

# Some Problems on Stability Analysis and Synchronization of Neural Networks



*The thesis submitted in partial fulfilment*

*for the Award of Degree*

**DOCTOR OF PHILOSOPHY**

*by*

Ankit Kumar

DEPARTMENT OF MATHEMATICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY

(BANARAS HINDU UNIVERSITY)

VARANASI -221005

**Roll No: 18121020**

**December 18, 2023**

CERTIFICATE

It is certified that the work contained in this thesis titled “*Some Problems on Stability Analysis and Synchronization of Neural Networks*” by *Ankit Kumar* has been carried out under our joint supervision and co-supervision, and this work has not been submitted elsewhere for a degree.

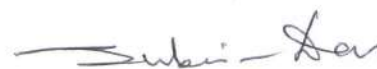
It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

  
21-08-23

**Dr. Rajeev**

(Supervisor)

Associate Professor  
Department of Mathematical Sciences  
Indian Institute of Technology  
(Banaras Hindu University)  
Varanasi-221005



**Prof. Subir Das**

(Co-Supervisor)

Professor  
Department of Mathematical Sciences  
Indian Institute of Technology  
(Banaras Hindu University)  
Varanasi-221005

### DECLARATION BY THE CANDIDATE

I, **Ankit Kumar**, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of **Dr. Rajeev** and co-supervision of **Prof. Subir Das** from **July, 2018** to **August, 2023** at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, thesis, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

Date: August 21, 2023

Place: Varanasi

  
(Ankit Kumar)

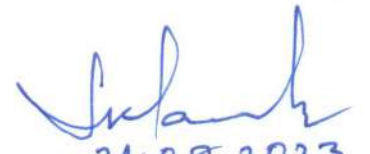
### CERTIFICATE BY THE SUPERVISOR

It is certified that the above statement made by the student is correct to the best of my/our knowledge.

  
21-08-23  
(Dr. Rajeev)

Associate Professor  
Department of Mathematical Sciences  
Indian Institute of Technology  
(Banaras Hindu University)  
Varanasi-221005

पर्यवेक्षक / Supervisor  
गणितीय विज्ञान विभाग  
Department of Mathematical Sciences  
भारतीय प्रौद्योगिकी संस्थान  
Indian Institute of Technology  
(काशी हिन्दू विश्वविद्यालय)  
(Banaras Hindu University)  
वाराणसी / Varanasi-221005

  
21.08.2023

(Prof. S.K. Pandey)

Professor and Head  
Department of Mathematical Sciences  
Indian Institute of Technology  
(Banaras Hindu University)  
Varanasi-221005

विभागाध्यक्ष / HEAD  
गणितीय विज्ञान विभाग  
Department of Mathematical Sciences  
भारतीय प्रौद्योगिकी संस्थान  
Indian Institute of Technology  
(काशी हिन्दू विश्वविद्यालय)  
(Banaras Hindu University)  
वाराणसी / Varanasi-221005

COPYRIGHT TRANSFER CERTIFICATE

Title of the Thesis: *Some Problems on Stability Analysis and Synchronization of Neural Networks.*

Name of the Student: *Ankit Kumar*

Copyright Transfer

The undersigned hereby assigns to the Indian Institute of Technology (Banaras Hindu University), Varanasi all rights under copyright that may exist in and for the above thesis submitted for the award of the Ph.D. degree.

Date: August 21, 2023

Place: Varanasi

  
(Ankit Kumar)

Note: However, the author may reproduce or authorize others to reproduce material extracted verbatim from the thesis or derivative of the thesis for author's personal use provided that the source and the Institute copyright notice are indicated.

---

To  
My Beloved Parents & Elder Brother  
Smt. Prema Devi  
&  
Shri Shyam Lal  
&  
Shri Irosh Kumar

---

## ACKNOWLEDGEMENTS

---

**L**ife has been a long memorable unforgettable journey with ups and downs. I would not be able to express my gratitude in words for the support, love, affection. I got throughout this journey, from so many people. It is mere attempt to name a few.

