Chapter 3

Approximation and Convergence of Generalized Fractional Sturm-Liouville Problem via Integral Form

This chapter deals with the study of GFSLP which is defined in terms of CGFD. The introduction part of this chapter is provided in Section 3.1. The conversion of GFSLP to its corresponding integral form using weighted Laplace transform is given in Section 3.2. Section 3.3 provides the numerical algorithm based on approximation of generalized fractional integral using Lagrange polynomials. In further sections, we have discussed the error analysis of the scheme and numerical examples. Further, the chapter is concluded.

3.1 Introduction

In [17], Agrawal reviewed the fractional calculus (FC) and discussed the generalized fractional integrals and derivatives. He also showed the application of generalized operators to integral equations. Further, generalized fractional integrals were also used to derive the generalized fractional derivatives of Caputo and R-L types. The advantage of the generalized fractional derivatives is that many existing integrals and derivatives, such as R-L, Caputo, Riesz, Hadamard, etc., are the particular case of generalized fractional derivatives. Initially, authors in [70] studied a class of new generalized advection diffusion equation. Further, authors in [71] presented three generalization of Van der Pol equation and also obtained its numerical solution. The equation of motion of an oscillator using generalized derivative is discussed in [72]. In 2023, Pandey et al. [73] discussed Sturm's theorems for generalized derivative and also defined generalized Sturm-Liouville problem. Motivated by this generalization, we have defined the GFSLP as the Caputo generalized fractional derivative and present a numerical approach to find solution to this problem.

The purpose of this study is to convert the GFSLP into an integral form and to give a numerical scheme based on the Lagrange polynomials. Using the weighted Laplace and inverse weighted Laplace transform [74], first the corresponding integral formulation is obtained and then an approximation of the generalized fractional integral is formulated using Lagrange interpolation. Further, the eigenvalues and the eigenfunctions are determined by the stated numerical algorithm.

3.1.1 Weighted Laplace Transform: Definitions and Preliminaries

The classical Laplace transform is difficult to utilize for the weighted fractional order. Therefore, Jarad et al. [74] proposed the weighted Laplace transform and some of its properties as follow:

Definition 3.1. Let Φ and w be functions defined on the interval $[a,\infty)$ and ξ be a strictly increasing function on $[a,\infty)$. Then the weighted Laplace transform of Φ is defined as

$$L_{\xi}^{w}\{\Phi\}(s)=\int_{a}^{\infty}e^{-s(\xi(x)-\xi(a))}w(x)\Phi(x)\xi'(x)dx, \tag{3.1}$$

for all values of s for which the integral given in Eq (3.1) is valid.

Definition 3.2. Let $\Phi, w(\neq 0):[a,\infty)\to\mathbb{R}$. Φ is called w-weighted ξ-exponential function if there exist constants M, p and N such that

$$|w(x)\Phi(x)|\leq Me^{p\ \xi(x)},\quad \text{for}\quad x>N. \tag{3.2}$$

Theorem 3.3. *[74] Let $\Phi, w(\neq 0):[a,\infty)\to\mathbb{R}$ be functions such that Φ is piecewise continuous function such that Φ is w-weighted ξ-exponential function. Then the weighted Laplace tranform of Φ exists for $s>c$.*

Theorem 3.4. *Let Φ be a piecewise continuous function on interval $[a,x]$ and satisfies Eq.(3.2). Then,*

$$L_{\xi}^{w}\{(I_{a+;[\xi,w]}^{\alpha}\Phi)(x)\}(s)=\frac{L_{\xi}^{w}\{\Phi(x)\}(s)}{s^{\alpha}}. \tag{3.3}$$

Corollary 3.5. *Let $\alpha > 0, \Phi \in AC_w^n[a,b]$ for any $b > 0, \xi \in C^n[a,b], \xi'(x) > 0$ and $\Phi^{(k)}, k = 0,1,2,...,n-1$ be w-weighted ξ-exponential function. Then,*

$$L_\xi^w\{({}^C_{a+}D^\alpha_{[\xi,w]}\Phi)(x)\}(s) = s^\alpha\left[L_\xi^w\{\ \Phi(x)\ \}(s) - \sum_{k=0}^{n-1} s^{-k-1}\Phi^{(k)}(a)\right], \tag{3.4}$$

where, $\Phi^{(k)}$ denotes the k-th order derivative of Φ.

Proposition 3.6. *1.* $L_\xi^w\{[w(x)]^{-1}e^{\rho(\xi(x)-\xi(a))}\}(s) = \frac{1}{s-\rho}, \rho \in \mathbb{R}, s > \rho.$
2. $L_\xi^w\{[w(x)]^{-1}(\xi(x)-\xi(a))^{\gamma-1}\}(s) = \frac{\Gamma(\gamma)}{s^\gamma}, \gamma > -1, s > 0.$

3.2 Transformation of Generalized Fractional Sturm-Liouville Problem into an Integral Form

We consider a GFSLP as follows:

$${}^C_{a+}D^\alpha_{[\xi,w]}y(x) + (\lambda r(x) - q(x))y(x) = 0, \quad x\in(0,1), \tag{3.5}$$

where $q(x)$ and $r(x)$ are real-valued continuous functions on the interval $[0,1]$ such that $r(x) \neq 0, 1 < \alpha \leq 2$ with boundary conditions,

$$c_1y(0) + c_2y'(0) = 0, \qquad d_1y(1) + d_2y'(1) = 0, \tag{3.6}$$

with $c_1, c_2, d_1, d_2 \in \mathbb{R}, c_1^2 + c_2^2 \neq 0, d_1^2 + d_2^2 \neq 0.$

Proof: We first take the weighted Laplace transform (using Eq.(3.1)) on both sides of Eq.(3.5) and get

$$L_\xi^w\{{}^C_{a+}D^\alpha_{[\xi,w]}y(x)\} + L_\xi^w\{(\lambda r(x) - q(x))y(x)\} = 0.$$

Using Eq.(3.4), we have

$$s^{\alpha} L_{\xi}^{w}[y(x);s] - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) + L_{\xi}^{w}[G(x);s] = 0,$$

where

$$G(x) = (\lambda r(x) - q(x))y(x).$$

Further, we get

$$L_{\xi}^{w}[y(x);s] = \frac{1}{s}y(0) + \frac{1}{s^2}y'(0) - \frac{1}{s^{\alpha}}L_{\xi}^{w}[G(x);s]. \tag{3.7}$$

Now, taking inverse weighted Laplace transform on both sides of Eq.(3.7) and using Propositions 3.6, and Theorem 3.4, we get

$$y(x) = [w(x)]^{-1}y(0) + [w(x)]^{-1}(\xi(x) - \xi(a))y'(0) - I_{a+;[\xi,w]}^{\alpha}G(x). \tag{3.8}$$

Now, we apply boundary conditions (3.6) to obtain $y(0)$ and $y'(0)$

Case-1 If $c_1 \neq 0, d_1 \neq 0$, i.e.

$$y(0) = \frac{-c_2}{c_1}y'(0), \qquad y(1) + \frac{d_2}{d_1}y'(1) = 0. \tag{3.9}$$

