Chapter 3

Quasi-Projective synchronization of non-identical time-varying delayed quaternion-valued neural networks with interaction terms: Direct method

3.1 Introduction

This chapter is concerned with the quasi-projective synchronization (QPS) of non-identical QVNNs with time-varying delays and interaction terms. Because of parameter mismatches between the drive and response systems, complete projective synchronization is not possible. The direct method is used to derive several general criteria to achieve QPS. The Lyapunov stability theory is employed to estimate the error bound. The numerical simulation results are presented graphically for various special cases to demonstrate the synchronization of time-varying delayed QVNNs with mismatched parameters and interaction terms, which confirm the efficiency of our presented research results. The second example showcases the application of QVNNs coupled with associative memory to illustrate its efficacy in accurately restoring original color image patterns.

3.2 Model description and preliminaries

The drive system for QVNN model is defined as

$$\begin{aligned}\dot{\epsilon}_{\vartheta}(\mathbf{t}) = &- a_{\vartheta}\epsilon_{\vartheta}(\mathbf{t}) + \sum_{\sigma=1}^{n} b_{\vartheta\sigma} f_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) + \sum_{\sigma=1}^{n} d_{\vartheta\sigma} g_{\sigma}(\epsilon_{\sigma}(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_{\sigma}(\tilde{\epsilon}_{\sigma}(\mathbf{t})) \\ &+ r_{\vartheta}(\mathbf{t}), \qquad (3.1)\end{aligned}$$

which can be written in the matrix form as

$$\dot{\epsilon}(\mathbf{t}) = -A\epsilon(\mathbf{t}) + Bf(\epsilon(\mathbf{t})) + Dg(\epsilon(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 W h(\tilde{\epsilon}(\mathbf{t})) + R(\mathbf{t}),$$

where $\epsilon(\mathbf{t}) = (\epsilon_1(\mathbf{t}), \epsilon_2(\mathbf{t}), \ldots, \epsilon_n(\mathbf{t}))^T \in \mathbb{Q}^n$ is the state vector of the neuron at time t, $A = diag(a_1, a_2, \ldots, a_n) \in \mathbb{R}^{n\times n}$ with $a_{\vartheta} > 0$ denotes the self-feedback connection matrix, $B = (b_{\vartheta\sigma})_{n\times n}, D = (d_{\vartheta\sigma})_{n\times n} \in \mathbb{Q}^{n\times n}$ are the connection weight matrix, $W = (w_{\vartheta\sigma})_{n\times n}$ is the interaction structures and x_1 is the outer interaction strength, with and without time varying delays activation functions are represented by $f(\epsilon(\mathbf{t})) = (f_1(\epsilon(\mathbf{t})), f_2(\epsilon(\mathbf{t})), \ldots, f_n(\epsilon(\mathbf{t})))^T \in \mathbb{Q}^n$, $g(\epsilon(\mathbf{t})) = (g_1(\epsilon(\mathbf{t} - \tau_0(\mathbf{t}))), g_2(\epsilon(\mathbf{t} - \tau_0(\mathbf{t}))), \ldots, g_n(\epsilon(\mathbf{t} - \tau_0(\mathbf{t}))))^T \in \mathbb{Q}^n$, respectively, and $h(\tilde{\epsilon}(\mathbf{t})) = (h_1(\tilde{\epsilon}(\mathbf{t})), h_2(\tilde{\epsilon}(\mathbf{t})), \ldots, h_n(\tilde{\epsilon}(\mathbf{t})))^T \in \mathbb{Q}^n$ is interaction function. $R(\mathbf{t}) = (r_1(\mathbf{t}), r_2(\mathbf{t}), \ldots, r_n(\mathbf{t}))^T \in \mathbb{Q}^n$ is the external input vector.

