Chapter 4

An Efficient Matrix Approach for Two-dimensional Diffusion and Telegraph Equations with Dirichlet Boundary Conditions

4.1 Introduction

Partial differential equations (PDEs) play an important role in the description of many central models in physical, biological, and social sciences [11–13]. Due to their key role in several areas of applied sciences, PDEs are studied extensively by many researchers. In this chapter, the feasibility of applying the Euler matrix method is examined for the following two-dimensional diffusion equation:

$$\frac{\partial \omega}{\partial \tau} = \frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial^2 \omega}{\partial \eta^2} + f(\xi, \eta, \tau), \quad 0 < \xi, \eta < 1, \ 0 < \tau \leq 1, \tag{4.1}$$

with the following initial condition as

$$\omega(\xi, \eta, 0) = k(\xi, \eta), \quad 0 \leq \xi, \eta \leq 1, \tag{4.2}$$

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and the Dirichlet boundary conditions as

$$\begin{cases} \omega(0,\eta,\tau) = f_1(\eta,\tau),\ \omega(1,\eta,\tau) = f_2(\eta,\tau), \quad 0 \le \eta \le 1, \quad 0 < \tau \le 1, \\ \\ \omega(\xi,0,\tau) = g_1(\xi,\tau),\ \omega(\xi,1,\tau) = g_2(\xi,\tau), \quad 0 < \xi < 1, \quad 0 < \tau \le 1. \end{cases} \tag{4.3}$$

Let us assume that ω and f are smooth enough. Among PDEs, two-dimensional diffusion equations are of special interest because of their wide applications in physical and applied sciences. They have been used to model chemical exchange reactions, the transport of ground water in an aquifer, pattern formation in the study of biology, chemistry and ecology etc. However, there are some analytical approaches such as homotopy perturbation method (HPM) [17], variational iteration method (VIM) [19], Laplace transform method (LTM) [20] and Adomian decomposition method (ADM) [18] for solving diffusion problems, all these methods are computationally deficient and expensive since they require the use of infinite power series. Therefore, some efficient numerical methods were developed for approximate solutions of diffusion equations. Two classical numerical methods are finite difference methods (FDMs) and finite element methods (FEMs)[23–25]. Moreover, a detail review of classical methods involving FDMs and FEMs is provided in [133]. Recently, some numerical methods such as Bernoulli matrix method [1], trigonometrically fitted method [134], reproducing kernel method [135] and modified Craig-Sneyd method [136] have developed for solving diffusion equations.

Among the PDEs, hyperbolic PDEs have their own importance due to their wide number of applications in science and engineering. They are useful in the modeling of the propagation of signal (digital and analog) through media [137] and also in the propagation of electromagnetic waves in the earth-ionosphere waveguide [101]. Telegraph equation is one of the important equations in this category. The use of several numerical methods for solving two-dimension telegraph equations can be found in [29, 116, 119–122, 124–126]. In this chapter, a new method called as Euler matrix method is proposed for finding the approximate solutions of two-dimensional diffusion and telegraph equations. The method uses the operational matrices of differentiation and integration. The advantage of using operational matrices is their sparsity containing many zeros, which makes computation easy and fast. Interested readers can see some applications of operational matrix techniques for solving PDEs in [1, 2, 138–141].

4.2 Definition and function approximation

4.2.1 Definition of Euler polynomials

The classical Euler polynomials $E_n(\xi)$ are usually defined in terms of the exponential generating functions [142] as

$$\frac{2e^{\xi\tau}}{e^{\tau}+1} = \sum_{n=0}^{\infty} E_n(\xi)\frac{\tau^n}{n!}, \quad |\tau| \leq \pi, \tag{4.4}$$

where $E_n(\xi)$ are the Euler polynomials of degree n and can be expressed by the Bernoulli polynomials via the following relation:

$$E_n(\xi) = \sum_{k=0}^{n} \frac{-2}{k+1}\binom{n}{k} E_{k+1}(0)B_{n-k}(\xi), \;\; n = 1, 2, ...; \tag{4.5}$$

where $B_k(\xi)$, $(k = 0, 1, ...)$ are the Bernoulli polynomials of order k which satisfy the following relations:

$$\begin{cases} B_n'(\xi) = nB_{n-1}(\xi), \quad \forall n \geq 1, \\ \int_0^1 B_n(\xi) = 0, \quad \forall n \geq 1, \\ B_0(\xi) = 1. \end{cases} \tag{4.6}$$

The first few Euler polynomials are

$$E_0(\xi) = 1, \quad E_1(\xi) = \xi - \frac{1}{2}, \quad E_2(\xi) = \xi^2 - \xi,$$
$$E_3(\xi) = \xi^3 - \frac{3}{2}\xi^2 - \frac{1}{4}, \quad E_4(\xi) = \xi^4 - 2\xi^3 + \xi.$$

They also satisfy the following useful properties:

$$\begin{cases} E_n'(\xi) = nE_{n-1}(\xi), \quad \forall n \geq 1, \\ \\ \sum_{k=0}^{n} \binom{n}{k} E_k(\xi) + E_n(\xi) = 2\xi^n, \quad \forall n \geq 1. \end{cases} \tag{4.7}$$

Theorem 4.2.1. *[143]* Assume that $F(\xi, \eta) \in L^2[0,1] \times L^2[0,1]$ be an arbitrary enough smooth function and also is approximated by the two variable truncated Euler series $\sum_{m=0}^{N} \sum_{n=0}^{N} f_{m,n} E_m(\xi) E_n(\eta)$, then the coefficients for all $m, n = 0, 1, 2, ..., N$ can be calculated from the following backward linear relation

$$\begin{aligned} \int_0^1 \int_0^1 \frac{\partial^{m+n} F(\xi,\eta)}{\partial \xi^m \partial \eta^n} d\xi d\eta = \sum_{i=m}^{N} \sum_{j=n}^{N} & \frac{4(m!n!) f_{i,j}}{(i-m+1)(j-n+1)} \\ & \binom{i}{i-m} \binom{j}{j-n} E_{i-m+1}(0) E_{j-n+1}(0). \end{aligned} \tag{4.8}$$

4.2.2 Euler operational matrices

Let $E(\xi) = [E_0(\xi)\ E_1(\xi)\ E_2(\xi) ... E_N(\xi)]^T$. Then, by using the first relation of equation (4.7), we have

$$\begin{bmatrix} E_0(\xi) \\ E_1(\xi) \\ E_2(\xi) \\ \vdots \\ E_N(\xi) \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & N & 0 \end{bmatrix}}_{=M} \begin{bmatrix} E_0(\xi) \\ E_1(\xi) \\ E_2(\xi) \\ \vdots \\ E_N(\xi) \end{bmatrix},$$

$$\int_0^{\xi} \begin{bmatrix} E_0(\xi') \\ E_1(\xi') \\ \vdots \\ E_{N-1}(\xi') \\ E_N(\xi') \end{bmatrix} d\xi' \approx \underbrace{\begin{bmatrix} -E_1(0) & 1 & 0 & \cdots & 0 \\ -\frac{E_1(0)}{2} & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{E_N(0)}{N} & 0 & 0 & \cdots & \frac{1}{N} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{=P} \begin{bmatrix} E_0(\xi) \\ E_1(\xi) \\ \vdots \\ E_{N-1}(\xi) \\ E_N(\xi) \end{bmatrix},$$

where M and P are the Euler operational matrices of differentiation and integration, respectively. For the two variables functions, we can write

$$E(\xi,\eta) = [E_0(\xi,\eta)\ E_1(\xi,\eta)\ \cdots\ E_N(\eta,\xi)]^T_{(N+1)^2\times 1}, \tag{4.9}$$

where $E_i(\eta,\xi) = [E_{i,0}\,(\eta,\xi)\ E_{i,1}(\eta,\xi)\ \cdots\ E_{i,N}(\xi,\eta)]^T$ for all $i = 0, 1, ..., N$ and $E_{r,s}(\xi,\eta) = E_r(\xi)E_s(\eta)$ for all $r, s = 0, 1, ..., N$. Evidently, $E(\xi,\eta) = E(\xi) \otimes E(\eta)$.

