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ORIGINAL RESEARCH PAPER

Generalized Hausdorff metric on S_b-metric space and some fixed point results

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Abstract

In this paper, a metric on S_b-metric space analogous to the Hausdorff metric has been introduced, and we have proved that the set of all bounded and closed subsets of any non-empty set $\mathcal{M}$ is a S_b metric space. We have presented here the fixed point results for the set-valued map in the framework of S_b metric space, which generalizes the famous Nadler's (Pac J Math 30(2):475–488, 1969) fixed point results for the set-valued map in the metric space. Furthermore, we have generalized Theorem 2 of Kikkawa and Suzuki (Nonlinear Anal Theory Methods Appl 69(9):2942–2949) in the setting of S_b metric space from the metric space. Illustrative examples and numerical calculations are given to support the obtained results.

Keywords Complete metric space · Hausdorff metric space · Contraction · S_b-metric space · Fixed point

Mathematics Subject Classification $54H10 \cdot 54H25 \cdot 47H10$

1 Introduction

The conceptualization of spaces, particularly the generalization of metric spaces and the examination of their properties, has consistently been a captivating field in mathematics. Additionally, the study of fixed point theory on such spaces has always been a fascinating area of research for mathematicians, given its relevance not only in other mathematical domains but also in various other disciplines. For instance, the concept of b-metric space came into the picture by Bakhtin [1], on which few researchers have developed some fixed point results. However, Czerwick [6] developed the extensions of the Banach contraction principle by considering different

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contractive conditions. For additional and related results see [4, 8, 12, 16, 22, 26–28, 33].

Various mathematicians have explored the S-metric space and manifested numerous consequences allied to the endurance of fixed points. We cite here some of the following contributions [10, 14, 21, 23, 25, 29].

Persuaded by the job of Bakhtin [1], Souayah and Mlaiki [32] have generalized the theory of b-metric space, presently known as S_b-metric space and proved certain fixed point results by taking different contractions in a complete S_b-metric space. Mlaiki [15] had further generalized the theory of S_b-metric space to extended S_b-metric space.

Our motive in this paper is to generalize the Hausdorff metric for S_b-metric space and to prove fixed point results for a set-valued operator on S_b-metric space. The fixed point of set-valued mapping on metric spaces has been generalized in several ways. An initial credential in this direction is because of Nadler [17], where Banach contraction principle [2] is extended to the domain of set-valued setting. In this paper, we have generalized the Nadler [17] fixed point result for set-valued mapping on S_b-metric space. For complementary and related results on the set-valued map in the metric space, we refer [19, 20, 30].

Throughout this paper, we denote the set of natural numbers by $\mathbb{N}$ and the collection of real numbers by $\mathbb{R}$.

Nadler [17] established a fundamental fixed point theorem for the set-valued map in the metric space. Let $(\mathcal{M}, d)$ be a metric space, then let us first recall the definition of Pompeiu-Hausdorff metric H on the set $\mathcal{CB}(\mathcal{M}) := \{K : K \text{ is a non-empty bounded and closed subset of } \mathcal{M}\}$ induced by the metric d.

For $A, B \in \mathcal{CB}(\mathcal{M})$, the Hausdorff metric H is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B),\ \sup_{y\in B} D(y,A)\right\},$$

where $D(x,B) = \inf\{d(x,y) : y \in B\}$. Moreover, if $(\mathcal{M}, d)$ is a complete metric space, then $(\mathcal{CB}(\mathcal{M}), H)$ is also a complete metric space.

1.1 Delineation

The paper is organized as follows: the next section contains the preliminary results required for the study. Section 3 is devoted to the development of the Hausdorff metric-like theory in the case of S_b-metric space. In Sect. 4, we have proved two fixed point results. First, one generalizes Nadler's fixed point theorem [17], and the other one is a conceptualization of Theorem 2 of Kikkawa and Suzuki [11]. Lastly, the paper is concluded in Sect. 5.

2 Preliminaries

In this section, we will give some preliminary definitions and results that will be required for our study. For any two non-empty sets X and Y with $X \cap Y \neq \emptyset$, a point $x \in X$ is designated as a fixed point of the mapping $T : X \to Y$, if $T(x) = x$.

Definition 1 Let $(\mathcal{M}, d)$ be a complete metric space and $(\mathcal{CB}(\mathcal{M}), H)$ be the Hausdorff metric space. Then, a map defined from $\mathcal{M}$ to $\mathcal{CB}(\mathcal{M})$, i.e., $T : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ is known as a set-valued map.

Definition 2 Let $(\mathcal{M}, d)$ be a complete metric space and T be a set-valued map on $\mathcal{M}$ such that Tx is a non-empty closed bounded subset of $\mathcal{M}$ $(Tx \in CB(\mathcal{M}))$ for any $x \in \mathcal{M}$. If there exists $k \in (0, 1)$ such that

$$H(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in \mathcal{M},$$

then T is known as a set-valued contraction map.

Theorem 1 [17] *Let $(\mathcal{M}, d)$ be a complete metric space and T be a set-valued map on $\mathcal{M}$ such that Tx is a non-empty closed bounded subset of $\mathcal{M}$ $(Tx \in CB(\mathcal{M}))$ for any $x \in \mathcal{M}$. If there exists $k \in (0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in \mathcal{M},$$

then T has a fixed point in $\mathcal{M}$, i.e., there exists $x^* \in \mathcal{M}$ such that $x^* \in Tx^*$.

