtained. In Section 4.3, $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - type Sobolev space of order r is defined and boundedness of the pseudo-differential operators $T_\sigma : H^{r,p}_\alpha\to H^{r-m,p}_\alpha$ and other properties are given.

4.2 An Integral Representation of Pseudo Differential Operators

An integral representation of the pseudo-differential operators T_σ associated with a symbol $\sigma \in S^m$ involving the Weinstein transform is obtained and its various properties studied.

Lemma 4.2.1. *Let $\alpha > -\frac{1}{2}$ and σ be a symbol in S^0. Define*

$$K(x,z) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle z',\xi'\rangle} \hat{J}_\alpha(z_{n+1}\xi_{n+1})\sigma(x,\xi)d\mu_\alpha(\xi), \tag{4.2.1}$$

in the distributional sense. Then

(i) for each fixed $x \in \mathbb{R}^{n+1}_+$, $K(x,\cdot)$ is a function defined on $\mathbb{R}^{n+1}_+$,

(ii) for large values of $k \in \mathbb{N}_0$, there exists a positive constant $C_{\alpha,k}$ such that

$$|K(x,z)| \leq C_{\alpha,k}(1+\|x\|^2)^{-q}(1+\|z\|^2)^{-k}. \tag{4.2.2}$$

Proof. For $k \in \mathbb{N}$, (4.2.1) can be written as

$$K(x,z) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle z',\xi'\rangle} \hat{J}_\alpha(z_{n+1}\xi_{n+1})(1+\|z\|^2)^{-k}(1-\Delta_{\alpha,n,\xi})^k\sigma(x,\xi)d\mu_\alpha(\xi).$$

Invoking Binomial Theorem, we get

$$\begin{aligned} K(x,z) = \int_{\mathbb{R}^{n+1}_+} & e^{i\langle z',\xi'\rangle} \hat{J}_\alpha(z_{n+1}\xi_{n+1})(1+\|z\|^2)^{-k} \\ & \times \left(\sum_{r=0}^{k} \binom{k}{r}(-1)^r\Delta^r_{\alpha,n,\xi}\sigma(x,\xi)\right)d\mu_\alpha(\xi). \end{aligned}$$

Therefore,

$$|K(x,z)| \leq (1+\|z\|^2)^{-k}\sum_{r=0}^{k}\binom{k}{r}\int_{\mathbb{R}^{n+1}_+}|\Delta^r_{\alpha,n,\xi}\sigma(x,\xi)|d\mu_\alpha(\xi).$$

In view of Lemma 3.3.2, the last expression becomes

$$\begin{aligned}|K(x,z)| \leq{}& (1+\|z\|^2)^{-k}\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j}\binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,l}\\ &\times\int_{\mathbb{R}^{n+1}_+}|\xi_{n+1}|^{l-r}\big|D^{2\delta'+l}_{\xi}\sigma(x,\xi)\big|d\mu_\alpha(\xi)\\ \leq{}& (1+\|z\|^2)^{-k}\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j}\binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,l}A_\alpha\\ &\times\int_{\mathbb{R}^{n+1}_+}|\xi_{n+1}|^{l-r+2\alpha+1}\big|D^{2\delta'+l}_{\xi}\sigma(x,\xi)\big|d\xi.\end{aligned}$$

Using the fact that $\sigma\in S^0$, we obtain

$$\begin{aligned}|K(x,z)| \leq{}& (1+\|z\|^2)^{-k}\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j}\binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,l}A_\alpha C_{2\delta'+l,0}\\ &\times\int_{\mathbb{R}^{n+1}_+}|\xi_{n+1}|^{l-r+2\alpha+1}(1+\|x\|^2)^{-q}(1+\|\xi\|^2)^{-2|\delta'|-l}d\xi.\end{aligned}$$

For large values of l, the above expression becomes

$$\begin{aligned}|K(x,z)| \leq{}& (1+\|z\|^2)^{-k}\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j}\binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,l}A_\alpha C_{2\delta'+l,0}\\ &\times\int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^{l-r+2\alpha+1}(1+\|x\|^2)^{-q}(1+\|\xi\|^2)^{-2|\delta'|-l}d\xi.\end{aligned}$$

Therefore,

$$|K(x,z)| \leq (1+\|z\|^2)^{-k}(1+\|x\|^2)^{-q}\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j}\binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n}$$
$$\times E'_{\alpha,l}A_\alpha C_{2\delta'+l,0}\int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^{-r-2|\delta'|+2\alpha+1}d\xi.$$

It gives

$$|K(x,z)| \leq (1+\|z\|^2)^{-k}(1+\|x\|^2)^{-q}\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\binom{k}{r}\binom{r}{j}$$
$$\times E'_{\alpha,l}A_\alpha C_{l,0}\int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^{-r+2\alpha+1}d\xi.$$

Choosing $r > 2\alpha + \frac{n}{2} + \frac{3}{2}$, there exists a positive constant $C_{\alpha,k}$ such that

$$|K(x,z)| \leq C_{\alpha,k}(1+\|x\|^2)^{-q}(1+\|z\|^2)^{-k}.$$

□

Theorem 4.2.2. *Let $\alpha > -\frac{1}{2}$ and $\sigma \in S^m$. Then for all $u \in S_*(\mathbb{R}^{n+1}_+)$, the pseudo-differential operator T_σ can be written as*

$$(T_\sigma u)(x) = \int_{\mathbb{R}^{n+1}_+}\left(\int_{\mathbb{R}^{n+1}_+}K(x,z)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z)\right)u(y)d\mu_\alpha(y), \tag{4.2.3}$$

in the distributional sense.