Using Eq.(3.8) and Eq.(3.9), we have

$$y'(0) = \frac{[I_{a+;[\xi,w]}^{\alpha}G(x)|_{x=1} + \frac{d_2}{d_1}DI_{a+;[\xi,w]}^{\alpha}G(x)]|_{x=1}}{K_1(x)}, \tag{3.10}$$

where,

$$\begin{aligned} K_1(x) = \frac{-c_2}{c_1}[w(1)]^{-1} + (\xi(1) - \xi(0))[w(1)]^{-1} - \frac{c_2 d_2}{c_1 d_1}([w(x)]^{-1})'|_{x=1} + \\ \frac{d_2}{d_1}([w(x)]^{-1}(\xi(x) - \xi(a)))'|_{x=1}. \end{aligned} \tag{3.11}$$

Thus, Eq.(3.8) implies

$$y(x) = [w(x)]^{-1}\frac{1}{K_1(x)}\left[\frac{-c_2}{c_1} + (\xi(x)-\xi(a))\right]\left[I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}+ \frac{d_2}{d_1}DI^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}\right] - I^{\alpha}_{a+;[\xi,w]}G(x). \tag{3.12}$$

Case-2 If $c_2 \neq 0, d_2 \neq 0$, i.e

$$y'(0) = \frac{-c_1}{c_2}y(0), \qquad \frac{d_1}{d_2}y(1) + y'(1) = 0. \tag{3.13}$$

Following the similar steps as in Case 1, we have

$$y(x) = [w(x)]^{-1}\frac{1}{K_2(x)}\left[1 - \frac{c_1}{c_2}(\xi(x)-\xi(a))\right]\left[\frac{d_1}{d_2}I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}+ DI^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}\right] - I^{\alpha}_{a+;[\xi,w]}G(x), \tag{3.14}$$

where,

$$K_2(x) = \frac{d_1}{d_2}[w(1)]^{-1} - \frac{c_1d_1}{c_2d_2}(\xi(1)-\xi(0))[w(1)]^{-1} + ([w(x)]^{-1})'|_{x=1}+ \frac{c_1}{c_2}([w(x)]^{-1}(\xi(x)-\xi(a)))'|_{x=1}. \tag{3.15}$$

Case-3 If $c_1 \neq 0, d_2 \neq 0$, i.e.

$$y(0) = \frac{-c_2}{c_1}y'(0), \qquad \frac{d_1}{d_2}y(1) + y'(1) = 0. \tag{3.16}$$

$$y(x) = [w(x)]^{-1}\frac{1}{K_3(x)}\left[\frac{-c_2}{c_1} + (\xi(x)-\xi(a))\right]\left[\frac{d_1}{d_2}I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}+ DI^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}\right] - I^{\alpha}_{a+;[\xi,w]}G(x), \tag{3.17}$$

where,

$$K_3(x) = \frac{-c_2 d_1}{c_1 d_2}[w(1)]^{-1} + \frac{d_1}{d_2}(\xi(1) - \xi(0))[w(1)]^{-1} - \frac{c_2}{c_1}([w(x)]^{-1})'|_{x=1} + ([w(x)]^{-1}(\xi(x) - \xi(a)))'|_{x=1}. \quad (3.18)$$

Case-4 If $c_2 \neq 0, d_1 \neq 0$, i.e.

$$y'(0) = \frac{-c_1}{c_2}y(0), \qquad y(1) + \frac{d_2}{d_1}y'(1) = 0. \quad (3.19)$$

$$y(x) = [w(x)]^{-1}\frac{1}{K_4(x)}\left[1 - \frac{c_1}{c_2}(\xi(x) - \xi(a))\right]\left[I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1} + \frac{d_2}{d_1}DI^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}\right] - I^{\alpha}_{a+;[\xi,w]}G(x), \quad (3.20)$$

where,

$$K_4(x) = [w(1)]^{-1}\left[1 - \frac{c_1}{c_2}(\xi(1) - \xi(0))\right] - \frac{d_2}{d_1}([w(x)]^{-1})'|_{x=1} - \frac{c_1 d_2}{c_2 d_1}([w(x)]^{-1} (\xi(x) - \xi(a)))'|_{x=1}. \quad (3.21)$$

Now, we discuss the numerical approximation of the generalized integral operator $I^{\alpha}_{a+;[\xi,w]}$.

3.3 Numerical Algorithm

We develop an approximation of the generalized integral operator appeared in Eq.(3.8) using quadratic polynomials. For this, suppose $P_N = \{a = x_0 < x_1 < x_2 < ... < t_N = b\}$ be the partition of $[a, b]$ into N parts with equal spacing $\delta = \frac{b-a}{N}$. For convenience, we denote $G(x_k) = G_k$, $w(x_k) = w_k$, and $\xi(x_k) = \xi_k$. We approximate the operator

$I^{\alpha}_{a+;[\xi,w]}G(x)$ at nodes $x_k, k = 0, 1, ..., N$ as follows:

$$I^{\alpha}_{a+;[\xi,w]}G(x) = \frac{[w(x)]^{-1}}{\Gamma(\alpha)} \int_a^x \frac{w(s)\xi'(s)G(s)}{[\xi(x)-\xi(s)]^{1-\alpha}} ds.$$

$$\begin{aligned}
I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=x_0} &= 0, \\
I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=x_k} &= \frac{[w(x_k)]^{-1}}{\Gamma(\alpha)} \sum_{l=1}^{k} \int_{x_{l-1}}^{x_l} \frac{w(s)\xi'(s)G(s)}{[\xi(x_k)-\xi(s)]^{1-\alpha}} ds \\
&= \frac{[w(x_k)]^{-1}}{\Gamma(\alpha)} \Bigg[\int_{x_0}^{x_1} \frac{w(s)\xi'(s)G(s)}{[\xi(x_k)-\xi(s)]^{1-\alpha}} ds + \sum_{l=2}^{k} \int_{x_{l-1}}^{x_l} \frac{w(s)\xi'(s)G(s)}{[\xi(x_k)-\xi(s)]^{1-\alpha}} ds \Bigg].
\end{aligned}$$

Let $v = \xi(s) \implies dv = \xi'(s)ds$,

$$\begin{aligned}
I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=x_k} = \frac{[w(x_k)]^{-1}}{\Gamma(\alpha)} \int_{\xi_0}^{\xi_1} \frac{w(\xi^{-1}(v))G(\xi^{-1}(v))}{[\xi(x_k)-v]^{1-\alpha}} dv + \\
\frac{[w(x_k)]^{-1}}{\Gamma(\alpha)} \sum_{l=2}^{k} \int_{\xi_{l-1}}^{\xi_l} \frac{w(\xi^{-1}(v))G(\xi^{-1}(v))}{[\xi(x_k)-v]^{1-\alpha}} dv. \qquad (3.22)
\end{aligned}$$