Remark 1 Note that, unlike the case of Banach's fixed point theorem for a single-valued map, Nadler's fixed point result for a set-valued map does not guarantee the fixed point's uniqueness. For instance, take $T : \mathbb{R} \to \mathcal{CB}(\mathcal{M})$ such that $T(x) = [0, 1]$ for each $x \in \mathbb{R}$. Then, observe that T is a constant set-valued map. Hence, it satisfies the contraction condition, therefore it has a fixed point. Notice that each $x \in [0, 1]$ will be a fixed point of T, i.e., each $x \in Tx$.

Theorem 2 [11] *Define a strictly decreasing function η from* [0,1) onto $(\frac{1}{2}, 1]$ by

$$\eta(r) = \frac{1}{1 + r}.$$

Let $(\mathcal{M}, d)$ be a complete metric space, and $T : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ be a set-valued map. Assume that there exists $r \in (0, 1)$ such that

$$\eta(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq rd(x, y),$$

for all $x, y \in \mathcal{M}$. Then, there exists $z \in \mathcal{M}$ such that $z \in Tz$.

Definition 3 [32] Let $\mathcal{M}$ be a non-empty set and $s \geq 1$ be any real number. Define a map $S_b : \mathcal{M}^3 \to [0, \infty)$ such that for all $u, w, v, t \in \mathcal{M}$, satisfies the properties

(i) $S_b(u, w, v) = 0$ if and only if $u = w = v$,
(ii) $S_b(u, w, v) \leq s[S_b(u, u, t) + S_b(w, w, t) + S_b(v, v, t)]$.

Then, the trio, $(\mathcal{M}, S_b, s)$, is known as S_b-metric space.

Examples

(i) Let $\mathcal{M} = \mathbb{R}$ and $S_b(u, w, v) = s(|u - v| + |w - v|)$. Then, $(\mathcal{M}, S_b, s)$ will be a S_b-metric space for $s \in \mathbb{R}$ with $s \geq 1$.

(ii) Let $\mathcal{M} = \mathbb{R}^n$ and $\|\cdot\|$ a norm on $\mathcal{M}$, then $S_b(u, w, v) = s(\|w + v - 2u\| + \|w - v\|)$ will be a S_b-metric on $\mathcal{M}$ for $s \in \mathbb{R}$ with $s \geq 1$.

(iii) Let $\mathcal{M}$ be a non-empty set, d be the usual metric on $\mathcal{M}$, then $S_b(u, w, v) = s(d(u, v) + d(w, v))$ is a S_b-metric space on $\mathcal{M}$ for $s \in \mathbb{R}$ with $s \geq 1$.

Remark 2 If $S_b(u, u, w) = S_b(w, w, u)$ for all $u, w \in \mathcal{M}$, then $(\mathcal{M}, S_b, s)$ is known as symmetric S_b-metric space.

Everywhere in this paper, $\mathcal{M}$ will be a symmetric S_b-metric space.

Definition 4 [7, 24] Assume $(\mathcal{M}, S_b, s)$ be a S_b-metric space, then for $h \in \mathcal{M}$ and $\bar{q} > 0$, open ball $B_{S_b}(h, \bar{q})$ and closed ball $B_{S_b}[h, \bar{q}]$ with center h and radius $\bar{q}$ are defined as below :

$$B_{S_b}(h, \bar{q}) = \{u \in \mathcal{M} \ : \ S_b(h, h, u) < \bar{q}\} \ \text{ and } \ B_{S_b}[h, \bar{q}] = \{t \in \mathcal{M} \ : \ S_b(h, h, u) \leq \bar{q}\}, \ \text{respectively.}$$

Definition 5 [24, 32] Let $(\mathcal{M}, S_b, s)$ is a S_b-metric space and $\{h_n\}$ is a sequence in $\mathcal{M}$. Then,

(i) $\{h_n\}$ is called convergent if and only if there exists $h \in \mathcal{M}$ such that $S_b(h_n, h_n, h) \to 0$ when $n \to \infty$. We put down $\lim\limits_{n \to \infty} h_n = h$.

(ii) $\{h_n\}$ is called Cauchy sequence if and only if $S_b(h_n, h_n, h_m) \to 0$ when $n, m \to \infty$.

iii) $(\mathcal{M}, S_b)$ is said to be complete S_b-metric space if every Cauchy sequence $\{h_n\}$ converges to a point $h \in \mathcal{M}$.

Definition 6 [32] Diameter of a subset, K, of S_b-metric space, $\mathcal{M}$, is defined as follows

$$D_m(K) = \sup\{S_b(h, u, v) \ : \ h, u, v \in K\}.$$

We shall require the following concepts:

When $D_m(K) < \infty$ for any subset K of a S_b-metric space, $\mathcal{M}$, then K is said to be a bounded subset of $\mathcal{M}$.