The response system for QVNN model is defined as

$$\dot{\tilde{\epsilon}}_\vartheta(\mathbf{t}) = - a'_\vartheta \tilde{\epsilon}_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b'_{\vartheta\sigma} f_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma} g_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 \sum_{\sigma=1}^{n} w'_{\vartheta\sigma} h_\sigma(\epsilon_\sigma(\mathbf{t})) + r'_\vartheta(\mathbf{t}) + l_\vartheta(\mathbf{t}), \tag{3.2}$$

and in the matrix form as

$$\dot{\tilde{\epsilon}}(\mathbf{t}) = -A'\tilde{\epsilon}(\mathbf{t}) + B'f(\tilde{\epsilon}(\mathbf{t})) + D'g(\tilde{\epsilon}(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 W' h(\epsilon(\mathbf{t})) + R'(\mathbf{t}) + L(\mathbf{t}),$$

where $\tilde{\epsilon}(\mathbf{t}) = (\tilde{\epsilon}_1(\mathbf{t}), \tilde{\epsilon}_2(\mathbf{t}), \ldots, \tilde{\epsilon}_\omega(\mathbf{t}))^T \in \mathbb{Q}^\omega$ is the state vector of the neuron at time $\mathbf{t}$, $A' = diag(a'_1, a'_2, \ldots, a'_\omega) \in \mathbb{R}^{\omega\times\omega}$ with $a'_\vartheta > 0$ denotes the self-feedback connection matrix, $B' = (b'_{\vartheta\sigma})_{\omega\times\omega}, D' = (d'_{\vartheta\sigma})_{\omega\times\omega} \in \mathbb{Q}^{\omega\times\omega}$ are the connection weight matrix, $W' = (w'_{\vartheta\sigma})_{\omega\times\omega} \in \mathbb{Q}^{\omega\times\omega}$ is the interaction structures and x_1 is the outer interaction strength, the activation functions are represented by $f(\tilde{\epsilon}(\mathbf{t})) = (f_1(\tilde{\epsilon}(\mathbf{t})), f_2(\tilde{\epsilon}(\mathbf{t})), \ldots, f_\omega(\tilde{\epsilon}(\mathbf{t})))^T \in \mathbb{Q}^\omega$, and $g(\tilde{\epsilon}(\mathbf{t})) = (g_1(\tilde{\epsilon}(\mathbf{t} - \tau_0(\mathbf{t}))), g_2(\tilde{\epsilon}(\mathbf{t} - \tau_0(\mathbf{t}))), \ldots, g_\omega(\tilde{\epsilon}(\mathbf{t} - \tau_0(\mathbf{t}))))^T \in \mathbb{Q}^\omega$ with and without time-varying delays, respectively, and $h(\epsilon(\mathbf{t})) = (h_1(\epsilon(\mathbf{t})), h_2(\epsilon(\mathbf{t})), \ldots, h_\omega(\epsilon(\mathbf{t})))^T \in \mathbb{Q}^\omega$ is interaction function. $R'(\mathbf{t}) = (r'_1(\mathbf{t}), r'_2(\mathbf{t}), \ldots, r'_\omega(\mathbf{t}))^T$
$\in \mathbb{Q}^\omega$ is the external input vector, $L(\mathbf{t})$ is the control vector.

Remark 3.2.1. The QVNN model with time-varying delay and interaction terms presents numerous advantages. This model offers a potent combination of representation power, adaptability to temporal dynamics, improved capturing of interactions, and potential efficiency gains, making it a compelling choice for various applications. QVNNs can capture more complex relationships than traditional RVNNs. By leveraging quaternion algebra, the model can represent multi-dimensional data with fewer parameters, potentially leading to more efficient and effective learning. Incorporating time-varying delays allows the model to adapt to dynamic temporal patterns in the data. This capability is crucial for applications where the relationships between variables evolve, such as in time series forecasting or dynamical system modeling. Including interaction terms it is enabled the model to capture dependencies and interactions between different variables accurately. This enhances the model's ability to capture complex relationships and patterns in the data, leading to improve predictive performance. Despite the increased complexity of the model due to its quaternion nature and inclusion of time-varying delay and interaction terms, it may still require fewer parameters as compared to the alternative approaches. This can reduce computational overhead and memory requirements, making the model more scalable and practical for real-world applications.