On $[\xi_0, \xi_1]$, we take linear interpolation polynomial $G_1(v)$ to approximate $G(x)$ and for other intervals ($k \geq 2$), the quadratic approximation is considered using points $(\xi_{l-2}, G_{l-2}), (\xi_{l-1}, G_{l-1})$ and (ξ_l, G_l), such that,

$$G_1(v) = \frac{(v-\xi_1)}{\xi_0-\xi_1} w_0 G_0 + \frac{(v-\xi_0)}{(\xi_1-\xi_0)} w_1 G_1, \qquad (3.23)$$

$$\begin{aligned}
G_l(v) = \frac{(v-\xi_{l-1})(v-\xi_l)}{(\xi_{l-2}-\xi_{l-1})(\xi_{l-2}-\xi_l)} w_{l-2} G_{l-2} + \frac{(v-\xi_{l-2})(v-\xi_l)}{(\xi_{l-1}-\xi_{l-2})(\xi_{l-1}-\xi_l)} w_{l-1} G_{l-1} \\
+ \frac{(v-\xi_{l-1})(v-\xi_{l-2})}{(\xi_l-\xi_{l-1})(\xi_l-\xi_{l-2})} w_l G_l. \qquad (3.24)
\end{aligned}$$

Now, taking the first term of Eq.(3.22), we get

$$\frac{[w(x_k)]^{-1}}{\Gamma(\alpha)} \int_{\xi_0}^{\xi_1} \frac{w(\xi^{-1}(v))G(\xi^{-1}(v))}{[\xi(x_k)-v]^{1-\alpha}} dv \approx a(k) w_0 G_0 + b(k) w_1 G_1, \qquad (3.25)$$

where,

$$a(k) = \frac{1}{(\xi_1 - \xi_0)}[(\alpha+1)(\xi_1 - \xi_0)(\xi_k - \xi_0)^{\alpha} + (\xi_k - \xi_1)^{\alpha+1} - (\xi_k - \xi_0)^{\alpha+1}], \quad (3.26)$$

$$b(k) = \frac{1}{(\xi_0 - \xi_1)}[-(\alpha+1)(\xi_0 - \xi_1)(\xi_k - \xi_1)^{\alpha} + (\xi_k - \xi_1)^{\alpha+1} - (\xi_k - \xi_0)^{\alpha+1}]. \quad (3.27)$$

And the second term of Eq.(3.22) gives

$$\frac{[w(x_k)]^{-1}}{\Gamma(\alpha)} \sum_{l=2}^{k} \int_{\xi_{l-1}}^{\xi_l} \frac{w(\xi^{-1}(v))G(\xi^{-1}(v))}{[\xi(x_k) - v]^{1-\alpha}} dv \approx \sum_{l=2}^{k} \bar{\alpha}(k,l) w_{l-2} G_{l-2} + \bar{\beta}(k,l) w_{l-1} G_{l-1} + \bar{\gamma}(k,l) w_l G_l, \quad (3.28)$$

where,

$$\bar{\alpha}(k,l) = \frac{1}{(\xi_{l-2} - \xi_{l-1})(\xi_{l-2} - \xi_l)} \Big[-(\alpha+2)(\xi_l - \xi_{l-1})(\xi_k - \xi_l)^{\alpha+1} - (\alpha+2) (\xi_l - \xi_{l-1})(\xi_k - \xi_{l-1})^{\alpha+1} - 2(\xi_k - \xi_l)^{\alpha+2} + 2(\xi_k - \xi_{l-1})^{\alpha+2} \Big], \quad (3.29)$$

$$\bar{\beta}(k,l) = \frac{-1}{(\xi_{l-1} - \xi_{l-2})(\xi_{l-1} - \xi_l)} \Big[(\alpha+1)(\alpha+2)(\xi_{l-1} - \xi_{l-2})(\xi_{l-1} - \xi_l) (\xi_k - \xi_{l-1})^{\alpha} - (\alpha+2)(\xi_l - \xi_{l-2})(\xi_k - \xi_l)^{\alpha+1} + (2\xi_{l-1} - \xi_{l-2}) (\xi_k - \xi_{l-1})^{\alpha+1}(\alpha+2) - 2(\xi_k - \xi_l)^{\alpha+2} + 2(\xi_k - \xi_{l-1})^{\alpha+2} \Big], \quad (3.30)$$

$$\bar{\gamma}(k,l) = \frac{1}{(\xi_{l-1} - \xi_{l-2})(\xi_{l-1} - \xi_l)} \Big[-(\alpha+1)(\alpha+2)(\xi_l - \xi_{l-2})(\xi_l - \xi_{l-1}) (\xi_k - \xi_l)^{\alpha} - (2\xi_l - \xi_{l-1} - \xi_{l-2})(\xi_k - \xi_l)^{\alpha+1}(\alpha+2) + (\xi_{l-1} - \xi_{l-2}) (\xi_k - \xi_{l-1})^{\alpha+1}(\alpha+2) - 2(\xi_k - \xi_l)^{\alpha+2} + 2(\xi_k - \xi_{l-1})^{\alpha+2} \Big]. \quad (3.31)$$

Now, combining Eq.(3.25) and (3.28), we obtain an approximate scheme for the generalized integral operator

$$I^{\alpha}_{a+;[\xi,w]}G(x_k) \approx \sum_{l=0}^{k} \psi_l^k G_l, \tag{3.32}$$

where, the coefficients ψ_l^k are given as follows,

for $k = 1$,

$$\begin{cases} \psi_0^1 = a(1), \\ \psi_1^1 = b(1), \end{cases} \tag{3.33}$$

for $k = 2$,

$$\begin{cases} \psi_0^2 = \bar{\alpha}(2,2) + a(2), \\ \psi_1^2 = \bar{\beta}(2,2) + b(2), \\ \psi_2^2 = \bar{\gamma}(2,2), \end{cases} \tag{3.34}$$

for $k = 3$,

$$\begin{cases} \psi_0^3 = \bar{\alpha}(3,2) + a(3), \\ \psi_1^3 = \bar{\alpha}(3,3) + \bar{\beta}(3,2) + b(3), \\ \psi_2^3 = \bar{\gamma}(3,3) + \bar{\gamma}(3,2), \\ \psi_3^3 = \bar{\gamma}(3,3), \end{cases} \tag{3.35}$$

and for $k > 3$,

$$
\begin{cases}
\psi_0^k &= \bar{\alpha}(k,2) + a(k),\\
\psi_1^k &= \bar{\alpha}(k,3) + \bar{\beta}(k,2) + b(k),\\
\psi_l^k &= \bar{\alpha}(k,k-l+2) + \bar{\beta}(k,k-l+1) + \bar{\gamma}(k,k-l), (2 \le l \le k-2)\\
\psi_{k-1}^k &= \bar{\beta}(k,k) + \bar{\gamma}(k,k-1),\\
\psi_k^k &= \bar{\gamma}(k,k).
\end{cases}
\tag{3.36}
$$