3 Set-valued S_b-metric spaces

Let $(\mathcal{M}, S_b, s)$ be a S_b-metric space for $s \geq 1$ and $\mathcal{CB}(\mathcal{M}) := \{K :$ K is a non-empty bounded and closed subset of $\mathcal{M}\}$. Let $h \in \mathcal{M}$ and for $P, R, Q \in \mathcal{CB}(\mathcal{M})$ define

$$D_b(h, R) = \inf\{S_b(h, h, r) : \ r \in R\}, \tag{1}$$

$$N(\epsilon, P, R) = \{h \in \mathcal{M} : \ D_b(h, P) \leq \epsilon \ \text{ and } \ D_b(h, R) \leq \epsilon\}, \tag{2}$$

$$\mathcal{L} = \{\epsilon > 0 : \ P \subset N(\epsilon, R, Q), \ R \subset N(\epsilon, P, Q), \ \text{ and } \ Q \subset N(\epsilon, P, R)\}, \tag{3}$$

$$\begin{aligned} H_b(P, Q, R) = \max\Bigg\{ &\max\left\{\sup_{\tilde{p}\in P} D_b(\tilde{p}, R), \sup_{\tilde{p}\in P} D_b(\tilde{p}, Q)\right\}, \max\left\{\sup_{\tilde{r}\in R} D_b(\tilde{r}, P), \sup_{\tilde{r}\in R} D_b(\tilde{r}, Q)\right\}, \\ &\max\left\{\sup_{\tilde{q}\in Q} D_b(\tilde{q}, P), \sup_{\tilde{q}\in Q} D_b(\tilde{q}, R)\right\}\Bigg\}. \end{aligned} \tag{4}$$

The following lemma proves that the set of all bounded and closed subsets of any non-empty set $\mathcal{M}$ is a S_b metric space.

Lemma 1 *Assume* $(\mathcal{M}, S_b, s)$ *is a* S_b*-metric space,* $H_b : (\mathcal{CB}(\mathcal{M}))^3 \to [0, \infty)$ *be a function defined in* (4), *then for some* $k \in \mathbb{R}$ *with* $k \geq 1$, H_b *be a* S_b*-metric on* $\mathcal{CB}(\mathcal{M})$, *i.e.,* $(\mathcal{CB}(\mathcal{M}), H_b, k)$ *is a* S_b*-metric space.*

Proof To prove that H_b is a S_b-metric space, for every $P, R, Q, M \in \mathcal{CB}(\mathcal{M})$, it needs to satisfy the following two conditions:

(a) $H_b(P, R, Q) = 0$ iff $P = R = Q$,
(b) $H_b(P, R, Q) \leq k\{H_b(P, P, M) + H_b(R, R, M) + H_b(Q, Q, M)\}$.

Proof of (a). Let $P = R = Q$, then $H_b(P, R, Q) = 0$ because every $\tilde{p} \in P$ satisfies $D_b(\tilde{p}, R) = 0$ and $D_b(\tilde{p}, Q) = 0$. Conversely, suppose $H_b(P, R, Q) = 0$, then from (4) we have

$$\begin{aligned} D_b(\tilde{p}, R) &= 0 \text{ and} \\ \Rightarrow P &\subseteq R \text{ and} \end{aligned} \tag{5}$$

By a similar argument, we obtain that

$$R \subseteq P \text{ and} \tag{6}$$

$$Q \subseteq P \text{ and} \tag{7}$$

Hence, from (5), (6), and (7) we conclude that $P = R = Q$.

Proof of (b). Since $D_b(h, R) = \inf_{r\in R}\{S_b(h, h, r)\}$ by (1) and $(\mathcal{M}, S_b, s)$ is a S_b-metric space. Therefore, for every $a \in M$, $\tilde{r} \in R$ and $\tilde{p} \in P$, we have

$$
\begin{aligned}
& S_b(\tilde{p},\tilde{p},\tilde{r}) \le s\{S_b(\tilde{p},\tilde{p},a) + S_b(\tilde{p},\tilde{p},a) + S_b(\tilde{r},\tilde{r},a)\} \\
& \quad \Rightarrow \inf_{\tilde{r}\in R} S_b(\tilde{p},\tilde{p},\tilde{r}) \le s\{2S_b(\tilde{p},\tilde{p},\tilde{r}) + \inf_{\tilde{r}\in R} S_b(\tilde{r},\tilde{r},a)\} \\
& \quad \Rightarrow D_b(\tilde{p},R) \le s\{2S_b(\tilde{p},\tilde{p},\tilde{r}) + 2\inf_{\tilde{r}\in R} S_b(\tilde{r},\tilde{r},a)\} \\
& \quad \Rightarrow D_b(\tilde{p},R) \le 2s\{S_b(\tilde{p},\tilde{p},a) + D_b(\tilde{r},A)\} \\
& \quad \Rightarrow D_b(\tilde{p},R) \le 2s\{S_b(\tilde{p},\tilde{p},a) + H_b(R,R,M)\} \text{ by (4)} \\
& \quad \Rightarrow D_b(\tilde{p},R) \le 2s\{\inf_{a\in A} S_b(\tilde{p},\tilde{p},a) + H_b(R,R,M)\} \\
& \quad \Rightarrow D_b(\tilde{p},R) \le 2s\{D_b(\tilde{p},A) + H_b(R,R,M)\} \\
& \quad \Rightarrow D_b(\tilde{p},R) \le 2s\{H_b(P,P,M) + H_b(R,R,M)\} \text{ by (4)} \\
& \quad \Rightarrow \sup_{\tilde{p}\in P} D_b(\tilde{p},R) \le 2s\{H_b(P,P,M) + H_b(R,R,M)\} \text{ since } \tilde{p} \text{ is arbitrary.}
\end{aligned}
\tag{8}
$$

By applying an analogous argument, we can prove that

$$\sup_{\tilde{p}\in P} D_b(\tilde{p},Q) \le 2s\{H_b(P,P,M) + H_b(Q,Q,M)\}. \tag{9}$$