Proof. Exploiting the concept of [10, p. 266], we can see that the Schwartz space $S_*(\mathbb{R}^{n+1}_+)$ is invariant under the translation operator τ^α_x, $x \in \mathbb{R}^{n+1}_+$. Then for all $u \in S_*(\mathbb{R}^{n+1}_+)$, we have $\mathcal{F}_\alpha(\tau^\alpha_x u) \in S_*(\mathbb{R}^{n+1}_+)$. Now, from (1.4.12), we have

$$(\tau^\alpha_x u)(z) = \int_{\mathbb{R}^{n+1}_+} u(y)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(y).$$

In view of (1.4.13), above expression becomes

$$(\tau^\alpha_x u)(z) = \int_{\mathbb{R}^{n+1}_+} u(y)\Bigg(\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})e^{-i\langle y',\xi'\rangle}\hat{J}_\alpha(y_{n+1}\xi_{n+1}) \\ \times e^{-i\langle z',\xi'\rangle}\hat{J}_\alpha(z_{n+1}\xi_{n+1})d\mu_\alpha(\xi)\Bigg)d\mu_\alpha(y).$$

Therefore,

$$\begin{aligned}(\tau^\alpha_x u)(z) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})e^{-i\langle z',\xi'\rangle}\hat{J}_\alpha(z_{n+1}\xi_{n+1}) \\ &\quad \times \Bigg(\int_{\mathbb{R}^{n+1}_+} u(y)e^{-i\langle y',\xi'\rangle}\hat{J}_\alpha(y_{n+1}\xi_{n+1})d\mu_\alpha(y)\Bigg)d\mu_\alpha(\xi) \\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})e^{-i\langle z',\xi'\rangle}\hat{J}_\alpha(z_{n+1}\xi_{n+1})(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi).\end{aligned}$$

Invoking (1.4.1), we get

$$(\tau^\alpha_x u)(z) = \mathcal{F}_\alpha\big[e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(\mathcal{F}_\alpha u)(\xi)\big](z).$$

Therefore, we get

$$(\mathcal{F}_\alpha u)(\xi) = \big[e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\big]^{-1}\mathcal{F}^{-1}_\alpha(\tau^\alpha_x u)(\xi) \in S_*(\mathbb{R}^{n+1}_+). \tag{4.2.4}$$

Now, we define the pseudo-differential operator in the distributional sense

$$(T_\sigma u)(x) = \Big\langle e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi), (\mathcal{F}_\alpha u)(\xi)\Big\rangle. \tag{4.2.5}$$

From (4.2.4), (4.2.5) becomes

$$\begin{aligned}(T_\sigma u)(x) &= \Big\langle e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi), [e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})]^{-1}\mathcal{F}^{-1}_\alpha(\tau^\alpha_x u)(\xi)\Big\rangle \\ &= \Big\langle \sigma(x,\xi), \mathcal{F}^{-1}_\alpha(\tau^\alpha_x u)(\xi)\Big\rangle.\end{aligned}$$

From Theorem 1.4.8, we obtain

$$(T_\sigma u)(x) = \Big\langle \mathcal{F}^{-1}_\alpha\big(\sigma(x,\xi)\big)(z), (\tau^\alpha_x u)(z)\Big\rangle.$$

Using (4.2.1), we get

$$\begin{aligned}(T_\sigma u)(x) &= \Big\langle K(x,z), (\tau^\alpha_x u)(z)\Big\rangle\\ &= \int_{\mathbb{R}^{n+1}_+} K(x,z)(\tau^\alpha_x u)(z)d\mu_\alpha(z).\end{aligned}$$

Invoking (1.4.12), the last expression yields

$$(T_\sigma u)(x) = \int_{\mathbb{R}^{n+1}_+} K(x,z)\bigg(\int_{\mathbb{R}^{n+1}_+} \mathcal{D}_\alpha(x,y,z)u(y)d\mu_\alpha(y)\bigg)d\mu_\alpha(z).$$

□

Theorem 4.2.3. *Let* $\alpha > -\frac{1}{2}$ *and* $\theta \in C^k(\mathbb{R}^{n+1}_+), k \in \mathbb{N}$. *Assume that for* $\beta \in \mathbb{N}^{n+1}_0$, *there exists a positive constant* B *such that*

$$\big|D^\beta_\xi \theta(\xi)\big| \le B(1+\|\xi\|^2)^{-|\beta|}, \qquad |\beta| \le k. \tag{4.2.6}$$

If

$$\psi(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\theta(\xi)d\mu_\alpha(\xi), \tag{4.2.7}$$

then $\psi \in L^p_\alpha(\mathbb{R}^{n+1}_+)$, *for* $1 \le p < \infty$.