The matrix form of linear algebraic equations thus obtained for the approximation of Eq.(3.32) for every node $x_k, k = 0,1,2,...,N$ as follows

$$Y_L^\alpha = G^\alpha Y, \tag{3.37}$$

where,

$$
G^\alpha = \begin{bmatrix}
\psi_0^0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\psi_0^1 & \psi_1^1 & 0 & \cdots & 0 & 0 & 0\\
\psi_0^2 & \psi_1^2 & \psi_2^2 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
\psi_0^k & \psi_1^k & \psi_2^k & \cdots & \psi_l^k & 0 & 0\\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots\\
\psi_0^N & \psi_1^N & \psi_2^N & \cdots & \psi_l^N & \cdots & \psi_N^N
\end{bmatrix},
Y_L^\alpha = \begin{bmatrix}
I^\alpha_{a+;[\xi,w]} y(x)|_{x=x_0}\\
I^\alpha_{a+;[\xi,w]} y(x)|_{x=x_1}\\
I^\alpha_{a+;[\xi,w]} y(x)|_{x=x_2}\\
\vdots\\
I^\alpha_{a+;[\xi,w]} y(x)|_{x=x_k}\\
\vdots\\
I^\alpha_{a+;[\xi,w]} y(x)|_{x=x_N}
\end{bmatrix},
Y = \begin{bmatrix}
y_0\\ y_1\\ y_2\\ \vdots\\ y_i\\ \vdots\\ y_N
\end{bmatrix}
$$

where G^α is a matrix of order $(N+1) \times (N+1)$.

Now, we will discuss the formation of the eigenvalue problems for all four cases having suitable boundary conditions.

3.3.1 Formation of the Eigenvalue Problem

Case 1: Consider the GFSLP with boundary conditions of Dirichlet type as $y(0) = 0, y(1) = 0$.

Thus, we have $(c_1 \neq 0, d_1 \neq 0)$ and Eq.(3.12) implies

$$y(x) = [w(x)]^{-1}\frac{(\xi(x)-\xi(a))}{(\xi(1)-\xi(0))[w(1)]^{-1}}[I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}] - I^{\alpha}_{a+;[\xi,w]}G(x). \tag{3.38}$$

The discrete form of Eq.(3.38) is as follows:

$$y(x_k) = [w(x_k)]^{-1}\frac{(\xi(x_k)-\xi(a))}{(\xi(1)-\xi(0))[w(1)]^{-1}}[I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=x_N}] - I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=x_k}. \tag{3.39}$$

Using the approximation of the generalized integral operator given by Eq.(3.32), we have

$$\begin{aligned} y_k+[w(x_k)]^{-1}&\frac{(\xi(x_k)-\xi(a))}{(\xi(x_N)-\xi(0))[w(1)]^{-1}}\sum_{l=1}^{N-1}\psi_l^N q(x_l)y_l - \sum_{l=1}^{k}\psi_l^k q(x_l)y_l \\ &= \lambda\left[[w(x_k)]^{-1}\frac{(\xi(x_k)-\xi(a))}{(\xi(x_N)-\xi(0))[w(1)]^{-1}}\sum_{l=1}^{N-1}\psi_l^N r(x_l)y_l - \sum_{l=1}^{k}\psi_l^k r(x_l)y_l\right], \\ y_k+S_k\sum_{l=1}^{N-1}\psi_l^N q(x_l)y_l &- \sum_{l=1}^{k}\psi_l^k q(x_l)y_l = \lambda\left[S_k\sum_{l=1}^{N-1}\psi_l^N r(x_l)y_l - \sum_{l=1}^{k}\psi_l^k r(x_l)y_l\right], \end{aligned} \tag{3.40}$$

where, $S_k = [w(x_k)]^{-1}\frac{(\xi(x_k)-\xi(a))}{(\xi(x_N)-\xi(0))[w(1)]^{-1}}$.

Now, rewriting Eq.(3.40) in matrix form, the generalized eigenvalue problem obtained as follows

$$(I+E)Y = \lambda FY. \tag{3.41}$$

where, $E = GQ,\ F = GR,\ Q = diag(q(x_1), q(x_2), ..., q(x_{N-1}))$,
$R = diag(r(x_1), r(x_2), ..., r(x_{N-1})), Y = [y_1, y_2, ..., y_{N-1}]$,

$$G = \begin{bmatrix} S_1\psi_1^N - \psi_1^1 & S_1\psi_2^N & S_1\psi_3^N & \cdots & S_1\psi_{N-1}^N \\ S_2\psi_1^N - \psi_1^2 & S_2\psi_2^N - \psi_2^2 & S_2\psi_3^N & \cdots & S_2\psi_{N-1}^N \\ S_3\psi_1^N - \psi_1^3 & S_3\psi_2^N - \psi_2^3 & S_3\psi_3^N - \psi_3^3 & \cdots & S_3\psi_{N-1}^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_k\psi_1^N - \psi_1^k & S_k\psi_2^N - \psi_2^k & S_k\psi_3^N - \psi_3^k & \ddots & S_k\psi_{N-1}^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{N-1}\psi_1^N - \psi_1^{N-1} & S_{N-1}\psi_2^N - \psi_2^{N-1} & S_{N-1}\psi_3^N - \psi_3^{N-1} & \cdots & S_{N-1}\psi_{N-1}^N \end{bmatrix},$$

and the matrix G is of order $(N-1)\times(N-1)$.

Case 2: Consider GFSLP with boundary conditions of mixed Dirichlet-Neumann type as $y'(0) = y(1) = 0$.

Here, we have $(c_2 \neq 0, d_1 \neq 0)$ and Eq.(3.20) implies

$$y(x) = \frac{[w(x)]^{-1}}{[w(1)]^{-1}}[I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=1}] - I^{\alpha}_{a+;[\xi,w]}G(x). \tag{3.42}$$

For every node $x_k, k = 0, 1, 2, ...N-1$, the discrete form of Eq.(3.42) is written as follows:

$$y(x_k) = \frac{[w(x_k)]^{-1}}{[w(1)]^{-1}}[I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=x_N}] - I^{\alpha}_{a+;[\xi,w]}G(x)|_{x=x_k}. \tag{3.43}$$

Using the approximation of generalized integral operator given by Eq.(3.32), we have

$$y_k + \frac{[w(x_k)]^{-1}}{[w(x_N)]^{-1}} \sum_{l=0}^{N-1} \psi_l^N q(x_l) y_l - \sum_{l=0}^{k} \psi_l^k q(x_l) y_l = \lambda \Bigg[\frac{[w(x_k)]^{-1}}{[w(x_N)]^{-1}} \sum_{l=0}^{N-1} \psi_l^N r(x_l) y_l - \sum_{l=0}^{k} \psi_l^k r(x_l) y_l \Bigg]$$

$$y_k + T_k \sum_{l=0}^{N-1} \psi_l^N q(x_l) y_l - \sum_{l=0}^{k} \psi_l^k q(x_l) y_l = \lambda \Bigg[T_k \sum_{l=0}^{N-1} \psi_l^N r(x_l) y_l - \sum_{l=0}^{k} \psi_l^k r(x_l) y_l \Bigg] \tag{3.44}$$

where $T_k = \frac{[w(x_k)]^{-1}}{[w(x_N)]^{-1}}$.

