Analysis of Discretization Methods for Time-Fractional Diffusion and Diffusion-Wave Equations with Weak Initial Singularity

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by

SARITA KUMARI

DEPARTMENT OF MATHEMATICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY

(BANARAS HINDU UNIVERSITY)

VARANASI-221005

Roll Number: 19121022

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Chapter 6

Conclusions and Future Scopes

In this chapter, we will summarize the conclusions of each chapter of this thesis and suggest potential future research possibilities of the work presented over here.

6.1 Conclusions

A primary feature of fractional derivatives in mathematics is their non-local behavior, leading to significant storage demands when numerically addressing fractional differential equations. Despite the availability of high-performance computers, conducting long-term simulations remains challenging. However, numerous studies in the literature demonstrate that, unlike integer-order derivatives, the numerical approximation of fractional differential operators can not be arbitrarily reduced using the single-step method. Certainly, developing an efficient and high-order numerical method for single-term/multi-term time-fractional diffusion-wave equations represents an effective strategy for tackling this challenge. This marks the exact point of our current research work presented through the chapters of this thesis.

Chapter 1 represents the introduction of frictional calculus with some basic definitions of fractional operators. There is an extensive literature review on time-fractional diffusion and diffusion-wave equations is included in this chapter. The chapter further explained the objective and motivation of the current research work.

In Chapter 2, we proposed two different schemes in solving the two-dimensional TFDW equation via ADI approach. To approximate the TFCD of order α $(1 < \alpha < 2)$ in equation (2.1), two approximation techniques were proposed: the Nonuniform $L1$ and the Nonuniform Crank-Nicolson $L1-2$ method. The outlined $L1$ and $L1-2$ difference schemes have OC two in the space direction and in the time direction, which are $\min(3-\alpha, \gamma\alpha)$ and second-order, respectively. The bounds on the local truncation errors were displayed for both approximation techniques. In order to solve the TFDWE with non-smooth exact solutions that have a weak singularity at $t = 0$, numerical approaches have an advantage because they include a nonuniform mesh. Finally, numerical results confirm the theoretical findings.

Chapter 3, we presented a nonuniform $L1$ approximation technique based on half point discretization for TFCDs of order $\beta \in (0, 1)$ and $\alpha \in (1, 2)$. To get a system of equations for TFMDWE, the proposed $L1$ method is used for time derivatives and central difference approximation in space direction. According to the theoretical analysis, the obtained scheme has a second order convergence rate in the space direction and in time $\min(2-\beta, 3-\alpha, \gamma\beta, \gamma(\alpha-1))$. The existing numerical technique deals with the weak singularity at $t = 0$, that is, when the governing problem (3.1)-(3.3) has an exact solution that is not enough smooth. In order to represent both smooth and non-smooth exact solutions of the considered problem, we provided numerical examples with both zero and non-zero boundary conditions.

In Chapter 4, we developed and examined two nonuniform discretization methods for computing numerical solutions of both single-term and multi-term nonlinear TFDW equation having variable coefficients. We utilized the nonuniform $L1$ method to approximate the Caputo derivative, alongside employing the central difference formula for approximating spatial derivatives. Subsequently, the provided model described by equations (4.1)-(4.3) was transformed into an equivalent system of equations. The truncation error bound is also shown. We described several experiments involving discontinuous initial conditions as well as smooth and nonsmooth exact solutions to achieve optimal accuracy with the current schemes of this chapter.

Chapter 5 deals with an high-order approximation of the GCFD of order $\beta \in (0,1)$. The characteristics of discretized coefficients in approximation of GCFD are examined. Order of accuracy for the difference scheme is $\mathcal{O}(\tau^{4-\beta} + h^2)$ where step-sizes τ in time and h in space direction. Moreover, the proposed scheme cab be directly employed to obtain the other time-fractional advection-diffusion equations for the different choices of scale $\zeta(t)$ and weight functions $\omega(t)$.

6.1.1 Scope for Future Work

In this section, we will discuss some future problems in the direction of the work presented in this thesis.

- In this thesis, we have provided linear interpolation polynomials to approximation of the fractional derivatives of order $\alpha \in (1,2)$ on nonuniform meshes. Can we extend this approach of discretization utilizing higher-degree interpolation polynomials to get high-order accuracy?
- For the spatial derivative discretization, we have used central difference approximation. We can apply Finite element method (FEM) instead of FDM and can verify CPU time or accuracy after using FEM.

- The lack of second or higher-order finite difference or finite element methods for the time-fractional linear/nonlinear diffusion-wave equation with nonsmooth solutions is still challenging and need attention to tackle this issue.

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