Lemma 4.2.1. If $M_\eta = I_{N+1} \otimes M, P_\eta = I_{N+1} \otimes P$ and $P_\xi = P \otimes I_{N+1}$, where I_{N+1} denotes the identity matrix of dimension $(N+1)$, then we have

(i) $\frac{\partial E(\xi,\eta)}{\partial\eta}$=$E_\eta(\xi,\eta)$=$M_\eta E(\xi,\eta)$,

(ii) $\frac{\partial^2 E(\xi,\eta)}{\partial\eta^2}$=$E_{\eta\eta}(\xi,\eta)$=$(M_\eta)^2 E(\xi,\eta)$,

(iii) $\int_0^\eta E(\xi,\eta')d\eta' \approx P_\eta E(\xi,\eta)$,

(iv) $\int_0^\eta \int_0^{\eta'} E(\xi,\eta'')d\eta''d\eta' \approx (P_\eta)^2 E(\xi,\eta)$,

(v) $\int_0^\xi E(\xi',\eta)d\xi' \approx P_\xi E(\xi,\eta)$,

(vi) $\int_0^\xi \int_0^{\xi'} E(\xi'',\eta)d\xi''d\xi' \approx (P_\xi)^2 E(\xi,\eta)$.

4.3 Description of Euler matrix method

4.3.1 Description of Euler matrix method for two-dimensional diffusion equations

In order to solve numerically equations (4.1)-(4.3), they are transformed into PIDEs. Rewrite the equation (4.1) in the following form

$$\omega_{\xi\xi}(\xi,\eta,\tau) = \omega_\tau(\xi,\eta,\tau) - \omega_{\eta\eta}(\xi,\eta,\tau) - f(\xi,\eta,\tau), \tag{4.10}$$

so that

$$\begin{aligned}\omega_\xi(\xi,\eta,\tau) &= \omega_\xi(0,\eta,\tau) + \int_0^\xi \omega_{\xi\xi}(\xi',\eta,\tau)d\xi' \\ &= \omega_\xi(0,\eta,\tau) + \int_0^\xi (\omega_\tau(\xi',\eta,\tau) - \omega_{\eta\eta}(\xi',\eta,\tau) - f(\xi',\eta,\tau))d\xi',\end{aligned}$$

and then

$$\begin{aligned}\omega(\xi,\eta,\tau) = {} & \underbrace{\omega(0,\eta,\tau)}_{=f_1(\eta,\tau)} + \int_0^\xi \omega_\xi(\xi',\eta,\tau)d\xi' \\ & = f_1(\eta,\tau) + \xi\omega_\xi(0,\eta,\tau) \\ & + \int_0^\xi \int_0^{\xi'} (\omega_\tau(\xi'',\eta,\tau) - \omega_{\eta\eta}(\xi'',\eta,\tau) - f(\xi'',\eta,\tau))d\xi''d\xi'.\end{aligned} \tag{4.11}$$

Imposing $\xi = 1$ on both side of equation (4.11), we get

$$\omega_\xi(0,\eta,\tau) = f_2(\eta,\xi) - f_1(\eta,\tau) - \int_0^1 \int_0^\xi (\omega_\tau(\xi',\eta,\tau) - \omega_{\eta\eta}(\xi',\eta,\tau) - f(\xi',\eta,\tau))d\xi'd\xi. \tag{4.12}$$

Substituting the above expression of $\omega_\xi(0,\eta,\tau)$ in equation (4.11), one can obtain

$$\begin{aligned}\omega(\xi,\eta,\tau) = {} & \xi f_2(\eta,\tau) + (1-\xi)f_1(\eta,\tau) \\ & -\xi \int_0^1 \int_0^\xi (\omega_\tau(\xi',\eta,\tau) - \omega_{\eta\eta}(\xi',\eta,\tau) - f(\xi',\eta,\tau))d\xi'd\xi \\ & + \int_0^\xi \int_0^{\xi'} (\omega_\tau(\xi'',\eta,\tau) - \omega_{\eta\eta}(\xi'',\eta,\tau) - f(\xi'',\eta,\tau))d\xi''d\xi'.\end{aligned} \tag{4.13}$$

In a similar way, one can write

$$\omega(\xi,\eta,\tau) = \eta g_2(\xi,\tau) + (1-\eta)g_1(\xi,\tau) - \eta\int_0^1\int_0^{\eta}\omega_{\eta\eta}(\xi,\eta',\tau)d\eta' d\eta$$
$$+\int_0^{\eta}\int_0^{\eta'}\omega_{\eta\eta}(\xi,\eta'',\tau)d\eta'' d\eta'. \tag{4.14}$$

Now, for making use of the initial condition (4.2), we integrate $\omega_\tau(\xi,\eta,\tau)$ in $[0,\tau]$ and obtain

$$\omega(\xi,\eta,\tau) = \underbrace{\omega(\xi,\eta,0)}_{=k(\xi,\eta)} + \int_0^{\tau}\omega_\tau(\xi,\eta,\tau')d\tau. \tag{4.15}$$

The obtained PIDEs which are expressed in equations (4.13)-(4.15) are equivalent to the basic problem described by (4.1)-(4.3). In order to find the solution of above PIDEs, all the known as well as unknown functions are approximated in terms of Euler polynomials. Before proceeding further, we first introduce an important Lemma needed for implementing the method.

Lemma 4.3.1. Suppose that $\varsigma = \int_0^1 E^T(\xi)d\xi,\ \varsigma_p = \varsigma P^T$ and e_i^T denotes the i-th row of $(N+1)\times(N+1)$ identity matrix, then the following relations hold:

(i) $E^T(\eta) = E^T(\xi,\eta)\gamma$,

(ii) $\xi E^T(\eta) = E^T(\xi,\eta)\overline{\gamma}$,

(iii) $\xi(\varsigma_p \otimes E^T(\eta)) = E^T(\xi,\eta)\lambda$,

(iv) $\eta(E^T(\xi)\otimes\varsigma_p) = \varsigma^T(\xi,\eta)\overline{\lambda}$,

(v) $E^T(\xi) = E^T(\xi,\eta)\zeta$,

(vi) $\eta E^T(\xi) = E^T(\xi,\eta)\overline{\zeta}$,

where

$$\gamma = \begin{bmatrix} I_{N+1} \\ 0_{N+1} \\ 0_{N+1} \\ \vdots \\ 0_{N+1} \end{bmatrix}_{(N+1)^2 \times (N+1)}, \ \overline{\gamma} = \begin{bmatrix} \frac{1}{2} I_{N+1} \\ 0_{N+1} \\ 0_{N+1} \\ \vdots \\ 0_{N+1} \end{bmatrix}_{(N+1)^2 \times (N+1)}, \ \overline{\lambda} = I_{N+1} \otimes A_{N+1}$$

$$A_{N+1} = \begin{bmatrix} \frac{1}{2}\varsigma_p \\ \varsigma_p \\ 0_{1\times N+1} \\ \vdots \\ 0_{1\times N+1} \end{bmatrix}_{(N+1)\times(N+1)}, \ \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{N+1} \end{bmatrix}_{(N+1)^2 \times (N+1)}, \ \overline{\zeta} = \begin{bmatrix} \overline{\zeta_1} \\ \overline{\zeta_2} \\ \vdots \\ \overline{\zeta_{N+1}} \end{bmatrix}_{(N+1)^2 \times (N+1)},$$

$$\zeta_i = \begin{bmatrix} e_i^T \\ 0_{1\times N+1} \\ 0_{1\times N+1} \\ \vdots \\ 0_{1\times N+1} \end{bmatrix}_{(N+1)\times(N+1)}, \ \overline{\zeta_i} = \begin{bmatrix} \frac{1}{2} e_i^T \\ e_i^T \\ 0_{1\times N+1} \\ \vdots \\ 0_{1\times N+1} \end{bmatrix}_{(N+1)\times(N+1)}, \ i = 0, 1, 2, ..., N,$$

$$\lambda = \begin{bmatrix} -\frac{E_1(0)}{2}I_{N+1} & -\frac{E_2(0)}{4}I_{N+1} & \cdots & -\frac{E_N(0)}{2N}I_{N+1} & 0_{N+1} \\ -E_1(0)I_{N+1} & -\frac{E_2(0)}{2}I_{N+1} & \cdots & -\frac{E_N(0)}{N}I_{N+1} & 0_{N+1} \\ 0_{N+1} & 0_{N+1} & \cdots & 0_{N+1} & 0_{N+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{N+1} & 0_{N+1} & \cdots & 0_{N+1} & 0_{N+1} \end{bmatrix}_{(N+1)^2\times(N+1)^2}.$$