Similarly, using the matrix approach, Eq.(3.44) can be written as in Eq.(3.41). Here, matrices Q and R are same as defined in Case 1 and matrix G is as follows and

$$G = \begin{bmatrix} T_0\psi_0^N - \psi_0^0 & T_0\psi_1^N & T_0\psi_2^N & \cdots & T_0\psi_{N-1}^N \\ T_1\psi_0^N - \psi_1^0 & T_1\psi_1^N - \psi_1^1 & T_1\psi_2^N & \cdots & T_1\psi_{N-1}^N \\ T_2\psi_0^N - \psi_2^0 & S_2\psi_1^N - \psi_1^2 & T_2\psi_2^N - \psi_2^2 & \cdots & T_2\psi_{N-1}^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_k\psi_0^N - \psi_k^0 & T_k\psi_1^N - \psi_1^k & T_k\psi_2^N - \psi_2^k & \ddots & T_k\psi_{N-1}^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{N-1}\psi_0^N - \psi_{N-1}^0 & T_{N-1}\psi_1^N - \psi_1^{N-1} & T_{N-1}\psi_2^N - \psi_2^{N-1} & \cdots & T_{N-1}\psi_{N-1}^N \end{bmatrix}.$$

Case 3: Consider GFSLP with boundary conditions of mixed Dirichlet-Neumann type as $y(0) = y'(0) \neq 0, y(1) = 0$.

In this case $(c_1 = -c_2 \neq 0, d_1 \neq 0)$ and

$$y(x) = \frac{[w(x)]^{-1}}{[w(1)]^{-1}} \frac{[1 + (\xi(x) - \xi(a))]}{[1 + (\xi(1) - \xi(a))]} [I^{\alpha}_{a+;[\xi,w]} G(x)|_{x=1}] - I^{\alpha}_{a+;[\xi,w]} G(x). \tag{3.45}$$

For every node $x_k, k = 0, 1, 2, ..., N-1$, the discrete form of Eq (3.45) is expressed as follows:

$$y(x_k) = \frac{[w(x_k)]^{-1}}{[w(1)]^{-1}} \frac{[1 + (\xi(x_k) - \xi(a))]}{[1 + (\xi(1) - \xi(a))]} [I^{\alpha}_{a+;[\xi,w]} G(x)|_{x=x_N}] - I^{\alpha}_{a+;[\xi,w]} G(x)|_{x=x_k}. \tag{3.46}$$

Using the approximation of the generalized integral operator, we have the same formulation as in Eq.(3.44) where $T_k = \frac{[w(x_k)]^{-1}}{[w(1)]^{-1}} \frac{[1+(\xi(x_k)-\xi(a))]}{[1+(\xi(1)-\xi(a))]}$ and same matrices as given in Case 2.

3.4 Error Analysis of the Scheme

In this section, we determine the rate of convergence of the proposed numerical method and discuss the error of approximation. The numerical calculations of the rate of convergence of the kth eigenvalue for the variable N and for fixed parameter α, is given by following formula

$$erc_{\lambda}(N, k, \alpha) = \log_2 \frac{\lambda_k^{(N,\alpha)} - \lambda_k^{(N/2,\alpha)}}{\lambda_k^{(2N,\alpha)} - \lambda_k^{(N,\alpha)}}. \tag{3.47}$$

3.4.1 An Error of Approximation for the Generalized Fractional Integral

To analyze the error of approximation for the generalized integral operator, we use the notation $U(s) = w(s)G(s)$ and $s = \xi^{-1}(v)$ for simplicity. Hence, we have $U(v) = w(\xi^{-1}(v))G(\xi^{-1}(v))$, $\xi(x_k) = \xi_k$ and $w(x_k) = w_k$.

We interpolate $U(v)$ on $[\xi_0, \xi_1]$ and $[\xi_{k-1}, \xi_k]$ for $k > 1$ by $\tilde{U_1}(v)$ and $\tilde{U_2}(v)$ respectively such that

$$U(v)-\tilde{U}_1(v)=\frac{U''(\xi_1)}{2!}(v-\xi_0)(v-\xi_1),\xi_1\in[\xi_0,\xi_1], \tag{3.48}$$

$$U(v)-\tilde{U}_2(v)=\frac{U'''(\xi_2)}{3!}(v-\xi_{k-2})(v-\xi_{k-1})(v-\xi_k),\xi_2\in[\xi_{k-1},\xi_k], \tag{3.49}$$

where $\tilde{U}_1(v)$ and $\tilde{U}_2(v)$ are piecewise linear and quadratic interpolation polynomials using the nodes $(\xi_0,U_0),(\xi_1,U_1)$ and $(\xi_{k-2},U_{k-2}),(\xi_{k-1},U_{k-1}),(\xi_k,U_k)$ respectively. The truncation error is given by

$$E^k=\frac{[w(x_k)]^{-1}}{\Gamma(\alpha)}\int_{\xi_0}^{\xi_1}\frac{U(v)-\tilde{U}_1(v)}{[\xi(x_k)-v]^{1-\alpha}}dv+\frac{[w(x_k)]^{-1}}{\Gamma(\alpha)}\sum_{l=2}^{k}\int_{\xi_{l-1}}^{\xi_l}\frac{U(v)-\tilde{U}_2(v)}{[\xi(x_k)-v]^{1-\alpha}}dv. \tag{3.50}$$

Consider first term of Eq.(3.50)

$$\begin{aligned}
\frac{[w_k]^{-1}}{\Gamma(\alpha)}\int_{\xi_0}^{\xi_1}\frac{U(v)-\tilde{U}_1(v)}{[\xi(x_k)-v]^{1-\alpha}}dv &= \frac{[w_k]^{-1}}{2\Gamma(\alpha)}\int_{\xi_0}^{\xi_1}\frac{U''(\xi_1)(v-\xi_0)(v-\xi_1)}{[\xi(x_k)-v]^{1-\alpha}}dv\\
&\leq \frac{[w_k]^{-1}}{2\Gamma(\alpha)}\max_{x_0\leq\xi_1\leq x_k}U''(\xi_1)\int_{\xi_0}^{\xi_1}(v-\xi_0)(v-\xi_1)[\xi(x_k)-v]^{\alpha-1}dv\\
&\leq \frac{[w_k]^{-1}}{8\Gamma(\alpha)}\max_{x_0\leq\xi_1\leq x_k}U''(\xi_1)(\xi_1-\xi_0)^2\int_{\xi_0}^{\xi_1}-[\xi(x_k)-v]^{\alpha-1}dv\\
&= \frac{[w_k]^{-1}}{8\Gamma(\alpha+1)}\max_{x_0\leq\xi_1\leq x_k}U''(\xi_1)(\xi_1-\xi_0)^2[(\xi_k-\xi_1)^\alpha-(\xi_k-\xi_0)^\alpha].
\end{aligned}$$

Since, scale function $\xi(x)$ satisfy Lipschitz condition with constant L i.e. $|\xi_l-\xi_{l-1}|<Lh$, therefore,

$$\begin{aligned}
&\frac{[w_k]^{-1}}{\Gamma(\alpha)}\int_{\xi_0}^{\xi_1}\frac{U(v)-\tilde{U}_1(v)}{[\xi(x_k)-v]^{1-\alpha}}dv\\
&\qquad\leq\frac{[w_k]^{-1}}{8\Gamma(\alpha+1)}\max_{x_0\leq\xi_1\leq x_k}U''(\xi_1)(Lh)^2[(\xi_k-\xi_1)^\alpha-(\xi_k-\xi_0)^\alpha].
\end{aligned} \tag{3.51}$$

