Chapter 5

Numerical solution of nonlinear reaction-advection-diffusion equation in sense of Caputo-Fabrizio derivative with finite difference and collocation method

5.1 Introduction

Fractional calculus is not only used in mathematics but is also used in many branches of science such as biology [123, 124, 125], visco-elastics materials [126, 127], porous media [128], engineering [129, 130], economics [131], financial system [132], etc. In our school, we studied integer order calculus, i.e integer order integration and derivatives. Fractional calculus theory extends this derivative to fractional order derivative. In last few years, scientists extended integer order derivative to derivative of fractional order which is known as derivative of fractional order. First, it follows the power kernel law. But after some time, scientists have generalized the derivative that follow the Mittag-leffler and exponential kernel law. The derivative that follows Mittag-Leffler kernel law is known as the Atangana-Baleanu derivative, and the derivative that follows the exponential kernel law is called the Caputo-Fabrizio (C-F) derivative.

Mathematically, the reaction-diffusion equation with fractional derivative is written as

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = D\nabla^{2}u(x,t) - k\frac{\partial u(x,t)}{\partial x} + f(x,t),$$

where $0 < \alpha \leq 1$, D denotes the diffusion coefficient, $\nabla^{2}u(x,t)$ represents the diffusion terms, $\frac{\partial u(x,t)}{\partial x}$ stands for advection term and $u(x,t)$ represents the solute concentration on position x and at time t and $f(x,t)$ is the forcing term. The advantage of a fractional order system over an integer order system is that the fractional differential operator is a non-local operator, whereas the integer order operator is local another advantage of the fractional operator is that it has hereditary property, i.e, a future state depends not only on the present state but also on a previous state. Moreover, a fractional derivative is related to systems with memory, which is used in many physical models.

There are many numerical methods available in the literature such as Atangana and Alqahtani [133] have solved advection diffusion equation after modifying it, Alizadeh et al. [134] used the Laplace transform to solve the transient response for a parallel LCR circuit, Qureshi et al. [135] approximated the C-F operator with the help of forward difference, Alshabanat et al. [136] have used the generalized C-F operator in electric circuit, Firoozjaee et al. [137] have used the Ritz method with the C-F derivative to solve the Fokker Planck equation, Ullah et al. [138] examined a mathematical model of hepatitis B virus with C-F derivative, Shah et al. [139] have developed a model for dengue with C-F derivative and make some discussion on it, Mirza and Vieru [140] have investigated numerical solutions for the diffusion equations, Dwivedi et al. [141] have solved two-dimensional nonlinear fractional order reaction-advection-diffusion equation (RADE), Dwivedi et al. [142] have used cubic B-spline method to find the numerical solution of non-linear fractional order

RADE, Dwivedi and Rajeev [143] have investigated numerical solution of RADE using Fibonacci neural network technique, Dwivedi and Das [144] have developed the non standard finite difference method for Fibonacci polynomials to solve nonlinear diffusion equation. Singh et al. [70] have used the collocation method to find the numerical solution of two-dimensional nonlinear RADE. Some research related to the present chapter can be seen in [145, 146, 147, 148, 149]. In this chapter, a new numerical scheme is developed with the help of shifted Legendre polynomials to investigate the approximate solution of a nonlinear model with a C-F derivative. In the proposed scheme, we discretized the time domain, used a finite difference scheme to deal with it, and used a shifted Legendre approximation to deal with spatial direction. On applying the finite difference and collocation methods, the system of PDEs is reduced to algebraic equations, which are solved with the help of initial conditions, and thus the numerical solution is obtained.

5.2 Preliminary definitions

In this section, some definitions and properties of fractional calculus theory are presented.

5.2.1 C-F derivative and integral

The C-F derivative of the function $g(x) \in H^1(a,b)$, $b > a$, of order α, where $q < \alpha \leq q+1$, is given by the following relation:

$$ {}_{0}^{CF}D_x^{\alpha} g(x) = \frac{M(\alpha)}{\lceil \alpha \rceil - \alpha} \int_0^x \exp\left[\frac{-\alpha}{\lceil \alpha \rceil - \alpha}(x-s)\right] \frac{\partial^{q+1} g(s)}{\partial s^{q+1}} ds, \tag{5.1}$$

where $M(\alpha)$ is normalized function.

Remark: Here $\alpha = q + \eta$ is considered where q represents the floor(α) (integral part) and η is the fractional part.

The C-F integral of order $q < \alpha \leq q + 1$ is given by the following formula

$$\begin{aligned} {}_0^{CF}I_x^{\alpha} g(x) = & \sum_{j=0}^{q} \frac{x^j}{j!} g^j(0) + \frac{(1-\eta)}{M(\eta)(q-1)!} \int_0^x (x-s)^{q-1} g(s) ds \\ & + \frac{\eta}{M(\eta)\eta!} \int_0^x (x-s)^q g(s) ds, \end{aligned} \tag{5.2}$$

when fractional part of α is zero (i.e., $\eta = 0$) then C-F integral is defined as

$${}_0^{CF}I_x^{\alpha} g(x) = \frac{(1-\alpha)}{M(\alpha)} g(x) + \frac{\alpha}{M(\alpha)} \int_0^x g(s) ds, \;\; x \geq 0. \tag{5.3}$$

5.2.2 Shifted Legendre polynomials

The Legendre polynomials ($l_i(z)$) from the interval $[-1, 1]$ is shifted to the interval $[0, 1]$ using the transformation $x = \frac{z+1}{2}$. Thus the shifted Legendre polynomials of degree i is given by

$$L_i(x) = \sum_{j=0}^{i} (-1)^{i+j} \frac{(i+j)! x^j}{(i-j)!(j!)^2}, \quad i = 0, 1, \cdots \tag{5.4}$$

The shifted Legendre polynomials are orthogonal with respect to the weight function $w(x) = 1$, and the orthogonality conditions is given by

$$\int_0^1 L_i(x) L_j(x) dx = \begin{cases} \frac{1}{1+2i}, & \text{for } \; i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

Let $u(x)$ be a function such that $u(x) \in L^2[0,1]$, then $u(x)$ can be expressed in combination of shifted Legendre polynomials as

$$u(x) = \sum_{i=0}^{\infty} c_i L_i(x). \tag{5.6}$$

Now, taking the first m terms of the approximation, we get

$$u(x) \simeq \sum_{i=0}^{m} c_i L_i(x), \tag{5.7}$$

where the unknown coefficients are determined by

$$c_i = (1+2i) \int_0^1 L_i(x) u(x) dx. \tag{5.8}$$

5.3 C-F approximation

Theorem 5.1. *The CF approximation of Eq.(5.7) of order $q < \alpha \leq q+1$ is defined as follows:*

$$ {}_0^{CF}D_x^{\alpha} u(x) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{j=\lceil\alpha\rceil}^{i} c_i \frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2} \frac{M(\alpha)\Gamma(j+1)}{\lceil\alpha\rceil - \alpha} \Pi_{s,j,\alpha}(x), \tag{5.9}$$

where

$$\Pi_{s,j,\alpha}(x) = \sum_{s=0}^{j-q-1} \frac{(-1)^s x^{j-q-s-1}}{\Gamma(j-q-s)\left(\frac{-\alpha}{\lceil\alpha\rceil-\alpha}\right)^{s+1}} + \frac{(-1)^{j-q}}{\left(\frac{-\alpha}{\lceil\alpha\rceil-\alpha}\right)^{j-q}} \exp\left(\frac{-\alpha}{\lceil\alpha\rceil-\alpha} x\right).$$