Proof. For $k \in \mathbb{N}$, (4.2.7) can be written as

$$\begin{aligned}\psi(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|x\|^2)^{-k}(1-\Delta_{\alpha,n,\xi})^k\theta(\xi)d\mu_\alpha(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|x\|^2)^{-k}\bigg(\sum_{r=0}^{k}\binom{k}{r}(-1)^r\Delta^r_{\alpha,n,\xi}\theta(\xi)\bigg)d\mu_\alpha(\xi).\end{aligned}$$

Therefore,

$$|\psi(x)| \leq (1+\|x\|^2)^{-k} \sum_{r=0}^{k} \binom{k}{r} \int_{\mathbb{R}^{n+1}_+} |\Delta^r_{\alpha,n,\xi}\theta(\xi)| d\mu_\alpha(\xi).$$

In view of Lemma 3.3.2, the last expression becomes

$$\begin{aligned}|\psi(x)| \leq (1+\|x\|^2)^{-k} &\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} \\ &\times \int_{\mathbb{R}^{n+1}_+} |\xi_{n+1}|^{l-r} \big|D^{2\delta'+l}_\xi \theta(\xi)\big| d\mu_\alpha(\xi).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|\psi(x)| \leq (1+\|x\|^2)^{-k} &\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha \\ &\times \int_{\mathbb{R}^{n+1}_+} |\xi_{n+1}|^{l-r+2\alpha+1} \big|D^{2\delta'+l}_\xi \theta(\xi)\big| d\xi.\end{aligned}$$

Using (4.2.6), we obtain

$$\begin{aligned}|\psi(x)| \leq (1+\|x\|^2)^{-k} &\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha B \\ &\times \int_{\mathbb{R}^{n+1}_+} |\xi_{n+1}|^{l-r+2\alpha+1} (1+\|\xi\|^2)^{-2|\delta'|-l} d\xi.\end{aligned}$$

For large values of l, above yields

$$\begin{aligned}|\psi(x)| \leq (1+\|x\|^2)^{-k} &\sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\leq r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n} E'_{\alpha,l} A_\alpha B \\ &\times \int_{\mathbb{R}^{n+1}_+} (1+\|\xi\|^2)^{l-r+2\alpha+1} (1+\|\xi\|^2)^{-2|\delta'|-l} d\xi.\end{aligned}$$

Therefore,

$$|\psi(x)| \le (1+\|x\|^2)^{-k} \sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j}\sum_{|\delta'|\le r-j} \binom{k}{r}\binom{r}{j}\binom{r-j}{\delta_1,\cdots,\delta_n}$$
$$\times E'_{\alpha,l}A_\alpha B\int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^{-r-2|\delta'|+2\alpha+1}d\xi.$$

It gives

$$|\psi(x)| \le (1+\|x\|^2)^{-k} \sum_{r=0}^{k}\sum_{j=0}^{r}\sum_{l=1}^{2j} \binom{k}{r}\binom{r}{j}$$
$$\times E'_{\alpha,l}A_\alpha B\int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^{-r+2\alpha+1}d\xi.$$

Choosing $r > 2\alpha + \frac{n}{2} + \frac{3}{2}$, there exists a positive constant $B_{\alpha,k}$ such that

$$|\psi(x)| \le B_{\alpha,k}(1+\|x\|^2)^{-k}.$$

Therefore,

$$\|\psi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \le B_{\alpha,k}\|(1+\|x\|^2)^{-k}\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)},$$

for large values of $k \in \mathbb{N}$. Therefore $\psi \in L^p_\alpha(\mathbb{R}^{n+1}_+)$. $\square$

Theorem 4.2.4. *Let* $\alpha > -\frac{1}{2}$ *and* $\theta \in C^k(\mathbb{R}^{n+1}_+), k \in \mathbb{N}$ *which satisfies* (4.2.6). *Then for* $1 \le p < \infty$, *there exists a positive constant* $C_{\alpha,k}$ *such that*

$$\|T_\theta u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \le C_{\alpha,k}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}, \qquad u \in S_*(\mathbb{R}^{n+1}_+) \tag{4.2.8}$$

where

$$\big(T_\theta u\big)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\theta(\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi). \tag{4.2.9}$$

Proof. Using (1.4.2), (4.2.9) can be expressed as

$$\big(T_\theta u\big)(x) = \mathcal{F}_\alpha^{-1}\big[\theta(\xi)(\mathcal{F}_\alpha u)(\xi)\big], \qquad u \in S_*(\mathbb{R}^{n+1}_+). \tag{4.2.10}$$

Now, assume that there exists ψ such that

$$(\psi *_w u)(x) = {\mathcal{F}_\alpha}^{-1}[\theta(\xi)(\mathcal{F}_\alpha u)(\xi)](x), \quad u \in S_*(\mathbb{R}^{n+1}_+). \tag{4.2.11}$$

From (4.2.11), (4.2.10) becomes

$$\big(T_\theta u\big)(x) = (\psi *_w u)(x).$$

In view of Theorem 4.2.3 and (1.4.16), we have

$$\|T_\theta u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq \|\psi\|_{L^1_\alpha(\mathbb{R}^{n+1}_+)}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Since $\psi \in L^1_\alpha(\mathbb{R}^{n+1}_+)$, therefore, we can find a positive constant $C_{\alpha,k}$ such that

$$\|T_\theta u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq C_{\alpha,k}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

□

Theorem 4.2.5. *Let $\alpha > -\frac{1}{2}$ and $\sigma \in S^0$. Then for $1 < p < \infty$, the pseudo-differential operator $T_\sigma : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is a bounded linear operator.*

Proof. Let us denote

$$\mathbb{Z}^n \times \mathbb{N}_0 = \{(x_1, x_2, \cdots, x_n, x_{n+1}) : x_j \in \mathbb{Z}, 1 \leq j \leq n, x_{n+1} \in \mathbb{N}_0\},$$

and $M = (m, m_1)$, for $m \in \mathbb{Z}^n, m_1 \in \mathbb{N}_0$. Then we write $\mathbb{R}^{n+1}_+$ as a union of Q_M with disjoint interiors, i.e.,

$$\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty) = \bigcup_{M \in \mathbb{Z}^n \times \mathbb{N}_0} Q_M,$$

where Q_M be the product of n-dimensional cube with center at m, edges of lenth one, parallel to the coordinate axes and the interval $[m_1, m_1 + 1]$.