Assumption 3.2.1. Assume that there are positive definite constants κ_σ, ρ_σ and δ_σ for any $\epsilon, \tilde{\epsilon} \in \mathbb{Q}$ such that the $f_\sigma(.), g_\sigma(.), h_\sigma(.) \in \mathbb{Q}$ satisfy the following conditions:

$$|f_\sigma(\epsilon) - f_\sigma(\tilde{\epsilon})| \leq \kappa_\sigma |\epsilon - \tilde{\epsilon}|.$$

$$|g_\sigma(\epsilon) - g_\sigma(\tilde{\epsilon})| \leq \rho_\sigma |\epsilon - \tilde{\epsilon}|.$$

$$|h_\sigma(\epsilon) - h_\sigma(\tilde{\epsilon})| \leq \delta_\sigma |\epsilon - \tilde{\epsilon}|.$$

Assumption 3.2.2. There exist positive real numbers $\alpha_\sigma, \beta_\sigma, \gamma_\sigma$ for any $\epsilon \in \mathbb{Q}$ s.t.

$$|f_\sigma(\epsilon)| \leq \alpha_\sigma, \; |g_\sigma(\epsilon)| \leq \beta_\sigma, \; |h_\sigma(\epsilon)| \leq \gamma_\sigma.$$

Assumption 3.2.3. $\tau_0(\mathbf{t}) \geq 0$ is a differential function with $0 \leq \tau_0(\mathbf{t}) \leq \tilde{\tau}$ and $\dot{\tau}_0(\mathbf{t}) \leq \xi < 1$, $\forall t$, where $\tilde{\tau}$ and ξ are constants.

Lemma 3.2.1. [89] Assume that $\hat{Z}(\mathbf{t}) : [\mathbf{t}_0 - \mu, \infty) \to [0, \infty)$ is the continuous function that satisfies the following inequality:

$$\dot{\hat{Z}}(\mathbf{t}) \leq -\tilde{y}_1 \hat{Z}(\mathbf{t}) + \tilde{y}_2 Z(\mathbf{t} - \tau_0(\mathbf{t})) + C, \mathrm{or} \mathbf{t} \geq \mathbf{t}_0,$$

where $\tilde{y}_1 > \tilde{y}_2 > 0,\ C > 0,\ \tau_0(\mathbf{t}) \leq \xi$, then $\hat{Z}(\mathbf{t}) \leq \sup_{-\xi\leq\epsilon\leq 0}(Z(\epsilon))e^{-\chi t} + \frac{C}{\chi}$, where $\chi > 0$ is the unique solution of $\tilde{y}_1 - \tilde{y}_2 e^{\chi\xi} - \chi = 0$.

Lemma 3.2.2. [91] The following inequality holds for any pair of quaternion vectors $\nu,\ \tilde{\nu} \in \mathbb{Q}^n$, a constant $s > 0$, and positive definite Hermitian matrix $M \in \mathbb{Q}^{n\times n}$:

$$\nu^*\tilde{\nu} + \tilde{\nu}^*\nu \leq s\nu^* M\nu + s^{-1}\tilde{\nu}^* M^{-1}\tilde{\nu}.$$

Lemma 3.2.3. [91] For any $\epsilon,\ \tilde{\epsilon} \in \mathbb{Q}$ the following inequality holds.

$$(\epsilon + \tilde{\epsilon})^*(\epsilon + \tilde{\epsilon}) \leq 2\epsilon^*\epsilon + 2\tilde{\epsilon}^*\tilde{\epsilon}.$$