Now, we approximate all the known as well as unknown functions in terms of Euler polynomials as

$$\begin{cases} f(\xi,\eta,\tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N}\sum_{p=0}^{N} f_{mnp}E_m(\xi)E_n(\eta)E_p(\tau) = E^T(\xi,\eta)FE(\tau), \\ \omega(\xi,\eta,\tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N}\sum_{r=0}^{N} \omega_{mnp}E_m(\xi)E_n(\eta)E_r(\tau) = E^T(\xi,\eta)WE(\tau), \\ k(\xi,\eta) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} k_{mn}E_m(\xi)E_n(\eta) = E^T(\xi)KE(\eta), \\ f_1(\eta,\tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} f_{1,mn}E_m(\eta)E_n(\tau) = E^T(\eta)F_1E(\tau), \\ f_2(\eta,\tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} f_{2,mn}E_m(\eta)E_n(\tau) = E^T(\eta)F_2E(\tau), \\ g_1(\xi,\tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} g_{1,mn}E_m(\xi)E_n(\tau) = E^T(\xi)G_1E(\tau), \\ g_2(\xi,\tau) \approx \sum_{m=0}^{N}\sum_{n=0}^{N} g_{2,mn}E_m(\xi)E_n(\tau) = E^T(\xi)G_2E(\tau), \end{cases} \tag{4.16}$$

where

$$F = \begin{bmatrix} f_{000} & f_{001} & \cdots & f_{00N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{0N0} & f_{0N1} & \cdots & f_{0NN} \\ f_{100} & f_{101} & \cdots & f_{10N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1N0} & f_{1N1} & \cdots & f_{0NN} \\ \vdots & \vdots & \vdots & \vdots \\ f_{N00} & f_{N01} & \cdots & f_{N0N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{NN0} & f_{NN1} & \cdots & f_{NNN} \end{bmatrix}_{(N+1)^2\times(N+1)},$$

in which the entries of F can be calculated via the formula given as

$$\int_0^1\int_0^1\int_0^1 \frac{\partial^{m+n+r}F(\xi,\eta,\tau)}{\partial\xi^m\partial\eta^n\partial\tau^r}d\xi d\eta d\tau = \sum_{i=m}^{N}\sum_{j=n}^{N}\sum_{k=r}^{N}\frac{8(m!n!r!)f_{i,j,k}}{(i-m+1)(j-n+1)(k-r+1)} \times\binom{i}{i-m}\binom{j}{j-n}\binom{k}{k-r}E_{i-m+1}(0)E_{j-n+1}(0)E_{k-r+1}(0). \tag{4.17}$$

$$W = \begin{bmatrix} \omega_{000} & \omega_{001} & \cdots & \omega_{00N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{0N0} & \omega_{0N1} & \cdots & \omega_{0NN} \\ \omega_{100} & \omega_{101} & \cdots & \omega_{10N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{1N0} & \omega_{1N1} & \cdots & \omega_{0NN} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{N00} & \omega_{N01} & \cdots & \omega_{N0N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{NN0} & \omega_{NN1} & \cdots & \omega_{NNN} \end{bmatrix}_{(N+1)^2\times(N+1)}, \; K = \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix},$$

$$F_1 = \begin{bmatrix} f_{1,00} & f_{1,01} & \cdots & f_{1,0N} \\ f_{1,10} & f_{1,11} & \cdots & f_{1,1N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1,N0} & f_{1,N1} & \cdots & f_{1,NN} \end{bmatrix}, \; F_2 = \begin{bmatrix} f_{2,00} & f_{2,01} & \cdots & f_{2,0N} \\ f_{2,10} & f_{2,11} & \cdots & f_{2,1N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{2,N0} & f_{2,N1} & \cdots & f_{2,NN} \end{bmatrix},$$

$$G_1 = \begin{bmatrix} g_{1,00} & g_{1,01} & \cdots & g_{1,0N} \\ g_{1,10} & g_{1,11} & \cdots & g_{1,1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1,N0} & g_{1,N1} & \cdots & g_{1,NN} \end{bmatrix}, \; G_2 = \begin{bmatrix} g_{2,00} & g_{2,01} & \cdots & g_{2,0N} \\ g_{2,10} & g_{2,11} & \cdots & g_{2,1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{2,N0} & g_{2,N1} & \cdots & g_{2,NN} \end{bmatrix}.$$

The entries of all the other matrices K, F_1, F_2, G_1 and G_2 can be calculated via the linear relation (4.8). The purpose is to evaluate the entries of matrix W. Now, Substituting the approximate values of the known functions $f(\xi,\eta,\tau), f_1(\eta,\tau), f_2(\eta,\tau)$ and the unknown function $\omega(\xi,\eta,\tau)$ which are given in (4.16) in the right hand side of (4.13)

$$\begin{aligned} \omega(\xi,\eta,\tau) &\approx \xi(E^T(\eta)F_2E(\tau)) + (1-\xi)(E^T(\eta)F_1E(\tau)) \\ &- \xi\int_0^1\int_0^{\xi} E^T(\xi',\eta)(WM - (M_\eta^T)^2W - F)E(\tau)d\xi' d\xi \\ &+ \int_0^{\xi}\int_0^{\xi'} E^T(\xi'',\eta)(WM - (M_\eta^T)^2W - F)E(\tau)d\xi'' d\xi' \\ &\approx \xi E^T(\eta)(F_2 - F_1)E(\tau) + E^T(\eta)F_1E(\tau) \\ &- \xi(\varsigma_p \otimes E^T(\eta))(WM - (M_\eta^T)^2W - F)E(\tau) \\ &+ E^T(\xi,\eta)(P_\xi^T)^2(WM - (M_\eta^T)^2W - F)E(\tau). \end{aligned} \tag{4.18}$$

By making the use of Lemma 4.3.1, we can rearrange the above relation as

$$\omega(\xi,\eta,\tau) \approx E^T(\xi,\eta)ZE(\tau), \tag{4.19}$$

where

$$Z = \overline{\gamma}(F_2 - F_1) + \gamma F_1 + ((P_\xi^T)^2 - \lambda)(WM - (M_\eta^T)^2 W - F).$$

Now, one can use (4.19) as an approximation of $\omega(\xi, \eta, \tau)$, and $g_1(\xi, \tau), g_2(\xi, \tau)$ introduced in equation (4.16) are inserted in the right hand side of (4.14) to obtain

$$\omega(\xi, \eta, \tau) \approx E^T(\xi, \eta) Y E(\tau), \tag{4.20}$$

in which

$$Y = \overline{\zeta}(G_2 - G_1) + \zeta G_1 + ((P_\eta^T)^2 - \overline{\lambda})(M_\eta^T)^2 Z.$$

The obtained relation (4.20) can now be used as an approximation of $\omega(\xi, \eta, \tau)$. Moreover, the approximation of $\omega(\xi, \eta, \tau)$ along with the approximation of $k(\xi, \eta)$ given in equation (4.16) can be used in the equation (4.15) to enable us to get

$$E^T(\xi, \eta)\widetilde{W}E(\tau) = E^T(\xi, \eta) K E(\tau) + E^T(\xi, \eta)\widetilde{Y} M P E(\tau), \tag{4.21}$$

where $\widetilde{W}$ is an approximation of W and

$$\begin{aligned} \widetilde{Y} &= \overline{\zeta}(G_2 - G_1) + \zeta G_1 + ((P_\eta^T)^2 - \overline{\lambda})(M_\eta^T)^2 \widetilde{Z}, \\ \widetilde{Z} &= \overline{\gamma}(F_2 - F_1) + \gamma F_1 + ((P_\xi^T)^2 - \lambda)(WM - (M_\eta^T)^2 \widetilde{W} - F). \end{aligned}$$

The equation (4.21) gives rise to the following algebraic generalized Sylvester equation

$$\widetilde{W} + A_1 \widetilde{W} B_1 + A_2 \widetilde{W} B_2 = C, \tag{4.22}$$

where

$$\begin{aligned} A_1 &= ((P_\eta^T)^2 - \overline{\lambda})(M_\eta^T)^2((P_\xi^T)^2 - \lambda), \quad B_1 = -M^2 P, \\ A_2 &= ((P_\eta^T)^2 - \overline{\lambda})(M_\eta^T)^2((P_\xi^T)^2 - \lambda)(M_\eta^T)^2, \quad B_2 = MP, \\ C &= K + \zeta G_1 MP + \overline{\zeta}(G_2 - G_1)MP + ((P_\eta^T)^2 - \overline{\lambda})(M_\eta^T)^2 \\ & \quad (\overline{\gamma}(F_2 - F_1) + \gamma F_1 - ((P_\xi^T)^2 - \lambda)F)MP. \end{aligned}$$

Finally, the Sylvester equation (4.22) can be solved for $\widetilde{W}$ via a robust Krylov subspace iterative method (i.e., generalized BICGSTAB, see [39]).