Let η be the smooth function defined on $\mathbb{R}^{n+1}_+$ such that

$$\eta(x) = 1, \qquad \forall x \in Q_0$$

and

$$\left|D^\gamma_x \eta(x)\right| \leq C_\gamma, \qquad \forall \gamma \in \mathbb{N}^{n+1}_0 \tag{4.2.12}$$

where $C_\gamma > 0$ is constant depends on γ.

Now, define

$$\sigma_m(x, \xi) = \eta(x - m)\sigma(x, \xi), \quad \forall x, \xi \in \mathbb{R}^{n+1}_+. \tag{4.2.13}$$

Then, from (2.2.5) we have

$$\big(T_{\sigma_m} u\big)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma_m(x, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi). \tag{4.2.14}$$

Using (4.2.13), we get

$$\begin{aligned}\big(T_{\sigma_m} u\big)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\eta(x - m)\sigma(x, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi) \\ &= \eta(x - m) \int_{\mathbb{R}^{n+1}_+} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi).\end{aligned}$$

Taking (2.2.5), we can get

$$\big(T_{\sigma_m}u\big)(x) = \eta(x-m)\big(T_\sigma u\big)(x). \tag{4.2.15}$$

Now, we have

$$\int_{Q_M} \big|\big(T_\sigma u\big)(x)\big|^p d\mu_\alpha(x) \leq \int_{\mathbb{R}^{n+1}_+} \big|\eta(x-m)\big(T_\sigma u\big)(x)\big|^p d\mu_\alpha(x).$$

Therefore, from (4.2.15) we find

$$\int_{Q_M} \big|\big(T_\sigma u\big)(x)\big|^p d\mu_\alpha(x) \leq \int_{\mathbb{R}^{n+1}_+} \big|\big(T_{\sigma_m} u\big)(x)\big|^p d\mu_\alpha(x). \tag{4.2.16}$$

Since $\sigma_m(x,\xi)$ has compact support in variable x and applying inversion formula of the Weinstein transform (1.4.2), we obtain

$$\sigma_m(x,\xi) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\lambda'\rangle} \hat{J}_\alpha(x_{n+1}\lambda_{n+1})\big(\mathcal{F}_\alpha \sigma_m\big)(\lambda,\xi) d\mu_\alpha(\lambda), \tag{4.2.17}$$

where

$$\big(\mathcal{F}_\alpha \sigma_m\big)(\lambda,\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\lambda'\rangle} \hat{J}_\alpha(x_{n+1}\lambda_{n+1})\sigma_m(x,\xi) d\mu_\alpha(x). \tag{4.2.18}$$

Invoking (4.2.17) in (4.2.14), we get

$$\begin{aligned}\big(T_{\sigma_m}u\big)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \\ &\quad \times \bigg(\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\lambda'\rangle} \hat{J}_\alpha(x_{n+1}\lambda_{n+1})\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi) d\mu_\alpha(\lambda)\bigg)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi) \\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\lambda'\rangle} \hat{J}_\alpha(x_{n+1}\lambda_{n+1}) \\ &\quad \times \bigg(\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi)\bigg) d\mu_\alpha(\lambda).\end{aligned}$$

Hence,

$$\big(T_{\sigma_m}u\big)(x) = \int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})(T_\lambda u)(x)d\mu_\alpha(\lambda), \tag{4.2.19}$$

where

$$\big(T_\lambda u\big)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi). \tag{4.2.20}$$

For the proof of the present theorem we need the following Lemmas.

Lemma 4.2.6. *Let $\alpha > -\frac{1}{2}$ and σ_m be defined in (4.2.13). Then for $\beta \in \mathbb{N}_0^{n+1}, N \in \mathbb{N}_0$, there exists a positive constant $C_{\alpha,N}$ such that*

$$\big|D^\beta_\xi\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi)\big| \le C_{\alpha,N}(1+\|\lambda\|^2)^{-N}(1+\|\xi\|^2)^{-|\beta|}, \qquad |\beta| \le k. \tag{4.2.21}$$

Proof. Using (4.2.18), we have

$$D^\beta_\xi\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})D^\beta_\xi\sigma_m(x,\xi)d\mu_\alpha(x).$$

Invoking (4.2.13), we get

$$D^\beta_\xi\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})\eta(x-m)D^\beta_\xi\sigma(x,\xi)d\mu_\alpha(x).$$

Using (1.4.3) for $N \in \mathbb{N}_0$, we have

$$\begin{aligned}D^\beta_\xi\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi) = \int_{\mathbb{R}^{n+1}_+} & e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})(1+\|\lambda\|^2)^{-N} \\ & \times (1-\Delta_{\alpha,n,x})^N\Big(\eta(x-m)D^\beta_\xi\sigma(x,\xi)\Big)d\mu_\alpha(x).\end{aligned}$$

Using Binomial Theorem, we get

$$\begin{aligned} D^\beta_\xi\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi) = \int_{\mathbb{R}^{n+1}_+} & e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})(1+\|\lambda\|^2)^{-N} \\ & \times \sum_{r=0}^{N}\binom{N}{r}(-1)^r\Delta^r_{\alpha,n,x}\Big(\eta(x-m)D^\beta_\xi\sigma(x,\xi)\Big)d\mu_\alpha(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \Big|D^\beta_\xi\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi)\Big| \le (1+\|\lambda\|^2)^{-N} & \sum_{r=0}^{N}\binom{N}{r} \\ & \times \int_{\mathbb{R}^{n+1}_+}\Big|\Delta^r_{\alpha,n,x}\Big(\eta(x-m)D^\beta_\xi\sigma(x,\xi)\Big)\Big|d\mu_\alpha(x). \end{aligned} \tag{4.2.22}$$