3.3 Main results

Assume that the synchronization error between the systems (3.1) and (3.2) is defined as $E_\vartheta(\mathbf{t}) = \tilde{\epsilon}_\vartheta(\mathbf{t}) - \lambda\epsilon_\vartheta(\mathbf{t})$, where λ is a projective coefficient. By using the error function formulation and using the equations (3.1) and (3.2), we can determine the error system as

$$\begin{aligned}
\dot{E}_\vartheta(\mathbf{t}) =& - a'_\vartheta\tilde{\epsilon}_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b'_{\vartheta\sigma} f_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma} g_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 \sum_{\sigma=1}^{n} w'_{\vartheta\sigma} \\
& \times h_\sigma(\epsilon_\sigma(\mathbf{t})) + r'_\vartheta(\mathbf{t}) + l_\vartheta(\mathbf{t}) - \lambda\Big(- a_\vartheta\epsilon_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b_{\vartheta\sigma} f_\sigma(\epsilon_\sigma(\mathbf{t})) \\
& + \sum_{\sigma=1}^{n} d_{\vartheta\sigma} g_\sigma(\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) + r_\vartheta(\mathbf{t})\Big) \\
=& - a'_\vartheta\tilde{\epsilon}_\vartheta(\mathbf{t}) + \lambda a_\vartheta\epsilon_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b'_{\vartheta\sigma} f_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) - \lambda\sum_{\sigma=1}^{n} b_{\vartheta\sigma} f_\sigma(\epsilon_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma} \\
& \times g_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - \lambda\sum_{\sigma=1}^{n} d_{\vartheta\sigma} g_\sigma(\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1\sum_{\sigma=1}^{n} w'_{\vartheta\sigma} h_\sigma(\epsilon_\sigma(\mathbf{t}))
\end{aligned}$$

$$
\begin{aligned}
&- \lambda x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) + r'_\vartheta(\mathbf{t}) - \lambda r_\vartheta(\mathbf{t}) + l_\vartheta(\mathbf{t}) \\
=& - a'_\vartheta E_\vartheta(\mathbf{t}) + \lambda \Delta a_\vartheta \epsilon_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b'_{\vartheta\sigma} f_\sigma(E_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma} g_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) \\
&- \lambda x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(E_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} b'_{\vartheta\sigma} f_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) - \lambda \sum_{\sigma=1}^{n} b_{\vartheta\sigma} f_\sigma(\epsilon_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma} \\
&\times g_\sigma(\lambda\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - \lambda \sum_{\sigma=1}^{n} d_{\vartheta\sigma} g_\sigma(\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 \sum_{\sigma=1}^{n} w'_{\vartheta\sigma} h_\sigma(\epsilon_\sigma(\mathbf{t})) \\
&- \lambda x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) + r'_\vartheta(\mathbf{t}) - \lambda r_\vartheta(\mathbf{t}) + l_\vartheta(\mathbf{t}),
\end{aligned}
\tag{3.3}
$$

where $\Delta a_\vartheta = a_\vartheta - a'_\vartheta, f_\sigma(E_\sigma(\mathbf{t})) = f_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) - f_\sigma(\lambda\epsilon_\sigma(\mathbf{t})), g_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) = g_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - g_\sigma(\lambda\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))), h_\sigma(E_\sigma(\mathbf{t})) = h_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) - h_\sigma(\lambda\epsilon_\sigma(\mathbf{t}))$.

Theorem 3.3.1. Let us suppose Assumptions (2.2.1)-(2.2.2) hold, and if there exist positive constants $\tilde{y}_1, \tilde{y}_2, m_{\vartheta\sigma}, n_{\vartheta\sigma}, o_{\vartheta\sigma}$ such that

