# Some Problems on Stability and Synchronization of Integer and Fractional order Neural Networks

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# Chapter 8

# Conclusions and future work

## 8.1 Conclusion

This thesis addresses the challenges surrounding stability and synchronization in integer and fractional order neural networks, particularly emphasizing higher dimensional networks with diverse controllers. It underscores the importance of conducting thorough mathematical analyses to tackle synchronization issues effectively. A key aspect of neural network synchronization involves analyzing the stability of nonlinear differential equations. Consequently, the study explores and derives methods for stability analysis of error systems in synchronization problems, aiming to produce less conservative results than existing approaches. The primary focus lies in investigating synchronization in general neural network models and extending stability analyses to higher dimensional models. The first chapter provides a comprehensive overview of the origin, motivation, and mathematical modeling of artificial neural networks. DDE's basic definitions and characteristics and a physical explanation of Hopfield, Cohen-Grossberg, and hypercomplex neural networks are introduced. These models are expanded to incorporate time delays. The concepts of fractional calculus with Caputo and Mittag-Leffler definitions are covered. The first chapter covers the Lyapunov and Lagrange theories, which are essential tools throughout the thesis for analyzing stability and error.

Chapter 2 explores quasi-projective synchronization within Cohen-Grossberg CVNNs while accounting for parameter mismatches. The methodology followed a direct rather than a real decomposition approach, resulting in more lenient outcomes. Addressing non-identical parameters with time delays in both the drive and response systems, reflects practical scenarios where mismatches are common. This chapter defines the error as the difference between the response and a scaling factor multiplied by the drive system. It thoroughly analyzes the error system and establishes sufficient criteria for quasi-projective synchronization using Lyapunov theory. Various forms of synchronization are identified by investigating different values of the scaling factor. Additionally, the chapter investigates error bounds in quasi-projective synchronization, uncovering fluctuations in system error. The numerical section includes graphical representations illustrating these fluctuations across different scenarios.

Chapter 3 investigates the quasi-projective synchronization of QVNNs, where the master-response systems demonstrate non-identical characteristics. The QVNN model incorporates time-varying delay and interaction terms, offering several advantages. Within the context of neural network, the applicability of Lyapunov stability becomes uncertain. Multistable dynamics are indispensable for managing the intricate neural computations demanded in many scenarios where monostable NNs reach computational limits. LS evaluates the overall system's stability without necessitating knowledge of the equilibrium points. Furthermore, the globally attractive exponential set is acquired, delineating the areas of convergence within the system. To illustrate the efficacy of Lagrange stability in handling multistable systems, the chapter presents three numerical examples focusing on CVNNs, QVNNs and OVNNs.

Chapter 4 focuses on addressing the fixed-time synchronization of HCNNs. All components of HCNNs, including the state vector, weight matrices, activation functions, and input vectors, are hypercomplex numbers. Direct application of techniques from CVNNs and QVNNs is not feasible due to their limitations in lower dimensions. Instead, a decomposition method divides HCNNs into $(n+1)$ RVNNs, leveraging the distributive law to handle non-commutativity and non-associativity. Synchronization is facilitated through a well-designed controller coupled with a Lyapunov function and the application of the fixed-time lemma effectively addressing challenges posed by non-commutativity. Considering the impact of time delays discussed in Chapter 1, the study extends its analysis to mixed-type delays within this synchronization framework. Numerical examples for CVNNs, QVNNs and OVNNs are provided to offer empirical support, demonstrating the attainment of synchronization within a predetermined time frame. This specific form of synchronization, achieved within a fixed time interval, is termed as fixed-time synchronization.

Chapter 5 delves into the analysis of Lagrange stability in HCNNs with time delays. In the realm of dynamical systems, the concept of global stability in the Lyapunov sense is clear-cut for monostable systems, where a single equilibrium point attracts all trajectories asymptotically. However, when dealing with multistable dynamics featuring multiple equilibrium points, some of which might be unstable, the applicability of Lyapunov stability becomes uncertain. Multistable dynamics are indispensable for managing the intricate neural computations demanded in many scenarios where monostable NNs reach computational limits. It is worth highlighting that LS offers a distinct advantage over Lyapunov stability. LS evaluates the overall system's stability without necessitating knowledge of the equilibrium points.

