

Chapter 6

Global dissipativity analysis for a class of quaternion-valued neural networks with inertial term and mixed time delays:

Non-separation approach

6.1 Introduction

In the year 1970s, Willems proposed dissipativity since it has been used in numerous areas, including chaos, robust control, state estimation, and stability theory. Certainly, dissipativity can be viewed as a generalization or extension of Lyapunov

stability, as it encompasses not only the stability of equilibrium points but also a broader range of dynamical behaviors [150]. The most significant advantage of the dissipative theory is that it can be used to discuss complicated systems and offer a reliable framework for stability analysis. One must identify the attractive global set in the subject of dissipativity, an estimated range of periodic states. The equilibrium point and chaos may be calculated once the set has been found. Many researchers are interested in the dissipativity analysis of neural networks, and many significant conclusions have been obtained [136, 151, 134, 134, 152, 153, 154]. Unfortunately, these works are mainly based on traditional neural networks. To date, Still, up to now, there are very few results of global dissipativity on INNs [134, 155, 156, 157]. The present chapter directed their attention toward the INNs and considered the global dissipativity of INNs with time delays in the quaternion domain. This chapter dealt with the global and exponential dissipativity of QVINNs with mixed time-varying delays and without utilizing order reduction of INNs and the quaternion separation method. Using innovative Lyapunov functional and inequality techniques, several fruitful sufficient criteria with multi-parameters are derived for QVINNs to ensure the global dissipativity (GD) results, which generalize and refine existing results. This chapter has estimated the attractive sets and exponentially attractive sets globally. Unlike the existing one in which QVNNs are separated into RVNNs and reduced the INNs into first-order systems, the foundation of this chapter rests upon approaches that diverge from the traditional methods of separation and order reduction. Unlike existing results about the global dissipativity of traditional neural networks with bounded discrete time delays, this chapter concerns INNs with unbounded discrete time-varying delay, which is more realistic because neurons relate all their entire past rather than partial history in bounded time delays. In general, in exponential stability, synchronization, and dissipativity results, the researchers have bounded the rate of convergence λ , but the present chapter investigates the

dissipativity criteria, which have no upper bound on the convergence rate λ in global exponential dissipativity (GED). Finally, to demonstrate the efficiency of the present theoretical work, three consecutive examples are proposed to show the effectiveness of the obtained results. The first two examples are used to validate the proposed results, and the third one, related to QVNNs, is used to show the efficiency of storing the true color image pattern.

6.2 Model Description and Preliminaries

In this chapter, the class of QVINNs with unbounded discrete time-varying delays is considered as

$$\begin{aligned} \frac{d^2 r_i(t)}{dt^2} = & -\alpha_i \frac{dr_i(t)}{dt} - \beta_i r_i(t) + \sum_{j=1}^n a_{ij} f_j(r_j(t)) + \sum_{j=1}^n b_{ij} g_j(r_j(t - \tau_j(t))) \\ & + \sum_{j=1}^n d_{ij} \int_{t-\sigma_j(t)}^t h_j(r_j(s)) ds + u_i(t), \quad i \in I, \end{aligned} \quad (6.1)$$

where $r_i(t)$ is the state of the i th neurons, $\frac{d^2 r_i(t)}{dt^2}$ is the inertial terms, $\alpha_i, \beta_i \in \mathbf{R}^+$, a_{ij}, b_{ij} , and $d_{ij} \in \mathbf{H}$ are weight connection matrices respectively, $f_j(\cdot)$, $g_j(\cdot)$, and $h_j(\cdot)$ are the activation functions, $\tau_j(t), \sigma_j(t)$ denotes the discrete and distributed time-delays respectively from j -th neurons at a time t , $u_i(t)$ is the external input.

In this chapter, the following are taken into account.

Assumption 6.2.1. *The time-varying delays $\tau_j(t)$ are unbounded, and σ_j are differentiable and satisfy $\dot{\tau}_j(t) \leq \tau < 1$, and $\dot{\sigma}_j(t) \leq \sigma < 0.5$, respectively $\forall j \in I$. where τ and σ are constants.*

Assumption 6.2.2. *For any $j \in I$, $\forall s_1, s_2 \in \mathbf{H}$, there exist non-negative constants M_j, L_j and K_j so that the nonlinear activation functions satisfy*

$$|f_j(s_1) - f_j(s_2)| \leq M_j |s_1 - s_2|,$$

$$|g_j(s_1) - g_j(s_2)| \leq L_j |s_1 - s_2|,$$

$$|h_j(s_1) - h_j(s_2)| \leq K_j |s_1 - s_2|.$$

Assumption 6.2.3. For any $j \in I$, $\forall s_1, s_2 \in \mathbf{H}$, there exist non-negative constants M_j, L_j , and, K_j such that the nonlinear activation functions satisfy

$$|f_j(s_1) - f_j(s_2)| \leq M_j |s_1 - s_2|,$$

$$|g_j(s_1) - g_j(s_2)| \leq L_j |s_1 - s_2| e^{-a\tau_j(t)},$$

$|h_j(s_1) - h_j(s_2)| \leq K_j |s_1 - s_2|$, where $\tau_j(t)$ is the time delay in the system (6.1) and $a > 0$ is sufficiently large number.

Definition 6.2.1. If there is a compact set $\Upsilon \subset \mathbf{H}^n$, for $\forall \Psi(s), \Psi^*(s) \in \mathbf{H}^n - \Upsilon$ for $s \in (-\infty, t_0]$, $\exists T > 0$ such that $r(t, t_0, \Psi, \Psi^*) \subset \Upsilon$ for $t \geq T + t_0$, then QVINN (6.1) is referred to as a global dissipativity system. The set Υ is the globally attractive set (GAS) of (6.1). A set Υ is said to be the positive invariant set if for $\forall \Psi(s), \Psi^*(s) \in \Upsilon$ for $s \in (-\infty, t_0]$ which implies that $r(t, t_0, \Psi, \Psi^*) \subset \Upsilon$ for $t \geq t_0$.

Definition 6.2.2. Let Υ be a global attractive set of the QVINNs model (6.1). Then the model (6.1) is said to be a globally exponentially dissipative system if there exists a compact set $\Upsilon \subset \mathbf{H}^n$, for all $\Psi(s), \Psi^*(s) \in \mathbf{H}^n - \Upsilon$, $s \in (-\infty, t_0]$, there exist constants $K(\Psi, \Psi^*) > 0$, $\lambda > 0$ such that

$\inf_{u \in \mathbf{H}^n - \Upsilon} \{ \|r(t, t_0, \Psi, \Psi^*) - u^*\| \mid u^* \in \Upsilon \} \leq K(\Psi, \Psi^*) e^{-\lambda(t-t_0)}$. Meanwhile, the set Υ is a globally exponential attractive set of (6.1).

Remark 6.2.1. Most of the results regarding global dissipativity and stability for QVINNs and QVNNs rely on the method of quaternion separation as given in [152, 158, 159, 149, 160, 161]. As a result, the number of variables and dimensions will rise, and the complexity of the computation will rise. Instead of taking this

technique, let's carry out the task directly, which better fits the requirements of actual applications while maintaining the originality of the addressed systems.

Remark 6.2.2. *It's worth highlighting that Assumption 6.2.3 covers a range of time delay types as specific instances. For instance, it encompasses constant discrete delays as discussed in references like [127, 128], proportional delays as explored in [136], and bounded discrete-time delays, as examined in works such as [162, 163, 164, 127, 128, 129, 161, 160, 134]. This characteristic renders the considered model more comprehensive and adaptable. Conversely, investigating global exponential dissipativity commonly poses challenges when involving unbounded time delays. However, this chapter introduces an innovative assumption for the activation function $g_l(\cdot)$, simplifying the resolution of this challenge.*

Remark 6.2.3. *According to the majority of the existing research, the most popular approach for analyzing the dynamic characteristics of INNs is transforming the INNs into first-order differential equations by appropriate selection of variable substitutions. However, this work directly analyses the global dissipativity of the proposed QVINNs by utilizing some new Lyapunov functional without converting the suggested system into a conventional one. The considered Lyapunov functional, quite from the usual Lyapunov functional, directly includes both the state variables and their derivatives. It is more convenient for us to examine the asymptotic characteristics of different INNs directly because this reduces the amount of redundant processing.*

6.3 Main Results

This section will analyze global dissipativity results by employing Lyapunov theory along with various analytic techniques, which are alternatives to the conventional

reduced order and separation techniques.

Theorem 6.3.1. *Let us presume the Assumptions (6.2.1)-(6.2.2), there exist $\delta_i > 0$ and the system (6.1), such that $X_i < 0$, $Y_i < 0$, $Z_i^2 \leq X_i Y_i$, $i \in I$ hold, where $Z_i = \delta_i - \alpha_i - \beta_i + 1$, $Y_i = 2 - 2\alpha_i + \epsilon_5 + \frac{n}{\epsilon_3} + \sum_{j=1}^n \left(\frac{1}{\epsilon_1} a_{ij} \bar{a}_{ij} + \frac{\sigma_2}{\epsilon_7} K_j^2 d_{ij} \bar{d}_{ij} \right)$ and $X_i = \epsilon_6 + \frac{n}{\epsilon_4} + \tau \gamma_i M_i^2 - 2\beta_i + n\epsilon_1 M_i^2 + n\epsilon_2 M_i^2 + \sum_{j=1}^n \left(\frac{1}{\epsilon_2} a_{ij} \bar{a}_{ij} + \frac{\sigma_2}{\epsilon_7} K_j^2 d_{ij} \bar{d}_{ij} + \frac{(\epsilon_3 + \epsilon_4) |b_{ij}|^2 L_j^2}{(1-\tau)} \right)$.*

Then, the QVINN model (6.1) is a global dissipative system, and the set $\Upsilon_1 = \left\{ r(t) \in \mathbf{Q}^n \mid \bar{r}_i(t) r_i(t) \leq \frac{\epsilon^}{\kappa} \bar{u}_i(t) u_i(t), \quad i \in I \right\}$ is a global attractive set and a positively invariant set of QVINNs (6.1) where $\kappa = \min_{1 \leq i \leq n} \left\{ \frac{Z_i^2}{Y_i} - X_i \right\}$.*

Proof. To establish the proof for this theorem, let's employ a Lyapunov functional in the following manner:

$$\begin{aligned}
 V(t) &= \sum_{i=1}^n \delta_i \bar{r}_i(t) r_i(t) + \sum_{i=1}^n \left(\overline{\dot{r}_i(t) + r_i(t)} \right) (\dot{r}_i(t) + r_i(t)) \\
 &\quad + \sum_{i=1}^n \sum_{j=1}^n \int_{t-\tau_j(t)}^t \frac{(\epsilon_3 + \epsilon_4) |b_{ij}|^2}{(1-\tau)} \bar{g}_j(r_j(s)) g_j(r_j(s)) ds \\
 &\quad + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \int_{-\sigma_j(t)}^0 \int_{t+s}^t \bar{r}_i(\vartheta) r_i(\vartheta) ds d\vartheta \\
 &\quad + \sum_{i=1}^n \int_{-\tau}^0 \int_{t+\theta}^t \gamma_i \bar{f}_i(r_i(s)) f_i(r_i(s)) ds d\theta. \tag{6.2}
 \end{aligned}$$