Now, the second term of Eq.(3.50) gives,

$$\frac{[w_k]^{-1}}{\Gamma(\alpha)}\sum_{l=2}^{k}\int_{\xi_{l-1}}^{\xi_l}\frac{U(v)-\tilde{U_2}(v)}{[\xi(x_k)-v]^{1-\alpha}}dv$$
$$=\frac{[w_k]^{-1}}{3!\Gamma(\alpha)}\sum_{l=2}^{k}\int_{\xi_{l-1}}^{\xi_l}\frac{U'''(\xi_2)(v-\xi_{l-2})(v-\xi_{l-1})(v-\xi_l)}{[\xi(t_k)-v]^{1-\alpha}}dv.$$

Following the similar steps as in ref. [75], we have

$$\frac{[w_k]^{-1}}{\Gamma(\alpha)}\sum_{l=2}^{k}\int_{\xi_{l-1}}^{\xi_l}\frac{U(v)-\tilde{U_2}(v)}{[\xi(x_k)-v]^{1-\alpha}}\leq\frac{[w_k]^{-1}}{12\Gamma(\alpha+1)}\max_{x_0\leq\xi_2\leq x_k}U'''(\xi_2)(\xi_k-\xi_1)^{\alpha}L^3h^3. \quad (3.52)$$

Thus, Eq. (3.50) implies

$$E^k\leq\frac{[w_k]^{-1}}{8\Gamma(\alpha+1)}\max_{x_0\leq\xi_1\leq x_k}U''(\xi_1)(Lh)^2[(\xi_k-\xi_1)^{\alpha}-(\xi_k-\xi_0)^{\alpha}]$$
$$+\frac{[w_k]^{-1}}{12\Gamma(\alpha+1)}\max_{x_0\leq\xi_2\leq x_k}U'''(\xi_2)(\xi_k-\xi_1)^{\alpha}L^3h^3. \quad (3.53)$$

Now, we will discuss some test examples to validate the theory.

3.5 Numerical Examples

Here, we take some test examples from the literature to describe the simulation results.

Example 1. Consider the GFSLP with Dirichlet boundary conditions

$${}^{C}_{a+}D^{\alpha}_{[\xi,w]}y(x)+\lambda y(x)=0, x\in[0,1], 1<\alpha<2, y(0)=0, y(1)=0. \quad (3.54)$$

In this example, the proposed algorithm is performed for GFSLP given by Eq. (3.54). We have calculated the eigenvalues, its convergence rate and eigenfunctions.

From Tables 3.1 and 3.2, we observe that the scheme is convergent and achieve the convergence order which is close to 3. We studied few more cases by varying the scale and weight functions. Firstly, we fixed the weight function $w(x) = 1$, and examine the numerical results by varying the scale function $\xi(x) = \{x, 2x, 3x\}$. Further, we fixed the scale function $\xi(x) = x$, and analysed the numerical results by varying the weight function $w(x) = x + 1, e^x, e^{x/2}$. In Figure 3.1, we have plotted the graphs of the eigenfunctions corresponding to the first four eigenvalues for different order α. From Table 3.1, it is observed that the eigenvalues decrease as we expand the scale function $\xi(x)$.

TABLE 3.1: Numerical values of the first three eigenvalues for example 1 and for $\alpha = 1.8$ and its convergence rate roc_λ varying scale functions $\xi(x)$ and N

k	N	$\xi(x) = x$		$\xi(x) = 2x$		$\xi(x) = 3x$	
		λ_k	roc_λ	λ_k	roc_λ	λ_k	roc_λ
1.	50	9.456766		2.715742		1.308954	
	100	9.456845	2.9848	2.715765	2.9848	1.308964	2.9848
	200	9.456855	2.9940	2.715768	2.9936	1.308966	2.9938
	400	9.456856	2.9973	2.715768	3.0081	1.308966	2.9760
	800	9.456856		2.715768		1.308966	
2.	50	28.475672		8.177489		3.941447	
	100	28.476721	2.9310	8.177790	2.9310	3.941592	2.9310
	200	28.476859	2.9940	8.177830	2.9691	3.941611	2.9689
	400	28.476876	2.9973	8.177835	2.9740	3.941614	2.9849
	800	28.476878		8.117783		3.941614	
3.	50	62.185426		17.858074		8.607367	
	100	62.198443	2.9460	17.868123	2.9460	8.609169	2.9460
	200	62.200132	2.9777	17.862297	2.9777	8.609403	2.9778
	400	62.200346	3.0029	17.862359	2.9963	8.609432	2.9849
	800	62.200373		17.862366		8.609436	

TABLE 3.2: The first three eigenvalues for example 1 and its convergence rate roc_λ varying weight functions $w(x)$ and for $\alpha = 1.8$

k	N	$w(x) = 1$[56]		$w(x) = x + 1$		$w(x) = e^x$	
		λ_k	roc_λ	λ_k	roc_λ	λ_k	roc_λ
1.	50	9.456766		3.826091		2.556448	
	100	9.456845	2.9848	3.826109	3.0416	2.556462	3.0376
	200	9.456855	2.9940	3.826111	3.0216	2.556464	3.0189
	400	9.456856	2.9973	3.826112	3.0239	2.556464	3.0023
	800	9.456868		3.826112		2.556464	
2.	50	28.456726		18.408637		16.084645	
	100	28.476721	2.9310	18.409204	2.9445	16.085035	2.8930
	200	28.476859	2.9689	18.409277	2.9738	16.085088	2.9491
	400	28.476876	2.9728	18.409287	2.9760	16.085094	2.9634
	800	28.476878	2.9728	18.409288		16.085095	
3.	50	62.185426		37.653423		32.588607	
	100	62.198442	2.9460	37.656159	2.8362	32.589681	2.6048
	200	62.200132	2.9777	37.656543	2.9267	32.589858	2.8365
	400	62.200346	3.0029	37.656593	2.9843	32.589883	2.9524
	800	62.200373		37.656999		32.589886	