First of all, I would like to express my sincere thanks to my respected supervisor and co-supervisor, *Dr. Rajeev* and *Prof. Subir Das*, for their excellent guidance, continuous encouragement, patience and advice during the whole span of Ph.D. I have been very lucky to have such supervisors who cared so much about my work, have shaped my understanding on the subject that has given me confidence to work independently. I also learnt from them which is much more than one could have expected from the supervisor.

I express my cordial thanks to *Prof. S.K. Pandey, Head of Department* and *Dr. Anuradha Banerjee, Convener, DPGC, of the Department of Mathematical Sciences* for their supports throughout my research work. I also express my deep sense of gratitude to all faculty members of the department, specially *Prof. T. Som, Prof. S. Mukhopadhyay, Prof. L. P. Singh, Dr. V. K. Singh, Dr. R. K. Pandey, Dr. Debdas Ghosh, Dr. Sunil Kumar* and RPEC member *Dr. B. Biswas, Department of Computer Science*, for their constant moral supports, suggestions, and encouragement.

I am very grateful to have supportive seniors and lab-mates Dr. Vijay Kumar Yadav, Dr. Rakesh Kumar, Dr. Anuwedita Singh, Mr. Abhishek Kumar, Mr. Umesh

Kumar, Mr. Manpal Singh, during my Ph.D. Journey. I would also like to thanks to sincere juniors of my lab Mr. Sunny Singh, Ms. Anshika, Mr. Shivshankar Chauhan, Ms. Sapna Baluni, Mr. Shivshankar Das, Mr. Shubham Tiwari, Ms. Anjuman, Ms. Rashmi Sharma, Ms. Jaya Joshi, and Ms. Arpita Charan.

I have very special thanks to my colleagues *Dr. Sumit, Dr. Harendra Kumar, Dr. R.S. Chauhan, Mr. Shashikant, Mr. Aman, Mr. Yashveer, Sandeep* and all the research scholars of the department for their moral supports.

I am also grateful to my Institute, IIT(BHU), for providing necessary resources throughout my research. I express my thanks to all non teaching staff members of the department for their supports.

I gratefully acknowledge Council of Scientific Industrial Research, India for providing the fellowship in form of Junior Research Fellowship and Senior Research fellowship.

I express my sincere and cordial gratitude to my father ***Shri Shyam Lal***, my mother ***Smt. Prema Devi***, my elder brother ***Shri Irosh Kumar***, my elder sisters ***Smt. Surela & Smt. Reena*** who always stood by my decisions and provided all kinds of supports, moral as well as financial. It was their love, care and patience which encouraged me to move on. Special love to my sweet niece and nephew ***Ishu & Krishu***.

This acknowledgement would be incomplete if the name of great visionary ***Pt. Madan Mohan Malaviya*** is not mentioned, who made this divine centre of knowledge. Deepest regards to him.

Above all, praises and thanks to the God, the Almighty, for His showers of blessings throughout my research work, who has made everything possible.

**Ankit Kumar**

# Contents

|  |           |
|--|-----------|
| List of Figures  | xv        |
| List of Tables   | xix       |
| Preface  | xxi       |
| <b>1 Introduction</b>  | <b>1</b>  |
| 1.1 Biological Neural Network  | 2         |
| 1.1.1 Biological Model   | 6         |
| 1.2 Artificial Neural Networks(ANNs)   | 11        |
| 1.3 Some Additive Neural Networks  | 22        |
| 1.3.1 Hopfield Neural Networks   | 24        |
| 1.3.2 Complex-valued Neural Networks   | 28        |
| 1.3.3 Memristor based Neural Networks  | 30        |
| 1.4 An Overview of Mathematical Concepts   | 32        |
| 1.4.1 Delay Differential Equations   | 32        |
| 1.4.2 Matrix Measure Theory  | 37        |
| 1.5 Synchronization  | 41        |
| 1.6 Summary of the thesis  | 44        |
| <b>2 Global exponential synchronization of complex valued recurrent neural networks in presence of uncertainty along with time-varying bounded and unbounded delay terms</b> | <b>47</b> |
| 2.1 Some preliminaries and Problem formulation   | 48        |
| 2.1.1 Preliminaries  | 48        |
| 2.1.2 Problem formulation  | 49        |
| 2.2 Main results   | 53        |
| 2.3 Numerical Examples   | 59        |
| 2.4 Conclusion   | 73        |
| <b>3 Global quasi-synchronization of complex-valued recurrent neural networks with time-varying delay and interaction terms</b>  | <b>75</b> |