The chapter provides three numerical examples involving different types of neural networks viz., CVNNs, QVNNs, and OVNNs to validate the findings obtained.

Chapter 6 shifts its focus to fractional-order neural networks, representing a generalization of integer-order neural networks. The chapter investigates the function projective Mittag-Leffler synchronization of non-identical networks. Global exponential synchronization can be seen as a particular case of GMLS when the fractional order approaches 1, gradually resembling the characteristic form of global exponential synchronization. FONNs inherently incorporate memory effects, indicating that the current state depends on the entire historical trajectories of the system. The Mittag-Leffler function is well-suited for representing fractional-order dynamics, providing a more accurate depiction of memory effects than traditional exponential functions. By defining the Caputo derivative, utilizing the Mittag-Leffler function, and applying Lyapunov stability theory, the chapter elucidates stability outcomes regarding the function projective Mittag-Leffler synchronization scheme for FONNs. The Mittag-Leffler PS and AS are examined as specific cases of FPMLS of non-identical FONNs with appropriate error function formulations. Finally, the proposed technique is applied to a numerical example to confirm its effectiveness and the robustness of the various applied synchronization conditions.

Chapter 7 delves into synchronizing non-identical fractional-order complex-valued neural networks using Lagrange $\alpha$-exponential techniques. It investigates the conditions necessary for achieving Lagrange $\alpha$-exponential synchronization and $\alpha$-exponential convergence of FONNs, utilizing additional inequalities and the Lyapunov method. Furthermore, the chapter elucidates the structure of the $\alpha$-exponential convergence ball, illustrating how the system's characteristics and the differential order influence the convergence rate. A notable aspect of the study is the graphical demonstration of the effectiveness ofthe proposed method through numerical simulations.

## 8.2 Future work

In future research endeavors, attention may be given toward exploring the impact of second-order differential equations, specifically focusing on inertial neural network (INN) models. Traditionally, studies on INNs have tended to employ order reduction techniques, converting second-order systems into first-order ones through variable substitution. However, such methods often increase the complexity within the original systems, involving more variables. Departing from these conventional approaches, it is proposed to employ a non-reduction order strategy. A particular emphasis will be placed on investigating the effects of impulsivity on projective synchronization within hypercomplex inertial neural networks (HCINNs) that feature unbounded time delays. The research will examine impulsive control mechanisms, incorporating both continuous and discontinuous components, to effectively manage the response system during impulsive events across various magnitudes of impulses. Building upon the insights gleaned from previous studies, particularly in the realm of higher-dimensional HCNNs and HCINNs, the superiority of the practical performance have been demonstrated compared to other neural network models, such as RVNNs, CVNNs, QVNNs, and OVNNs. In near future, the focus will be given on extensively exploring higher-dimensional and higher-order neural network models. Furthermore, the investigations will consider unbounded time delays as a critical factor.

- This thesis mainly focuses on employing linear and nonlinear feedback controllers across various integer and fractional-order neural network models to achieve specific objectives. Future research will delve into various dynamical studies, mainly aimed at assessing the efficacy of adaptive, sliding mode, event trigger, and impulsive controllers in accomplishing desired outcomes across different neural network models. This shift in emphasis aims to offer a thorough understanding of alternative control methodologies and their relevance within the research domain.

- The forthcoming research will employ the matrix measure method to achieve the desired synchronization scheme. Compared to the Lyapunov function approach, which necessitates lengthy integral computations, the criteria derived from the matrix measure perspective are concise, and Halanay inequality streamlines the analytical process. While a norm can only be computed for non-negative values, a matrix measure can encompass positive and negative values. This sign sensitivity arises because $\|A\| = \|-A\|$ generally, but $\mu(A) \neq \mu(-A)$. These distinctive characteristics render the results obtained through matrix measures more precise than those derived from matrix norms. The matrix measure method offers advantages over the Lyapunov method by providing a more efficient and nuanced analytical approach, particularly in scenarios where sign sensitivity and concise criteria are crucial for considerations.

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