Taking derivative along the equation (6.1), one can get

$$\begin{aligned}
 \frac{dV(t)}{dt} &= \sum_{i=1}^n \delta_i \bar{\dot{r}}_i(t) r_i(t) + \sum_{i=1}^n \delta_i \bar{r}_i(t) \dot{r}_i(t) + \sum_{i=1}^n (\bar{\ddot{r}}_i(t) + \bar{\dot{r}}_i(t)) (\dot{r}_i(t) + r_i(t)) \\
 &\quad + \sum_{i=1}^n (\bar{\dot{r}}_i(t) + \bar{r}_i(t)) (\ddot{r}_i(t) + \dot{r}_i(t)) + \sum_{i=1}^n \sum_{j=1}^n \frac{(\epsilon_3 + \epsilon_4) |b_{ij}|^2}{(1-\tau)} \\
 &\quad \times (\bar{g}_j(r_j(t)) g_j(r_j(t)) - \bar{g}_j(r_j(t-\tau_j)) g_j(r_j(t-\tau_j)) (1 - \dot{\tau}_j(t)))
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j(t) \bar{r}_j(t) r_j(t) - (1 - \dot{\sigma}_j(t)) \int_{t-\sigma_j(t)}^t \bar{r}_j(s) r_j(s) ds \right) \\
 & + \sum_{i=1}^n \tau \gamma_i \bar{f}_i(r_i(t)) f_i(r_i(t)) - \sum_{i=1}^n \gamma_i \int_{t-\tau}^t \bar{f}_i(r_i(s)) f_i(r_i(s)) ds, \\
 \frac{dV(t)}{dt} \leq & \sum_{i=1}^n \delta_i \bar{r}_i(t) r_i(t) + \sum_{i=1}^n \delta_i \bar{r}_i(t) \dot{r}_i(t) + \sum_{i=1}^n (\bar{r}_i(t) \dot{r}_i(t) + \bar{r}_i(t) r_i(t)) \\
 & + \sum_{i=1}^n (\bar{r}_i(t) \dot{r}_i(t) + \bar{r}_i(t) \dot{r}_i(t)) + \sum_{i=1}^n \left\{ -\alpha_i \bar{r}_i(t) - \beta_i \bar{r}_i(t) \right. \\
 & + \sum_{j=1}^n \bar{f}_j(r_j(t)) \bar{a}_{ij} + \sum_{j=1}^n \bar{g}_j(r_j(t - \tau_j(t))) \bar{b}_{ij} \\
 & + \left. \sum_{j=1}^n \int_{t-\sigma_j(t)}^t \bar{h}_j(r_j(s)) \bar{d}_{ij} ds + \bar{u}_i(t) \right\} (\dot{r}_i(t) + r_i(t)) \\
 & + \sum_{i=1}^n (\bar{r}_i(t) + \bar{r}_i(t)) \left\{ -\alpha_i \dot{r}_i(t) - \beta_i r_i(t) + \sum_{j=1}^n a_{ij} f_j(r_j(t)) \right. \\
 & + \left. \sum_{j=1}^n b_{ij} g_j(r_j(t - \tau_j(t))) + \sum_{j=1}^n \int_{t-\sigma_j(t)}^t h_j(r_j(s)) d_{ij} ds + u_i(t) \right\} \\
 & + \sum_{i=1}^n \tau \gamma_i \bar{f}_i(r_i(t)) f_i(r_i(t)) + \sum_{i=1}^n \sum_{j=1}^n \frac{(\epsilon_3 + \epsilon_4) |b_{ij}|^2}{(1 - \tau)} \bar{g}_j(r_j(t)) g_j(r_j(t)) \\
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j(t) \bar{r}_j(t) r_j(t) - (1 - \dot{\sigma}_j(t)) \int_{t-\sigma_j(t)}^t \bar{r}_j(s) r_j(s) ds \right) \\
 & - \sum_{i=1}^n \sum_{j=1}^n (\epsilon_3 + \epsilon_4) |b_{ij}|^2 \bar{g}_j(r_j(t - \tau_j(t))) g_j(r_j(t - \tau_j(t))),
 \end{aligned}$$

$$\begin{aligned}
 \frac{dV(t)}{dt} \leq & \sum_{i=1}^n \delta_i \bar{r}_i(t) r_i(t) + \sum_{i=1}^n \delta_i \bar{r}_i(t) \dot{r}_i(t) + \sum_{i=1}^n (\bar{r}_i(t) \dot{r}_i(t) + \bar{r}_i(t) r_i(t)) \\
 & + \sum_{i=1}^n (\bar{r}_i(t) \dot{r}_i(t) + \bar{r}_i(t) \dot{r}_i(t)) + \sum_{i=1}^n \left(-2\alpha_i \bar{r}_i(t) \dot{r}_i(t) \right. \\
 & - \left. \alpha_i \bar{r}_i(t) r_i(t) - \beta_i \bar{r}_i(t) \dot{r}_i(t) - 2\beta_i \bar{r}_i(t) r_i(t) - \alpha_i \bar{r}_i(t) \dot{r}_i(t) - \beta_i \bar{r}_i(t) r_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n (\bar{f}_j(r_j(t)) \bar{a}_{ij} \dot{r}_i(t) + \bar{r}_i(t) a_{ij} f_j(r_j(t)))
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^n \sum_{j=1}^n (\bar{f}_j(r_j(t)) \bar{a}_{ij} r_i(t) + \bar{r}_i(t) a_{ij} f_j(r_j(t))) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\bar{g}_j(r_j(t - \tau_j(t))) \bar{b}_{ij} \dot{r}_i(t) + \bar{r}_i(t) b_{ij} g_j(r_j(t - \tau_j(t))) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\bar{g}_j(r_j(t - \tau_j(t))) \bar{b}_{ij} r_i(t) + \bar{r}_i(t) b_{ij} g_j(r_j(t - \tau_j(t))) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\bar{r}_i(t) d_{ij} \int_{t-\sigma_j(t)}^t h_j(r_j(s)) ds + \int_{t-\sigma_j(t)}^t \bar{h}_j(r_j(s)) ds \bar{d}_{ij} \dot{r}_i(t) ds \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\bar{r}_i(t) d_{ij} \int_{t-\sigma_j(t)}^t h_j(r_j(s)) ds + \int_{t-\sigma_j(t)}^t \bar{h}_j(r_j(s)) ds \bar{d}_{ij} r_i(t) ds \right) \\
 & + \sum_{i=1}^n (\bar{r}_i(t) u_i(t) + \bar{u}_i(t) \dot{r}_i(t)) + \sum_{i=1}^n (\bar{r}_i(t) u_i(t) + \bar{u}_i(t) r_i(t)) \\
 & + \sum_{i=1}^n \tau \bar{f}_i(r_i(t)) f_i(r_i(t)) + \sum_{i=1}^n \sum_{j=1}^n \frac{(\epsilon_3 + \epsilon_4) |b_{ij}|^2}{(1 - \tau)} \bar{g}_j(r_j(t)) g_j(r_j(t)) \\
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j(t) \bar{r}_j(t) r_j(t) - (1 - \dot{\sigma}_j(t)) \int_{t-\sigma_j(t)}^t \bar{r}_j(s) r_j(s) ds \right) \\
 & - \sum_{i=1}^n \sum_{j=1}^n (\epsilon_3 + \epsilon_4) |b_{ij}|^2 \bar{g}_j(r_j(t - \tau_j(t))) g_j(r_j(t - \tau_j(t))).
 \end{aligned}$$

On behalf of Lemma 5.2.2, one can obtain

$$\begin{aligned}
 \frac{dV(t)}{dt} & \leq \sum_{i=1}^n (\delta_i - \alpha_i - \beta_i + 1) \bar{r}_i(t) r_i(t) + \sum_{i=1}^n (\delta_i - \alpha_i - \beta_i + 1) \bar{r}_i(t) \dot{r}_i(t) \\
 & + \sum_{i=1}^n (2 - 2\alpha_i) \bar{r}_i(t) \dot{r}_i(t) + \sum_{i=1}^n (-2\beta_i) \bar{r}_i(t) r_i(t) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_1 \bar{f}_j(r_j(t)) f_j(r_j(t)) + \frac{1}{\epsilon_1} \bar{r}_i(t) a_{ij} \bar{a}_{ij} \dot{r}_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_2 \bar{f}_j(r_j(t)) f_j(r_j(t)) + \frac{1}{\epsilon_2} \bar{r}_i(t) a_{ij} \bar{a}_{ij} r_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_3 \bar{g}_j(r_j(t - \tau_j(t))) \bar{b}_{ij} b_{ij} g_j(r_j(t - \tau_j(t))) + \frac{1}{\epsilon_3} \bar{r}_i(t) \dot{r}_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_4 \bar{g}_j(r_j(t - \tau_j(t))) \bar{b}_{ij} b_{ij} g_j(r_j(t - \tau_j(t))) + \frac{1}{\epsilon_4} \bar{r}_i(t) r_i(t) \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^n \left(\epsilon_5 \bar{r}_i(t) \dot{r}_i(t) + \frac{1}{\epsilon_5} \bar{u}_i(t) u_i(t) \right) \\
 & + \sum_{i=1}^n \left(\epsilon_6 \bar{r}_i(t) r_i(t) + \frac{1}{\epsilon_6} \bar{u}_i(t) u_i(t) \right) + \sum_{i=1}^n \tau \gamma_i \bar{f}_i(r_i(t)) f_i(r_i(t)) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \frac{(\epsilon_3 + \epsilon_4) |b_{ij}|^2}{(1 - \tau)} \bar{g}_j(r_j(t)) g_j(r_j(t)) \\
 & - \sum_{i=1}^n \sum_{j=1}^n (\epsilon_3 + \epsilon_4) |b_{ij}|^2 \bar{g}_j(r_j(t - \tau_j(t))) g_j(r_j(t - \tau_j(t))) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\int_{t-\sigma_j(t)}^t \epsilon_7 \bar{h}_j(r_j(s)) h_j(r_j(s)) ds + \frac{\sigma_j(t)}{\epsilon_7} \bar{r}_i(t) d_{ij} \bar{a}_{ij} \dot{r}_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\int_{t-\sigma_j(t)}^t \epsilon_8 \bar{h}_j(r_j(s)) h_j(r_j(s)) ds + \frac{\sigma_j(t)}{\epsilon_8} \bar{r}_i(t) d_{ij} \bar{a}_{ij} r_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j(t) \bar{r}_j(t) r_j(t) - (1 - \dot{\sigma}_j(t)) \int_{t-\sigma_j(t)}^t \bar{r}_j(s) r_j(s) ds \right).
 \end{aligned}$$

Following Assumptions 6.2.1 and 6.2.2 and rearranging the terms, we get

$$\begin{aligned}
 \frac{dV(t)}{dt} & \leq \sum_{i=1}^n (\delta_i - \alpha_i - \beta_i + 1) \bar{r}_i(t) r_i(t) + \sum_{i=1}^n (\delta_i - \alpha_i - \beta_i + 1) \bar{r}_i(t) \dot{r}_i(t) \\
 & + \sum_{i=1}^n (2 - 2\alpha_i) \bar{r}_i(t) \dot{r}_i(t) + \sum_{i=1}^n (-2\beta_i) \bar{r}_i(t) r_i(t) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_1 M_j^2 \bar{r}_j(t) r_j(t) + \frac{1}{\epsilon_1} \bar{r}_i(t) a_{ij} \bar{a}_{ij} \dot{r}_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_2 M_j^2 \bar{r}_j(t) r_j(t) + \frac{1}{\epsilon_2} \bar{r}_i(t) a_{ij} \bar{a}_{ij} r_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_3 L_j^2 \bar{r}_j(t - \tau_j(t)) \bar{b}_{ij} b_{ij} r_j(t - \tau_j(t)) + \frac{1}{\epsilon_3} \bar{r}_i(t) \dot{r}_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_4 L_j^2 \bar{r}_j(t - \tau_j(t)) \bar{b}_{ij} b_{ij} r_j(t - \tau_j(t)) + \frac{1}{\epsilon_4} \bar{r}_i(t) r_i(t) \right) \\
 & + \sum_{i=1}^n \left(\epsilon_5 \bar{r}_i(t) \dot{r}_i(t) + \frac{1}{\epsilon_5} \bar{u}_i(t) u_i(t) \right) + \sum_{i=1}^n \left(\epsilon_6 \bar{r}_i(t) r_i(t) + \frac{1}{\epsilon_6} \bar{u}_i(t) u_i(t) \right) \\
 & + \sum_{i=1}^n \tau \gamma_i M_i^2 \bar{r}_i(t) r_i(t) + \sum_{i=1}^n \sum_{j=1}^n \frac{(\epsilon_3 + \epsilon_4) |b_{ij}|^2}{(1 - \tau)} L_j^2 \bar{r}_j(t) r_j(t)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\int_{t-\sigma_j(t)}^t \epsilon_7 K_j^2 \bar{r}_j(s) r_j(s) ds + \frac{\sigma_j \bar{r}_i(t) d_{ij} \bar{d}_{ij} \dot{r}_i(t)}{\epsilon_7} \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\int_{t-\sigma_j(t)}^t \epsilon_8 K_j^2 \bar{r}_j(s) r_j(s) ds + \frac{\sigma_j \bar{r}_i(t) d_{ij} \bar{d}_{ij} r_i(t)}{\epsilon_8} \right) \\
 & - \sum_{i=1}^n \sum_{j=1}^n (\epsilon_3 + \epsilon_4) |b_{ij}|^2 L_j^2 \bar{r}_j(t - \tau_j(t)) r_j(t - \tau_j(t)) \\
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j \bar{r}_j(t) r_j(t) - 0.5 \int_{t-\sigma_j(t)}^t \bar{r}_j(s) r_j(s) ds \right),
 \end{aligned}$$