By using (8) and (9), we obtain

$$
\begin{aligned}
\max\left\{\sup_{\tilde{p}\in P} D_b(\tilde{p},R), \sup_{\tilde{p}\in P} D_b(\tilde{p},Q)\right\} & \le \max\{2s\{H_b(P,P,M) + H_b(R,R,M)\}, \\
& \quad 2s\{H_b(P,P,M) + H_b(Q,Q,M)\}\} \\
& \le 2s\{H_b(P,P,M) + H_b(R,R,M)\} \\
& \quad + 2s\{H_b(P,P,M) + H_b(Q,Q,M)\} \\
& \le 4s\{H_b(P,P,M) + H_b(R,R,M) + H_b(Q,Q,M)\}, \\
\textit{i.e.},\ \max\left\{\sup_{\tilde{p}\in P} D_b(\tilde{p},R), \sup_{\tilde{p}\in P} D_b(\tilde{p},Q)\right\} & \le 4s\{H_b(P,P,M) + H_b(R,R,M) \\
& \quad + H_b(Q,Q,M)\}.
\end{aligned}
\tag{10}
$$

Similarly, we can show that

$$
\begin{aligned}
\max\left\{\sup_{\tilde{r}\in R} D_b(\tilde{r},P), \sup_{\tilde{r}\in R} D_b(\tilde{r},Q)\right\} & \le 4s\{H_b(P,P,M) + H_b(R,R,M) \\
& \quad + H_b(Q,Q,M)\}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\text{and , } \max\left\{\sup_{\tilde{q}\in Q} D_b(\tilde{q},P), \sup_{\tilde{q}\in Q} D_b(\tilde{q},R)\right\} & \le 4s\{H_b(P,P,M) + H_b(R,R,M) \\
& \quad + H_b(Q,Q,M)\}.
\end{aligned}
\tag{12}
$$

Thus, the inequalities (4), (10), (11), and (12) completes the proof of (b). □

Remark 3 If $(\mathcal{M}, S_b, s)$ is a S_b-metric space, then $(\mathcal{CB}(\mathcal{M}), H_b, k)$ will be a S_b-metric with $k = 4s$.

Lemma 2 *Consider* $(\mathcal{M}, S_b, s)$ *be a* S_b*-metric space, then for every* $P, R, Q \in \mathcal{CB}(\mathcal{M})$, $H_b(P, R, Q)$ *defined by* (4), *is equivalent to* $H_b(P, R, Q) = \inf_{\epsilon > 0} \mathcal{L}$.

Proof First assume that $H_b(P, R, Q)$ is defined by (4), then for all $\tilde{p} \in P$, we have

$$\begin{aligned} & D_b(\tilde{p}, R) \leq H_b(P, R, Q) \text{ and } D_b(\tilde{p}, R) \leq H_b(P, R, Q) \\ & \Rightarrow P \subset N(H_b(P, R, Q), R, Q). \end{aligned} \tag{13}$$

Similarly, we have

$$R \subset N(H_b(P, R, Q), P, Q) \text{ and } Q \subset N(H_b(P, R, Q), P, R). \tag{14}$$

Thus, from (13) and (14), we get $H_b(P, R, Q) \in \mathcal{L}$.

Now for any $\epsilon \in \mathcal{L}$, we have

$$\begin{aligned} & A \subset N(\epsilon, R, Q) \\ & \Rightarrow D_b(\tilde{p}, R) \leq \epsilon \text{ and } D_b(\tilde{p}, Q) \leq \epsilon \text{ for all } \tilde{p} \in P \\ & \Rightarrow \sup_{\tilde{p} \in P} D_b(\tilde{p}, R) \leq \epsilon \quad \text{and} \quad \sup_{\tilde{p} \in P} D_b(\tilde{p}, Q) \leq \epsilon \\ & \Rightarrow \max\left\{ \sup_{\tilde{p} \in P} D_b(\tilde{p}, R), \ \sup_{\tilde{p} \in P} D_b(\tilde{p}, Q) \right\} \leq \epsilon. \end{aligned} \tag{15}$$

Also by definition of $\mathcal{L}$ in (3), we have $R \subset N(\epsilon, P, Q)$ and $Q \subset N(\epsilon, P, R)$ this implies

$$\max\left\{ \sup_{\tilde{r} \in R} D_b(\tilde{r}, P), \ \sup_{\tilde{r} \in R} D_b(\tilde{r}, Q) \right\} \leq \epsilon \text{ and} \tag{16}$$

$$\max\left\{ \sup_{\tilde{q} \in Q} D_b(\tilde{q}, P), \ \sup_{\tilde{q} \in Q} D_b(\tilde{q}, R) \right\} \leq \epsilon. \tag{17}$$

Thus, by using the (15), (16), (17) and $H_b(P, R, Q) \in \mathcal{L}$, we get $H_b(P, R, Q) = \inf \mathcal{L}$.

Conversely, assume $H_b(P, R, Q) = \inf \mathcal{L}$, then by (3), we have

$$\begin{aligned} & D_b(\tilde{p}, R) \leq \epsilon \text{ and } D_b(\tilde{p}, Q) \leq \epsilon \text{ for each } \tilde{p} \in P \text{ and for all } \epsilon \in \mathcal{L} \\ & \Rightarrow D_b(\tilde{p}, R) \leq \inf \mathcal{L} = H_b(P, R, Q) \text{ and } D_b(p, Q) \leq \inf \mathcal{L} \\ & \Rightarrow \sup_{\tilde{p} \in P} D_b(\tilde{p}, R) \leq H_b(P, R, Q) \quad \text{and} \quad \sup_{\tilde{p} \in P} D_b(\tilde{p}, Q) \leq H_b(P, R, Q) \\ & \Rightarrow \max\left\{ \sup_{\tilde{p} \in P} D_b(\tilde{p}, R), \ \sup_{\tilde{p} \in P} D_b(\tilde{p}, Q) \right\} \leq H_b(P, R, Q). \end{aligned} \tag{18}$$