---

|          |  |            |
|----------|--|------------|
| 3.1      | Some preliminaries and Problem formulation . . . . .   | 76         |
| 3.1.1    | Problem formulation . . . . .  | 77         |
| 3.2      | Main results . . . . .   | 81         |
| 3.3      | Numerical Example . . . . .  | 87         |
| 3.4      | Conclusion . . . . .   | 93         |
| <b>4</b> | <b>Quasi-projective synchronization of memristor-based complex valued recurrent neural network with time-varying delay and mismatched parameters</b> | <b>95</b>  |
| 4.1      | Preliminaries and Model Formulation . . . . .  | 96         |
| 4.2      | Main results . . . . .   | 105        |
| 4.3      | Numerical Simulation . . . . .   | 117        |
| 4.4      | Conclusion . . . . .   | 131        |
| <b>5</b> | <b>Quasi-projective synchronization of inertial complex-valued recurrent neural networks with mixed time-varying delay and mismatched parameters</b> | <b>133</b> |
| 5.1      | Preliminaries and System Formulation . . . . .   | 134        |
| 5.2      | Main results . . . . .   | 144        |
| 5.3      | Numerical Simulation . . . . .   | 154        |
| 5.4      | Conclusion . . . . .   | 163        |
| <b>6</b> | <b>Conclusion and Future Works</b>   | <b>165</b> |
|          | <b>Bibliography</b>  | <b>169</b> |

# List of Figures

|      |   |    |
|------|---|----|
| 1.1  | Schematic structure of a typical neuron [1]. . . . .  | 4  |
| 1.2  | Schematic diagram of a network of neurons. . . . .  | 7  |
| 1.3  | A diagram of an artificial neuron of the network. . . . .   | 13 |
| 1.4  | (a) A Step function, (b) Piecewise linear function, and (c) Sigmoid activation function for $\beta = 3, 4$ and 5. . . . .   | 14 |
| 1.5  | A feedforward network without hidden layer. . . . .   | 17 |
| 1.6  | Multilayer feedforward neural network with one hidden layer. . . . .  | 18 |
| 1.7  | A recurrent neural network without a hidden layer . . . . .   | 19 |
| 1.8  | A recurrent network with one hidden layer of neurons. . . . .   | 20 |
| 1.9  | Partially connected neural network. . . . .   | 21 |
| 1.10 | Additive model of a neuron. . . . .   | 22 |
| 1.11 | Hopfield neural network for $n=4$ neurons. . . . .  | 24 |
| 2.1  | Plots of the state trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with respect to time $t$ in 3-dimensional space without uncertain parameters $\Delta A$ and $\Delta B$ of the system (2.3.1) and bounded time varying delay term $\tau(t) = (2 + \cos(t))$ . . . . .  | 62 |
| 2.2  | Plots of the state trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with respect to time $t$ in 3-dimensional space with uncertain parameters $\Delta A$ and $\Delta B$ and bounded time varying delay term $\tau(t) = (2 + \cos(t))$ . . . . .   | 63 |
| 2.3  | Plots of the state trajectories $u_1(t)$ and $\tilde{u}_1(t), v_1(t)$ and $\tilde{v}_1(t), u_2(t)$ and $\tilde{u}_2(t), v_2(t)$ and $\tilde{v}_2(t)$ with respect to time $t$ , of master system (2.3.1) and response system (2.3.2) without uncertain parameters $\Delta A$ and $\Delta B$ and bounded time varying delay term $\tau(t) = (2 + \cos(t))$ . . . . . | 64 |
| 2.4  | Plots of the global exponential synchronization errors $e_i(t), i = 1, 2, 3, 4$ of the systems (2.3.1) and (2.3.2) without uncertain parameters $\Delta A$ and $\Delta B$ and bounded time varying delay term $\tau(t) = (2 + \cos(t))$ . . . . .   | 65 |
| 2.5  | Plots of the state trajectories $u_1(t)$ and $\tilde{u}_1(t), v_1(t)$ and $\tilde{v}_1(t), u_2(t)$ and $\tilde{u}_2(t), v_2(t)$ and $\tilde{v}_2(t)$ with respect to time $t$ , of master system (2.3.1) and response system (2.3.2) with uncertain parameters $\Delta A$ and $\Delta B$ and bounded time varying delay term $\tau(t) = (2 + \cos(t))$ . . . . .    | 65 |
| 2.6  | Plots of the global exponential synchronization errors $e_i(t), i = 1, 2, 3, 4$ of the systems (2.3.1) and (2.3.2) with uncertain parameters $\Delta A$ and $\Delta B$ and bounded time varying delay term $\tau(t) = (2 + \cos(t))$ . . . . .  | 66 |