4.3.2 *Description of Euler matrix method for solving two-dimensional telegraph equations*

The proposed method can also be implemented for the following two-dimensional hyperbolic telegraph equation:

$$\frac{\partial^2 \omega(\xi,\eta,\tau)}{\partial \tau^2} + 2\alpha \frac{\partial \omega(\xi,\eta,\tau)}{\partial \tau} \beta^2 \omega(\xi,\eta,\tau) = \frac{\partial^2 \omega(\xi,\eta,\tau)}{\partial \xi^2} + \frac{\partial^2 \omega(\xi,\eta,\tau)}{\partial \eta^2} + f(\xi,\eta,\tau), \tag{4.23}$$

where α and β are constants. The initial conditions are given by

$$\begin{cases} \omega(\xi,\eta,0) = k_1(\eta,\xi), \\ \omega_\tau(\xi,\eta,0) = k_2(\xi,\eta), \end{cases} \tag{4.24}$$

and the Dirichlet boundary conditions are

$$\begin{cases} \omega(0,\eta,\tau) = f_1(\eta,\tau),\ \omega(1,\eta,\tau) = f_2(\eta,\tau), \\ \omega(\xi,0,\tau) = g_1(\xi,\tau),\ \omega(\xi,1,\tau) = g_2(\xi,\tau). \end{cases} \tag{4.25}$$

Proceeding in a similar way as it is done for diffusion equation, the PIDEs corresponding to the above equations (4.23)-(4.25) are

$$\omega(\xi,\eta,\tau) = \xi g_2(\eta,\tau) + (1-\xi) g_1(\eta,\tau) - \xi \int_0^1 \int_0^{\xi} \omega_{\xi\xi}(\xi',\eta,\tau) d\xi' d\xi + \int_0^{\xi} \int_0^{\xi'} \omega_{\xi\xi}(\xi'',\eta,\tau) d\xi'' d\xi', \tag{4.26}$$

$$\text{or,} \quad \omega(\xi,\eta,\tau) = \eta h_2(\xi,\tau) + (1-\eta) h_1(\xi,\tau) - \eta \int_0^1 \int_0^{\eta} \omega_{\eta\eta}(\xi,\eta',\tau) d\eta' d\eta + \int_0^{\eta} \int_0^{\eta'} \omega_{\eta\eta}(\xi,\eta'',\tau) d\eta'' d\eta', \tag{4.27}$$

the above equation can be rewritten as

$$\omega(\xi,\eta,\tau) = k(\xi,\eta,\tau) + \int_0^{\tau} \int_0^{\tau'} \omega_{\tau\tau}(\xi,\eta,\tau'') d\tau'' d\tau', \tag{4.28}$$

where $k(\xi,\eta,\tau) = k_1(\xi,\eta) + \tau k_2(\xi,\eta)$. In order to solve the above integro-PDEs (4.26)-(4.28), all the known as well unknown functions should be approximated as follows:

$$\begin{cases} f(\xi,\eta,\tau) \approx \sum_{m=0}^{N} \sum_{n=0}^{N} \sum_{p=0}^{N} f_{mnp} E_m(\xi) E_n(\eta) E_p(\tau) = E^T(\xi,\eta) F E(\tau), \\ k(\xi,\eta,\tau) \approx \sum_{m=0}^{N} \sum_{n=0}^{N} \sum_{p=0}^{N} k_{mnp} E_m(\xi) E_n(\eta) E_p(\tau) = E^T(\xi,\eta) K E(\tau), \\ \omega(\xi,\eta,\tau) \approx \sum_{m=0}^{N} \sum_{n=0}^{N} \sum_{r=0}^{N} \omega_{mnp} E_m(\xi) E_n(\eta) E_r(\tau) = E^T(\xi,\eta) W E(\tau), \\ f_1(\eta,\tau) \approx \sum_{m=0}^{N} \sum_{n=0}^{N} f_{1,mn} E_m(\eta) E_n(\tau) = E^T(\eta) F_1 E(\tau), \\ f_2(\eta,\tau) \approx \sum_{m=0}^{N} \sum_{n=0}^{N} f_{2,mn} E_m(\eta) E_n(\tau) = E^T(\eta) F_2 E(\tau), \\ g_1(\xi,\tau) \approx \sum_{m=0}^{N} \sum_{n=0}^{N} g_{1,mn} E_m(\xi) E_n(\tau) = E^T(\xi) G_1 E(\tau), \\ g_2(\xi,\tau) \approx \sum_{m=0}^{N} \sum_{n=0}^{N} g_{2,mn} E_m(\xi) E_n(\tau) = E^T(\xi) G_2 E(\tau). \end{cases} \tag{4.29}$$

Now, proceeding similarly as we have done for diffusion equations, we finally get the following algebraic generalized Sylvester equation as

$$\widetilde{W} + A_1\widetilde{W}B_1 + A_2\widetilde{W}B_2 = C, \tag{4.30}$$

where

$$\begin{aligned}
A_1 &= ((P_\eta^T)^2 - \overline{\lambda})(M_\eta^T)^2((P_\xi^T)^2 - \lambda), \quad B_1 = -(M^2 + 2\alpha M + \beta^2)M^2P^2, \\
A_2 &= ((P_\eta^T)^2 - \overline{\lambda})(M_\xi^T)^2((P_\xi^T)^2 - \lambda)(M_\eta^T)^2, \quad B_2 = M^2P^2, \\
C &= K + \zeta G_1 M^2P^2 + \overline{\zeta}(G_2 - G_1)M^2P^2 + ((P_\eta^T)^2 - \overline{\lambda})(M_\eta^T)^2 \\
&\quad (\overline{\gamma}(F_2 - F_1) + \gamma F_1 - ((P_\xi^T)^2 - \lambda)F)M^2P^2.
\end{aligned}$$

The Sylvester equation (4.30) can now be solved for $\widetilde{W}$ via a robust Krylov subspace iterative method.

4.4 Numerical experiments

The proposed method is applied here to solve four test functions. The results obtained via the Euler matrix methods are provided through Tables 4.1-4.4 and are compared with the results obtained in [1] and [2]. The graphs of errors are shown through Figures 4.1-4.8.

Example 4.4.1. *[1]* We consider the diffusion equation (4.1)-(4.3) with the following assumptions as

$$\begin{cases}
f(\xi,\eta,\tau) = 0, \quad k(\xi,\eta) = e^{\xi+\eta}, \\
f_1(\eta,\tau) = e^{\eta+2\tau}, \quad f_2(\eta,\tau) = e^{1+\eta+2\tau}, \\
g_1(\xi,\tau) = e^{\xi+2\tau}, \quad g_2(\xi,\tau) = e^{1+\xi+2\tau},
\end{cases} \tag{4.31}$$

which has the exact solution as

$$\omega(\xi,\eta,\tau) = e^{\xi+\eta+2\tau}.$$

Table 4.1: Results of absolute errors $e_N(\xi, \eta, \tau)$ of example 4.4.1.

(ξ, η)	$N = 5$	$N = 7$	$N = 9$	$N = 11$
(0.1,0.1)	5.0e-04	4.7e-06	6.4e-07	4.6e-08
(0.2,0.2)	1.3e-04	6.2e-05	4.8e-06	6.3e-07
(0.3,0.3)	5.1e-04	3.9e-05	2.2e-05	3.2e-06
(0.4,0.4)	1.6e-03	1.5e-04	5.1e-05	1.9e-06
(0.5,0.5)	1.4e-03	3.3e-04	1.3e-05	2.8e-06
(0.6,0.6)	2.6e-03	4.6e-03	6.6e-04	5.5e-05
(0.7,0.7)	4.3e-03	4.3e-04	3.3e-05	6.7e-05
(0.8,0.8)	7.4e-03	2.1e-04	7.0e-06	4.9e-06
(0.9,0.9)	1.7e-04	3.7e-05	2.5e-06	1.8e-07

This example is solved for $N = 5, 7, 9$ and 11 via Euler matrix method. Absolute value of errors for this example at different points of computational domain $D = [0, 1] \times [0, 1]$ are given in Table 4.1 . Moreover, the error graphs for $N = 5, 7$ are depicted through Figures 4.1 and 4.2.