$$
\begin{aligned}
T_\vartheta =& 2(a'_\vartheta + \hat{k}_\vartheta) - \sum_{\sigma=1}^{n} \Big(m_{\vartheta\sigma}^{-1} b'_{\vartheta\sigma} (b'_{\vartheta\sigma})^* + 2 m_{\sigma\vartheta} \kappa_\vartheta^2 + n_{\vartheta\sigma}^{-1} d'_{\vartheta\sigma} (d'_{\vartheta\sigma})^* - \lambda x_1 o_{\vartheta\sigma}^{-1} w_{\vartheta\sigma} (w_{\vartheta\sigma})^* \\
&- 2\lambda x_1 o_{\sigma\vartheta} \delta_\vartheta^2 + \hat{m}_{\vartheta\sigma}^{-1} (b'_{\vartheta\sigma})^* b'_{\vartheta\sigma} + \hat{n}_{\vartheta\sigma}^{-1} b^*_{\vartheta\sigma} b_{\vartheta\sigma} + \hat{u}_{\vartheta\sigma}^{-1} (d'_{\vartheta\sigma})^* d'_{\vartheta\sigma} + \hat{v}_{\vartheta\sigma}^{-1} d^*_{\vartheta\sigma} d_{\vartheta\sigma} \\
&+ x_1 \hat{i}_{\vartheta\sigma}^{-1} (w'_{\vartheta\sigma})^* w'_{\vartheta\sigma} + x_1 \hat{j}_{\vartheta\sigma}^{-1} w^*_{\vartheta\sigma} w_{\vartheta\sigma} \Big) - \hat{o}_\vartheta^{-1} - \tilde{o}_\vartheta^{-1} - \tilde{y}_1 > 0,
\end{aligned}
$$

$$
\begin{aligned}
\hat{T}_\vartheta =& \sum_{\sigma=1}^{n} 2 n_{\sigma\vartheta} \rho_\vartheta^2 - \tilde{y}_2 < 0, \\
&\text{and } \tilde{y}_1 - \tilde{y}_2 > 0,
\end{aligned}
\tag{3.4}
$$

and the controller is proportional to the error state $E_\vartheta(\mathbf{t})$ is defined as

$$
l_\vartheta(\mathbf{t}) = -\hat{k}_\vartheta E_\vartheta(\mathbf{t}) - \lambda \Delta a_\vartheta \epsilon_\vartheta(\mathbf{t}),
\tag{3.5}
$$

then QVNNs (3.1) and (3.2) are QPS. Moreover, QVNNs (3.1) and (3.2) are globally exponentially converge to the region

$$\mathcal{B} = \left\{ E(\mathbf{t}) \in \mathbb{Q}^n : \|E(\mathbf{t})\|^2 \leq \frac{C}{\chi} \right\}, \tag{3.6}$$

where $C = \sum_{\vartheta=1}^{n} \sum_{\sigma=1}^{n} \Big((\hat{m}_{\vartheta\sigma} + \hat{n}_{\vartheta\sigma}\lambda^2)\alpha_\sigma^2 + (\hat{u}_{\vartheta\sigma} + \hat{v}_{\vartheta\sigma}\lambda^2)\beta_\sigma^2 + x_1(\hat{i}_{\vartheta\sigma} + \hat{j}_{\vartheta\sigma}\lambda^2)\gamma_\sigma^2 \Big) + \sum_{\vartheta=1}^{n} \Big(\hat{o}_\vartheta r_\vartheta'^*(\mathbf{t}) r_\vartheta'(\mathbf{t}) + \lambda^2 \sum_{\vartheta=1}^{n} \tilde{o}_\vartheta r_\vartheta^*(\mathbf{t}) r_\vartheta(\mathbf{t}) \Big)$, and χ is the unique solution of $\tilde{y}_1 - \tilde{y}_2 E^{\chi\delta} - \chi = 0$.