Proof. :

On employing C-F derivative of order $q < \alpha \leq q+1$ on both the sides of Eq.(5.7), we have

$$ {}_{0}^{CF}D_{x}^{\alpha} u(x) = \sum_{i=0}^{m} c_i \, {}_{0}^{CF}D_{x}^{\alpha} L_i(x). \tag{5.10}$$

From the definition of C-F derivative, it is clear that

$$ {}_{0}^{CF}D_{x}^{\alpha} L_i(x) = 0 \quad \text{for} \quad i = 0, 1, \cdots, \lceil \alpha \rceil - 1. \tag{5.11}$$

Therefore for $i \geq \lceil \alpha \rceil$, we get

$$ {}_{0}^{CF}D_{x}^{\alpha} L_i(x) = \sum_{j=\lceil \alpha \rceil}^{i} \frac{(-1)^{i+j}(j+i)!}{(i-j)!(j!)^2} \; {}_{0}^{CF}D_{x}^{\alpha} x^j \tag{5.12}$$

Now, we have

$$ {}_{0}^{CF}D_{x}^{\alpha} x^j = \frac{M(\alpha)}{\lceil \alpha \rceil - \alpha} \int_0^x \exp\Big(\frac{-\alpha}{\lceil \alpha \rceil - \alpha}(x-t) \Big) \frac{\partial^{q+1}}{\partial t^{q+1}} t^j \, dt, $$

$$ = \frac{M(\alpha)}{\lceil \alpha \rceil - \alpha} \int_0^x \exp\Big(\frac{-\alpha}{\lceil \alpha \rceil - \alpha}(x-t) \Big) \frac{\Gamma(j+1)}{\Gamma(j-q)} t^{j-q-1} dt, $$

$$ = \frac{M(\alpha)}{\lceil \alpha \rceil - \alpha} \frac{\Gamma(j+1)}{\Gamma(j-q)} \exp\Big(\frac{-\alpha}{\lceil \alpha \rceil - \alpha} x \Big) \int_0^x t^{j-q-1} \exp\Big(\frac{\alpha}{\lceil \alpha \rceil - \alpha} t \Big) dt, $$

$$ = \frac{M(\alpha)}{\lceil \alpha \rceil - \alpha} \frac{\Gamma(j+1)}{\Gamma(j-q)} \exp\Big(\frac{-\alpha}{\lceil \alpha \rceil - \alpha} x \Big) $$
$$ \times \left[\exp\Big(\frac{\alpha}{\lceil \alpha \rceil - \alpha} x \Big) \sum_{s=0}^{j-q-1} \frac{(-1)^s \Gamma(j-q) x^{j-q-s-1}}{\Gamma(j-q-s)(\frac{-\alpha}{\lceil \alpha \rceil - \alpha})^{s+1}} - \frac{(-1)^{j-q-1}\Gamma(j-q)}{(\frac{-\alpha}{\lceil \alpha \rceil - \alpha})^{j-q}} \right], $$

$$ {}_{0}^{CF}D_{x}^{\alpha} x^j = \frac{M(\alpha)\Gamma(j+1)}{\lceil \alpha \rceil - \alpha} \left[\sum_{s=0}^{j-q-1} \frac{(-1)^s x^{j-q-s-1}}{\Gamma(j-q-s)(\frac{-\alpha}{\lceil \alpha \rceil - \alpha})^{(s+1)}} + \frac{(-1)^{j-q}}{(\frac{-\alpha}{\lceil \alpha \rceil - \alpha})^{j-q}} \exp\Big(\frac{-\alpha}{\lceil \alpha \rceil - \alpha} x \Big) \right]. \tag{5.13}$$

From Eqs.(5.12) and Eq.(5.13), we obtain

$$
{}_0^{CF}D_x^{\alpha} L_i(x) = \sum_{j=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2} \frac{M(\alpha)\Gamma(j+1)}{\lceil\alpha\rceil-\alpha} \Pi_{s,j,\alpha}(x). \tag{5.14}
$$

Now from Eq.(5.10), the desired result is obtained.

□

5.4 Numerical technique

In this section, the numerical technique is used to solve the following nonlinear time-space fractional order reaction-diffusion equation with the C-F derivative given by

$$
\begin{aligned}
{}_0^{C}D_t^{\alpha} u(x,t) = {}_0^{CF}D_x^{\beta} u(x,t) - a u^{\delta}(x,t)\frac{\partial u(x,t)}{\partial x} + b u(x,t)(u^{\delta}(x,t)-\eta) \\
\times (1-u^{\delta}(x,t)) + f(x,t),
\end{aligned} \tag{5.15}
$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$, η is constant, and a, b, δ is a non negative integer. The initial and boundary conditions associated with Eq.(5.15) are

$$
\begin{aligned}
u(x,0) &= \rho_1(x), \\
u(0,t) &= \rho_2(t), \\
u(1,t) &= \rho_3(t),
\end{aligned} \tag{5.16}
$$

where $\rho_1(x)$, $\rho_2(t)$, and $\rho_3(t)$ are continuous function.

Now, to explore the numerical solution of the above nonlinear model, the spectral method is used together with the finite difference method. For this purpose, let us take an approximation of $u(x,t)$ as

$$u(x,t) \simeq u_m(x,t) = \sum_{i=0}^{m} c_i(t) L_i(x), \tag{5.17}$$

where unknown coefficients c_i's are function of t.