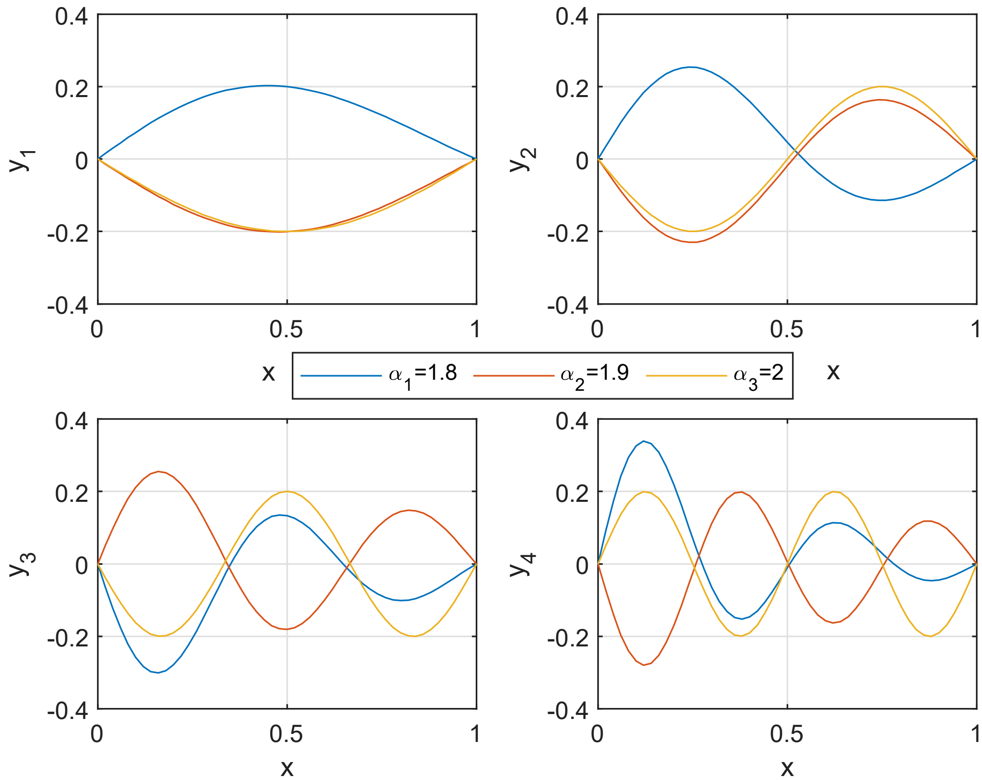

FIGURE 3.1: Plot of first four eigenfunctions for example 1 with fixed $\xi(x) = x, w(x) = 1$ and different values of α.

Example 2. Consider the GFSLP,

$$ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}y(x) + (\lambda + 10\ sin\pi x)y(x) = 0, x \in [0,1], y(0) = 0, y(1) = 0. \tag{3.55}$$

The set of first three approximated eigenvalues with its convergence rate is presented in Table 3.3 and Table 3.4 for different scale functions $\xi(x) = \{x, x^2, x^3\}$ and $\xi(x) = \{x, x/2, x/4\}$ and for $N \in \{50, 100, 200, 400, 800\}$. The graphs of first four eigenfunctions for different values of α are shown in Figure 3.2.

Table 3.3: Numerical values of the first three eigenvalues for example 2 and its convergence rate roc_λ varying scale functions $\xi(x)$ and N

k	N	$\xi(x) = x$		$\xi(x) = x/2$		$\xi(x) = x/4$	
		λ_k	roc_λ	λ_k	roc_λ	λ_k	roc_λ
1.	50	0.9037		26.9275		123.9481	
	100	0.9037	2.9150	26.9275	2.9396	123.9485	2.9410
	200	0.9037	2.9602	26.9276	2.9714	123.9485	2.9720
	400	0.9037	3.0589	26.9276	2.9755	123.9486	2.9933
	800	0.9037		26.9276		123.9486	
2.	50	26.7046		118.5177		461.0824	
	100	26.7040	2.8413	118.5198	2.8419	461.0905	2.8417
	200	26.7047	2.9284	118.5201	2.9285	461.0916	2.9284
	400	26.7047	2.9714	118.5202	2.9690	461.0918	2.9697
	800	26.7047		118.5202		461.0918	
3.	50	66.4985		266.0325		1010.7282	
	100	66.5035	2.7977	266.0509	2.7971	1010.7968	2.7968
	200	66.5042	2.9121	266.0535	2.9118	1010.8067	2.9119
	400	66.5043	2.9509	266.0539	2.9602	1010.8080	2.9581
	800	66.5043		266.0539		1010.8082	

TABLE 3.4: The first three eigenvalues for example 2 and its convergence rate roc_λ varying scale functions $\xi(x)$ and for $\alpha = 1.9$

k	N	$\xi(x) = x$		$\xi(x) = x^2$		$\xi(x) = x^3$	
		λ_k	roc_λ	λ_k	roc_λ	λ_k	roc_λ
1.	50	0.903655		2.088803		3.585319	
	100	0.903673	2.9150	2.089334	3.0033	3.586420	2.9965
	200	0.903675	2.9602	2.089400	3.0027	3.586556	2.9645
	400	0.903675	3.0589	2.089409	3.0127	3.586655	3.0156
	800	0.903675		2.089410		3.615882	
2.	50	25.704040		26.738072		27.658247	
	100	26.704616	2.8413	26.742562	2.9141	27.669503	2.9845
	200	26.704696	2.9284	26.743157	2.9613	27.670993	2.8734
	400	26.704707	2.9714	26.743234	2.9712	27.670993	2.9959
	800	26.704708		26.743244		27.633309	
3.	50	66.498507		66.421077		67.178806	
	100	66.503460	2.9977	66.444290	2.7915	67.234893	2.9949
	200	66.504172	2.9121	66.447643	2.9105	67.243744	3.0426
	400	66.504267	2.9509	66.448089	2.9673	67.245383	3.0568
	800	66.504279		66.448146		67.199237	

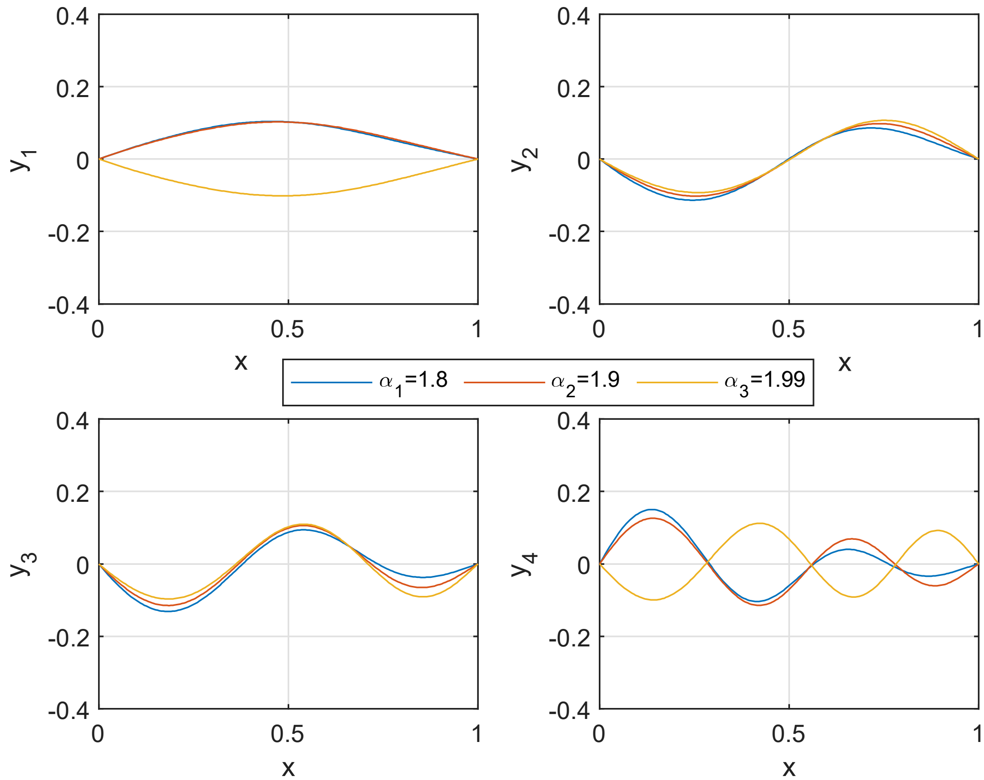

FIGURE 3.2: Plot of first four eigenfunctions for example 2 for fixed $\xi(x) = x, w(x) = exp(x)$ and for different values of α.