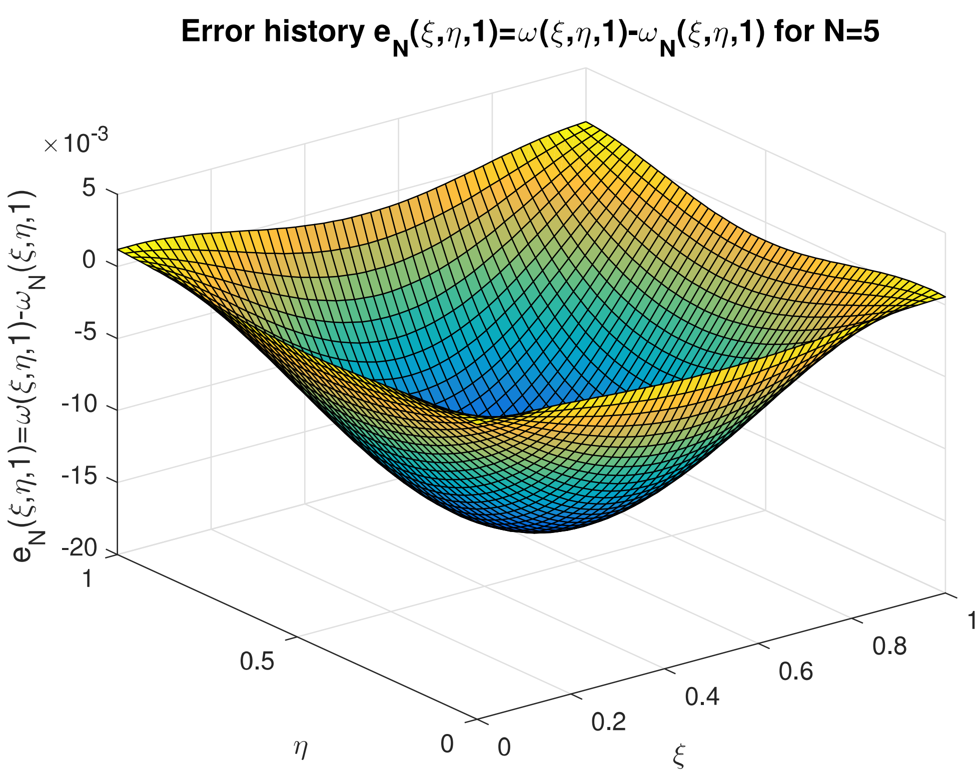

Figure 4.1: Error graph of example 4.4.1 for $N = 5$.

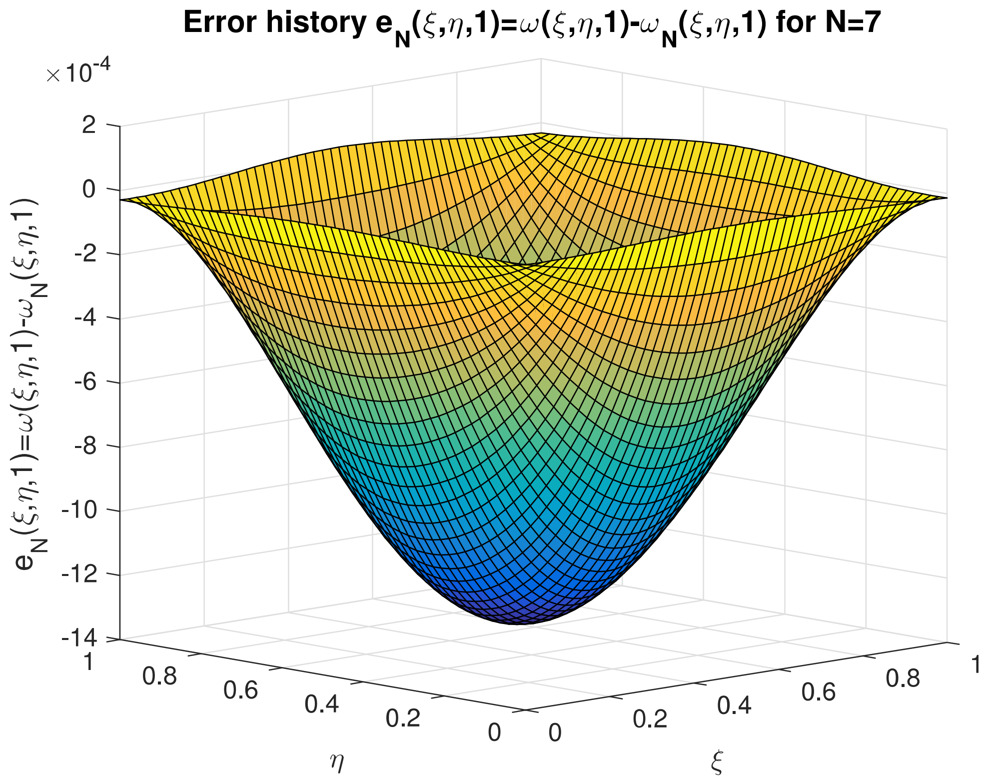

Figure 4.2: Error graph of example 4.4.1 for $N = 7$.

Example 4.4.2. *[1]* We consider (4.1)-(4.3) with the following assumptions as

$$\begin{cases} f(\xi,\eta,\tau) = 0, \quad k(\xi,\eta) = \cos(\pi\xi)\sin(\pi\eta), \\ f_1(\eta,\tau) = \sin(\pi\eta)e^{-2\pi^2\eta}, \quad f_2(\eta,\tau) = -\sin(\pi\eta)e^{-2\pi^2\eta}, \\ g_1(\xi,\tau) = 0, \quad g_2(\xi,\tau) = 0. \end{cases} \tag{4.32}$$

The exact solution is given by

$$\omega(\xi,\eta,\tau) = \cos(\pi\xi)\sin(\pi\eta)e^{-2\pi^2\eta}.$$

Again, the approximate solutions of this example for $N = 5, 7, 9$ and 11 are obtained via Euler matrix method. The Absolute value of errors for this example at different points of computational domain $D = [0,1] \times [0,1]$ are given in Table 4.3. Error graphs for $N = 7, 9$ are depicted through Figures 4.3 and 4.4, respectively.

Table 4.2: Results of absolute errors $e_N(\xi, \eta, \tau)$ of example 4.4.2.

(ξ, η)	$N = 5$	$N = 7$	$N = 9$	$N = 11$
(0.1,0.1)	4.8e-03	3.9e-03	6.0e-04	8.2e-05
(0.2,0.2)	7.3e-03	2.0e-04	5.3e-04	8.2e-05
(0.3,0.3)	4.9e-02	3.1e-04	2.0e-04	4.0e-04
(0.4,0.4)	1.6e-02	4.2e-04	5.9e-04	1.0e-04
(0.5,0.5)	9.1e-15	7.0e-15	7.3e-15	8.8e-16
(0.6,0.6)	3.8e-05	1.8e-05	2.6e-05	5.0e-05
(0.7,0.7)	3.3e-03	3.5e-04	4.4e-05	4.8e-05
(0.8,0.8)	5.4e-02	7.1e-03	6.1e-04	4.9e-06
(0.9,0.9)	4.1e-03	4.1e-04	2.5e-04	3.0e-05