Selecting $\rho_{ij} = (\epsilon_7 + \epsilon_8) K_j^2$, and $\epsilon^* = \left(\frac{1}{\epsilon_5} + \frac{1}{\epsilon_6} \right)$, we obtain

$$\begin{aligned}
 \frac{dV(t)}{dt} & \leq \sum_{i=1}^n (\delta_i - \alpha_i - \beta_i + 1) \bar{r}_i(t) r_i(t) + \sum_{i=1}^n (\delta_i - \alpha_i - \beta_i + 1) \bar{r}_i(t) \dot{r}_i(t) \\
 & + \sum_{i=1}^n \left(2 - 2\alpha_i + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{j=1}^n \left(\frac{1}{\epsilon_1} a_{ij} \bar{a}_{ij} + \frac{\sigma_j}{\epsilon_7} K_j^2 d_{ij} \bar{d}_{ij} \right) \right) \bar{r}_i(t) \dot{r}_i(t) \\
 & + \sum_{i=1}^n \left(\epsilon_6 + \frac{n}{\epsilon_4} + \tau \gamma_i M_i^2 - 2\beta_i + n\epsilon_1 M_i^2 + n\epsilon_2 M_i^2 \right. \\
 & \left. + \sum_{j=1}^n \left(\frac{1}{\epsilon_2} a_{ij} \bar{a}_{ij} + \frac{\sigma_j}{\epsilon_8} d_{ij} \bar{d}_{ij} + \frac{(\epsilon_3 + \epsilon_4) |b_{ij}|^2 L_j^2}{(1 - \tau)} \right) \right) \bar{r}_i(t) r_i(t) + \sum_{i=1}^n \epsilon^* \bar{u}_i(t) u_i(t). \\
 & \leq \sum_{i=1}^n Z_i (\bar{r}_i(t) r_i(t) + \bar{r}_i(t) \dot{r}_i(t)) + \sum_{i=1}^n Y_i \bar{r}_i(t) \dot{r}_i(t) \\
 & + \sum_{i=1}^n X_i \bar{r}_i(t) r_i(t) + \sum_{i=1}^n \epsilon^* \bar{u}_i(t) u_i(t)
 \end{aligned}$$

$$\begin{aligned}
 \text{or} \quad \frac{dV(t)}{dt} & \leq \sum_{i=1}^n Y_i \overline{\left(\dot{r}_i(t) + \frac{Z_i}{Y_i} r_i(t) \right)} \left(\dot{r}_i(t) + \frac{Z_i}{Y_i} r_i(t) \right) \\
 & + \sum_{i=1}^n \left(X_i - \frac{Z_i^2}{Y_i} \right) \bar{r}_i(t) r_i(t) + \sum_{i=1}^n \epsilon^* \bar{u}_i(t) u_i(t)
 \end{aligned}$$

$$\text{or} \quad \frac{dV(t)}{dt} \leq \sum_{i=1}^n \left\{ \left(X_i - \frac{Z_i^2}{Y_i} \right) \bar{r}_i(t) r_i(t) + \epsilon^* \bar{u}_i(t) u_i(t) \right\}. \quad (6.3)$$

$$\text{or} \quad \frac{dV(t)}{dt} \leq -\kappa \|r(t)\|^2 + \epsilon^* \|u(t)\|^2 < 0, \quad (6.4)$$

for $\|r(t)\|^2 > \frac{\epsilon^* \|u(t)\|^2}{\kappa}$.

i.e., $r(t) \notin \Upsilon_1$. This implies that if $\Psi(s), \Psi^*(s) \in \Upsilon_1$, then $r(t) \subset \Upsilon_1$ for $t > t_0$, which means that the set Υ is a positive invariant set of (6.1). When $\Psi(s), \Psi^*(s) \notin \Upsilon_1$, there is a $T > 0$ such that $r(t) \subset \Upsilon_1$, for $t > T + t_0$. This means Υ_1 is a global attractive set, and then the QVINNs system is a global dissipativity system, and Υ_1 is a global attractive set, and it is a positively invariant set of (6.1). \square

Corollary 6.3.1. *If $d_{ij} = 0$ in system (6.1) and presume Assumptions (6.2.1)-(6.2.2), there exist $\delta_i > 0$ such that $X_i < 0, Y_i < 0$,*

$$Z_i^2 \leq X_i Y_i, \quad i \in I, \quad \text{where } Z_i = \delta_i - \alpha_i - \beta_i + 1, \quad Y_i = 2 - 2\alpha_i + \epsilon_5 + \frac{n}{\epsilon_3} + \sum_{j=1}^n \frac{1}{\epsilon_1} a_{ij} \bar{a}_{ij} \quad \text{and}$$

$$X_i = \epsilon_6 + \frac{n}{\epsilon_4} + \tau \gamma_i M_i^2 - 2\beta_i + n\epsilon_1 M_i^2 + n\epsilon_2 M_i^2 + \sum_{j=1}^n \left(\frac{1}{\epsilon_2} a_{ij} \bar{a}_{ij} + \frac{(\epsilon_3 + \epsilon_4) |b_{ji}|^2 L_i^2}{(1-\tau)} \right).$$

Then, the QVINN model (6.1) is a global dissipative system, and the set $\Upsilon_1 = \left\{ r(t) \in \mathbf{Q}^n \mid \bar{r}_i(t) r_i(t) \leq \frac{\epsilon^}{\kappa} \bar{u}_i(t) u_i(t), \quad i \in I \right\}$ is a global attractive set and a positively invariant set of QVINNs (6.1)*

Corollary 6.3.2. *Presume Assumptions (6.2.1)-(6.2.2), and the time delay is bounded, i.e., $\tau_j(t) \leq \tau$ and $\dot{\tau}(t) < 1$, where $\tau > 0$. Then there exist $\delta_i > 0$ and the systems (6.1), such that $X_i < 0, Y_i < 0$,*

$$Z_i^2 \leq 4X_i Y_i, \quad i \in I, \quad \text{where } Z_i = \delta_i - \alpha_i - \beta_i + 1, \quad Y_i = 2 - 2\alpha_i + \epsilon_5 + \frac{n}{\epsilon_3} + \sum_{j=1}^n \left(\frac{1}{\epsilon_1} a_{ij} \bar{a}_{ij} + \frac{\sigma_2 K_j^2 d_{ij} \bar{d}_{ij}}{\epsilon_7} \right) \quad \text{and}$$

$$X_i = \epsilon_6 + \frac{n}{\epsilon_4} + \tau \gamma_i M_i^2 - 2\beta_i + n\epsilon_1 M_i^2 + n\epsilon_2 M_i^2 + \sum_{j=1}^n \left(\frac{1}{\epsilon_2} a_{ij} \bar{a}_{ij} + \frac{\sigma_2 K_j^2 d_{ij} \bar{d}_{ij}}{\epsilon_7} + \frac{(\epsilon_3 + \epsilon_4) |b_{ji}|^2 L_i^2}{(1-\tau)} \right).$$

The system (6.1) is global dissipative under the same conditions of the Theorem 6.3.1.

Corollary 6.3.3. *Presume Assumptions (6.2.1)-(6.2.2), such that, if $\delta_i = \alpha_i + \beta_i - 1$ and $Y_i < 0, Z_i < 0$,*

then the system (6.1) is global dissipative and the set $\Upsilon_2 = \left\{ r(t) \in \mathbf{Q}^n \mid \bar{r}_i(t)r_i(t) \leq \left(\frac{\pi\epsilon^}{\min_{1 \leq i \leq n} |X_i|} \right) \bar{u}_i(t)u_i(t), \quad i \in I \right\}$ is the global attractive set.*

Theorem 6.3.2. *Followed from Assumptions 6.2.1, 6.2.3, the considered system is global exponential dissipativity for $\delta_i > 0$ and $\lambda < \sigma$ such that $X_i < 0, Z_i^2 \leq 4X_iY_i$ $i \in I$, where $Z_i = \lambda + \delta_i + 1 - \alpha_i - \beta_i, Y_i = \lambda + 2 - 2\alpha_i + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{j=1}^n \left(\frac{1}{\epsilon_1} a_{ij} \bar{a}_{ij} + \frac{\sigma_2}{\epsilon_7} d_{ij} \bar{d}_{ij} \right)$,*

$$X_i = \lambda(\delta_i + 1) - 2\beta_i + n\epsilon_1 M_i^2 + n\epsilon_2 M_i^2 + \frac{n}{\epsilon_4} + \epsilon_6 + \sum_{j=1}^n \left(\frac{1}{\epsilon_2} a_{ij} \bar{a}_{ij} + \frac{2\epsilon L_j^2 |b_{ij}|^2}{(1-\tau)} + \frac{\sigma_2}{\epsilon_8} d_{ij} \bar{d}_{ij} \right),$$

where σ is the sufficient large positive number. Then QVINN (6.1) is a global dissipative system, and the set $\Upsilon_1^ = \left\{ r(t) \in \mathbf{Q}^n \mid \bar{r}_i(t)r_i(t) \leq \frac{\epsilon^*}{\kappa_1} \bar{u}_i(t)u_i(t), \quad i \in I \right\}$, is a global attractive set and a positively invariant set of QVINN model (6.1), where*

$$\kappa_1 = \min_{1 \leq i \leq n} \left\{ \frac{Z_i^2}{Y_i} - X_i \right\}.$$

Proof. Let's consider the Lyapunov functional as

$$\begin{aligned} V(t) &= \sum_{i=1}^n \delta_i \bar{r}_i(t)r_i(t) e^{\lambda t} + \sum_{i=1}^n \left(\overline{\dot{r}_i(t) + r_i(t)} \right) (\dot{r}_i(t) + r_i(t)) e^{\lambda t} \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n \frac{2\epsilon L_j^2 |b_{ij}|^2}{(1-\tau)} \int_{t-\tau_j(t)}^t \bar{r}_j(s)r_j(s) e^{\lambda s} \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \int_{-\sigma_j(t)}^0 \int_{t+s}^t \bar{r}_i(\vartheta)r_i(\vartheta) e^{\lambda \vartheta} ds d\vartheta. \end{aligned} \quad (6.5)$$

Taking derivative along the equation (6.1), we have

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_{i=1}^n (\lambda \delta_i \bar{r}_i(t)r_i(t) + \delta_i \bar{r}_i(t)r_i(t) + \delta_i \bar{r}_i(t)\dot{r}_i(t)) e^{\lambda t} + \sum_{i=1}^n \lambda (\overline{\dot{r}_i(t) + r_i(t)}) \\ &\quad \times (\dot{r}_i(t) + r_i(t)) e^{\lambda t} + \sum_{i=1}^n (\overline{\dot{r}_i(t) + r_i(t)}) (\dot{r}_i(t) + r_i(t)) e^{\lambda t} + \sum_{i=1}^n (\overline{\dot{r}_i(t) + r_i(t)}) \end{aligned}$$

$$\begin{aligned}
 & \times (\ddot{r}_i(t) + \dot{r}_i(t)) e^{\lambda t} + \sum_{i=1}^n \sum_{j=1}^n \frac{2\epsilon L_j^2 |b_{ij}|^2}{(1-\tau)} \left(\bar{r}_j(t) r_j(t) e^{\lambda t} \right. \\
 & \left. - \bar{r}_j(t - \tau_j(t)) r_j(t - \tau_j(t)) (1 - \dot{\tau}_j(t)) e^{\lambda(t - \tau_j(t))} \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j(t) \bar{r}_j(t) r_j(t) e^{\lambda t} - (1 - \dot{\sigma}_j(t)) \int_{t - \sigma_j(t)}^t \bar{r}_j(s) r_j(s) e^{\lambda s} ds \right).
 \end{aligned} \tag{6.6}$$