From [49, Lemma 2.1], we have

$$\big|\Delta^r_{\alpha,n,x}f(x)\big| \le \sum_{|\gamma|\le r} C_{\alpha,r}\big|D^{2\gamma}_x f(x)\big|, \quad \forall f\in C^\infty_c(\mathbb{R}^{n+1}_+). \tag{4.2.23}$$

Using (4.2.23) we can write (4.2.22) in the following way

$$\begin{aligned} \Big|D^\beta_\xi\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi)\Big| \le (1+\|\lambda\|^2)^{-N} & \sum_{r=0}^{N}\sum_{|\gamma|\le r}\binom{N}{r}C_{\alpha,r} \\ & \times \int_{\mathbb{R}^{n+1}_+}\Big|D^{2\gamma}_x\Big(\eta(x-m)D^\beta_\xi\sigma(x,\xi)\Big)\Big|d\mu_\alpha(x). \end{aligned}$$

By Leibnitz formula, the last expression becomes

$$\begin{aligned} \Big|D^\beta_\xi\big(\mathcal{F}_\alpha\sigma_m\big)(\lambda,\xi)\Big| \le (1+\|\lambda\|^2)^{-N} & \sum_{r=0}^{N}\sum_{|\gamma|\le r}\sum_{\delta\le 2\gamma}\binom{N}{r}\binom{2\gamma}{\delta}C_{\alpha,r} \\ & \times \int_{\mathbb{R}^{n+1}_+}\Big|D^\delta_x\eta(x-m)\Big|\Big|D^{2\gamma-\delta}_x D^\beta_\xi\sigma(x,\xi)\Big|d\mu_\alpha(x). \end{aligned}$$

Since $\sigma \in S^0$ then from (4.2.12), we get

$$\left|D^\beta_\xi\left(\mathcal{F}_\alpha\sigma_m\right)(\lambda,\xi)\right| \leq (1+\|\lambda\|^2)^{-N}\sum_{r=0}^{N}\sum_{|\gamma|\leq r}\sum_{\delta\leq 2\gamma}\binom{N}{r}\binom{2\gamma}{\delta}C_{\alpha,r}C_\delta C_{2\gamma-\delta,\beta}$$
$$\times\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{-q}(1+\|\xi\|^2)^{-|\beta|}d\mu_\alpha(x).$$

For each $q \in \mathbb{N}$, there exists a positive constant $C_{\alpha,N}$ depends only on α and N such that

$$\left|D^\beta_\xi\left(\mathcal{F}_\alpha\sigma_m\right)(\lambda,\xi)\right| \leq C_{\alpha,N}(1+\|\lambda\|^2)^{-N}(1+\|\xi\|^2)^{-|\beta|}.$$

□

Hence, from Theorem 4.2.4 and Lemma 4.2.6, the operator $u \to T_\lambda u$ defined on $S_*(\mathbb{R}^{n+1}_+)$ by (4.2.20) can be extened to a bounded linear operator on $L^p_\alpha(\mathbb{R}^{n+1}_+)$. Moreover, using the similar steps of Theorem 4.2.3 and Lemma 4.2.6, we obtain

$$\left|{\mathcal{F}_\alpha}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]\right| \leq B_{\alpha,N,k}(1+\|\lambda\|^2)^{-N}(1+\|x\|^2)^{-k}, \quad \forall N,k\in\mathbb{N}_0. \quad (4.2.24)$$

Using (1.4.11) and (1.4.16), we get

$$\|T_\lambda u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} = \left(\int_{\mathbb{R}^{n+1}_+}\left|\left({\mathcal{F}_\alpha}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right] *_w u\right)(x)\right|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}$$
$$= \left\|\left({\mathcal{F}_\alpha}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right] *_w u\right)(x)\right\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Applying (1.4.16), we get

$$\|T_\lambda u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq \left\|{\mathcal{F}_\alpha}^{-1}\left[(\mathcal{F}_\alpha\sigma_m)(\lambda,\xi)\right]\right\|_{L^1_\alpha(\mathbb{R}^{n+1}_+)}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

From (4.2.24), the last inequality becomes

$$\|T_\lambda u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq B_{\alpha,N,k}(1+\|\lambda\|^2)^{-N}\|(1+\|x\|^2)^{-k}\|_{L^1_\alpha(\mathbb{R}^{n+1}_+)}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

For large $k \in \mathbb{N}$, we get a positive constant $C_{\alpha,N,k}$ such that

$$\|T_\lambda u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq C_{\alpha,N,k}(1+\|\lambda\|^2)^{-N}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}, \quad N \in \mathbb{N}_0. \tag{4.2.25}$$

From (4.2.19), we have

$$\begin{aligned}\|T_{\sigma_m}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} &= \left(\int_{\mathbb{R}^{n+1}_+}|T_{\sigma_m}u(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{R}^{n+1}_+}\left|\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})(T_\lambda u)(x)d\mu_\alpha(\lambda)\right|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}.\end{aligned}$$