Now, from Eqs.(5.15) and (5.17), we have

$$\begin{aligned}\sum_{i=0}^{m} \frac{\partial^\alpha c_i(t)}{\partial t^\alpha} L_i(x) = &\sum_{i=0}^{m} {}_0^{CF}D_x^\beta c_i(t) L_i(x) - a\Big(\sum_{i=0}^{m} c_i(t) L_i(x)\Big)^\delta \sum_{i=0}^{m} c_i(t) \frac{\partial L_i(x)}{\partial x} \\ &+ b\Big(\sum_{i=0}^{m} c_i(t) L_i(x)\Big)\left(\Big(\sum_{i=0}^{m} c_i(t) L_i(x)\Big)^\delta - \eta\right) \\ &\times \left(1 - \Big(\sum_{i=0}^{m} c_i(t) L_i(x)\Big)^\delta\right) + f(x,t). \end{aligned} \tag{5.18}$$

On using Theorem 5.1, in Eq.(5.18), we obtain

$$\begin{aligned}\sum_{i=0}^{m} \frac{\partial^\alpha c_i(t)}{\partial t^\alpha} L_i(x) = &\sum_{i=\lceil\beta\rceil}^{m} \sum_{j=\lceil\beta\rceil}^{i} \frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2} \frac{M(\beta)\Gamma(j+1)}{\lceil\beta\rceil} \Pi_{i,j,\beta}(x) c_i(t) \\ &- a\Big(\sum_{i=0}^{m} c_i(t) L_i(x)\Big)^\delta \sum_{i=0}^{m} \sum_{j=0}^{i} \frac{(-1)^{i+j}(j+i)! x^{-1+j}}{(j-1)!(i-j)!j!} c_i(t) \\ &+ b\Big(\sum_{i=0}^{m} c_i(t) L_i(x)\Big)\left(\Big(\sum_{i=0}^{m} c_i(t) L_i(x)\Big)^\delta - \eta\right) \\ &\times \left(1 - \Big(\sum_{i=0}^{m} c_i(t) L_i(x)\Big)^\delta\right) + f(x,t). \end{aligned} \tag{5.19}$$

Now, to find the unknown coefficients $c_i(t)$ at $i = 0, 1, \cdots, m$, we discretize time domain $[0, T]$ into n equal parts of length h such that $t_n = nh$.

Remark: The value of coefficients $c_i(t)$ at the point $t = t_n$ is represent by c_i^n.

Thus, by the definition of Caputo's derivative, we obtain

$$
\begin{aligned}
\frac{\partial^\alpha c_i(t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial c_i(s)}{\partial s}ds \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{s^\alpha}\frac{dc_i(t-s)}{ds}ds, \\
\frac{\partial^\alpha c_i(t_{n+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\int_{jh}^{(j+1)h}\frac{1}{s^\alpha}\frac{dc_i(t_{n+1}-s)}{ds}ds \\
&\approx \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\frac{c_i(t_{n+1}-jh)-c_i(t_{n+1}-(j+1)h)}{h}\int_{jh}^{(j+1)h}\frac{1}{s^\alpha} \\
&\approx \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n}\Big((j+1)^{1-\alpha}-j^{1-\alpha}\Big)\Big(c_i^{n+1-j}-c_i^{n-j}\Big).
\end{aligned}
$$

From Eqs.(5.19) and (5.20), we have

$$
\begin{aligned}
&\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=0}^{m}\sum_{j=0}^{n}\Big((j+1)^{1-\alpha}-j^{1-\alpha}\Big)\Big(c_i^{n+1-j}-c_i^{n-j}\Big)L_i(x) \\
&= \sum_{i=\lceil\beta\rceil}^{m}\sum_{j=\lceil\beta\rceil}^{i}\frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2}\frac{M(\beta)\Gamma(j+1)}{\lceil\beta\rceil}\Pi_{i,j,\beta}(x)c_i^n \\
&- a\Big(\sum_{i=0}^{m}c_i^nL_i(x)\Big)^\delta\sum_{i=0}^{m}\sum_{j=0}^{i}\frac{(-1)^{i+j}(j+i)!x^{-1+j}}{(j-1)!(i-j)!j!}c_i^n \\
&+ b\Big(\sum_{i=0}^{m}c_i^nL_i(x)\Big)\left(\Big(\sum_{i=0}^{m}c_i^nL_i(x)\Big)^\delta-\eta\right)\times\left(1-\Big(\sum_{i=0}^{m}c_i^nL_i(x)\Big)^\delta\right)+f(x,t_n).
\end{aligned}
$$

Collocating the above equation at time level $(n+1)$ at certain collocation points x_l, where $l = 1, 2, \cdots, m+1-\lceil\beta\rceil$ together with initial condition, we get

$$
\begin{aligned}
&\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=0}^{m}\sum_{j=0}^{n}\Big((j+1)^{1-\alpha}-j^{1-\alpha}\Big)\Big(c_i^{n+1-j}-c_i^{n-j}\Big)L_i(x_l) \\
&= \sum_{i=\lceil\beta\rceil}^{m}\sum_{j=\lceil\beta\rceil}^{i}\frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2}\frac{M(\beta)\Gamma(j+1)}{\lceil\beta\rceil}\Pi_{i,j,\beta}(x_l)c_i^n \\
&- a\Big(\sum_{i=0}^{m}c_i^nL_i(x_l)\Big)^\delta\sum_{i=0}^{m}\sum_{j=0}^{i}\frac{(-1)^{i+j}(j+i)!x_l^{-1+j}}{(j-1)!(i-j)!j!}c_i^n
\end{aligned}
$$

$$+\,b\Big(\sum_{i=0}^{m} c_i^n L_i(x_l)\Big)\left(\Big(\sum_{i=0}^{m} c_i^n L_i(x_l)\Big)^{\delta}-\eta\right)\times\left(1-\Big(\sum_{i=0}^{m} c_i^n L_i(x_l)\Big)^{\delta}\right)+f(x_l,t_{n+1}). \tag{5.20}$$

From Eq.(5.16), the initial and boundary conditions become

$$\begin{aligned}\sum_{i=0}^{m} c_i^0 L_i(x_l) &= \rho_1(x_l),\\ \sum_{i=0}^{m} c_i^n L_i(0) &= \rho_2(t_n),\\ \sum_{i=0}^{m} c_i^n L_i(1) &= \rho_3(t_n),\end{aligned} \tag{5.21}$$

Eqs.(5.20) and (5.21) generate $(m+1)$ algebraic equations at each time level. The initial approximation is obtained from Eq.(5.21) with the given boundary conditions; with the help of this approximation, we are able to find the values of unknown coefficients c_i^n. Now, by replacing the unknown values in Eq.(5.17), an approximate solution to the considered model is obtained.

5.5 Error analysis

In this section, for simplicity, the special case $(\delta = 0)$ is considered in the model (5.15) and determines the bound for the error function obtained by the proposed method.

For $\delta = 0$, the model (5.15) reduces as

$${}_0^C D_t^{\alpha} u(x,t) = {}_0^{CF} D_x^{\beta} u(x,t) - a\frac{\partial u(x,t)}{\partial x} - b\eta u(x,t) + f(x,t). \tag{5.22}$$

Consider,

$$L(u(x,t)) = {}_0^C D_t^{\alpha} u(x,t) - {}_0^{CF} D_x^{\beta} u(x,t) + a\frac{\partial u(x,t)}{\partial x} + b\eta u(x,t) - f(x,t).$$

To obtain the error bound of the model (5.22), some theorems are needed that are given below.