Example 3. Consider the GFSLP,

$$ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}y(x) + \lambda\ cos(x)\ y(x) = 0, x \in [0,1], y'(0) = 0, y(1) = 0. \tag{3.56}$$

For this case, the absolute errors corresponding to the first eigenvalues for weight $w(x) = 1$, scale $\xi(x) = x$, varying values of α are shown in Table 3.5.

Table 3.5: Absolute error calculation for fixed λ ($h = 1/200$) for example 3 with $\xi(x) = x, w(x) = 1$ and different values of α.

α/h	$\alpha = 1.8$	$\alpha = 1.85$	$\alpha = 1.9$	$\alpha = 1.95$
1/50	0.0300	0.0313	0.0329	0.0346
1/100	0.0100	0.0104	0.0109	0.0114
1/400	0.0049	0.0052	0.0054	0.0057
1/800	0.0074	0.0078	0.0081	0.0085

Example 4. Consider the GFSLP,

$$ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}y(x) + \left(2\lambda e^{x} - 5sin(\pi x)\right) y(x) = 0, x \in [0,1], y(0) = y'(0) \neq 0, y(1) = 0. \tag{3.57} $$

First three eigenvalues and convergence rates for fixed scale and different weight function are given in Table 3.7. With this, graph of eigenfunctions associated with first four eigenvalues are shown in Figure 3.3. CPU time for each example is presented in Table 3.6.

Table 3.6: CPU time (in seconds) in calculating the eigenvalues for each example for $w(x) = 1, \xi(x) = x$ and $\alpha = 1.8$.

N	Ex-1	Ex-2	Ex-3	Ex-4
50	0.176283	0.196915	0.201188	0.407509
100	0.321716	0.301885	0.407181	0.533621
200	0.871611	0.865439	1.1674	1.323722
400	3.189992	3.129300	4.470506	5.017108
800	12.655291	12.986768	18.516797	18.872516

TABLE 3.7: Numerical values of the first three eigenvalues for example 4 and its convergence rate roc_λ varying weight functions $w(x)$ for $\alpha = 1.85$

k	N	$w(x) = x$		$w(x) = x^2$		$w(x) = x^3$	
		λ_k	roc_λ	λ_k	roc_λ	λ_k	roc_λ
1.	50	3.2498		4.1567		5.2466	
	100	3.2488	2.0018	4.1565	2.9795	5.2463	2.9685
	200	3.2485	2.0010	4.1565	2.9904	5.2462	2.9851
	400	3.2484	2.0005	4.1565	2.9939	5.2462	2.9851
	800	3.2484		4.1565		5.2462	
2.	50	12.9919		21.1692		31.0555	
	100	12.9845	2.0756	21.1657	3.0668	31.0566	3.0687
	200	12.9827	2.0366	21.1637	3.0404	31.0555	3.0446
	400	12.9823	2.0179	21.1653	3.0225	31.0555	3.0254
	800	12.9821		21.1653		31.0554	
3.	50	30.2815		50.2914		74.4311	
	100	30.2602	2.1335	50.2741	3.1912	74.3842	3.2957
	200	30.2553	2.0529	50.2741	3.1912	74.3789	3.2023
	400	30.2541	2.0221	50.2739	3.1107	74.3789	3.1207
	800	30.2538		50.2739		74.3788	

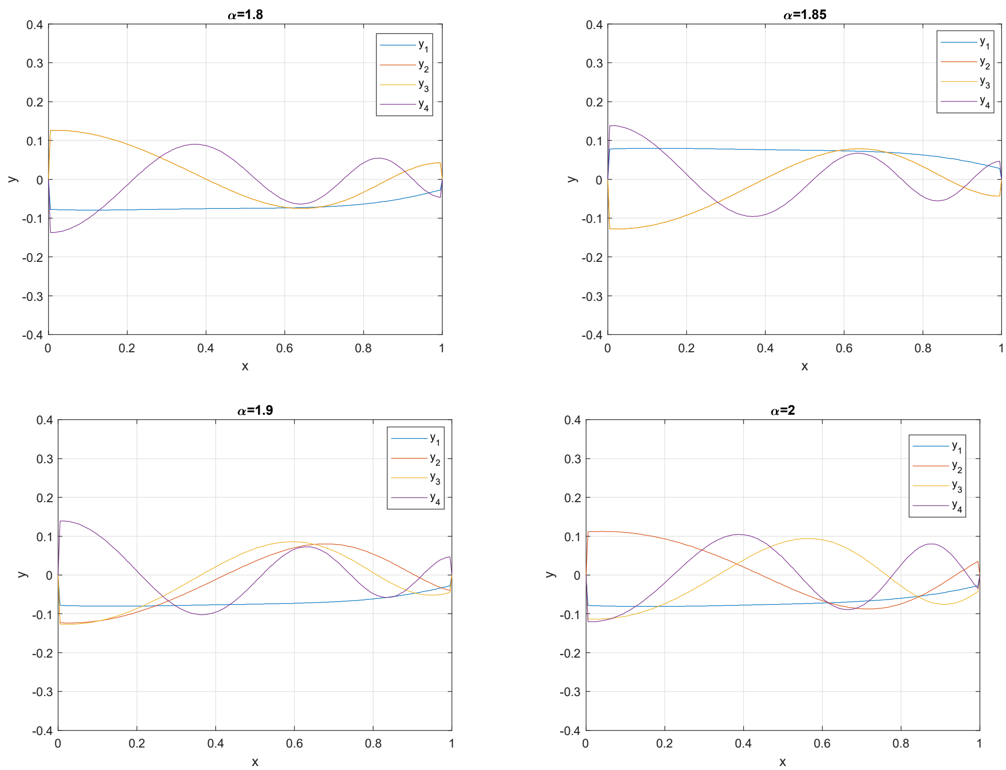

FIGURE 3.3: Plot of first four eigenfunctions for example 4 for $\xi(x) = x, w(x) = exp(x)$.

3.6 Conclusions

We have studied a numerical technique to approximate the eigenvalues and eigenfunctions of GFSLP. For this, the GFSLP is transformed into an integral equation. Further, the generalized fractional integral is discretized using the Lagrange interpolation method, and a system of an algebraic equation is obtained. Solving the obtained system, we get the approximate eigenvalues and eigenfunctions. The rate of convergence of the proposed scheme is calculated numerically. It is observed that the solutions of GFSLP converge to the classical case as $\alpha \to 2$.

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