The last expression becomes after applying Minkowski's inequality

$$\begin{aligned}\|T_{\sigma_m}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} &\leq \int_{\mathbb{R}^{n+1}_+}\left(\int_{\mathbb{R}^{n+1}_+}\left|(T_\lambda u)(x)\right|^p d\mu_\alpha(x)\right)^{\frac{1}{p}} d\mu_\alpha(\lambda)\\ &\leq \int_{\mathbb{R}^{n+1}_+}\|T_\lambda u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}d\mu_\alpha(\lambda).\end{aligned}$$

In view of (4.2.25), we get

$$\|T_{\sigma_m}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq C_{\alpha,N,k}\left(\int_{\mathbb{R}^{n+1}_+}(1+\|\lambda\|^2)^{-N}d\mu_\alpha(\lambda)\right)\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

For sufficiently large natural number N, we get a positive constant $E_{\alpha,N,k}$ such that

$$\|T_{\sigma_m}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq E_{\alpha,N,k}\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{4.2.26}$$

After using (4.2.16), we have

$$\int_{Q_M} \left|(T_\sigma u)(x)\right|^p d\mu_\alpha(x) \le \|T_{\sigma_m} u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

From (4.2.26), the last inequality becomes

$$\int_{Q_M} \left|(T_\sigma u)(x)\right|^p d\mu_\alpha(x) \le E^p_{\alpha,N,k}\|u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{4.2.27}$$

Let Q_M, Q^*_M and Q^{**}_M be the cubes with same center respectively. The length of Q^{**}_M is the twice of length of Q_M. Let Q^*_M be another concentric cube with Q_M and Q^{**}_M satisfying $Q_M \subset Q^*_M \subset Q^{**}_M$.

Let ψ be the smooth function defined on $\mathbb{R}^{n+1}_+$, with compact support and satisfies the following properties:

(i) $0 \le \psi(x) \le 1, \quad \forall x \in \mathbb{R}^{n+1}_+$

(ii) $supp(\psi) \subseteq Q^{**}_M$,

(iii) and $\psi(x) = 1$ for all x in a neighbourhood of Q^*_M.

If we write $u = u_1 + u_2$, where $u_1 = \psi u$ and $u_2 = (1-\psi)u$. Then we have

$$(T_\sigma u)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)\big(\mathcal{F}_\alpha(u_1+u_2)\big)(\xi)d\mu_\alpha(\xi).$$

Thus, we get

$$\begin{aligned}\big(T_\sigma u\big)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)\big(\mathcal{F}_\alpha u_1\big)(\xi)d\mu_\alpha(\xi)\\ &\quad+\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)\big(\mathcal{F}_\alpha u_2\big)(\xi)d\mu_\alpha(\xi)\\ &= \big(T_\sigma u_1\big)(x)+\big(T_\sigma u_2\big)(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{Q_M}\big|\big(T_\sigma u\big)(x)\big|^p d\mu_\alpha(x) &= \int_{Q_M}\big|\big(T_\sigma u_1\big)(x)+\big(T_\sigma u_2\big)(x)\big|^p d\mu_\alpha(x)\\ &\le 2^p\int_{Q_M}\big|\big(T_\sigma u_1\big)(x)\big|^p d\mu_\alpha(x)\\ &\quad+2^p\int_{Q_M}\big|\big(T_\sigma u_2\big)(x)\big|^p d\mu_\alpha(x).\end{aligned}$$

Taking (4.2.27), we get

$$\begin{aligned}\int_{Q_M}\big|\big(T_\sigma u\big)(x)\big|^p d\mu_\alpha(x) &\le 2^p E^p_{\alpha,N,k}\|u_1\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}\\ &\quad+2^p\int_{Q_M}\big|\big(T_\sigma u_2\big)(x)\big|^p d\mu_\alpha(x).\end{aligned}\tag{4.2.28}$$

Since $u_1=\psi u$ and $supp(\psi)\subseteq Q^{**}_M$, then from (4.2.28), we obtain

$$\begin{aligned}\int_{Q_M}\big|\big(T_\sigma u\big)(x)\big|^p d\mu_\alpha(x) &\le 2^p E^p_{\alpha,N,k}\int_{Q^{**}_M}|u(x)|^p d\mu_\alpha(x)\\ &\quad+2^p\int_{Q_M}\big|\big(T_\sigma u_2\big)(x)\big|^p d\mu_\alpha(x).\end{aligned}\tag{4.2.29}$$

From Theorem 4.2.2, we have

$$(T_\sigma u_2)(x)=\int_{\mathbb{R}^{n+1}_+}\Big(\int_{\mathbb{R}^{n+1}_+}K(x,y)\mathcal{D}_\alpha(x,y,z)d\mu_\alpha(y)\Big)u_2(z)d\mu_\alpha(z).$$

Since $u_2(z) = 0$ in a neighbourhood of $z \in Q^*_M$, therefore

$$\big|(T_\sigma u_2)(x)\big| \leq \int_{\mathbb{R}^{n+1}_+ - Q^*_M} \Big(\int_{\mathbb{R}^{n+1}_+} \big|K(x,y)\big| \mathcal{D}_\alpha(x,y,z) d\mu_\alpha(y)\Big) |u_2(z)| d\mu_\alpha(z).$$

In view of Lemma 4.2.1, above expression becomes

$$\begin{aligned}\big|(T_\sigma u_2)(x)\big| \leq C_{\alpha,k} \int_{\mathbb{R}^{n+1}_+ - Q^*_M} & \Big(\int_{\mathbb{R}^{n+1}_+} (1+\|x\|^2)^{-q}(1+\|y\|^2)^{-k} \mathcal{D}_\alpha(x,y,z) d\mu_\alpha(y)\Big) \\ & \times |u_2(z)| d\mu_\alpha(z).\end{aligned}$$