Theorem 5.2. *[150] Let* $(u(x,t))_N$ *be the approximation of a sufficiently smooth function* $u(x,t)$ *defined in* $[0,1] \times [0,1]$*. Then, there is a real constant* C *such that*

$$||u(x,t) - (u(x,t))_N||_2 \leq \frac{C}{(N+1)!2^{2N+1}}. \tag{5.23}$$

Theorem 5.3. *[151] Let* $\left(\frac{\partial u(x,t)}{\partial t}\right)_N$ *be the approximation of* $\frac{\partial u(x,t)}{\partial t}$ *such that* $\left|\frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2}\right| < C_1$ *for some positive constant* C_1*. Then, we have*

$$\Big|\Big|\frac{\partial u(x,t)}{\partial t} - \Big(\frac{\partial u(x,t)}{\partial t}\Big)_N\Big|\Big|_2^2 \leq \frac{3C_1}{8(2N-3)}. \tag{5.24}$$

Theorem 5.4. *[152] Let* $\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)_N$ *be the approximation of* $\frac{\partial^2 u(x,t)}{\partial t^2}$ *such that* $\left|\frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2}\right| < C_2$ *for some positive constant* C_2*. Then, we have*

$$\Big|\Big|\frac{\partial^2 u(x,t)}{\partial t^2} - \Big(\frac{\partial^2 u(x,t)}{\partial t^2}\Big)_N\Big|\Big|_2^2 < \frac{C_2^2}{65536}\Big(F_3(N-\frac{3}{2})\Big)^2, \tag{5.25}$$

where $F_n(t)$ *is the polygamma function defined by* $F_n(t) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(t+k)^{n+1}}$*.*

Theorem 5.5. *[151] Let* $\left(\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}\right)_N$ *be the approximation of* $\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}$ *such that* $\left|\frac{\partial^5 u(x,t)}{\partial x^2 \partial t^3}\right| < C_3$ *for some positive constant* C_3*, where* $\alpha \in (0,1)$ *Then, we have*

$$\Big|\Big|\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} - \Big(\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}\Big)_N\Big|\Big|_2 \leq \frac{3C_3}{8(2N-3)\Gamma(2-\alpha)}. \tag{5.26}$$

Theorem 5.6. *Let* $\Big({}_{0}^{CF}D_x^{\beta}u(x,t)\Big)_N$ *be the approximation of* $\Big({}_{0}^{CF}D_x^{\beta}u(x,t)\Big)$, *where* $\beta \in (1,2)$. *Also, let us assume that there is a positive constant* C_4 *such that* $\left|\frac{\partial^6 u(x,t)}{\partial x^4 \partial t^2}\right| < C_4$. *Then, we have*

$$\Big|\Big|\,{}_{0}^{CF}D_x^{\beta}u(x,t) - \Big({}_{0}^{CF}D_x^{\beta}u(x,t)\Big)_N\Big|\Big|_2 \leq \frac{M(\beta)C_4}{\beta\sqrt{65536}}\Big(\exp\Big[\frac{\beta}{\beta-\lceil\beta\rceil}\Big]-1\Big)F_3(N-\frac{3}{2}). \tag{5.27}$$

Proof.

$$\begin{aligned}\Big|\Big|\,{}_{0}^{CF}D_x^{\beta}u(x,t) - \Big({}_{0}^{CF}D_x^{\beta}u(x,t)\Big)_N\Big|\Big|_\infty = \Big|\Big|\frac{M(\beta)}{\lceil\beta\rceil-\beta}\int_0^x \exp\Big[\frac{-\beta}{\lceil\beta\rceil-\beta}(x-s)\Big]\\ \times\Big(\frac{\partial^2 u(x,t)}{\partial x^2} - \Big(\frac{\partial^2 u(x,t)}{\partial x^2}\Big)_N\Big)ds\Big|\Big|_\infty,\end{aligned}$$

$$\begin{aligned}&\leq \frac{M(\beta)}{\lceil\beta\rceil-\beta}\int_0^x \exp\Big[\frac{-\beta}{\lceil\beta\rceil-\beta}(x-s)\Big]\Big|\Big|\frac{\partial^2 u(x,t)}{\partial x^2} - \Big(\frac{\partial^2 u(x,t)}{\partial x^2}\Big)_N\Big|\Big|_\infty ds,\\ &\leq \frac{M(\beta)}{\lceil\beta\rceil-\beta}\times\frac{C_4}{\sqrt{65536}}F_3(N-\frac{3}{2})\int_0^x \exp\Big[\frac{-\beta}{\lceil\beta\rceil-\beta}(x-s)\Big]ds,\\ &\leq \frac{M(\beta)C_4}{(\lceil\beta\rceil-\beta)\sqrt{65536}}F_3(N-\frac{3}{2})\times\frac{(\beta-\lceil\beta\rceil)\Big(\exp\Big[\frac{-\beta x}{\lceil\beta\rceil-\beta}\Big]-1\Big)}{\beta},\end{aligned}$$

Hence,

$$\Big|\Big|\,{}_{0}^{CF}D_x^{\beta}u(x,t) - \Big({}_{0}^{CF}D_x^{\beta}u(x,t)\Big)_N\Big|\Big|_2 = \Big(\int_0^1\int_0^1\Big[\,{}_{0}^{CF}D_x^{\beta}u(x,t) -{}_{0}^{CF} D_x^{\beta}u(x,t)\Big]dxdt\Big)^{\frac{1}{2}},$$

$$\begin{aligned}&\leq\Big(\int_0^1\int_0^1\Big|\Big|\,{}_{0}^{CF}D_x^{\beta}u(x,t) - \Big({}_{0}^{CF}D_x^{\beta}u(x,t)\Big)_N\Big|\Big|_\infty^2 dxdt\Big)^{\frac{1}{2}},\\ &\leq \Big|\Big|\,{}_{0}^{CF}D_x^{\beta}u(x,t) - \Big({}_{0}^{CF}D_x^{\beta}u(x,t)\Big)_N\Big|\Big|_\infty,\end{aligned}$$

Therefore,

$$\Big|\Big| {}_0^{CF}D_x^{\beta}u(x,t) - \Big({}_0^{CF}D_x^{\beta}u(x,t)\Big)_N\Big|\Big|_2 \leq \frac{M(\beta)C_4}{\beta\sqrt{65536}}\Big(\exp\Big[\frac{-\beta}{\lceil\beta\rceil-\beta}\Big]-1\Big)F_3(N-\frac{3}{2}).$$

□

Theorem 5.7. *Let* $u(x,t)$ *be the exact solution of the given problem and* $u_N(x,t)$ *be the approximate solution estimated from the proposed method. Then, we have*

$$\begin{aligned}||E_N(x,t)||_2 \leq \frac{M(\beta)C_4}{\beta\sqrt{65536}}\Big(\Big|\exp\Big[\frac{-\beta}{\lceil\beta\rceil-\beta}\Big]-1\Big|\Big)F_3(N-\frac{3}{2}) + |a|\frac{3C_1}{8(2N-3)}\\ + (|b\eta|+1)\frac{C}{(N+1)!2^{2N+1}} + \frac{3C_3}{8(2N-3)\Gamma(2-\alpha)}.\end{aligned} \tag{5.28}$$