Proof. Consider the Lyapunov functional as

$$\hat{Z}(\mathbf{t}) = \sum_{\vartheta=1}^{n} E_\vartheta^*(\mathbf{t}) E_\vartheta(\mathbf{t}). \tag{3.7}$$

Calculating the derivative of (3.7) and using controller given in (3.5), we get

$$\begin{aligned}
\dot{\hat{Z}}(\mathbf{t}) =& 2\sum_{\vartheta=1}^{n} E_\vartheta^*(\mathbf{t}) \dot{E}_\vartheta(\mathbf{t}) \\
=& 2\sum_{\vartheta=1}^{n} E_\vartheta^*(\mathbf{t}) \Big\{ -a_\vartheta' E_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b_{\vartheta\sigma}' f_\sigma(E_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d_{\vartheta\sigma}' g_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) \\
& - \lambda x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(E_\sigma(\mathbf{t})) + \lambda \Delta a_\vartheta \epsilon_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b_{\vartheta\sigma}' f_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) - \lambda \sum_{\sigma=1}^{n} b_{\vartheta\sigma} \\
& \times f_\sigma(\epsilon_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d_{\vartheta\sigma}' g_\sigma(\lambda\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - \lambda \sum_{\sigma=1}^{n} d_{\vartheta\sigma} g_\sigma(\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) \\
& + x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma}' h_\sigma(\epsilon_\sigma(\mathbf{t})) - \lambda x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) + r_\vartheta'(\mathbf{t}) - \lambda r_\vartheta(\mathbf{t}) + l_\vartheta(\mathbf{t}) \Big\} \\
=& -2\sum_{\vartheta=1}^{n} (a_\vartheta' + \hat{k}_\vartheta) E_\vartheta^*(\mathbf{t}) E_\vartheta(\mathbf{t}) + 2\sum_{\vartheta=1}^{n} E_\vartheta^*(\mathbf{t}) \Big\{ \sum_{\sigma=1}^{n} b_{\vartheta\sigma}' f_\sigma(E_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d_{\vartheta\sigma}' \\
& \times g_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - \lambda x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(E_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} b_{\vartheta\sigma}' f_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) \\
& - \lambda \sum_{\sigma=1}^{n} b_{\vartheta\sigma} f_\sigma(\epsilon_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d_{\vartheta\sigma}' g_\sigma(\lambda\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - \lambda \sum_{\sigma=1}^{n} d_{\vartheta\sigma} g_\sigma(\epsilon_\sigma(\mathbf{t}
\end{aligned}$$

$$
\begin{aligned}
&- \tau_0(\mathbf{t}))) + x_1 \sum_{\sigma=1}^{n} w'_{\vartheta\sigma} h_\sigma(\epsilon_\sigma(\mathbf{t})) - \lambda x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) + r'_\vartheta(\mathbf{t}) - \lambda r_\vartheta(\mathbf{t})\Big\} \\
\leq &- 2\sum_{\vartheta=1}^{n}(a'_\vartheta + \hat{k}_\vartheta)E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t}) + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} m^{-1}_{\vartheta\sigma} b'_{\vartheta\sigma}(b'_{\vartheta\sigma})^* E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t}) \\
&+ \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} m_{\vartheta\sigma} f^*_\sigma(E_\sigma(\mathbf{t})) f_\sigma(E_\sigma(\mathbf{t})) + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} n^{-1}_{\vartheta\sigma} d'_{\vartheta\sigma}(d'_{\vartheta\sigma})^* E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t}) \\
&+ \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} n_{\vartheta\sigma} g^*_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) g_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - \lambda x_1 \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} o^{-1}_{\vartheta\sigma} w_{\vartheta\sigma}(w_{\vartheta\sigma})^* \\
&\times E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t}) - \lambda x_1 \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} o_{\vartheta\sigma} h^*_\sigma(E_\sigma(\mathbf{t})) h_\sigma(E_\sigma(\mathbf{t})) + 2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E^*_\vartheta(\mathbf{t}) b'_{\vartheta\sigma} \\
&\times f_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) - 2\lambda \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E^*_\vartheta(\mathbf{t}) b_{\vartheta\sigma} f_\sigma(\epsilon_\sigma(\mathbf{t})) + 2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E^*_\vartheta(\mathbf{t}) d'_{\vartheta\sigma} g_\sigma(\lambda\epsilon_\sigma(t \\
&- \tau_0(\mathbf{t}))) - 2\lambda \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E^*_\vartheta(\mathbf{t}) d_{\vartheta\sigma} g_\sigma(\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) + 2x_1 \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E^*_\vartheta(\mathbf{t}) w'_{\vartheta\sigma} \\
&\times h_\sigma(\epsilon_\sigma(\mathbf{t})) - 2\lambda x_1 \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E^*_\vartheta(\mathbf{t}) w_{\vartheta\sigma} h_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) + 2\sum_{\vartheta=1}^{n} E^*_\vartheta(\mathbf{t}) r'_\vartheta(\mathbf{t}) \\
&- 2\lambda \sum_{\vartheta=1}^{n} E^*_\vartheta(\mathbf{t}) r_\vartheta(\mathbf{t}). \qquad (3.8)
\end{aligned}
$$