$$\begin{aligned}
 & = \sum_{i=1}^n \left(\lambda(\delta_i + 1) \bar{r}_i(t) r_i(t) + (\lambda + \delta_i + 1) \bar{\dot{r}}_i(t) r_i(t) + (\lambda + \delta_i + 1) \bar{r}_i(t) \dot{r}_i(t) \right. \\
 & \left. + (\lambda + 2) \bar{\dot{r}}_i(t) \dot{r}_i(t) \right) e^{\lambda t} + \sum_{i=1}^n \left\{ -\alpha_i \bar{\dot{r}}_i(t) - \beta_i \bar{r}_i(t) + \sum_{j=1}^n \bar{f}_j(r_j(t)) \bar{a}_{ij} \right. \\
 & \left. + \sum_{j=1}^n \bar{g}_j(r_j(t - \tau_j(t))) \bar{b}_{ij} \sum_{j=1}^n \int_{t - \sigma_j(t)}^t \bar{h}_j(r_j(s)) \bar{d}_{ij} ds + \bar{u}_i(t) \right\} (\dot{r}_i(t) + r_i(t)) e^{\lambda t} \\
 & + \sum_{i=1}^n (\bar{\dot{r}}_i(t) + \bar{r}_i(t)) \left\{ -\alpha_i \dot{r}_i(t) - \beta_i r_i(t) + \sum_{j=1}^n a_{ij} f_j(r_j(t)) \right. \\
 & \left. + \sum_{j=1}^n b_{ij} g_j(r_j(t - \tau_j(t))) + \sum_{j=1}^n \int_{t - \sigma_j(t)}^t d_{ij} h_j(r_j(s)) ds + u_i(t) \right\} e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n \frac{2L_j^2 \epsilon |b_{ij}|^2}{(1-\tau)} \left(\bar{r}_j(t) r_j(t) e^{\lambda t} - \bar{r}_j(t - \tau_j(t)) r_j(t - \tau_j(t)) (1 - \dot{\tau}_j(t)) e^{\lambda(t - \tau_j(t))} \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j(t) \bar{r}_j(t) r_j(t) e^{\lambda t} - (1 - \dot{\sigma}_j(t)) \int_{t - \sigma_j(t)}^t \bar{r}_j(s) r_j(s) e^{\lambda s} ds \right), \\
 & = \sum_{i=1}^n \left((\lambda(\delta_i + 1) - 2\beta_i) \bar{r}_i(t) r_i(t) + (\lambda + \delta_i + 1 - \alpha_i - \beta_i) \bar{\dot{r}}_i(t) r_i(t) \right. \\
 & \left. + (\lambda + \delta_i + 1 - \alpha_i - \beta_i) \bar{r}_i(t) \dot{r}_i(t) + (\lambda + \delta_i + 1 - 2\alpha_i) \bar{\dot{r}}_i(t) \dot{r}_i(t) \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n (\bar{f}_j(r_j(t)) \bar{a}_{ij} \dot{r}_i(t) + \bar{\dot{r}}_i(t) a_{ij} f_j(r_j(t))) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n (\bar{f}_j(r_j(t)) \bar{a}_{ij} r_i(t) + \bar{r}_i(t) a_{ij} f_j(r_j(t))) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n (\bar{g}_j(r_j(t - \tau_j(t))) \bar{b}_{ij} \dot{r}_i(t) + \bar{\dot{r}}_i(t) b_{ij} \bar{g}_j(r_j(t - \tau_j(t)))) e^{\lambda t}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^n \sum_{j=1}^n (\bar{g}_j(r_j(t - \tau_j(t))) \bar{b}_{ij} r_i(t) + \bar{r}_i(t) b_{ij} \bar{g}_j(r_j(t - \tau_j(t)))) e^{\lambda t} \\
 & + \sum_{i=1}^n (\bar{u}_i(t) \dot{r}_i(t) + \bar{r}_i(t) u_i(t)) e^{\lambda t} + \sum_{i=1}^n (\bar{u}_i(t) r_i(t) + \bar{r}_i(t) u_i(t)) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n \frac{2\epsilon L_j^2 |b_{ij}|^2}{(1 - \tau)} \left(\bar{r}_j(t) r_j(t) e^{\lambda t} - \bar{r}_j(t - \tau_j(t)) r_j(t - \tau_j(t)) (1 - \dot{\tau}_j(t)) e^{\lambda(t - \tau_j(t))} \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\bar{r}_i(t) d_{ij} \int_{t - \sigma_j(t)}^t h_j(r_j(s)) ds + \int_{t - \sigma_j(t)}^t \bar{h}_j(r_j(s)) ds \bar{d}_{ij} r_i(t) ds \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\bar{r}_i(t) d_{ij} \int_{t - \sigma_j(t)}^t h_j(r_j(s)) ds + \int_{t - \sigma_j(t)}^t \bar{h}_j(r_j(s)) ds \bar{d}_{ij} \dot{r}_i(t) ds \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j(t) \bar{r}_j(t) r_j(t) e^{\lambda t} - (1 - \dot{\sigma}_j(t)) \int_{t - \sigma_j(t)}^t \bar{r}_j(s) r_j(s) e^{\lambda s} ds \right).
 \end{aligned}$$

Now by using Lemma 5.2.2 and Assumptions 6.2.1, 6.2.3, one can have

$$\begin{aligned}
 \frac{dV}{dt} & \leq \sum_{i=1}^n \left((\lambda(\delta_i + 1) - 2\beta_i) \bar{r}_i(t) r_i(t) + (\lambda + 1 + \delta_i - \alpha_i - \beta_i) \bar{r}_i(t) \dot{r}_i(t) r_i(t) \right. \\
 & \quad \left. + (\lambda + 1 + \delta_i - \alpha_i - \beta_i) \bar{r}_i(t) \dot{r}_i(t) + (\lambda + 2 - 2\alpha_i) \bar{r}_i(t) \dot{r}_i(t) \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_1 M_j^2 \bar{r}_j(t) r_j(t) + \frac{1}{\epsilon_1} \bar{r}_i(t) a_{ij} \bar{a}_{ij} \dot{r}_i(t) \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_2 M_j^2 \overline{r_j(t)} r_j(t) + \frac{1}{\epsilon_2} \bar{r}_i(t) a_{ij} \bar{a}_{ij} r_i(t) \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_3 L_j^2 \bar{r}_j(t - \tau_j(t)) \bar{b}_{ij} b_{ij} r_j(t - \tau_j(t)) e^{-2\sigma\tau_j(t)} + \frac{1}{\epsilon_3} \bar{r}_i(t) \dot{r}_i(t) \right) \\
 & + \sum_{i=1}^n \sum_{j=1}^n \left(\epsilon_4 L_j^2 \bar{r}_j(t - \tau_j(t)) \bar{b}_{ij} b_{ij} r_j(t - \tau_j(t)) e^{-2\sigma\tau_j(t)} + \frac{1}{\epsilon_4} \bar{r}_i(t) r_i(t) \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \left(\epsilon_5 \bar{r}_i(t) \dot{r}_i(t) + \frac{1}{\epsilon_5} \bar{u}_i(t) u_i(t) \right) e^{\lambda t} + \sum_{i=1}^n \left(\epsilon_6 \bar{r}_i(t) r_i(t) + \frac{1}{\epsilon_6} \bar{u}_i(t) u_i(t) \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \sum_{j=1}^n \frac{2\epsilon L_j^2 |b_{ij}|^2}{(1 - \tau)} \bar{r}_j(t) r_j(t) e^{\lambda t} + \sum_{i=1}^n \sum_{j=1}^n \left(\int_{t - \sigma_j(t)}^t \epsilon_7 K_j^2 \bar{r}_j(s) r_j(s) ds \right. \\
 & \quad \left. + \frac{\sigma_j}{\epsilon_7} \bar{r}_i(t) d_{ij} \bar{d}_{ij} \dot{r}_i(t) \right) e^{\lambda t} + \sum_{i=1}^n \sum_{j=1}^n \left(\int_{t - \sigma_j(t)}^t \epsilon_8 K_j^2 \bar{r}_j(s) r_j(s) ds \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\sigma_j \bar{r}_i(t) d_{ij} \bar{d}_{ij} r_i(t)}{\epsilon_8} \Big) e^{\lambda t} - \sum_{i=1}^n \sum_{j=1}^n 2\epsilon L_j^2 |b_{ij}|^2 \bar{r}_j(t - \tau_j(t)) r_j(t - \tau_j(t)) e^{\lambda(t - \tau_j(t))} \\
 & + \sum_{i=1}^n \sum_{j=1}^n 2\rho_{ij} \left(\sigma_j(t) \bar{r}_j(t) r_j(t) e^{\lambda t} - 0.5 \int_{t - \sigma_j(t)}^t \bar{r}_j(s) r_j(s) e^{\lambda s} ds \right).
 \end{aligned}$$

Since $\frac{e^{\lambda t - 2\sigma_j(t)}}{e^{\lambda(t - \tau_j(t))}} = e^{\tau_j(t)(\lambda - 2\sigma)} < 1$, choose $\rho_{ij} = (\epsilon_7 + \epsilon_8)K_j^2$, and $\epsilon = \text{Max}(\epsilon_3, \epsilon_4)$

$$\begin{aligned}
 \frac{dV(t)}{dt} & \leq \sum_{i=1}^n \left((\lambda + \delta_i + 1 - \alpha_i - \beta_i) (\bar{r}_i(t) r_i(t) + \bar{r}_i(t) \dot{r}_i(t)) \right) e^{\lambda t} \\
 & + \sum_{i=1}^n \left(\lambda(\delta_i + 1) - 2\beta_i + n\epsilon_1 M_i^2 + n\epsilon_2 M_i^2 + \frac{n}{\epsilon_4} + \epsilon_6 \right. \\
 & + \left. \sum_{j=1}^n \left(\frac{1}{\epsilon_2} a_{ij} \bar{a}_{ij} + \frac{2\epsilon L_j^2 |b_{ij}|^2}{(1 - \tau)} + \frac{\sigma_j d_{ij} \bar{d}_{ij}}{\epsilon_8} \right) \right) e^{\lambda t} \bar{r}_i(t) r_i(t) \\
 & + \sum_{i=1}^n \left(\lambda + 2 - 2\alpha_i + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{j=1}^n \left(\frac{1}{\epsilon_1} a_{ij} \bar{a}_{ij} + \frac{\sigma_j d_{ij} \bar{d}_{ij}}{\epsilon_7} \right) \right) \\
 & \times e^{\lambda t} \bar{r}_i(t) \dot{r}_i(t) + \sum_{i=1}^n \left(\frac{1}{\epsilon_5} + \frac{1}{\epsilon_6} \right) e^{\lambda t} \bar{u}_i(t) u_i(t). \tag{6.7}
 \end{aligned}$$

$$\begin{aligned}
 & = e^{\lambda t} \sum_{i=1}^n \left\{ Z_i (\bar{r}_i(t) r_i(t) + \bar{r}_i(t) \dot{r}_i(t)) + Y_i \bar{r}_i(t) \dot{r}_i(t) + X_i \bar{r}_i(t) r_i(t) + \epsilon^* \bar{u}_i(t) u_i(t) \right\}, \\
 & = e^{\lambda t} \sum_{i=1}^n \left\{ Y_i \overline{\left(\dot{r}_i(t) + \frac{Z_i}{X_i} r_i(t) \right)} \left(\dot{r}_i(t) + \frac{Z_i}{X_i} r_i(t) \right) + \left(X_i - \frac{Z_i^2}{Y_i} \right) \bar{r}_i(t) r_i(t) \right. \\
 & \quad \left. + \epsilon^* \bar{u}_i(t) u_i(t) \right\}. \tag{6.8}
 \end{aligned}$$

$$\text{or} \quad \frac{dV(t)}{dt} \leq e^{\lambda t} \sum_{i=1}^n \left\{ \left(X_i - \frac{Z_i^2}{Y_i} \right) \bar{r}_i(t) r_i(t) + \epsilon^* \bar{u}_i(t) u_i(t) \right\}. \tag{6.9}$$

$$\text{or} \quad \frac{dV(t)}{dt} \leq e^{\lambda t} (-\kappa \|r(t)\|^2 + \epsilon^* \|u(t)\|^2) < 0. \tag{6.10}$$

For $\|r(t)\|^2 > \frac{\epsilon^* \|u(t)\|^2}{\kappa}$ i.e., $r(t) \notin \Upsilon_2$. From equation (6.10), one has $V(t) \leq V(t_0)$, then from equation (6.5), we have

$$\sum_{i=1}^n \delta_i \bar{r}_i(t) r_i(t) \leq e^{-\lambda t} V(t) \leq e^{-\lambda t} V(t_0).$$

Let, $\delta = \min_{1 \leq i \leq n}$, $M(\Psi, \Psi^*) = \frac{1}{\delta} \sup_{s \in (-\infty, t_0]} V(s)$, then

$$\inf_{r^* \in \Upsilon_2} \{\|r(t) - r^*\|\} \leq \|r(t) - 0\| \leq M(\Psi, \Psi^*) e^{-\lambda(t-t_0)}, \quad t \geq t_0. \quad (6.11)$$