Proof. Suppose $E_N(x,t) = L(u(x,t)) - L(u_N(x,t))$, be the error function, here, $u(x,t)$ and $u_N(x,t)$ represents the exact and numerical solution respectively. Now, we have

$$\begin{aligned}||E_N(x,t)||_2 = \Big|\Big| {}_0^{C}D_t^{\alpha}u(x,t) - {}_0^{CF}D_x^{\beta}u(x,t) + a\frac{\partial u(x,t)}{\partial x} + b\eta u(x,t) - f(x,t)\\ - \Big({}_0^{C}D_t^{\alpha}u_N(x,t) - {}_0^{CF}D_x^{\beta}u_N(x,t) + a\frac{\partial u_N(x,t)}{\partial x} + b\eta u_N(x,t) - f_N(x,t)\Big)\Big|\Big|_2.\end{aligned}$$

$$\begin{aligned}||E_N(x,t)||_2 &= \Big|\Big|\Big({}_0^{C}D_t^{\alpha}u(x,t) - {}_0^{C}D_t^{\alpha}u_N(x,t)\Big) - \Big({}_0^{CF}D_x^{\beta}u(x,t) - {}_0^{CF}D_x^{\beta}u_N(x,t)\Big)\\ &+ a\Big(\frac{\partial u(x,t)}{\partial t} - \frac{\partial u_N(x,t)}{\partial t}\Big) + b\eta\Big(u(x,t) - u_N(x,t)\Big) - \Big(f(x,t) - f_N(x,t)\Big)\Big|\Big|_2,\\ &\leq \Big|\Big| {}_0^{C}D_t^{\alpha}u(x,t) - {}_0^{C}D_t^{\alpha}u_N(x,t)\Big|\Big|_2 + \Big|\Big| {}_0^{CF}D_x^{\beta}u(x,t) - {}_0^{CF}D_x^{\beta}u_N(x,t)\Big|\Big|_2\\ &+ |a|\Big|\Big|\frac{\partial u(x,t)}{\partial t} - \frac{\partial u_N(x,t)}{\partial t}\Big|\Big|_2 + |b\eta|||u(x,t) - u_N(x,t)||_2\\ &+ ||f(x,t) - f_N(x,t)||_2.\end{aligned}$$

Applying Theorem 5.2-5.6, it is clear that,

$$||E_N(x,t)||_2 \leq \frac{M(\beta)C_4}{\beta\sqrt{65536}}\Big(\Big|\exp\Big[\frac{-\beta}{\lceil\beta\rceil-\beta}\Big]-1\Big|\Big)F_3(N-\frac{3}{2})+|a|\frac{3C_1}{8(2N-3)} + (|b\eta|+1)\frac{C}{(N+1)!2^{2N+1}}+\frac{3C_3}{8(2N-3)\Gamma(2-\alpha)},$$

which completes the proof. □

5.6 Numerical results and discussion

The order of convergence is given by

$$\text{order of convergence} = \frac{\log\left(\frac{E_{h_1}}{E_{h_2}}\right)}{\log\left(\frac{h_1}{h_2}\right)},$$

where E_h is the absolute error with h interval of discretization.

Example 5.1 On selecting $\eta = 0.5$, $\alpha = 0.9$, $\beta = 1.5$, $a = 1$, $\delta = 1$ and $b = 1$, the considered model reduces to

$$ {}_0^C D_x^{0.9} u(x,t) = {}_0^{CF} D_x^{1.5} u(x,t) - u(x,t)\frac{\partial u(x,t)}{\partial x} + u(x,t)(u(x,t)-0.5)(1-u(x,t)) \tag{5.29}$$

$$+f(x,t),$$

with the prescribed conditions

$$\begin{aligned} u(x,0) &= 0, \\ u(0,t) &= t, \\ u(1,t) &= 0. \end{aligned} \tag{5.30}$$

For the proper choice of $f(x,t)$, the exact solution of Eqs.(5.29)-(5.30) become $u(x,t) = (1-x)^2 t$.

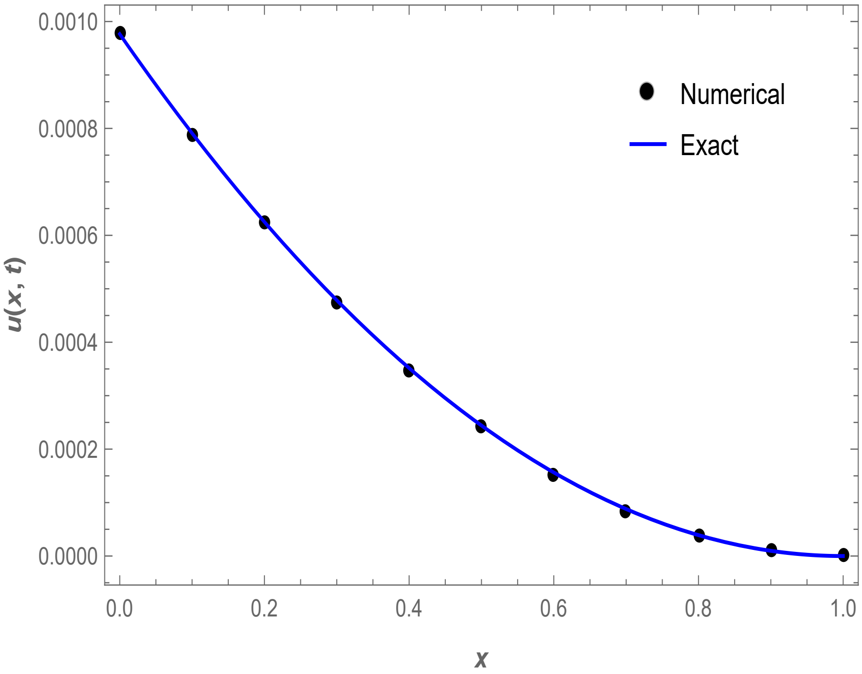

Figure 5.1: Plot of $u(x,t)$ versus x for Example 5.1 in case of Exact and Numerical solution.

The graph of exact and numerical solution for Example 5.1 with $n = 9,\ h = \frac{1}{1024}$ is shown in Fig.5.1 and the absolute error between exact and approximate solution is shown in Table 5.1 at $n = 9$, $x = 0.5$ and for various values of h, which clearly indicates that the numerical solution is in the good agreement with the exact one.