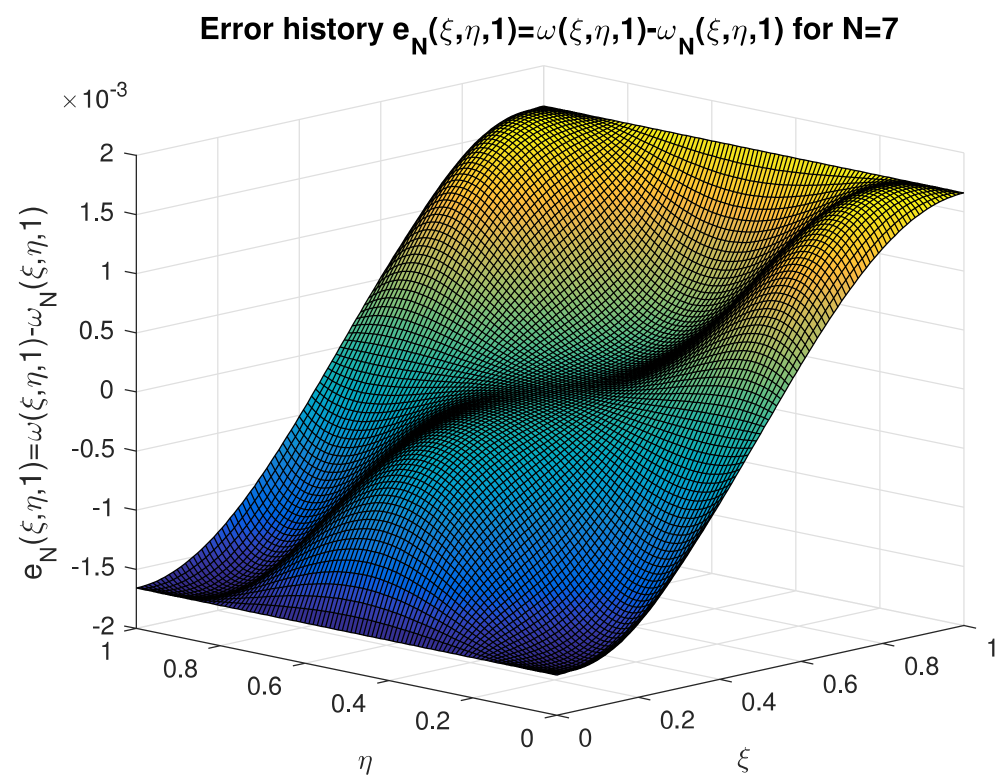

Figure 4.3: Error graph of example 4.4.2 for $N = 7$.

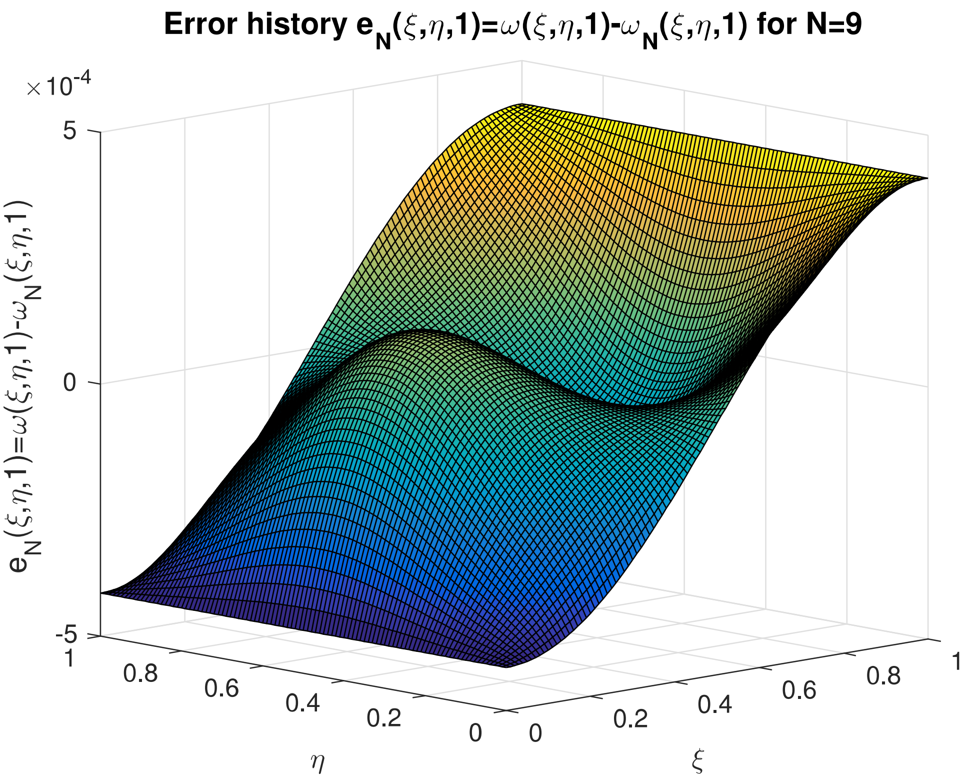

Figure 4.4: Error graph of example 4.4.2 for $N = 9$.

Example 4.4.3. *[2]* As the third example, we consider the telegraph equations (4.23)-(4.25) with $\alpha = 50, \beta = 5$. The initial and boundary conditions are given by

$$\begin{cases} \omega(\xi,\eta,0) = \sinh(\xi)\sinh(\eta), \\ \omega_\tau(\xi,\eta,0) = 0, \end{cases} \tag{4.33}$$

and

$$\begin{cases} \omega(0,\eta,\tau) = 0,\ \omega(1,\eta,\tau) = \cos(\tau)\sinh(1)\sinh(\eta), \\ \omega(\xi,0,\tau) = 0,\ \omega(\xi,1,\tau) = \cos(\tau)\sinh(\xi)\sinh(1). \end{cases} \tag{4.34}$$

The exact solution for this example is given by

$$\omega(\xi,\eta,\tau) = \cos\tau\sinh(\xi)\sinh(\eta).$$

Table 4.4 presents the results of absolute errors for $N = 4, 6, 8$ and 10. The graphs of

Table 4.3: Results of absolute errors $e_N(\xi, \eta, \tau)$ of example 4.4.3.

(ξ, η)	$N = 4$	$N = 6$	$N = 8$	$N = 10$
(0.1,0.1)	8.4e-03	4.3e-05	7.9e-06	3.6e-07
(0.2,0.2)	1.8e-02	7.3e-04	1.3e-05	1.6e-06
(0.3,0.3)	7.8e-02	6.1e-04	3.7e-05	2.0e-05
(0.4,0.4)	5.8e-01	3.0e-04	2.6e-06	4.7e-05
(0.5,0.5)	3.8e-02	6.0e-04	4.0e-05	3.8e-06
(0.6,0.6)	3.4e-02	1.7e-04	5.9e-06	1.6e-05
(0.7,0.7)	2.7e-02	3.0e-05	2.6e-05	8.2e-05
(0.8,0.8)	4.7e-02	1.5e-04	3.0e-05	4.1e-05
(0.9,0.9)	3.3e-03	9.5e-05	5.4e-06	5.0e-07

error are depicted through Figures 4.5 and 4.6 for N = 4 and 6, respectively.

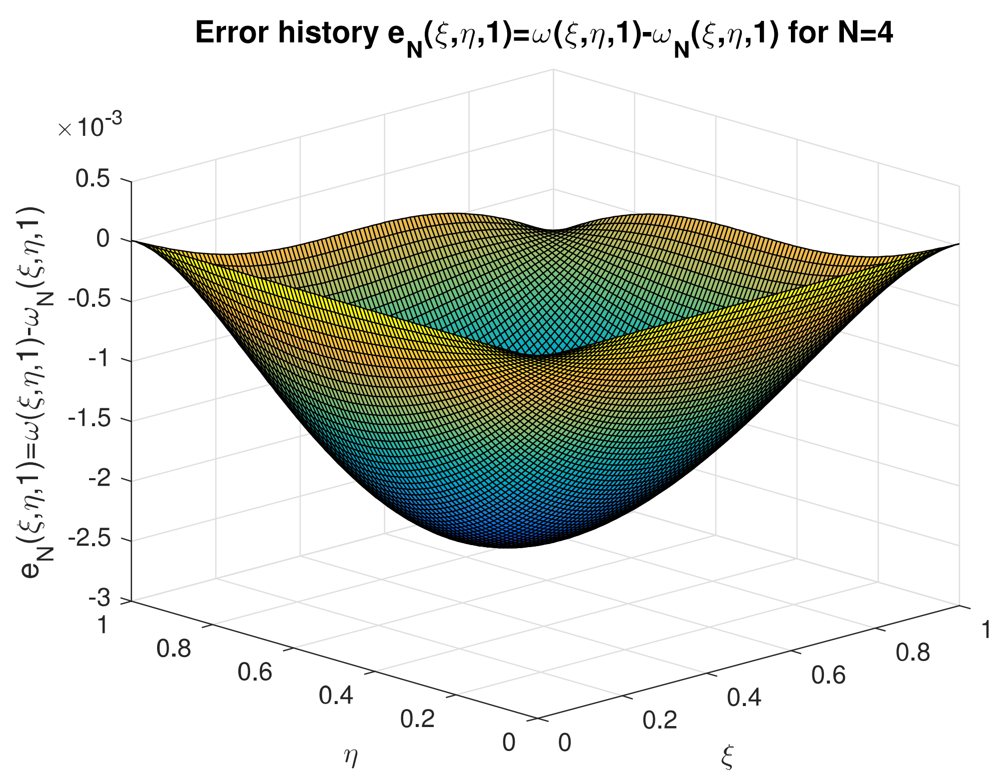

Figure 4.5: Error graph of example 4.4.3 for $N = 4$.