|      |   |    |
|------|---|----|
| 2.7  | Plots of the state trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with respect to time $t$ in 3-dimensional space without uncertain parameters $\Delta A$ and $\Delta B$ of the system (2.3.3) and unbounded time varying delay term $\tau(t) = \text{ceil}(3t/2)$ . . . . .  | 69 |
| 2.8  | Plots of the state trajectories $u_1(t), v_1(t), u_2(t)$ and $v_2(t)$ with respect to time $t$ in 3-dimensional space with uncertain parameters $\Delta A$ and $\Delta B$ and unbounded time varying delay term $\tau(t) = \text{ceil}(3t/2)$ . . . . .   | 70 |
| 2.9  | Plots of the state trajectories $u_1(t)$ and $\tilde{u}_1(t), v_1(t)$ and $\tilde{v}_1(t), u_2(t)$ and $\tilde{u}_2(t), v_2(t)$ and $\tilde{v}_2(t)$ with respect to time $t$ , of master system (2.3.3) and response system (2.3.4) without uncertain parameters $\Delta A$ and $\Delta B$ and unbounded time varying delay term $\tau(t) = \text{ceil}(3t/2)$ . . . . . | 71 |
| 2.10 | Plots of the global exponential synchronization errors $e_i(t), i = 1, 2, 3, 4$ of the systems (2.3.3) and (2.3.4) without uncertain parameters $\Delta A$ and $\Delta B$ and unbounded time varying delay term $\tau(t) = \text{ceil}(3t/2)$ . . . . .   | 72 |
| 2.11 | Plots of the state trajectories $u_1(t)$ and $\tilde{u}_1(t), v_1(t)$ and $\tilde{v}_1(t), u_2(t)$ and $\tilde{u}_2(t), v_2(t)$ and $\tilde{v}_2(t)$ with respect to time $t$ , of master system (2.3.3) and response system (2.3.4) with uncertain parameters $\Delta A$ and $\Delta B$ and unbounded time varying delay term $\tau(t) = \text{ceil}(3t/2)$ . . . . .    | 72 |
| 2.12 | Plots of the global exponential synchronization errors $e_i(t), i = 1, 2, 3, 4$ of the systems (2.3.3) and (2.3.4) with uncertain parameters $\Delta A$ and $\Delta B$ and unbounded time varying delay term $\tau(t) = \text{ceil}(3t/2)$ . . . . .  | 73 |
| 3.1  | Plots of the state trajectories $u_1(t), v_1(t)$ with respect to time $t$ in 3-dimensional space of the system (3.3.1) with time varying delay term $\tau(t) = 0.1\cos(t)$ . . . . .  | 88 |
| 3.2  | Plots of the state trajectories $u_2(t)$ and $v_2(t)$ with respect to time $t$ in 3-dimensional space of the system (3.3.2) with time varying delay term $\tau(t) = 0.1\cos(t)$ . . . . .   | 89 |
| 3.3  | Plots of the state trajectories $u_1(t)$ and $\tilde{u}_1(t)$ with respect to time $t$ , of master system (3.3.1) and response system (3.3.2) with time varying delay term $\tau(t) = 0.1\cos(t)$ . . . . .   | 90 |
| 3.4  | Plots of the state trajectories $v_1(t)$ and $\tilde{v}_1(t)$ with respect to time $t$ , of master system (3.3.1) and response system (3.3.2) with time varying delay term $\tau(t) = 0.1\cos(t)$ . . . . .   | 91 |
| 3.5  | Plots of the state trajectories $u_2(t)$ and $\tilde{u}_2(t)$ with respect to time $t$ , of master system (3.3.1) and response system (3.3.2) with time varying delay term $\tau(t) = 0.1\cos(t)$ . . . . .   | 91 |
| 3.6  | Plots of the state trajectories $v_2(t)$ and $\tilde{v}_2(t)$ with respect to time $t$ , of master system (3.3.1) and response system (3.3.2) with time varying delay term $\tau(t) = 0.1\cos(t)$ . . . . .   | 92 |
| 3.7  | Plots of the global quasi-synchronization error $e(t)$ of the systems (3.3.1) and (3.3.2) with time varying delay term as $\tau(t) = 0.1\cos(t)$ having error bound = 0.3131. . . . .   | 92 |