Similarly, we can show

$$\max\left\{ \sup_{\tilde{r} \in R} D_b(\tilde{r}, P), \ \sup_{\tilde{r} \in R} D_b(\tilde{r}, Q) \right\} \leq H_b(P, R, Q) \text{ and} \tag{19}$$

$$\max\left\{\sup_{\tilde{q}\in Q} D_b(\tilde{q},P),\ \sup_{\tilde{q}\in Q} D_b(\tilde{q},R)\right\}\leq H_b(P,R,Q). \tag{20}$$

Then, inequalities (18), (19) and (20) together, implies that

$$\begin{aligned}\max\Bigg\{&\max\left\{\sup_{\tilde{p}\in P} D_b(\tilde{p},R),\sup_{\tilde{p}\in P} D_b(\tilde{p},Q)\right\},\max\left\{\sup_{\tilde{r}\in R} D_b(\tilde{r},P),\sup_{\tilde{r}\in R} D_b(\tilde{r},Q)\right\},\\ &\max\left\{\sup_{\tilde{q}\in Q} D_b(\tilde{q},P),\sup_{\tilde{q}\in Q} D_b(\tilde{q},R)\right\}\Bigg\}\leq H_b(P,R,Q).\end{aligned} \tag{21}$$

By (3), it follows that

$$\begin{aligned}\max\Bigg\{&\max\left\{\sup_{\tilde{p}\in P} D_b(\tilde{p},R),\sup_{\tilde{p}\in P} D_b(\tilde{p},Q)\right\},\max\left\{\sup_{\tilde{r}\in R} D_b(\tilde{r},P),\sup_{\tilde{r}\in R} D_b(\tilde{r},Q)\right\},\\ &\max\left\{\sup_{\tilde{q}\in Q} D_b(\tilde{q},P),\sup_{\tilde{q}\in Q} D_b(\tilde{q},R)\right\}\Bigg\}\in\mathcal{L}.\end{aligned} \tag{22}$$

Hence, (21) and (22) together accomplishes the proof of Lemma 2. □

4 Main results

Theorem 3 (*Generalization of Nadler's fixed point theorem on S_b-metric space*). *Let $(\mathcal{M},S_b,s)$ be a complete metric space and $\mathcal{CB}(\mathcal{M})$ be the collection of closed and bounded subsets of $\mathcal{M}$ with metric H_b and suppose that $\mathfrak{T}:\mathcal{M}\to\mathcal{CB}(\mathcal{M})$ be a continuous set-valued map satisfying the condition,*

$$H_b(\mathfrak{T}h,\mathfrak{T}h,\mathfrak{T}g)\leq\alpha S_b(h,h,g)\ \text{for all}\ h,g\in\mathcal{M}\ \text{ and }\ \alpha\in\left[0,\frac{1}{s}\right). \tag{23}$$

Then, $\mathfrak{T}$ has a fixed point in $\mathcal{M}$.

Proof Consider $h_0\in\mathcal{M}$ and choose $h_1\in\mathfrak{T}h_0$. Since $\mathfrak{T}h_0$, $\mathfrak{T}h_1\in\mathcal{CB}(\mathcal{M})$, therefore by Lemma 2 there exists $h_2\in\mathfrak{T}h_1$ such that

$$\begin{aligned}&S_b(h_1,h_1,h_2)\leq H_b(\mathfrak{T}h_0,\mathfrak{T}h_0,\mathfrak{T}h_1)+\alpha,\\ \Rightarrow&S_b(h_1,h_1,h_2)\leq\alpha S_b(h_0,h_0,h_1)+\alpha\ \text{by (23)}.\end{aligned} \tag{24}$$

In the same way, there exists $h_3\in\mathfrak{T}h_2$, such that

$$
\begin{aligned}
S_b(h_2,h_2,h_3) &\leq H_b(\mathfrak{T}h_1,\mathfrak{T}h_1,\mathfrak{T}h_2)+\alpha^2\\
&\leq \alpha S_b(h_1,h_1,h_2)+\alpha^2\\
&\leq \alpha[\alpha S_b(h_0,h_0,h_1)+\alpha]+\alpha^2 \text{ by (24)}\\
\Rightarrow S_b(h_2,h_2,h_3) &\leq \alpha^2 S_b(h_0,h_0,h_1)+2\alpha^2.
\end{aligned}
$$

Ongoing in this fashion, we get a sequence $\{h_i\}_{i\in\mathbb{N}}\subset\mathcal{M}$ such that $h_{i+1}\in\mathfrak{T}h_i$ and $S_b(h_i,h_i,h_{i+1})\leq H_b(\mathfrak{T}h_{i-1},\mathfrak{T}h_{i-1},\mathfrak{T}h_i)+\alpha^i$ for all $i\geq 1$.