Using the fact

$$(1+\|x\|^2)^{-q} \leq (1+\|m\|^2)^{-q}, \qquad \forall x \in Q_M$$

the last inequality becomes

$$\begin{aligned}\big|(T_\sigma u_2)(x)\big| &\leq C_{\alpha,k}(1+\|m\|^2)^{-q} \int_{\mathbb{R}^{n+1}_+} \Big(\int_{\mathbb{R}^{n+1}_+} (1+\|y\|^2)^{-k} \mathcal{D}_\alpha(x,y,z) d\mu_\alpha(y)\Big) \\ &\quad \times |u(z)| d\mu_\alpha(z) \\ &\leq C_{\alpha,k}(1+\|m\|^2)^{-q} \big(h *_w |u|\big)(x),\end{aligned}$$

where $h(y) = (1+\|y\|^2)^{-k}$.

Therefore,

$$\int_{Q_M} \big|\big(T_\sigma u_2\big)(x)\big|^p d\mu_\alpha(x) \leq C^p_{\alpha,k}(1+\|m\|^2)^{-qp} \int_{Q_M} \big|\big(h *_w |u|\big)(x)\big|^p d\mu_\alpha(x).$$

From (1.4.16), we get

$$\int_{Q_M} \big|\big(T_\sigma u_2\big)(x)\big|^p d\mu_\alpha(x) \leq C^p_{\alpha,k}(1+\|m\|^2)^{-qp} \|h\|^p_{L^1_\alpha(\mathbb{R}^{n+1}_+)} \|u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

For large $k \in \mathbb{N}$, we get a positive constant $C_{\alpha,k,p}$ such that

$$\int_{Q_M} \big|\big(T_\sigma u_2\big)(x)\big|^p d\mu_\alpha(x) \leq C_{\alpha,k,p}(1+\|m\|^2)^{-qp}\|u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{4.2.30}$$

Therefore, from (4.2.29) and (4.2.30) we have

$$\begin{aligned}\int_{Q_M} \big|\big(T_\sigma u\big)(x)\big|^p d\mu_\alpha(x) \leq 2^p E^p_{\alpha,N,k} \int_{Q^{**}_M} |u(x)|^p d\mu_\alpha(x)\\ + 2^p C_{\alpha,k,p}(1+\|m\|^2)^{-qp}\|u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.\end{aligned}$$

Summing over all $m \in \mathbb{Z}^n \times \mathbb{N}_0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} \big|\big(T_\sigma u\big)(x)\big|^p d\mu_\alpha(x) &\leq 2^p E^p_{\alpha,N,k} \int_{\mathbb{R}^{n+1}_+} |u(x)|^p d\mu_\alpha(x)\\ &\quad + 2^p C_{\alpha,k,p}\Big(\sum_{m\in\mathbb{Z}^n\times\mathbb{N}_0}(1+\|m\|^2)^{-qp}\Big)\|u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}\\ &\leq 2^p\Big(E^p_{\alpha,N,k} + C_{\alpha,k,p}\sum_{m\in\mathbb{Z}^n\times\mathbb{N}_0}(1+\|m\|^2)^{-qp}\Big)\|u\|^p_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.\end{aligned}$$

For $q > 1$ and $1 < p < \infty$, we can find a positive constant $C = C(\alpha, k, N, p, q)$ such that

$$\|T_\sigma u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq C\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}, \quad u \in S_*(\mathbb{R}^{n+1}_+). \tag{4.2.31}$$

Since $S_*(\mathbb{R}^{n+1}_+)$ is dense in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ by Theorem 1.3.3, therefore from (4.2.31), T_σ can be extended to a bounded linear operator on $L^p_\alpha(\mathbb{R}^{n+1}_+)$. □

4.3 Sobolev Spaces

In this section, we introduce the Bessel potential and $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - Sobolev space of order r. Using $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness properties, we find various properties and boundedness results in Sobolev space.

Definition 4.3.1. $\big($**Bessel Potential**$\big)$
Let $r \in \mathbb{R}$ and $\sigma(\xi) = (1 + \|\xi\|^2)^{-r/2}$ be a symbol in S^{-r}. Then for $u \in S'_*(\mathbb{R}^{n+1}_+)$, the Weinstein potential of order r is defined by

$$(J_{r,\alpha}u)(x) = {\mathcal{F}_\alpha}^{-1}\big[(1 + \|\xi\|^2)^{-r/2}(\mathcal{F}_\alpha u)(\xi)\big](x). \tag{4.3.1}$$

Lemma 4.3.2. *Let* $u \in S'_*(\mathbb{R}^{n+1}_+)$*, then we have*

(i) $J_{0,\alpha}u = u$*,*

(ii) $J_{r,\alpha}J_{t,\alpha}u = J_{r+t,\alpha}u$*.*

Proof. (i) Taking $r = 0$ in (4.3.1), we get

$$(J_{0,\alpha}u)(x) = {\mathcal{F}_\alpha}^{-1}\big[(\mathcal{F}_\alpha u)(\xi)\big](x).$$