|     |   |     |
|-----|---|-----|
| 4.1 | The state trajectories of the system (4.3.1) in three-dimensional space for <b>(a)</b> $v_1(t)$ , $\varrho_1(t)$ , and <b>(b)</b> $v_2(t)$ and $\varrho_2(t)$ . . . . .   | 119 |
| 4.2 | The state trajectories of the systems (4.3.1) and (4.3.2) for <b>(a)</b> $v_1(t)$ , $\tilde{v}_1(t)$ , and <b>(b)</b> $v_2(t)$ , $\tilde{v}_2(t)$ with time $t$ for the Example 4.3.1 . . . . .                         | 123 |
| 4.3 | The state trajectories of the systems (4.3.1) and (4.3.2) for <b>(a)</b> $\varrho_1(t)$ , $\tilde{\varrho}_1(t)$ , and <b>(b)</b> $\varrho_2(t)$ , $\tilde{\varrho}_2(t)$ with time $t$ for the Example 4.3.1 . . . . . | 123 |
| 4.4 | Plots of the system (4.3.1) for <b>(a)</b> without controllers and <b>(b)</b> with controllers for the Example 4.3.1 . . . . .  | 124 |
| 4.5 | The state trajectories of the system (4.3.1) in three-dimensional space for <b>(a)</b> $v_1(t)$ , $\varrho_1(t)$ , and <b>(b)</b> $v_2(t)$ and $\varrho_2(t)$ . . . . .   | 126 |
| 4.6 | The state trajectories of the systems (4.3.1) and (4.3.2) for <b>(a)</b> $v_1(t)$ , $\tilde{v}_1(t)$ , and <b>(b)</b> $v_2(t)$ , $\tilde{v}_2(t)$ with time $t$ for the Example 4.3.2 . . . . .                         | 129 |
| 4.7 | The state trajectories of the systems (4.3.1) and (4.3.2) for <b>(a)</b> $\varrho_1(t)$ , $\tilde{\varrho}_1(t)$ , and <b>(b)</b> $\varrho_2(t)$ , $\tilde{\varrho}_2(t)$ with time $t$ for the Example 4.3.2 . . . . . | 130 |
| 4.8 | Plots of the system (4.3.1) for <b>(a)</b> without controllers and <b>(b)</b> with controllers for the Example 4.3.2 . . . . .  | 130 |
| 5.1 | The plots of state trajectories $\omega_1(t)$ and $\omega_2(t)$ of the system (5.3.1) for Example 5.3.1. . . . .  | 155 |
| 5.2 | The plots of real state trajectories $\alpha_1(t)$ , $\alpha_2(t)$ and $\tilde{\alpha}_1(t)$ , $\tilde{\alpha}_2(t)$ of drive system (5.3.1) and response system (5.3.2) for Example 5.3.1. . . . .                     | 157 |
| 5.3 | The plots of imaginary state trajectories $\beta_1(t)$ , $\beta_2(t)$ and $\tilde{\beta}_1(t)$ , $\tilde{\beta}_2(t)$ of drive system (5.3.1) and response system (5.3.2) for Example 5.3.1. . . . .                    | 158 |
| 5.4 | Plots of the error system (5.3.1) without controllers and with controllers having error bound = 0.9294 for Example 5.3.1. . . . .   | 158 |
| 5.5 | The state trajectories $\omega_1(t)$ and $\omega_2(t)$ of the system (5.3.1) for Example 5.3.2. . . . .   | 160 |
| 5.6 | The plots of real state trajectories $\alpha_1(t)$ , $\alpha_2(t)$ and $\tilde{\alpha}_1(t)$ , $\tilde{\alpha}_2(t)$ of drive system (5.3.1) and response system (5.3.2) for Example 5.3.2. . . . .                     | 161 |
| 5.7 | The plots of imaginary state trajectories $\beta_1(t)$ , $\beta_2(t)$ and $\tilde{\beta}_1(t)$ , $\tilde{\beta}_2(t)$ of drive system (5.3.1) and response system (5.3.2) for Example 5.3.2. . . . .                    | 162 |
| 5.8 | Plots of the error system (5.3.1) without controllers and with controllers having error bound = 0.8892 for Example 5.3.2. . . . .   | 162 |