Since

$$
\begin{aligned}
S_b(h_i,h_i,h_{i+1}) &\leq H_b(\mathfrak{T}h_{i-1},\mathfrak{T}h_{i-1},\mathfrak{T}h_i)+\alpha^i\\
&\leq \alpha S_b(h_{i-1},h_{i-1},h_i)+\alpha^i\\
&\leq \alpha[H_b(\mathfrak{T}h_{i-2},\mathfrak{T}h_{i-2},\mathfrak{T}h_{i-1})+\alpha^{i-1}]+\alpha^i\\
&= \alpha H_b(\mathfrak{T}h_{i-2},\mathfrak{T}h_{i-2},\mathfrak{T}h_{i-1})+2\alpha^i\\
&\leq \alpha^2 S_b(h_{i-2},h_{i-2},h_{i-1})+2\alpha^i\\
&\vdots\\
\Rightarrow S_b(h_i,h_i,h_{i+1}) &\leq \alpha^i S_b(h_0,h_0,h_1)+i\alpha^i \text{ for all } i\in\mathbb{N}.
\end{aligned}
\tag{25}
$$

Therefore, for $m\geq n$, we deduce that

$$
\begin{aligned}
S_b(h_n,h_n,h_m) &\leq s[2S_b(h_n,h_n,h_{n+1})+S_b(h_{n+1},h_{n+1},h_m)] \quad \text{(by Remark 3)}\\
&\leq 2sS_b(h_n,h_n,h_{n+1})+s[s[2S_b(h_{n+1},h_{n+1},h_{n+2})+S_b(h_{n+2}+h_{n+2}+h_m)]]\\
&= 2sS_b(h_n,h_n,h_{n+1})+2s^2S_b(h_{n+1},h_{n+1},h_{n+2})+s^2S_b(h_{n+2},h_{n+2},h_m)\\
&\vdots\\
&\leq 2sS_b(h_n,h_n,h_{n+1})+\cdots+2s^{m-n}S_b(h_{m-1},h_{m-1},h_m)\\
&\leq 2\Big[s\{\alpha^nS_b(h_0,h_0,h_1)+n\alpha^n\}+s^2\{\alpha^{n+1}S_b(h_0,h_0,h_1)+(n+1)\alpha^{n+1}\}\\
&\quad+\cdots+s^{m-n}\{\alpha^{m-1}S_b(h_0,h_0,h_1)+(m-1)\alpha^{m-1}\}\Big]\\
&= 2\Big[\{s\alpha^n+\cdots+s^{m-n}\alpha^{m-1}\}S_b(h_0,h_0,h_1)+\{sn\alpha^n+s^2(n+1)\alpha^{n+1}\\
&\quad+\cdots+s^{m-n}(m-1)\alpha^{m-1}\}\Big]\\
&= \frac{2(s\alpha^n)(1-(s\alpha)^{m-n-1})}{1-s\alpha}S_b(h_0,h_0,h_1)+s\alpha^n\{n+(n+1)s\alpha\\
&\quad+\cdots+(n+(m-n-1))(s\alpha)^{m-n-1}\}\\
&= \frac{2s\alpha^n(1-(s\alpha)^{m-n-1})}{1-s\alpha}S_b(h_0,h_0,h_1)+s\alpha^n\sum_{i=1}^{m-n-1}(n+i)(s\alpha)^i.
\end{aligned}
$$

This implies that

$$
S_b(h_n,h_n,h_m)\leq\frac{2s\alpha^n(1-(s\alpha)^{m-n-1})}{1-s\alpha}S_b(h_0,h_0,h_1)+s\alpha^n\sum_{i=1}^{m-n-1}(n+i)(s\alpha)^i. \tag{26}
$$

From the inequality (26), it follows that $\{h_i\}$ is a Cauchy sequence. As $(\mathcal{M},S_b,s)$ is complete, $\{h_i\}$ converges to a point $h\in\mathcal{M}$. Therefore, the sequence $\{\mathfrak{T}h_i\}$

converges to $\mathfrak{T}h$ and, since $h_i \in \mathfrak{T}h_{i-1}$ for all i, it follows that $h \in \mathfrak{T}h$. This establishes the theorem. □

Example 1 Let $\mathcal{M} = \mathbb{R}$ with $S_b(h,u,w) = s\{|h-w| + |u-w|\}$ for some $s \in \mathbb{R}$ with $s \geq 1$ and $\mathfrak{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ such that

$$\mathfrak{T}h = \begin{cases} \left[\dfrac{h}{24s}, \dfrac{3h}{24s} + 1\right], & h \geq 0 \\ \{0\}, & h < 0. \end{cases}$$

Observe that

$$H_b(\mathfrak{T}h, \mathfrak{T}h, \mathfrak{T}u) \leq \frac{1}{4s} S_b(h,h,u) \quad \text{for every } h,u \in \mathcal{M}$$

and $\mathfrak{T}$ is continuous. Hence, $\mathfrak{T}$ satisfies all the constrains of Theorem 3, therefore $\mathfrak{T}$ has a fixed point in $\mathcal{M}$. More precisely the set of all fixed points of $\mathfrak{T}$ is $\left[0, \frac{24s}{24s-3}\right]$.

Theorem 4 *Consider* $(\mathcal{M}, S_b, s)$ *be a complete* S_b*-metric space. Define a strictly decreasing function* $\eta : [0, \frac{1}{s}) \to (\frac{1}{2s+1}, \frac{1}{2s}]$

$$\eta(r) = \frac{1}{s(2+r)},$$

and let $\mathfrak{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ *be a mapping. Assume that there exists some* $r \in [0,1)$ *such that*

$$\eta(r) D_b(h, \mathfrak{T}h) \leq S_b(h,h,t) \text{ implies } H_b(\mathfrak{T}h, \mathfrak{T}h, \mathfrak{T}t) \leq r S_b(h,h,t) \text{ for all } h,t \in \mathcal{M}. \tag{27}$$

Then, there exists $z \in \mathcal{M}$ *such that* $z \in \mathfrak{T}z$.