Since, $u \in S'_*(\mathbb{R}^{n+1}_+)$, therefore we obtain

$$(J_{0,\alpha}u)(x) = u(x).$$

(ii) From the Definition 4.3.1, we have

$$\begin{aligned}(J_{r,\alpha}J_{t,\alpha}u)(x) &= \mathcal{F}_\alpha{}^{-1}\Big[(1+\|\xi\|^2)^{-r/2}\big(\mathcal{F}_\alpha(J_{t,\alpha}u)\big)(\xi)\Big](x)\\ &= \mathcal{F}_\alpha{}^{-1}\Big[(1+\|\xi\|^2)^{-r/2}(1+\|\xi\|^2)^{-t/2}(\mathcal{F}_\alpha u)(\xi)\Big](x)\\ &= \mathcal{F}_\alpha{}^{-1}\Big[(1+\|\xi\|^2)^{-(r+t)/2}(\mathcal{F}_\alpha u)(\xi)\Big](x)\\ &= (J_{r+t,\alpha}u)(x).\end{aligned}$$

□

Definition 4.3.3. For $r \in \mathbb{R}$ and $1 < p < \infty$, then following space is defined

$$H^{r,p}_\alpha = \big\{u \in S'_*(\mathbb{R}^{n+1}_+) : J_{-r,\alpha}u \in L^p_\alpha(\mathbb{R}^{n+1}_+)\big\}. \tag{4.3.2}$$

Then in the space $H^{r,p}_\alpha$ the following norm is given

$$\begin{aligned}\|u\|_{H^{r,p}_\alpha} &= \|J_{-r,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}\\ &= \Big(\int_{\mathbb{R}^{n+1}_+}\big|(J_{-r,\alpha}u)(\xi)\big|^p d\mu_\alpha(\xi)\Big)^{1/p}.\end{aligned} \tag{4.3.3}$$

The space $H^{r,p}_\alpha$ is called the $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - Sobolev space of order r.

If $r = 0$, then (4.3.2) becomes

$$H^{0,p}_\alpha = L^p_\alpha(\mathbb{R}^{n+1}_+).$$

Theorem 4.3.4. *Let $r, t \in \mathbb{R}$ and $1 < p < \infty$, then the Bessel potential $J_{t,\alpha}$ is an isometry of $H^{r,p}_\alpha$ onto $H^{r+t,p}_\alpha$. Moreover,*

$$\|J_{t,\alpha}u\|_{H^{r+t,p}_\alpha} = \|u\|_{H^{r,p}_\alpha}, \quad \forall\ u \in H^{r,p}_\alpha. \tag{4.3.4}$$

Proof. Let $u \in H^{r,p}_\alpha$. Then from (4.3.3), we have

$$\|J_{t,\alpha}u\|_{H^{r+t,p}_\alpha} = \|J_{-r-t,\alpha}J_{t,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Using Lemma 4.3.2, we find

$$\|J_{t,\alpha}u\|_{H^{r+t,p}_\alpha} = \|J_{-r,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

By invoking (4.3.3), above yields

$$\|J_{t,\alpha}u\|_{H^{r+t,p}_\alpha} = \|u\|_{H^{r,p}_\alpha}, \quad \forall\ u \in H^{r,p}_\alpha.$$

The last expression shows that, $J_{t,\alpha} : H^{r,p}_\alpha \to H^{r+t,p}_\alpha$ is an isometry.

For each $v \in H^{r+t,p}_\alpha$, take $u = J_{-t,\alpha}v$. Then, by Lemma 4.3.2, we get $v = J_{t,\alpha}u$. Therefore, $u \in H^{r,p}_\alpha$. Hence, $J_{t,\alpha} : H^{r,p}_\alpha \to H^{r+t,p}_\alpha$ is an onto isometry. □

Theorem 4.3.5. *Let $\sigma(x,\xi)$ be a symbol in $S^m, m \in \mathbb{R}$. Then for $r \in \mathbb{R}$ and $1 < p < \infty$, the pseudo-differential operator $T_\sigma : H^{r,p}_\alpha \to H^{r-m,p}_\alpha$ is a bounded linear operator.*

Proof. First, we consider the following operators:

$$J_{-r,\alpha} : H^{r,p}_\alpha \to H^{0,p}_\alpha,$$

$$T_\sigma J_{m,\alpha} : H^{0,p}_\alpha \to H^{0,p}_\alpha,$$

and

$$J_{r-m,\alpha} : H^{0,p}_\alpha \to H^{r-m,p}_\alpha,$$

which are linear. Then, from Theorem 4.3.4, the operators $J_{-r,\alpha} : H^{r,p}_\alpha \to H^{0,p}_\alpha$ and $J_{r-m,\alpha} : H^{0,p}_\alpha \to H^{r-m,p}_\alpha$ are bounded. Also, by Theorem 4.2.5, $T_\sigma J_{m,\alpha} : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is the bounded linear operator. Hence, $T_\sigma : H^{r,p}_\alpha \to H^{r-m,p}_\alpha$ is a bounded linear operator. □

Conclusion: Taking concepts of the papers of Fefferman [17], Illner [30], Cato [31], Nagase [43], Hwang-Lee [29], Wong [81] and Pathak and Upadhyay [57], the $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators associated with the Weinstein transform on a certain class of symbol S^0 is discussed. Applying the theory of $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness results, various properties of pseudo-differential operators associated with $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - Sobolev space are investigated. The Weinstein transform has rich calculus and well established theory. The results of $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators contains useful applications in Sobolev spaces which are given in the present chapter. The aforesaid theory acts as the bridge between pseudo-differential operators and maximal - minimal pseudo-differential operators. This theory will also be useful in functional analysis, partial-differential equations and other areas of mathematics.

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