Based on the definition provided in Definition 6.2.2, it can be deduced that the system described by equation (6.1) qualifies as a global exponential dissipative system, and the set Υ_1^* is a global exponentially attractive set. \square

Corollary 6.3.4. *Following from Assumptions 6.2.1, 6.2.3, the considered system is global exponential dissipativity without distributive delay term for $\lambda < \sigma$ such that $X_i < 0$, $Z_i^2 \leq 4X_i Y_i$ $\iota \in I$, where $Z_i = \lambda + \delta_i + 1 - \alpha_i - \beta_i$, $Y_i = \lambda + 2 - 2\alpha_i + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{j=1}^n \frac{1}{\epsilon_1} a_{ij} \bar{a}_{ij}$, $X_i = \lambda(\delta_i + 1) - 2\beta_i + n\epsilon_1 M_i^2 + n\epsilon_2 M_i^2 + \frac{n}{\epsilon_4} + \epsilon_6 + \sum_{j=1}^n \left(\frac{1}{\epsilon_2} a_{ij} \bar{a}_{ij} + \frac{2\epsilon L_i^2 |b_{ij}|^2}{(1-\tau)} \right)$, where σ is the sufficient large positive number. Then QVINN (6.1) is a global dissipative system, and the set $\Upsilon_1^* = \left\{ r(t) \in \mathbf{Q}^n \mid \bar{r}_i(t) r_i(t) \leq \frac{\epsilon^*}{\kappa_1} \bar{u}_i(t) u_i(t), \quad \iota \in I \right\}$, is a global attractive set and a positively invariant set of QVINN model (6.1) without distributive delay.*

Corollary 6.3.5. *Followed from Assumptions 6.2.1 and 6.2.3, the considered system (6.1) is global exponential dissipativity, for $\lambda > 0$ such that $X_i < 0$, $Z_i^2 \leq 4X_i Y_i$, $\iota \in I$, where $Z_i = \lambda + \delta_i + 1 - \alpha_i - \beta_i$, $Y_i = \lambda + 2 - 2\alpha_i + \frac{n}{\epsilon_3} + \epsilon_5 + \sum_{j=1}^n \frac{1}{\epsilon_1} a_{ij} \bar{a}_{ij}$, $X_i = \lambda(\delta_i + 1) - 2\beta_i + n\epsilon_1 M_i^2 + n\epsilon_2 M_i^2 + \frac{n}{\epsilon_4} + \epsilon_6 + \sum_{j=1}^n \left(\frac{1}{\epsilon_2} a_{ij} \bar{a}_{ij} + \frac{2\epsilon L_i^2 |b_{ij}|^2}{(1-\tau)} \right)$, and $\Upsilon_1^* = \left\{ r(t) \in \mathbf{Q}^n \mid \bar{r}_i(t) r_i(t) \leq \frac{\pi \epsilon^*}{\kappa_1} \bar{u}_i(t) u_i(t) \quad \iota \in I \right\}$, is a global attractive set, which is a positively invariant set of QVINN model (6.1).*

Corollary 6.3.6. *Followed from Assumptions 6.2.1 and 6.2.2, if $\delta_i = (\lambda + 1 - \alpha_i - \beta_i)$ and $X_i < 0, Y_i < 0$, then the system (6.1) is global dissipative system and the set $\Upsilon_2^* = \left\{ r(t) \in \mathbf{Q}^n \mid \bar{r}_i(t)r_i(t) \leq \left(\frac{\pi\epsilon^*}{\min_{1 \leq i \leq n} |X_i|} \right) \bar{u}_i(t)u_i(t), \quad i \in I \right\}$ is the global attractive set.*

Remark 6.3.1. *In many studies, the focus was on exponential stability and synchronization [162, 113, 165, 157], where the researchers have commonly imposed a bound on the rate of convergence denoted by λ . Specifically, they consider $\lambda \in (0, 1)$, indicating that a lower convergence rate corresponds to a slower stabilization speed. However, the problem presented in this chapter departed from this convention by eliminating the restriction on the convergence rate. Instead, criteria have been derived that allow the choice of any large value for λ without jeopardizing the validity of the results. This departure offers flexibility in selecting higher values for λ and provides potential benefits in terms of stabilization speed. By removing the constraint on λ , the range of possibilities has been expanded and opened up new avenues for research in the field of stability and synchronization. This approach allows researchers to explore scenarios with faster convergence rates, potentially leading to more efficient and practical applications in various domains.*

Remark 6.3.2. *The presented results yield interesting and having significant implications. Specifically, when the external inputs are set to zero ($u_i(t) = 0$), and the parameters of the system satisfy the conditions outlined in Theorem 6.3.1, it can be concluded that the equilibrium point $r = 0$ of the QVINN model (6.1) is globally asymptotically stable. This finding implies that the system will converge to the equilibrium point for any initial condition under these circumstances. Furthermore, if the system parameters fulfil the conditions specified in Theorem 6.3.2, the equilibrium point $r = 0$ exhibits global exponential stability. This means that the system converges to the equilibrium point and at an exponential rate, ensuring a faster and more*

robust stabilization. Hence, this chapter proposes a selective method for examining the Lyapunov stability of QVNNs. Identifying the conditions for global asymptotic and exponential stability, a valuable approach is given to analyze and understand the stability properties of QVNN models. These results have practical implications and contribute to the advancement of stability analysis in the field of NNs.

Remark 6.3.3. *This chapter introduces a novel perspective by considering the dissipativity of QVINNs, representing a broader concept of stability beyond the traditional Lyapunov stability. Unlike Lyapunov stability, dissipativity focuses on the dynamics of the entire system rather than solely on equilibrium points. It considers the behaviour of the system's orbits, which may not necessarily approach equilibrium points in some instances. Furthermore, dissipativity is applicable even in situations where certain networks do not possess equilibrium points at all. This characteristic significantly broadens the applicability scope for dissipativity as compared to the traditional stability analysis methods. By incorporating dissipativity analysis, the chapter expands the understanding and evaluation of stability in QVINNs, considering the system dynamics beyond equilibrium points. This approach offers valuable insights and a more comprehensive understanding of the behaviour and properties of QVINNs.*

Remark 6.3.4. *In addition to dissipativity analysis, this chapter presents a detailed estimation of the global attractive set for QVINNs. This estimation is highly valuable as it simplifies the study of QVNN dynamics to a significant extent. It enables researchers to investigate the dynamics within the identified global attractive set. The concept of the global attractive set, as explained by [166], encompasses all equilibriums, periodic solutions, and chaos attractors that exist within the QVINN systems. By confining the analysis to this attractive set, the researchers can comprehensively understand the system's dynamics without needing to explore regions*

outside this set. Consequently, the chapter provides an effective and practical approach to examining the dynamics of QVINNs. Researchers can concentrate their efforts on understanding and analyzing the behaviours occurring within the globally attractive set.

Remark 6.3.5. *The proof derivation process in Theorem 6.3.1 and Theorem 6.3.2 are relatively straightforward due to the utilization of the direct method. This approach involves the application of two inequalities and standard operations with real numbers. By employing this direct method, the proof's complexity is significantly reduced as compared to the separation technique. One notable advantage of the direct method is that it avoids decomposing the quaternion system into four real-valued systems. This decomposition process is typically required when utilizing the separation technique, which increases the computational complexity of the analysis. By circumventing this decomposition step, the direct method offers a more streamlined and efficient approach to provide the desired results. The simplicity and computational efficiency of the direct method make it an attractive choice for analyzing the stability and dissipative properties of QVINNs. By minimizing complexity and reducing computational burden, this method allows for more efficient analysis and facilitates the practical application of the derived results.*

Remark 6.3.6. *Up to this point, numerous outcomes have been obtained regarding the global dissipativity of QVNNs. These outcomes have been established for scenarios involving either constant delay or bounded delays, as referenced in [160, 134, 152, 161, 153, 167]. In contrast to these existing studies, the present chapter focuses on the global dissipativity of QVNNs with inertial terms. Specifically, it considers QVINNs with unbounded time delays and employs a non-separation and non-reduction order approach deemed more realistic and practical.*

Remark 6.3.7. Based on the conditions outlined in Theorems 6.3.1- 6.3.2 and Corollaries 6.3.1- 6.3.6, it can be concluded that the zero solutions of the model described by the equation (6.1), with $u_1(t) = u_2(t) = 0$, will demonstrate global attractiveness and global exponential attractiveness, respectively.

6.4 Numerical Examples

Example 6.4.1. The QVINNs with time delays in a two-dimensional setting are to be thought of as

$$\begin{aligned} \frac{d^2 r_i(t)}{dt^2} = & -\alpha_i \frac{dr_i(t)}{dt} - \beta_i r_i(t) + \sum_{j=1}^n a_{ij} f_j(s_j(t)) + \sum_{j=1}^n b_{ij} g_j(s_j(t - \tau_j(t))) \\ & + \sum_{j=1}^n d_{ij} \int_{t-\sigma_j(t)}^t h_j(r_j(s)) ds + u_i(t), \quad i \in I, \end{aligned} \quad (6.12)$$

where the parametric values are $\alpha_1 = 10, \alpha_2 = 11, \beta_1 = 15, \beta_2 = 16, \delta_1 = 12, \delta_2 = 15, \gamma_1 = \gamma_2 = 0.5, \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 = 1, f_1 = f_2 = \frac{1}{2} \tanh(r(t)), g_1 = g_2 = \frac{1}{2} \tanh(r(t)), \tau_1(t) = \tau_2(t) = \frac{t}{2}, u_1(t) = 0.1 \sin(t) + 0.6 \sin(t)i - 1.2 \sin(t)j + 0.2 \cos(t)k, u_2(t) = 1.7 \sin(t) - 0.5 \sin(t)i + 0.7 \sin(t)j - 0.4 \cos(t)k$

Now consider the weight connection matrices as

$$\begin{aligned} [a]_{2 \times 2} &= \begin{pmatrix} 0.5 + 1.2i - 0.4j + 0.6k & 0.6 + 0.9i + 0.6j + k \\ 1.1 + 0.9i - 0.5j + 0.8k & 0.7 - 1.2i - 0.9j + 0.5k \end{pmatrix}, \\ [b]_{2 \times 2} &= \begin{pmatrix} 0.8 + i - 0.9j - 0.5k & 0.9 - 1.1i - j + 0.1k \\ 0.7 - 0.8i + 1.1j - 1.2k & 1 - 0.5i - 1.4j - k \end{pmatrix}. \\ [d]_{2 \times 2} &= \begin{pmatrix} -0.25 - 0.2i + 0.3j + 0.12k & 0.1 - 0.3i - 0.5j - 0.1k \\ 0.15 + 0.25i - 0.2j - 0.3k & -0.1 + 0.2i - 0.4j - 0.5k \end{pmatrix}. \end{aligned}$$

Case I

When distributive delays are considered, the model becomes more general and complex. In the context of neural networks, distributive delays refer to delays that are not confined to a specific connection but spread out or distributed across the network. This more comprehensive model accounts for a broader range of temporal interactions within the system, making it more applicable to real-world scenarios where delays are not instantaneous. In summary, while the model with distributive delays is more general and realistic, it is also more complex, requiring advanced mathematical techniques for analysis and interpretation. Then, by simple computations, one can get

$L_1 = L_2 = M_1 = M_2 = \frac{1}{2}, K_1 = K_2 = 1, \tau = \frac{1}{2}, \epsilon^* = 2, \kappa = 1.0936$ and $Z_1 = -12 < 0, Y_1 = -10.118, X_1 = -15.32571$. It is obvious that $Z_1^2 = 144 < Y_1 X_1 = 155.065$ also $Z_2 = -11 < 0, Y_2 = -10.9312, X_2 = -13.8787$ and therefore, $Z_2^2 = 121 < Y_2 X_2 = 151.7120$. Hence Theorem 6.3.1 is satisfied, and the considered system (6.12) is globally dissipative, and the set $\Upsilon_1 = \left\{ r_1(t), r_2(t) \in \mathbf{H}^2 \mid \|r_1(t)\| \leq 1.819, \|r_2(t)\| \leq 2.6326 \right\}$ is a globally attractive set and a positive invariant set of (6.12), for the initial conditions $\{\Psi_1(s), \Psi_2(s), \Psi_1^*(s), \Psi_2^*(s)\} = \{0.1, 0.1, -0.8, 0.3\}$.