Proof Take a real number r_1 with $0 \leq r_1 < r < 1$. Then, for each $u = u_0 \in \mathcal{M}$ and $u_1 \in \mathfrak{T}u$, we have

$$\begin{aligned} &\eta(r) D_b(u, \mathfrak{T}u) \leq \eta(r) S_b(u,u,u_1) \leq S_b(u,u,u_1) \\ &\Rightarrow D_b(u_1, \mathfrak{T}u_1) \leq H_b(\mathfrak{T}u, \mathfrak{T}u, \mathfrak{T}u_1) \leq r S_b(u,u,u_1) \leq r_1 S_b(u,u,u_1) \text{ by (27).} \end{aligned}$$

Therefore, there exists $u_2 \in \mathfrak{T}u_1$ such that $S_b(u_1,u_1,u_2) \leq r_1 S_b(u,u,u_1)$. Thus, we get a sequence $\{u_n\}$ such that $u_n \in \mathfrak{T}u_{n-1}$ and

$$S_b(u_{n-1}, u_{n-1}, u_n) \leq r_1 S_b(u_{n-2}, u_{n-2}, u_{n-1}) \leq \cdots \leq r_1^{n-1} S_b(u,u,u_1). \tag{28}$$

Therefore, for $m \geq n$, we have

$$\begin{aligned}
S_b(u_n,u_n,u_m) &\le s[2S_b(u_n,u_n,u_{n+1}) + S_b(u_{n+1},u_{n+1},u_m)] \text{ (by Remark 3)}\\
&\le 2sS_b(u_n,u_n,u_{n+1}) + s[s[2S_b(u_{n+1},u_{n+1},u_{n+2}) + S_b(u_{n+2}+u_{n+2}+u_m)]]\\
&= 2sS_b(u_n,u_n,u_{n+1}) + 2s^2S_b(u_{n+1},u_{n+1},u_{n+2}) + s^2S_b(u_{n+2},u_{n+2},u_m)\\
&\vdots\\
&\le 2sS_b(u_n,u_n,u_{n+1}) + \cdots + 2s^{m-n}S_b(u_{m-1},u_{m-1},u_m)\\
&\le 2sr_1^nS_b(u,u,u_1) + 2s^2r_1^{n+1}S_b(u,u,u_1) + \cdots + 2s^{m-n}r_1^{m-1}S_b(u,u,u_1) \text{ by (28)}\\
&= 2sr_1^n[1 + sr_1 + (sr_1)^2 + \cdots + (sr_1)^{m-n-1}]S_b(u,u,u_1)\\
&= \frac{2sr_1^n(1-(sr_1)^{m-n})}{1-sr_1}S_b(u,u,u_1).
\end{aligned}$$

This implies

$$S_b(u_n,u_n,u_m) \le \frac{2sr_1^n(1-(sr_1)^{m-n})}{1-sr_1}S_b(u,u,u_1). \tag{29}$$

From (29), it follows that $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{M}$ is a complete S_b-metric space, $\{u_n\}$ converges to some point $z \in \mathcal{M}$.

We next show that

$$D_b(z,\mathfrak{T}h) \le rS_b(z,z,h) \text{ for each } h \in \mathcal{M}/\{z\}.$$

Since $u_n \to z$, there exists $k \in \mathbb{N}$ such that

$$S_b(u_n,u_n,z) \le \frac{1}{2s(1+4s)}S_b(h,h,z) \text{ for all } n \in \mathbb{N} \text{ and } n \ge k. \tag{30}$$

Then, we have

$$\begin{aligned}
\eta(r)D_b(u_n,\mathfrak{T}u_n) &\le D_b(u_n,\mathfrak{T}u_n) \le S_b(u_n,u_n,u_{n+1})\\
&\le s[2S_b(u_n,u_n,z) + S_b(u_{n+1},u_{n+1},z)]\\
&\le 2s[S_b(u_n,u_n,z) + S_b(u_{n+1},u_{n+1},z)]\\
&\le \frac{4s}{2s(1+4s)}S_b(h,h,z) \text{ by (30)}\\
&= \frac{1}{2s}S_b(h,h,z) - \frac{1}{2s(1+4s)}S_b(h,h,z)\\
&\le \frac{1}{2s}S_b(h,h,z) - S_b(u_n,u_n,z) \text{ by (30)}\\
&\le S_b(u_n,u_n,h)\\
\Rightarrow \eta(r)D_b(u_n,\mathfrak{T}u_n) &\le S_b(u_n,u_n,h).
\end{aligned}$$

Therefore, by assumption it follows that $H_b(\mathfrak{T}u_n,\mathfrak{T}u_n,\mathfrak{T}h) \le rS_b(u_n,u_n,h)$. This implies that $D_b(u_{n+1},\mathfrak{T}h) \le rS_b(u_n,u_n,h)$ for $n \in \mathbb{N}$ with $n \ge k$. Letting $n \to \infty$, we obtain

$$D_b(z,\mathfrak{T}h) \le rS_b(z,z,h) \text{ for every } h \in \mathcal{M}/\{z\}. \tag{31}$$

We next prove that

$$H_b(\mathfrak{T}h, \mathfrak{T}h, \mathfrak{T}z) \le rS_b(h, h, z) \text{ for every } h \in \mathcal{M}.$$

If $h = z$, then it holds obviously. Therefore, consider $h \ne z$. Then, for every $n \in \mathbb{N}$, there exists $y_n \in \mathfrak{T}h$ such that

$$S_b(z, z, y_n) \le D_b(z, \mathfrak{T}h) + \frac{1}{ns} S_b(h, h, z). \tag{32}$$