TABLE 5.1: variations of absolute errors at $x = 0.5$ and for different values of h for Example 5.1

h	E_h	order of convergence	CPU time (s)
$\frac{1}{128}$	1.48201×10^{-4}	$---$	3.313
$\frac{1}{256}$	3.88450×10^{-5}	1.93175	3.750
$\frac{1}{512}$	1.02573×10^{-5}	1.92107	3.813
$\frac{1}{1024}$	2.70832×10^{-6}	1.92118	3.872

Example 5.2 Considering some particular values of the parameters as $\eta = 1$, $\delta = 2$, $\alpha = 0.9$, $\beta = 1.5$, $a = 1$ and $b = 1$ in Eq.(5.15), we get

$$
{}_{0}^{C}D_{x}^{0.9}u(x,t) = {}_{0}^{CF}D_{x}^{1.5}u(x,t) - u^{2}(x,t)\frac{\partial u(x,t)}{\partial x} + u(x,t)(u^{2}(x,t)-1)(1-u^{2}(x,t)) \tag{5.31}
$$

$$
+f(x,t),
$$

with the prescribed conditions

$$
\begin{aligned}
u(x,0) &= 0,\\
u(0,t) &= 0,\\
u(1,t) &= t.
\end{aligned} \tag{5.32}
$$

For the suitable choice of $f(x,t)$, the exact solution of Eqs.(5.31)-(5.32) become $u(x,t) = x^2 t$.

We plot the graph of numerical and exact solution with $n = 9$, $h = \frac{1}{1024}$ which is depicted by Fig.5.2. Table 5.2 displays the absolute error of Example 5.2 at $n = 9$, $x = 0.5$, and for different values of h, which predicts that the approximate solution obtained by the proposed method is very close to the exact solution.

Table 5.2: variations of absolute errors at $x = 0.5$ and for different values of h for Example 5.2

h	E_h	order of convergence	CPU time (s)
$\frac{1}{128}$	1.07502×10^{-4}	$---$	7.641
$\frac{1}{256}$	2.83674×10^{-5}	1.92205	8.813
$\frac{1}{512}$	7.51891×10^{-6}	1.91563	8.873
$\frac{1}{1024}$	1.99089×10^{-6}	1.91711	8.876

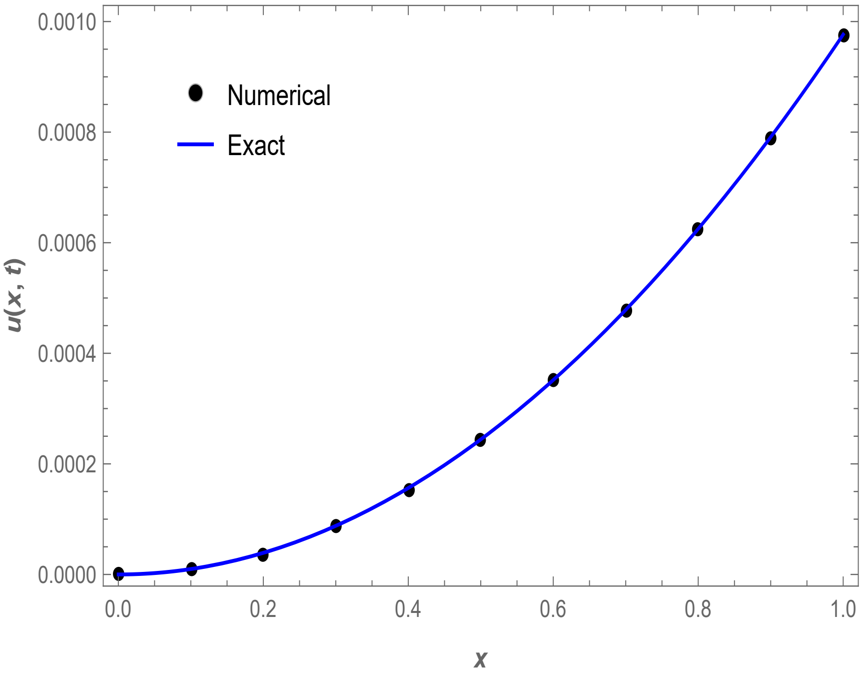

Figure 5.2: Plot of $u(x,t)$ versus x for Example 5.2 in case of Exact and Numerical solution.

Example 5.3 Consider the following nonlinear diffusion equation as a special case of model (5.15)

$$ {}_0^cD_t^{0.5}u(x,t) = {}_0^{CF}D_x^{1.5}u(x,t) + u(x,t)(1-u(x,t)) + f(x,t), \tag{5.33}$$

with the conditions

$$\begin{aligned} u(x,0) &= 0, \\ u(0,t) &= 0, \\ u(1,t) &= t\sin(1). \end{aligned} \tag{5.34}$$

For the appropriate choice of $f(x,t)$, the exact solution of Eqs.(5.33)-(5.34) becomes $u(x,t) = t\sin(x)$.

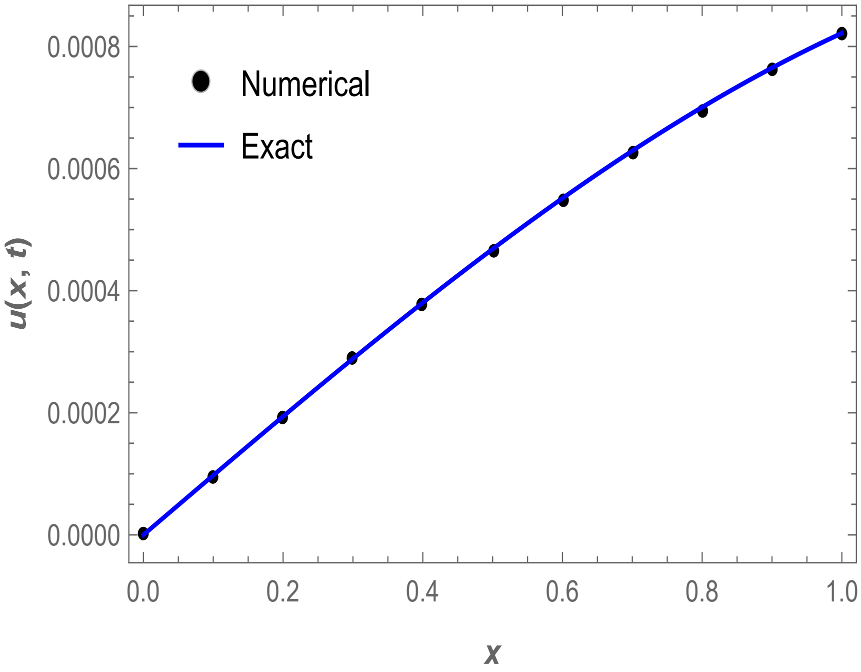

FIGURE 5.3: Plot of $u(x,t)$ versus x for Example 3 in case of Exact and Numerical solution.

The graph of exact and numerical solution with $n = 5,\ h = \frac{1}{1024}$ is displayed by Fig.5.3. Comparison between the approximate and numerical solution for Example 5.3 is displayed in Table 5.3 for $n = 5$, $x = 0.5$, and for several values of h, which indicates that the numerical solution agrees well with the exact solution. The convergence rate is also good.