Case II

When contemplating the scenario where $d_{ij} = 0$, it implies that there are no distributive delays in the considered model. This condition simplifies the model by eliminating the contribution of distributive delays between the nodes. Then, by simple computations, one can get $L_1 = L_2 = M_1 = M_2 = \frac{1}{2}, \tau = \frac{1}{2}, \epsilon^* = 2, \kappa = 1.432$ and $Z_1 = -12 < 0, Y_1 = -10.26, X_1 = -15.4675$. It is obvious that $Z_1^2 = 144 < Y_1 X_1 = 158.6965$ also $Z_2 = -11 < 0, Y_2 = -11.1, X_2 = -14.0475$ and therefore,

$Z_2^2 = 121 < Y_2 X_2 = 155.927$. Hence Corollary 6.3.1 is satisfied, and the considered system (6.12) is globally dissipative, and the set $\Upsilon_1 = \left\{ r_1(t), r_2(t) \in \mathbf{H}^2 \mid \|r_1(t)\| \leq 1.5899, \|r_2(t)\| \leq 2.2516 \right\}$ is a global attractive set and a positive invariant set of (6.12). For the initial conditions $\{\Psi_1(s), \Psi_2(s), \Psi_1^*(s), \Psi_2^*(s)\} = \{0.1, 0.1, -0.8, 0.3\}$. Figures 6.1 and 6.2 show the phase trajectories of the system (6.12), Figures 6.3 and 6.4 depict the time changes of the state variables $r_1(t), r_2(t)$. Figures 6.5 and 6.6 show the global attractiveness of the zero solution for the equation (6.12), which also validates the accuracy of Remark 6.3.7.

Example 6.4.2. The two-dimensional QVINNs with time delays are to be thought of as

$$\frac{d^2 r_i}{dt^2} = -\alpha_i \frac{dr_i}{dt} - \beta_i r_i(t) + \sum_{j=1}^n a_{ij} f_j(s_j(t)) + \sum_{j=1}^n b_{ij} g_j(s_j(t - \tau_j(t))) \quad (6.13)$$

$$+ \sum_{j=1}^n d_{ij} \int_{t-\sigma_j(t)}^t h_j(r_j(s)) ds + u_i(t), \quad i \in I, \quad (6.14)$$

where the parametric values are $\lambda = 1, \alpha_1 = 12, \alpha_2 = 14, \beta_1 = 17, \beta_2 = 19, \delta_1 = 13, \delta_2 = 15, f_1 = f_2 = \frac{1}{2} \tanh(r_j(t)),$
 $g_1 = g_2 = \frac{1}{2} \tanh(r_j(t)) e^{-2\tau_j(t)}, \quad \tau_1(t) = \tau_2(t) = 0.5 \ln(1+t), \sigma = 2, u_1(t) =$
 $-1.2 \sin(t) + 0.9 \cos(t)i + 0.9 \cos(t)j - 0.5 \sin(t)k, \quad u_2(t) = 0.8 \sin(t) - 0.7 \sin(t)i +$
 $0.3 \sin(t)j - 0.9 \cos(t)k.$

Now consider the weight connection matrices as

$$[a]_{2 \times 2} = \begin{pmatrix} 0.51 - 0.35i - 0.6j - 0.1k & 0.2 + 0.5i - 0.7j - 0.2k \\ 0.25 - 0.1i - 0.4j + 0.6k & 0.3 - 0.25i - 0.5j - 0.6k \end{pmatrix},$$

$$[b]_{2 \times 2} = \begin{pmatrix} 0.5 - 0.2i + 0.3j - 0.7k & -0.1 - 0.4i + 0.6j - 0.9k \\ 0.3 - 0.35i - 0.5j + 0.6k & -0.7 - 0.3i + 0.5j + 0.6k \end{pmatrix}.$$

$$[d]_{2 \times 2} = \begin{pmatrix} -0.1 - 0.3i - 0.5j + 0.2k & -0.7 + 0.3i + 0.4j - 0.6k \\ -0.2 + 0.5i + 0.1j - 0.2k & -0.1 + 0.4i - 0.8j - 0.2k \end{pmatrix}.$$

Case I

When distributive delays are considered, the model becomes more general. Then, by simple computations, one can get $M_1 = M_2 = L_1 = L_2 = \frac{1}{2}$, $K_1 = K_2 = 1$, $\tau = \frac{1}{2}$, $\sigma = 0.25$, $\dot{\sigma} = 0.5$, $\epsilon^* = 2$, $\kappa_1 = \min\{0.4894, 1.4519\} = 0.4894$, $Z_1 = -14 < 0$, $Y_1 = -16.32975 < 0$, $X_1 = -12.11975$, then it is obvious that $Z_1^2 = 196 < Y_1 X_1 = 197.9124$. Similarly, $Z_2 = -16 < 0$, $Y_2 = -20.3475 < 0$, $X_2 = -13.525 < 0$, which shows that $Z_2^2 = 256 < Y_2 X_2 = 275.1999$. Hence conditions of Theorem 6.3.2 are satisfied, and the considered system (6.14) is globally exponential dissipative. The set $\Upsilon_1^* = \left\{ r_1(t), r_2(t) \in \mathbf{H}^2 \mid \|r_1(t)\| \leq 2.9084, \|r_2(t)\| \leq 2.232 \right\}$ is a global attractive set and a positive invariant set of (6.14) for the initial conditions $\{\Psi_1(s), \Psi_2(s), \Psi_1^*(s), \Psi_2^*(s)\} = \{0.2, 0.3, 0.3, -0.5\}$.

Case II

When contemplating the scenario where $d_{i_j} = 0$, it implies that there are no distributive delays in the considered model, this condition simplifies the model by eliminating the contribution of distributive delays between the nodes. Then, by simple computations, one can get $M_1 = M_2 = L_1 = L_2 = \frac{1}{2}$, $\tau = \frac{1}{2}$, $\epsilon^* = 2$, $\kappa_1 = \min\{1.4324, 3.1465\} = 1.4324$, $Z_1 = -14 < 0$, $Y_1 = -16.7025 < 0$, $X_1 = -12.492$, then it is obvious that $Z_1^2 = 196 < Y_1 X_1 = 208.656$. Similarly, $Z_2 = -16 < 0$, $Y_2 =$

$-20.645 < 0$, $X_2 = -13.822 < 0$, which shows that $Z_2^2 = 256 < Y_2 X_2 = 285.365$. Hence conditions of Corollary 6.3.4 are satisfied, and the considered system (6.14) is globally exponential dissipative. The set $\Upsilon_1^* = \left\{ r_1(t), r_2(t) \in \mathbf{H}^2 \mid \|r_1(t)\| \leq 1.5360, \|r_2(t)\| \leq 1.3471 \right\}$ is a global attractive set and a positively invariant set of (6.14). For the initial conditions $\{\Psi_1(s), \Psi_2(s), \Psi_1^*(s), \Psi_2^*(s)\} = \{0.2, 0.3, 0.3, -0.5\}$, Figures 6.7 and 6.8 are the phase trajectory of the system (6.14), Figures 6.9 and 6.10 are the time changes of the state variables $r_1(t), r_2(t)$. Figures 6.11 and 6.12 depict the global exponential attractiveness of the zero solution for equation (6.14), which also validates the correctness of Remark 6.3.7.

Example 6.4.3. To illustrate the application of traditional QVNNs, let's focus on the example of a 12×12 pixel image pattern denoted as "T," and its corresponding color image representation is shown in Figure 6.13. In this context, a set of QVNNs in the format of (6.15) are configured, comprising 144 neurons. These QVNNs possess 144-dimensional equilibrium points, which facilitate the storage of colored "T" patterns. Therefore, to proceed, let's now consider the traditional QVNNs as

$$\frac{dr_i}{dt} = -\beta_i r_i(t) + \sum_{j=1}^n a_{ij} f_j(r_j(t)) + \sum_{j=1}^n b_{ij} g_j(r_j(t - \tau_j(t))) + u_i, \quad i \in I, \quad (6.15)$$

where $\beta_j > 0$, and a_{ij} and b_{ij} represent weight connection matrices, while u_i denotes the external input of (6.15). The chosen parameter values for the QVNNs defined by equation (6.15) are taken as

$$\beta_j = 1, \quad (6.16)$$

$$a_{ij} = \begin{cases} 4.0 + 4.0 \times 10^{-1}i - 3.0 \times 10^{-1}j + 5.0 \times 10^{-1}k, & i = j \\ 4.0 \times 10^{-1} - 5.0 \times 10^{-1}i + 5.0 \times 10^{-1}j - 3.0 \times 10^{-1}k, & i \neq j, \end{cases} \quad (6.17)$$

$$b_{\iota j} = \begin{cases} -2.0 \times 10^{-1} + 2.0 \times 10^{-1}i - 5.0 \times 10^{-1}j + 4.0 \times 10^{-1}k, & \iota < j \\ 2 + 3.0 \times 10^{-1}i + 2.0 \times 10^{-1}j - 3.0 \times 10^{-1}k, & \iota = j \\ -1.0 \times 10^{-1} + 2.0 \times 10^{-1}i + 3.0 \times 10^{-1}j - 5.0 \times 10^{-1}k, & \iota > j, \end{cases} \quad (6.18)$$

$$f_j^x(r) = \tanh(r), \quad (6.19)$$

where $\iota, j = 1, 2, \dots, 144$ and $x = 0, 1, 2, 3$. In order to recall the image pattern "T," The equilibrium point of the designed QVNNs should be $r = (r_1, r_2, \dots, r_{144})^T \in \mathbf{H}^{144}$ where $r_1 = 0 + 15.0 \times 10^{-2}i + 3.0 \times 10^{-1}j + 15.0 \times 10^{-2}k$, $r_2 = 0 + 15.0 \times 10^{-2}i + 3.0 \times 10^{-1}j + 14.0 \times 10^{-2}k$, ..., $r_{144} = 0 + 15.0 \times 10^{-2}i + 0j + 0k$, which correspond to the color $(15.0 \times 10^{-2}, 3.0 \times 10^{-1}, 0.150)$, $(15.0 \times 10^{-2}, 3.0 \times 10^{-1}, 14.0 \times 10^{-2})$, ..., $(15.0 \times 10^{-2}, 0, 0)$ of the pixels in color image pattern "T". Figure 6.14 shows a simulation with random initial values. Based on the equilibrium points r , calculate the external input parameter u as

$$U = (u_1, u_2, \dots, u_{144}) \in \mathbf{H}^{144},$$

where $u_1 = -628.0 \times 10^{-1} + 2275.0 \times 10^{-2}i - 905.0 \times 10^{-1}j - 725.0 \times 10^{-2}k$, $u_2 = -63 + 2095.0 \times 10^{-2}i - 905.0 \times 10^{-1}j - 566.0 \times 10^{-2}k$, ..., $u_{144} = -914.0 \times 10^{-1} - 2346.0 \times 10^{-1}i - 908.0 \times 10^{-1}j + 2214.0 \times 10^{-1}k$. Owing to spatial constraints, only three elements of r and u have been listed. One simulation result, conducted with random initial data, is presented in Figure 6.14. The parameters extracted from equations (6.16)-(6.19) indicate that the examined system (6.15) possesses the capability to recover the aforementioned "T" pattern consistently.

Remark 6.4.1. Example 6.4.3 demonstrates that the proposed system (6.15), defined by equations (6.16)-(6.19) for its parametric values, necessitates the use of 144.00 neurons to store a 12×12 pixel image pattern. In contrast, according to the findings

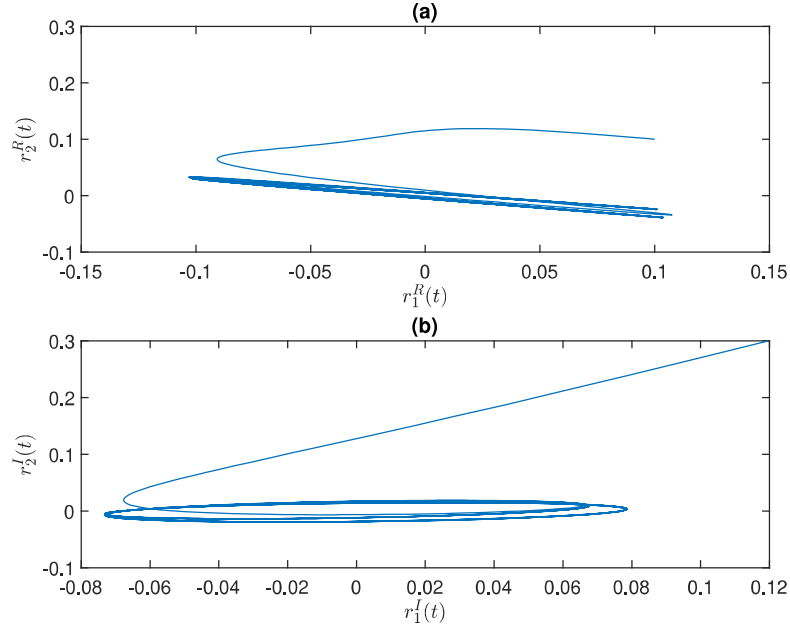


FIGURE 6.1: (a) Phase plots of state variables r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (6.12).

in the article [145], a comparable image stored in a CVNN would require 432 neurons, a significantly larger quantity. This distinction highlights the greater storage capacity of QVNNs over CVNNs. This chapter deals specifically with QVNNs employing the 'tanh' activation functions. Remarkably, the image "T" reconstruction can be approximated to occur within a time span of $t = 2.5$.