# List of Tables

|     |  |    |
|-----|--|----|
| 1.1 | Comparison between a human brain and an artificial neural network. | 12 |
| 1.2 | Complex valued activation functions. . . . .                       | 30 |

## PREFACE

---

**A**n artificial neural network is usually a computational network which is based on biological neural networks that construct the structure of the human brain. These networks are composed of layers of inter-connected nodes, with each node representing a neuron. Every node is connected to other nodes in the network, allowing information to be transmitted from one to another. The nodes are also able to learn and adapt based on input data. Artificial neural networks(ANNs) also have neurons those are inter-connected to one another in various layers of the networks. In the mathematical implementation of neural networks, the stability and synchronization analyses of the mathematical models are very crucial. Moreover, ANNs based on machine learning models and especially deep learning models have been widely applied in computer vision, signal processing, wireless communications, and many other domains, where complex numbers occur either naturally or by design. However, most of the current implementations of ANNs and machine learning frameworks are using real numbers rather than complex numbers. There are growing interests in building ANNs using complex numbers, and exploring the potential advantages of the so-called complex-valued neural networks (CVNNs) over their real-valued counterparts.

There are six chapters in the present thesis. ANNs are introduced from motivation to designate and model in the beginning of this thesis. The architectures of the distinct neural networks such as Hopfield, Complex-valued, and Memristor based neural networks, are briefly discussed along with their modeling into mathematical equations. To conclude the introductory chapter, all the mathematical tools are introduced, which will be used throughout the chapters. The first problem is studied in second chapter in which the global exponential synchronization of complex-valued

recurrent neural network(CVRNN) is investigated in presence of uncertainty. In the third chapter, the problem is extended to the global quasi-synchronization of complex-valued recurrent neural networks with interaction terms. In the fourth chapter, a study of quasi-projective synchronization on memristor based CVRNN model has been presented. Again, in the fifth chapter, same synchronization analysis on second ordered model of CVRNN is studied.

In the last chapter, the future work is discussed which will be the extension of the results obtained in this thesis.

\*\*\*\*\*