Table 5.3: variations of absolute errors at $x = 0.5$ and for different values of h for Example 5.3

h	E_h	order of convergence	CPU time (s)
$\frac{1}{128}$	2.38453×10^{-4}	$---$	4.499
$\frac{1}{256}$	4.20239×10^{-5}	2.50443	4.922
$\frac{1}{512}$	7.73066×10^{-6}	2.44252	4.890
$\frac{1}{1024}$	1.63538×10^{-6}	2.24104	5.016

Example 5.4 Consider the following nonlinear time fractional reaction-advection-diffusion equation as

$$ {}_0^cD_t^{0.7}u(x,t) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + u(x,t)(1-u(x,t)) + f(x,t), \tag{5.35}$$

with forcing term as

$$f(x,t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - (x+t)(1-(x+t)) + 1,$$

under the following initial & boundary conditions

$$\begin{aligned} u(x,0) &= x, \\ u(0,t) &= t, \\ u(1,t) &= 1+t. \end{aligned} \tag{5.36}$$

Advice is given to solve Example 5.4 for $n = 3$ and $h = \frac{1}{512}$ with the proposed method and compare the obtained numerical results with the existing method [153] through finding absolute errors whose results are displayed in Table 5.4.

TABLE 5.4: Comparison of absolute errors at $h = \frac{1}{512}$ and for different values of x for Example 5.4

		Method in [153]		Present Method	
x	COM [153]	Absolute error	CPU time (s)	Absolute error	CPU time (s)
0.2	2.2607×10^{-4}	4.7198×10^{-5}	1.672	1.1351×10^{-5}	1.218
0.4	3.1930×10^{-4}	7.0136×10^{-5}	1.828	5.6478×10^{-6}	1.235
0.6	3.2813×10^{-4}	7.3044×10^{-5}	1.672	5.7312×10^{-6}	1.265
0.8	2.4792×10^{-4}	5.3593×10^{-5}	1.704	1.1406×10^{-5}	1.282

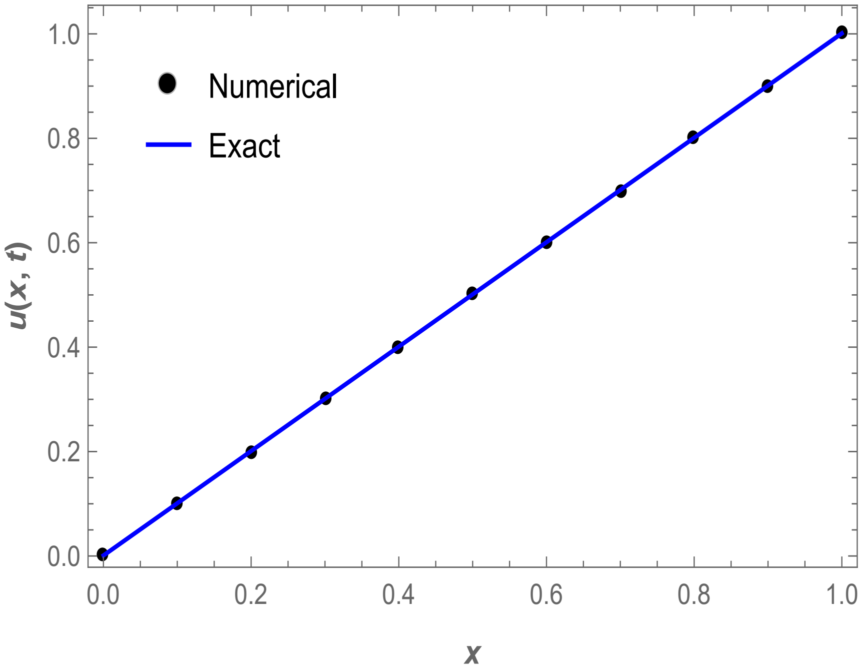

FIGURE 5.4: Plot of $u(x,t)$ versus x for Example 4 in case of Exact and Numerical solution.

Table 5.4 shows that the absolute errors obtained between the exact results and approximate result obtained by our proposed numerical method are smaller for a small value of approximation ($n = 3$) as compared to errors obtained by using the existing numerical method [153] for $n = 6$. One can also see that the computational time of the proposed method is less than the existing method while solving problem 5.4. The exactness of the given example for our method is displayed through Fig.5.4

for $n = 3$ and $h = \frac{1}{512}$. It can be seen from Fig.5.4 that the numerical results are in agreement with the exact one.

Example 5.5 Now, the considered model defined in Eq.(5.15) at some particular values $a = 1$, $b = 1$, $\eta = 0.5$, $\delta = 1$ and $f(x, t) = 0$ for various values of α and β at $n = 9$ is solved under the following initial and boundary conditions:

$$u(x, 0) = \tanh\left(\frac{x}{4}\right),$$

and

$$u(0, t) = \tanh\left(\frac{5t}{8}\right), \qquad u(1, t) = \tanh\left[\frac{1}{4}\left(1 + \frac{5t}{2}\right)\right].$$

The variations in solute profile $u(x, t)$ due to change in α at $\beta = 1.5$ and $\beta = 1.9$ are depicted in Fig.5.5 to Fig.5.6. It can be seen from the figures that on increasing the time derivative α, the value of the solute concentration $u(x, t)$ increases. Fig.5.7 to Fig.5.8 are drawn between $u(x, t)$ versus x for various values of β at $\alpha = 0.5$ and $\alpha = 0.9$, respectively. From the figures, one can observe that on increasing the values of β solute concentration $u(x, t)$ decreases in both cases.

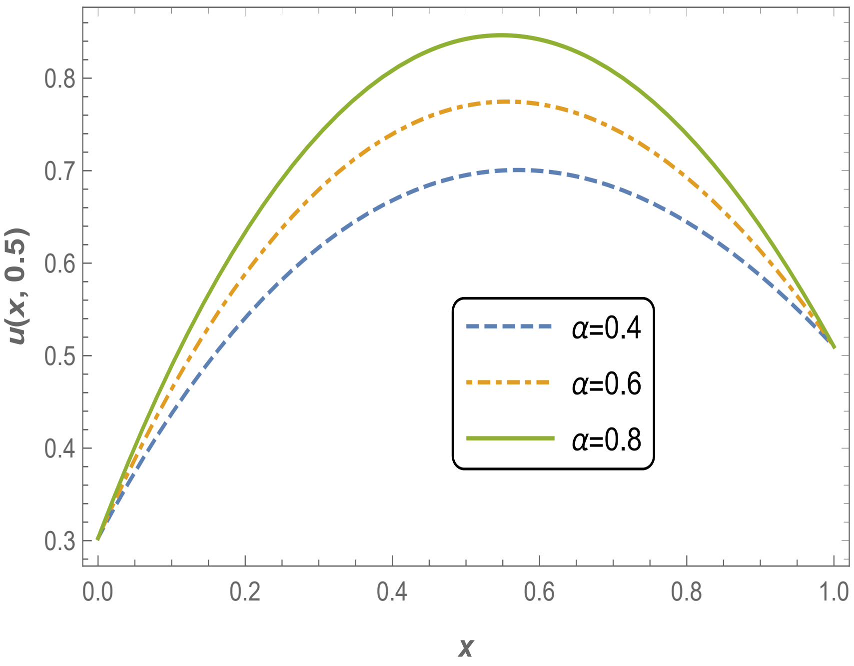

FIGURE 5.5: Plot between $u(x, 0.5)$ versus x for Example 5.5 at $\beta = 1.5$ for various values of α.