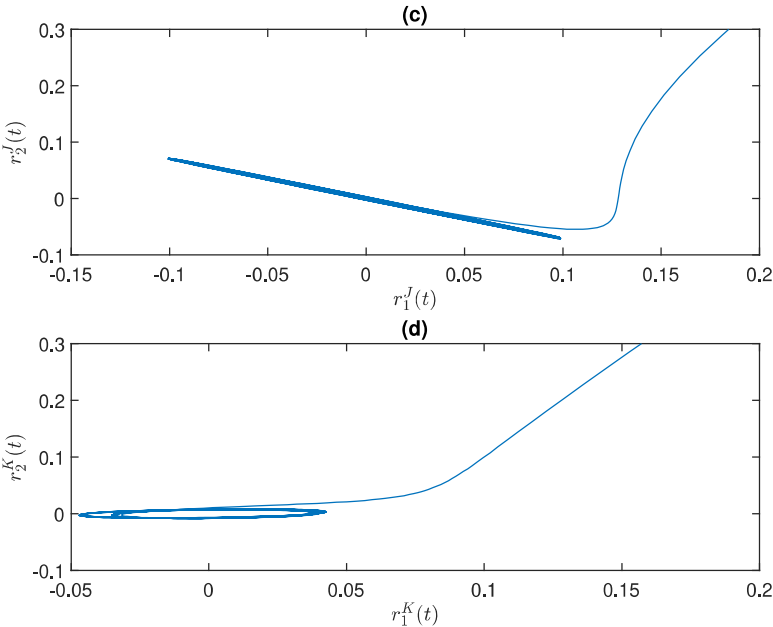


FIGURE 6.2: (c) Phase plots of state variables r_1^I, r_2^I and (d) r_1^K, r_2^K of the system (6.12).

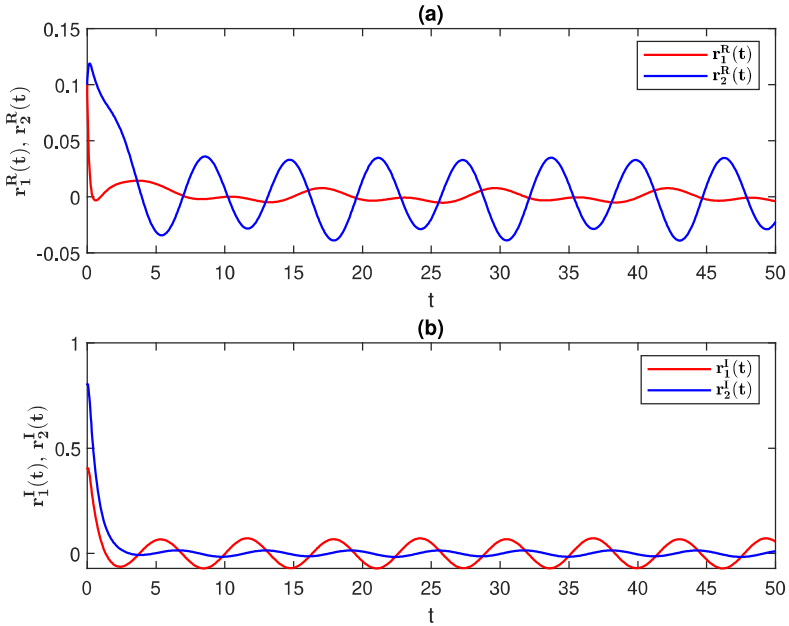


FIGURE 6.3: (a), (b) Time response of state variables r_1^R, r_2^R and r_1^I, r_2^I of the system (6.12) respectively.

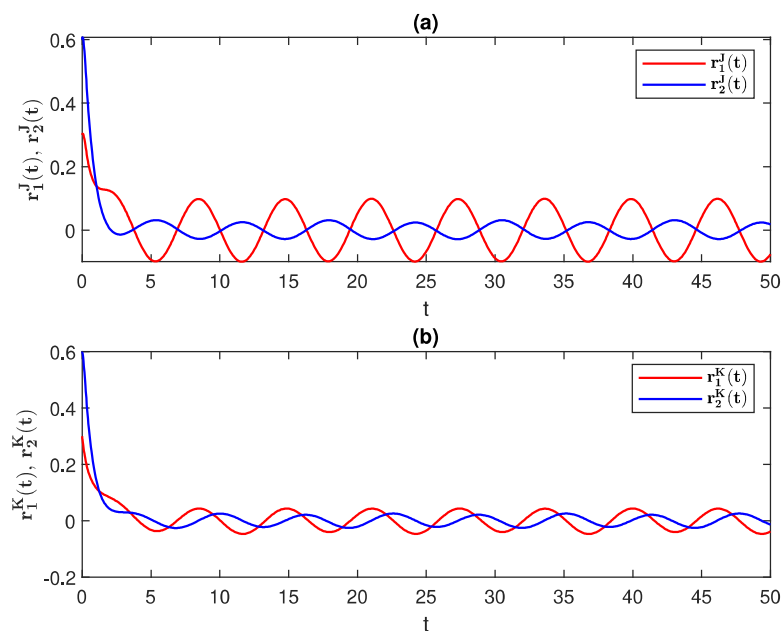


FIGURE 6.4: (a), (b) Time response of state variables r_1^J, r_2^J and r_1^K, r_2^K of the system (6.12) respectively.

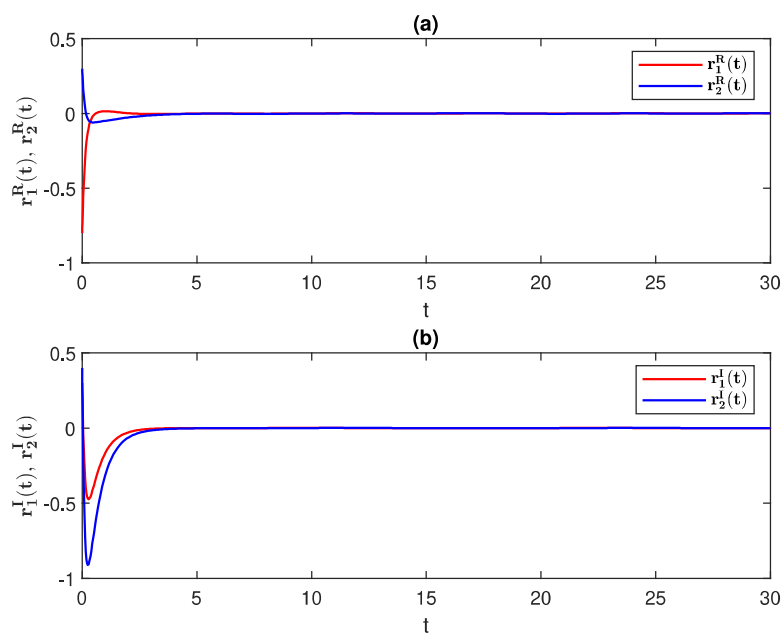


FIGURE 6.5: (a), (b) Time response of state variables r_1^R, r_2^R and r_1^I, r_2^I of the system (6.12) with u_1^R, u_2^R, u_1^I and $u_2^I = 0$.

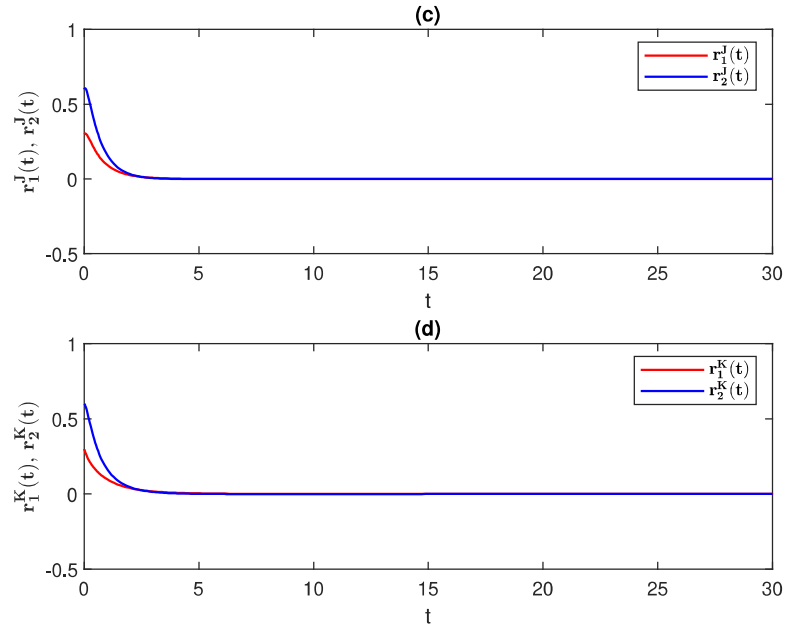


FIGURE 6.6: (a), (b) Time response of state variables r_1^J, r_2^J and r_1^K, r_2^K of the system (6.12) with u_1^J, u_2^J, u_1^K and $u_2^K = 0$.

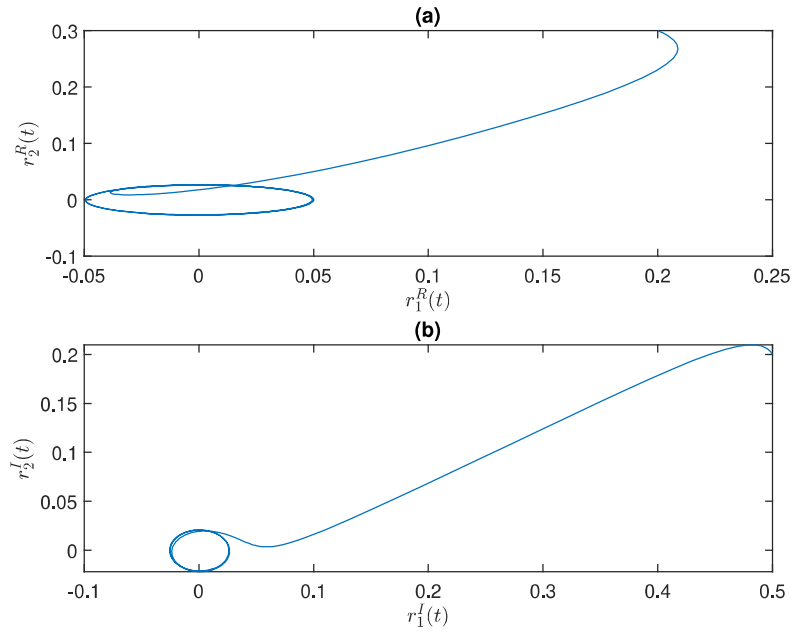


FIGURE 6.7: (a) Phase plots of state variables r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (6.14).

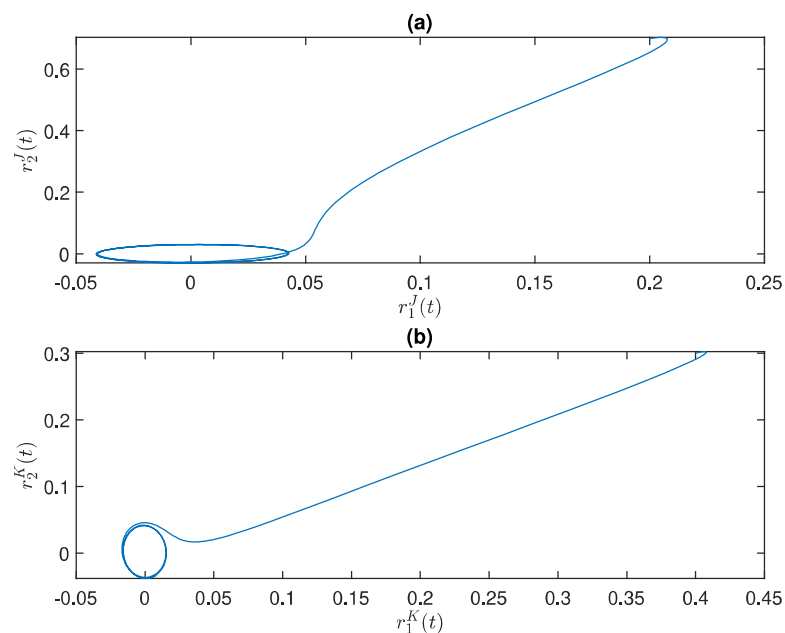


FIGURE 6.8: (c) Phase plots of state variables r_1^J, r_2^J and (d) r_1^K, r_2^K of the system (6.14).