We have

$$\begin{aligned} D_b(h, \mathfrak{T}h) &\le S_b(h, h, y_n) \\ &\le s[2S_b(h, h, z) + S_b(z, z, y_n)] \\ &\le 2sS_b(h, h, z) + sD_b(z, \mathfrak{T}h) + \frac{1}{n} S_b(h, h, z) \text{ by (30)} \\ &\le 2sS_b(h, h, z) + srS_b(h, h, z) + \frac{1}{n} S_b(h, h, z) \text{ by (30)} \\ \Rightarrow D_b(h, \mathfrak{T}h) &\le \left(2s + sr + \frac{1}{n}\right) S_b(h, h, z) \text{ for each } n \in \mathbb{N} \end{aligned}$$

and hence $\frac{1}{s(2+r)} D_b(h, \mathfrak{T}h) \le S_b(h, h, z)$. From (27), we have $H_b(\mathfrak{T}h, \mathfrak{T}h, \mathfrak{T}h) \le rS_b(h, h, z)$.

Since

$$D_b(z, \mathfrak{T}z) = \lim_{n\to\infty} D_b(u_{n+1}, \mathfrak{T}z) \le \lim_{n\to\infty} H_b(\mathfrak{T}u_n, \mathfrak{T}u_n, \mathfrak{T}z) \le \lim_{n\to\infty} rS_b(u_n, u_n, z) = 0$$

and $\mathfrak{T}z \in \mathcal{CB}(\mathcal{M})$, we obtain $z \in \mathfrak{T}z$. This completes the proof. ☐

Example 2 Let $\mathcal{M} = \mathbb{R}$ such that $S_b(h, u, w) = s\{|h - w| + |u - w|\}$, for $s \in \mathbb{R}$ and $s \ge 1$, be a S_b-metric on $\mathcal{M}$. Consider $\mathfrak{T} : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ be a set-valued map defined on $\mathcal{M}$ such that

$$\mathfrak{T}h = \begin{cases} \left[\dfrac{h-1}{48s} - 1, \dfrac{3(h-1)}{48s} + 2\right], & h \ge 0 \\ \{0\}, & h < 0. \end{cases}$$

Then, observe that for all $h, u \in \mathcal{M}$, $H_b(\mathfrak{T}h, \mathfrak{T}h, \mathfrak{T}u) \le \frac{1}{8s} S_b(h, h, u)$. Hence, it satisfies the condition of Theorem 4. Therefore, T has a fixed point. Actually the set of all fixed points of T is $\left[0, \frac{96s+3}{48s-3}\right]$.

Remark Notice that using Theorem 4 there exists $r \in \left[0, \frac{1}{s}\right)$ such that,

$$H_b(\mathfrak{T}h, \mathfrak{T}h, \mathfrak{T}t) \le rS_b(h, h, t) \text{ for all } h, t \in \mathcal{M}$$

which implies that $\mathfrak{T}$ has a fixed point in $\mathcal{M}$. This shows Theorem 3 is a generalization of Theorem 4.

5 Conclusion and future scopes

We have developed the initial theory for set-valued S_b-metric space. We have generalized the Hausdorff metric for S_b metric space to obtain a fixed point of a set-valued operator. In this article, we have used the generalization of the Banach contraction mapping principle in the domain of set-valued settings. In Sect. 2, we have defined the Hausdorff metric in S_b metric space and proved that $(\mathcal{CB}(\mathcal{M}), H_b, k)$ is a S_b-metric space.

We have proved two fixed point results in Sect. 3. The first one is Theorem 3, the fixed point results for the set-valued map in the framework of S_b metric space, which generalizes the famous Nadler's fixed point result [17] for the set-valued map in the metric space. The second one is Theorem 4, which is the generalization of the theorem in article [11] in the setting of S_b metric space from the metric space.

In the future, we may try to develop the generalized Hausdorff metric space defined in this paper and generalize the well-known fixed point results for the case of S_b metric space. Furthermore, it will be interesting to see if one can try to generate fractals using the Theorem 3 and the iterated function system and Hutchinson mapping in the following way:

Let $F_1, F_2, \ldots, F_m : \mathcal{M} \to \mathcal{CB}(\mathcal{M})$ be set-valued operators. The system $F = (F_1, F_2, \ldots, F_m)$ is called an iterated function system if $F = (F_1, F_2, \ldots, F_m)$ is such that $F_i : \mathcal{M} \to \mathcal{K}(\mathcal{M})$, $i \in \{1, 2, \ldots, m\}$ are continuous, (where $\mathcal{K}(\mathcal{M})$ denotes the set of all compact subset of $\mathcal{M}$) then the Hutchinson operator T_F is defined as

$$T_F(X) = \cup_{i=1}^{m} F_i(X), \quad \text{for each} \quad X \in \mathcal{K}(\mathcal{M}).$$

A nonempty compact subset $A^* \subset X$ is said to be a fractal generated by iterated function system $F = (F_1, F_2, \ldots, F_m)$ if and only if it is a fixed point of T_F, i.e. $T_F(A^*) = A^*$. For more details on fractal and set-valued fractals, we refer [3, 5, 9, 13, 18, 31]. Further, it has been noticed that results in S_b-metric spaces could be applied to problems that are inaccessible to applications based on results that are their metric space counterparts. This is also supposed to be true for the fixed point theorems proved in S_b metric spaces. The above is supposed to form a basis for our future work.

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Declarations

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