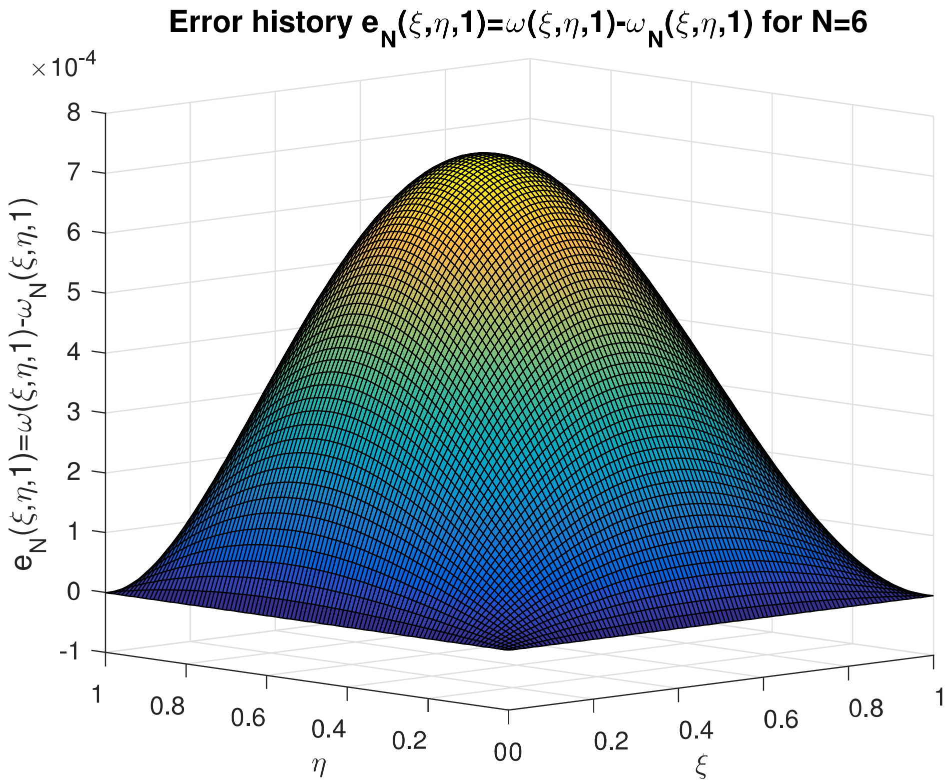

Figure 4.6: Error graph of example 4.4.3 for $N = 6$.

Example 4.4.4. *[2]* Consider the telegraph equations (4.23)-(4.25) with $\alpha = 1, \beta = 1$ along with the following initial and boundary conditions as

$$\begin{cases} \omega(\xi, \eta, 0) = \xi^2 + \eta^2, \\ \omega_\tau(\xi, \eta, 0) = \xi^2 + \eta^2 + 1, \end{cases} \tag{4.35}$$

and

$$\begin{cases} \omega(0, \eta, \tau) = \eta^2 + \tau,\ \omega(1, \eta, \tau) = 1 + \eta^2 + \tau, \\ \omega(\xi, 0, \tau) = \xi^2 + \tau,\ \omega(\xi, 1, \tau) = 1 + \xi^2 + \tau. \end{cases} \tag{4.36}$$

The exact solution of the above problem is

$$\omega(\xi, \eta, \tau) = \xi^2 + \eta^2 + \tau.$$

The numerical results of absolute errors $e_N(\xi, \eta, 1)$ for various values of N are presented in Table 4.5. The error graphs for $N = 8$ and 10 are depicted in Figures 4.7 and 4.8.

Table 4.4: Results of absolute errors $e_N(\xi, \eta, \tau)$ of example 4.4.4.

(ξ, η)	$N = 4$	$N = 6$	$N = 8$	$N = 10$
(0.1,0.1)	3.0e-04	3.0e-05	5.0e-06	7.0e-07
(0.2,0.2)	7.2e-03	4.9e-04	1.0e-05	1.0e-06
(0.3,0.3)	1.8e-03	3.5e-04	2.0e-05	2.0e-06
(0.4,0.4)	2.0e-03	4.0e-04	4.4e-05	4.1e-05
(0.5,0.5)	2.0e-03	7.2e-04	5.1e-05	2.2e-05
(0.6,0.6)	4.1e-03	4.4e-04	3.0e-04	3.0e-05
(0.7,0.7)	2.3e-03	4.6e-04	6.0e-05	5.2e-06
(0.8,0.8)	1.0e-03	2.9e-05	1.4e-05	4.3e-05
(0.9,0.9)	3.0e-04	8.0e-05	5.0e-07	7.0e-08

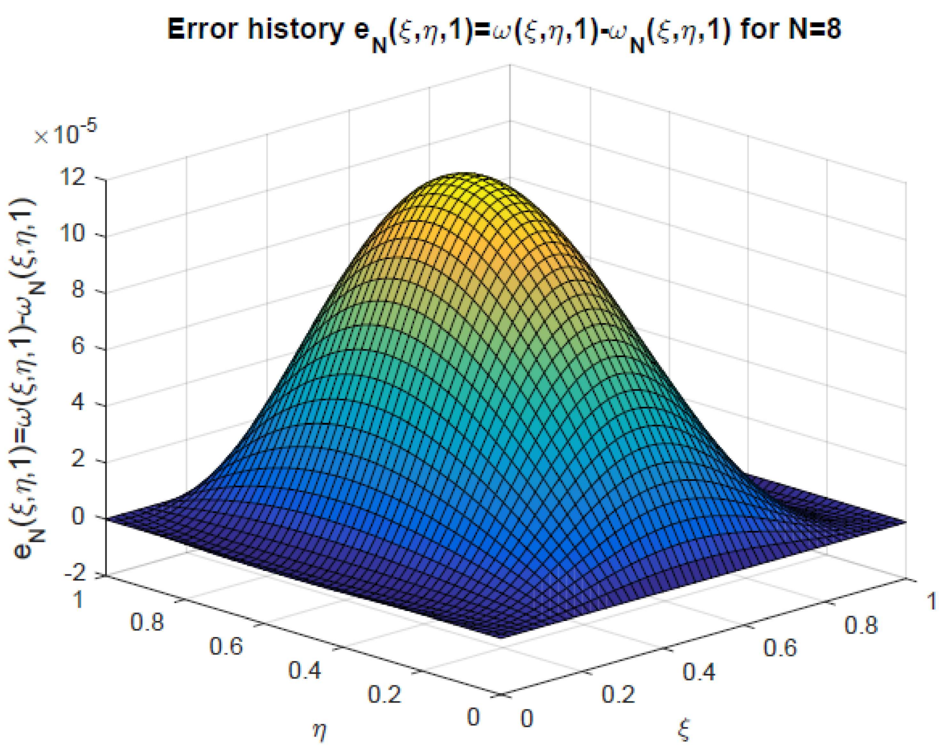

Figure 4.7: Error graph of example 4.4.4 for $N = 8$.