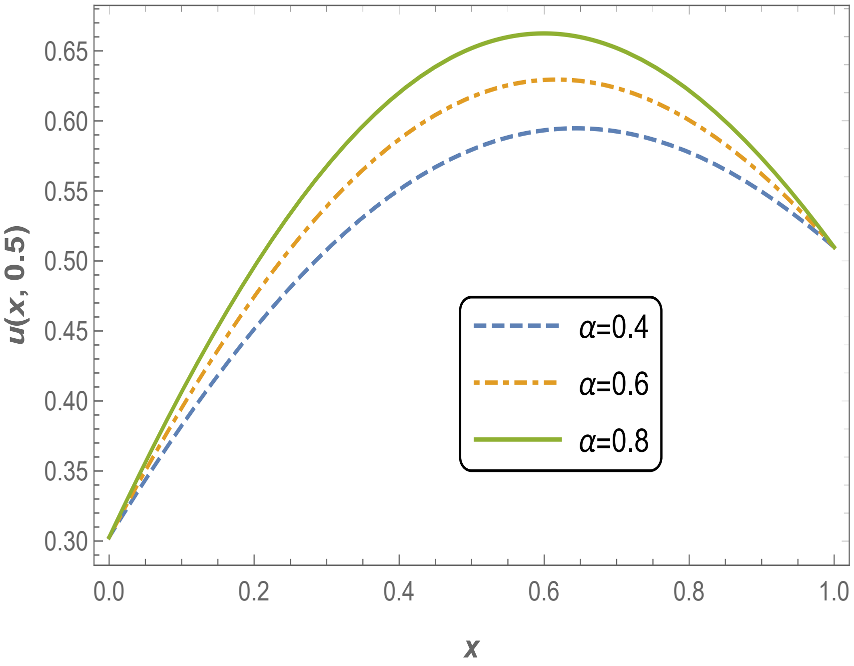

FIGURE 5.6: Plot between $u(x, 0.5)$ versus x for Example 5.5 at $\beta = 1.9$ for various values of α.

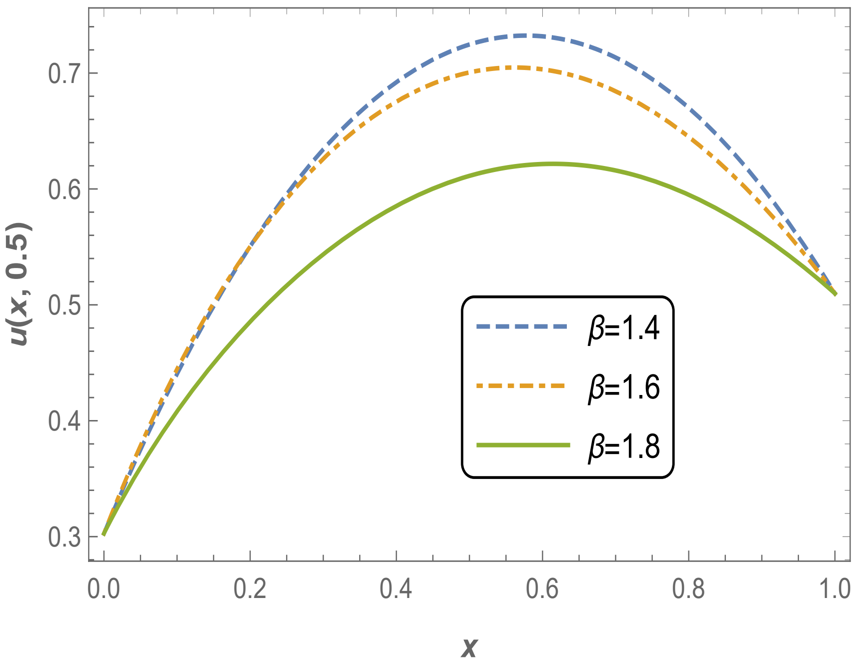

FIGURE 5.7: Plot between $u(x, 0.5)$ versus x for Example 5.5 at $\alpha = 0.5$ for various values of β.

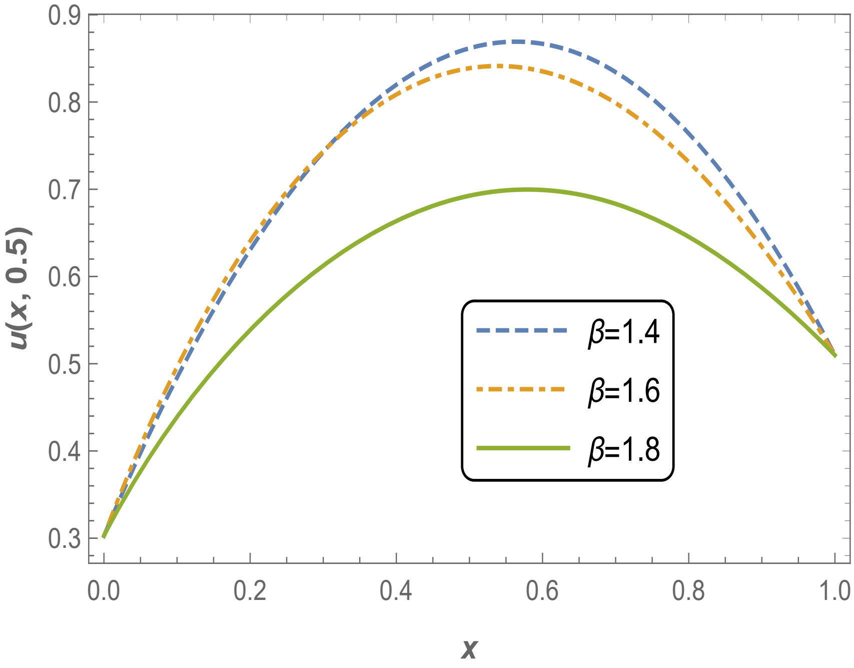

FIGURE 5.8: Plot between $u(x, 0.5)$ versus x for Example 5.5 at $\alpha = 0.9$ for various values of β.

5.7 Conclusion

In this chapter, the C-F approximation of shifted Legendre polynomials using the definition of the C-F derivative. Initially, the spectral collocation method is applied to the considered model, which reduces the problem to a system of fractional PDEs, which is solved by the provided finite difference scheme, and thus a numerical solution is obtained. To show the accuracy of the proposed method some numerical examples are solved. The absolute error is provided in the error table along with its convergence rate. From the error table, it is concluded that the proposed method works well with an excellent convergence rate.

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