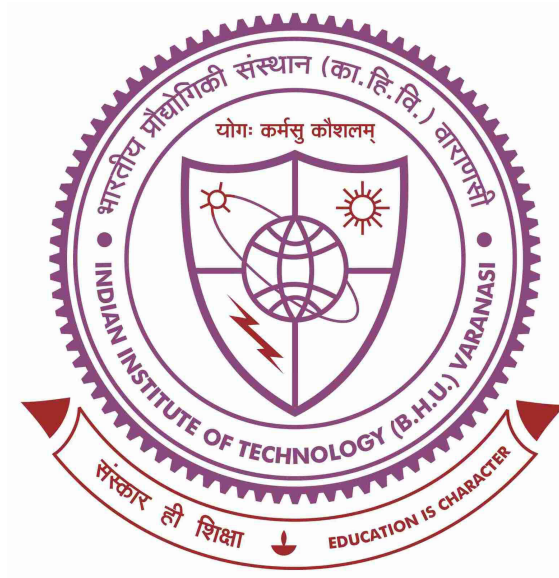


**STABILITY ANALYSIS OF APPROXIMATION  
TECHNIQUES AND COMPUTATIONAL ALGORITHMS  
FOR GENERALIZED FRACTIONAL ORDER  
MATHEMATICAL MODELS**



Thesis submitted in partial fulfillment for the  
Award of Degree

**DOCTOR OF PHILOSOPHY**

By

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*Dedicated to*  
*My lovely Parents*  
*“Mr. Ravi Kedia”*  
*∞*  
*“Mrs. Rani Kedia”*







## CERTIFICATE

It is certified that the work contained in this thesis titled “Stability Analysis of Approximation Techniques and Computational Algorithms for Generalized Fractional Order Mathematical Models” by *Nikki Kedia* has been carried out under our joint supervision and co-supervision, and this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all the requirements of Comprehensive Examination, Candidacy and SOTA for the award of Ph.D. degree.

  
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## DECLARATION BY THE CANDIDATE

I, **Nikki Kedia**, certify that the work embodied in this thesis is my own bona fide work and carried out by me under the supervision of **Dr. Vineet Kumar Singh** and co-supervision of **Prof. Anatoly A. Alikhanov** from **January 2019** to **December 2023** at the Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not wilfully copied any other's work, paragraphs, text, data, results, etc., reported in journals, books, magazines, reports, dissertations, thesis, etc., or available at websites and have not included them in this thesis and have not cited as my own work.


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