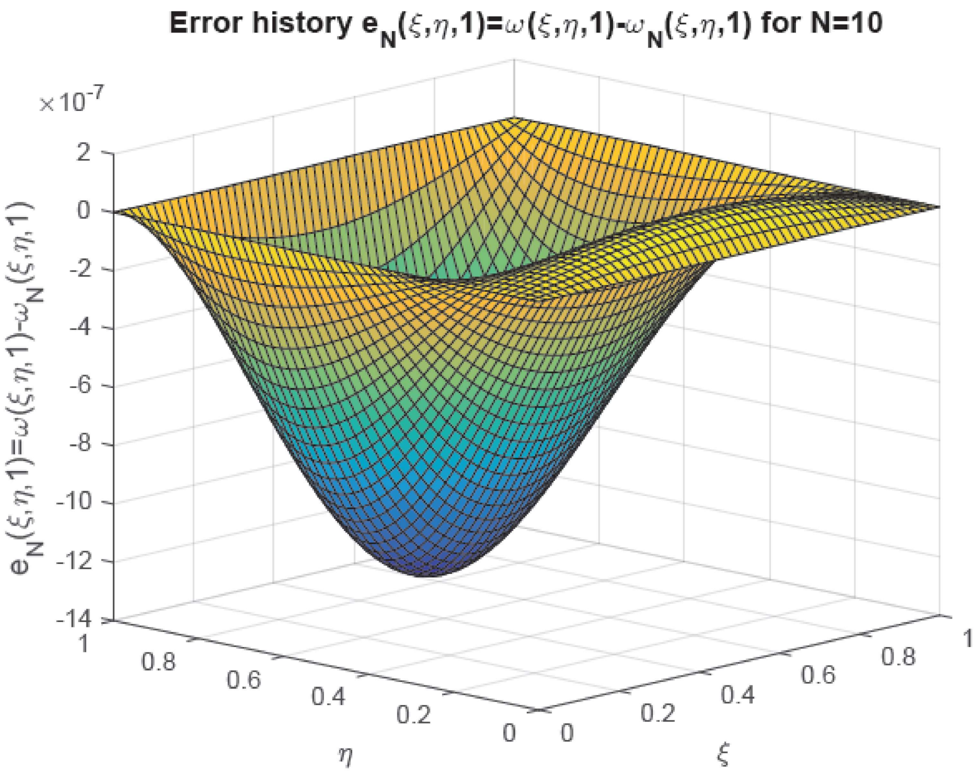

Figure 4.8: Error graph of example 4.4.4 for $N = 10$.

Table 4.5: Comparison of numerical results of EMM and BMM [1] for example 4.4.1.

Grid Points	EMM			BMM [1]		
	$N = 6$	$N = 8$	$N = 10$	$N = 6$	$N = 8$	$N = 10$
(0.1,0.1)	8.1e-05	4.3e-06	7.8e-07	8.0e-05	2.3e-06	4.2e-07
(0.2,0.2)	2.6e-05	6.4e-06	4.0e-08	1.5e-05	6.9e-06	7.0e-08
(0.3,0.3)	4.8e-04	2.6e-05	2.0e-06	4.3e-05	3.9e-05	2.6e-06
(0.4,0.4)	2.2e-04	1.4e-05	6.0e-06	1.1e-03	9.0e-05	6.8e-06
(0.5,0.5)	1.2e-03	3.2e-05	1.2e-06	1.8e-03	1.4e-04	1.1e-05
(0.6,0.6)	6.3e-03	1.7e-04	1.1e-05	2.2e-03	1.7e-04	1.4e-05
(0.7,0.7)	1.1e-03	1.3e-04	1.1e-05	2.1e-03	1.5e-04	1.3e-05
(0.8,0.8)	1.6e-03	1.2e-04	4.7e-06	1.5e-03	1.0e-04	8.9e-06
(0.9,0.9)	3.6e-04	1.8e-05	3.0e-06	6.6e-04	3.4e-05	3.3e-06

Table 4.6: Comparison of numerical results of EMM and BMM [1] for example 4.4.2.

Grid Points	EMM			BMM [1]		
	$N = 6$	$N = 8$	$N = 10$	$N = 6$	$N = 8$	$N = 10$
(0.1,0.1)	2.6e-03	8.3e-04	4.1e-04	2.2e-02	6.1e-03	1.4e-03
(0.2,0.2)	3.7e-02	4.3e-03	6.3e-04	1.0e-02	3.9e-03	8.0e-04
(0.3,0.3)	1.2e-03	1.7e-03	1.1e-04	1.3e-03	1.4e-03	1.0e-04
(0.4,0.4)	4.5e-03	9.1e-05	1.0e-04	4.2e-03	8.7e-05	2.0e-04
(0.5,0.5)	5.0e-15	6.3e-15	4.2e-16	5.4e-15	8.4e-15	1.3e-16
(0.6,0.6)	1.3e-03	3.5e-06	3.1e-06	4.2e-03	9.1e-06	2.0e-06
(0.7,0.7)	1.3e-03	2.3e-03	1.3e-04	1.3e-03	1.4e-03	1.0e-04
(0.8,0.8)	2.3e-02	4.3e-03	7.3e-04	1.0e-02	3.9e-03	8.0e-04
(0.9,0.9)	4.1e-03	7.2e-04	6.2e-04	2.2e-02	6.1e-03	1.4e-03

Table 4.7: Comparison of numerical results of EMM and BMM [1] for example 4.4.3.

Grid Points	EMM			BMM [1]		
	$N = 6$	$N = 8$	$N = 10$	$N = 6$	$N = 8$	$N = 10$
(0.1,0.1)	4.3e-05	7.9e-06	3.6e-07	6.7e-05	3.6e-06	5.0e-07
(0.2,0.2)	7.3e-04	1.3e-05	1.6e-06	2.3e-04	7.1e-05	1.5e-06
(0.3,0.3)	6.1e-04	3.7e-05	2.0e-05	4.3e-04	4.9e-05	5.1e-05
(0.4,0.4)	3.0e-04	2.6e-06	4.7e-05	6.1e-04	5.9e-06	9.1e-5
(0.5,0.5)	6.0e-04	4.0e-05	3.8e-06	7.0e-04	3.7e-05	1.1e-05
(0.6,0.6)	1.7e-04	5.9e-06	1.6e-05	6.9e-04	6.9e-06	1.0e-05
(0.7,0.7)	3.0e-05	2.6e-05	8.2e-05	5.5e-04	6.6e-05	6.8e-05
(0.8,0.8)	1.5e-04	3.0e-05	4.1e-05	3.3e-04	1.1e-04	2.4e-05
(0.9,0.9)	9.5e-05	5.4e-06	5.0e-07	6.6e-05	6.6e-05	9.0e-07

Table 4.8: Comparison of numerical results of EMM and BMM [1] for example 4.4.4.

Grid Points	EMM			BMM [1]		
	$N = 6$	$N = 8$	$N = 10$	$N = 6$	$N = 8$	$N = 10$
(0.1,0.1)	3.0e-05	5.0e-06	7.0e-07	5.4e-05	5.0e-06	7.8e-07
(0.2,0.2)	4.9e-04	1.0e-05	1.0e-06	1.9e-04	1.5e-05	1.9e-06
(0.3,0.3)	3.5e-04	2.0e-05	2.0e-06	3.5e-04	5.1e-05	2.4e-05
(0.4,0.4)	4.0e-04	4.4e-05	4.1e-05	4.9e-04	9.1e-05	2.4e-05
(0.5,0.5)	7.2e-04	5.1e-05	2.2e-05	5.7e-04	1.1e-05	2.8e-05
(0.6,0.6)	4.4e-04	3.0e-04	3.0e-05	5.6e-04	1.1e-04	3.2e-05
(0.7,0.7)	4.6e-04	6.0e-05	5.2e-06	4.6e-04	6.8e-04	2.6e-05
(0.8,0.8)	2.9e-05	1.4e-05	4.3e-05	2.9e-05	2.4e-05	1.4e-05
(0.9,0.9)	8.0e-05	5.0e-07	7.0e-08	9.7e-5	9.0e-07	9.0e-07

4.5 Conclusion

A new computational method named as Euler matrix method (EMM) is presented in this chapter for solving two-dimensional diffusion and telegraph equations with Dirichlet boundary conditions. The method uses the operational matrices of Euler polynomials to transform the main problem into the associated algebraic generalized Sylvester equations. Some robust Krylov subspace iterative solvers (i.e., generalized BICGSTAB) can be implemented to solve these Sylvester equations. To show the spectral accuracy of the proposed method, the numerical experiments are performed by considering four examples and comparison is made of the obtained numerical results with BMM. Numerical experiments conclude that the proposed method is more accurate and provide satisfactory results in comparison to the results obtained by BMM.

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