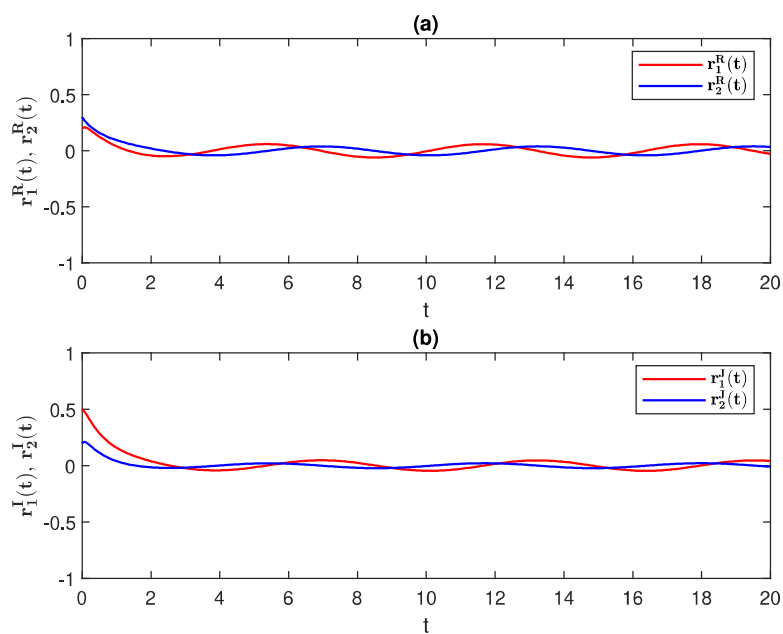


FIGURE 6.9: (a), (b) Time response of state variables r_1^R, r_2^R and r_1^I, r_2^I of the system (6.14) respectively.

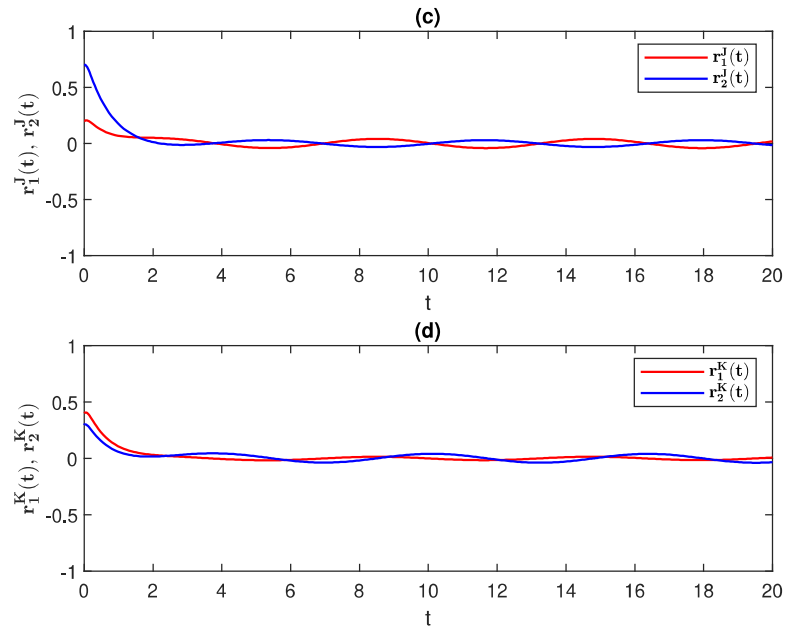


FIGURE 6.10: (c), (d) Time response of state variables r_1^J, r_2^J and r_1^K, r_2^K of the system (6.14) respectively.

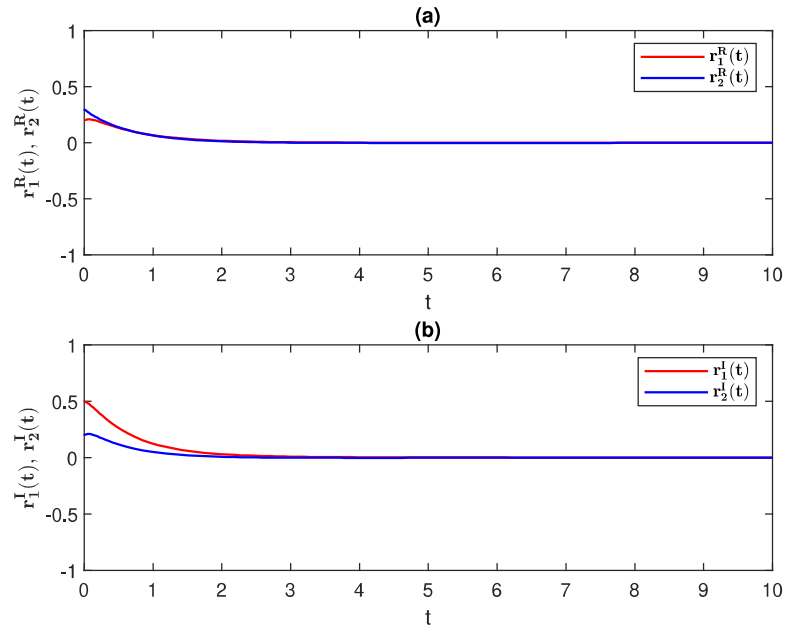


FIGURE 6.11: (a), (b) Time response of state variables r_1^R, r_2^R and r_1^I, r_2^I of the system (6.12) with u_1^R, u_2^R, u_1^I and $u_2^I = 0$.

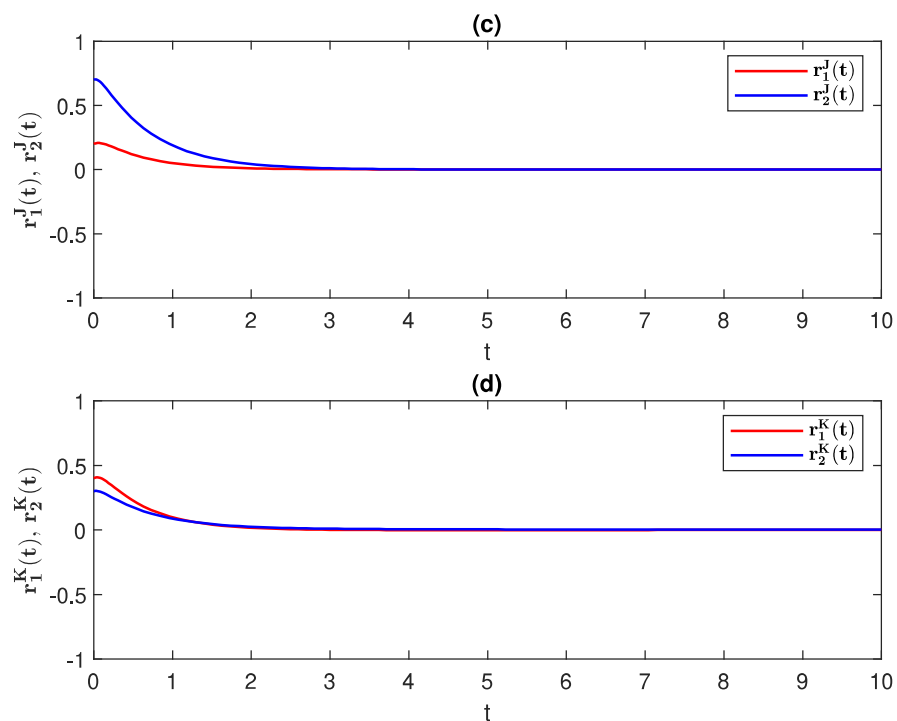


FIGURE 6.12: (c), (d) Time response of state variables r_1^J, r_2^J and r_1^K, r_2^K of the system (6.12) with u_1^J, u_2^J, u_1^K and $u_2^K = 0$.

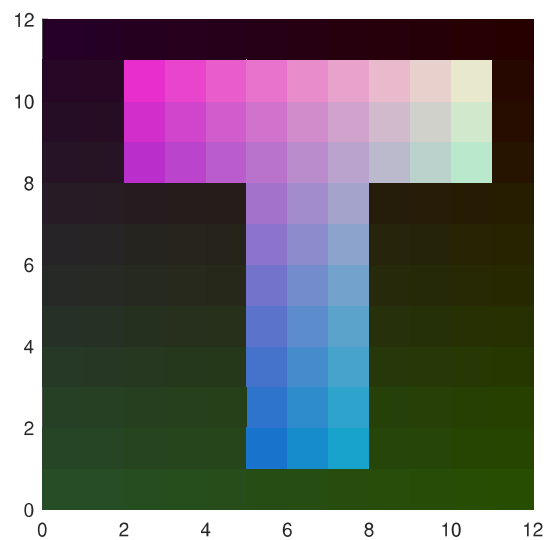


FIGURE 6.13: Original color image of pattern "T."

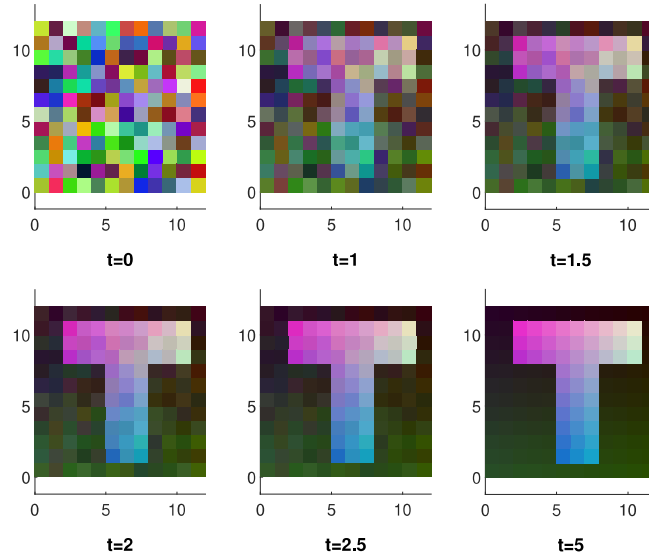


FIGURE 6.14: Simulation of retrieving image "T" with random initial values of time t .

6.5 Conclusion

In this chapter, the global dissipativity and global exponential dissipativity of QVINNs with unbounded time-varying delays have been comprehensively examined. The Lyapunov function and inequality approaches have been employed to investigate these properties. The focus is on the without-separation approach of QVINNs and the non-reduction order method for inertial terms, providing a novel perspective for analyzing these complex systems. The first contribution of this chapter, as presented in Theorem 6.3.1, pertains to the QVINNs. By utilizing inequality techniques, a global attractive set for the system (6.1) is established. Additionally, three corollaries were derived to refine and extend the findings of the theorem. Moving on to Theorem 6.3.2, the global exponential dissipativity and global exponential attractive set of QVINNs have been investigated. This Theorem highlights the unbounded nature of the convergence rate, which further enhances the understanding of the system's stability properties. Three corollaries have been proposed to provide additional insights

based on these results. Furthermore, the chapter has presented several algebraic criteria for assessing the global dissipativity of QVINNs with unbounded time-varying delays. This consideration of unbounded time delays is particularly relevant as it aligns with real-world scenarios more accurately than bounded delays. To illustrate the practical implications of the proposed results, three consecutive examples have been provided. The first two examples, which satisfy the conditions outlined in the first two Theorems and Corollaries, demonstrate the applicability of the derived criteria. The third example, showcasing the potential of QVNNs in true color image retrieval, reveals the versatility and practical utility of the NNs. Overall, this chapter contributes to the field of QVINNs by offering novel insights into global dissipativity and exponential dissipativity, providing algebraic criteria, and demonstrating their practical applications through illustrative examples.