The following inequality can be derived from Assumption 3.2.1 and Lemma 3.2.1 as

$$
\begin{aligned}
f^*_\sigma(E_\sigma(\mathbf{t})) f_\sigma(E_\sigma(\mathbf{t})) &\leq 2\kappa^2_\sigma E^*_\sigma(\mathbf{t}) E_\sigma(\mathbf{t}) \\
g^*_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) g_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) &\leq 2\rho^2_\sigma E^*_\sigma(\mathbf{t} - \tau_0(\mathbf{t})) E_\sigma(\mathbf{t} - \tau_0(\mathbf{t})) \\
h^*_\sigma(E_\sigma(\mathbf{t})) h_\sigma(E_\sigma(\mathbf{t})) &\leq 2\delta^2_\sigma E^*_\sigma(\mathbf{t}) E_\sigma(\mathbf{t}). \qquad (3.9)
\end{aligned}
$$

Furthermore, from Assumption 3.2.2, and by Lemma 3.2.2 if there exist real constants $\hat{m}_{\vartheta\sigma}, \hat{n}_{\vartheta\sigma}, \hat{u}_{\vartheta\sigma}, \hat{v}_{\vartheta\sigma}, \hat{i}_{\vartheta\sigma}, \hat{j}_{\vartheta\sigma}, \tilde{o}_\vartheta, \hat{o}_\vartheta > 0$, then one can deduce the following inequality as

$$
2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E^*_\vartheta(\mathbf{t}) b'_{\vartheta\sigma} f_\sigma(\lambda\epsilon_\sigma(\mathbf{t})) - 2\lambda \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E^*_\vartheta(\mathbf{t}) b_{\vartheta\sigma} f_\sigma(\epsilon_\sigma(\mathbf{t}))
$$

$$\leq \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\hat{m}_{\vartheta\sigma}^{-1}(b'_{\vartheta\sigma})^{*}b'_{\vartheta\sigma}E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\hat{m}_{\vartheta\sigma}f_{\sigma}^{*}(\lambda\epsilon_{\sigma}(\mathbf{t}))f_{\sigma}(\lambda\epsilon_{\sigma}(\mathbf{t}))$$
$$+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\hat{n}_{\vartheta\sigma}^{-1}b_{\vartheta\sigma}^{*}b_{\vartheta\sigma}E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\hat{n}_{\vartheta\sigma}\lambda^{2}f_{\sigma}^{*}(\epsilon_{\sigma}(\mathbf{t}))f_{\sigma}(\epsilon_{\sigma}(\mathbf{t}))\leq\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}$$
$$\Big(\hat{m}_{\vartheta\sigma}^{-1}(b'_{\vartheta\sigma})^{*}b'_{\vartheta\sigma}+\hat{n}_{\vartheta\sigma}^{-1}b_{\vartheta\sigma}^{*}b_{\vartheta\sigma}\Big)E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{m}_{\vartheta\sigma}+\hat{n}_{\vartheta\sigma}\lambda^{2}\Big)(\alpha_{\sigma})^{2}. \quad (3.10)$$

Similarly,

$$2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}E_{\vartheta}^{*}(\mathbf{t})d'_{\vartheta\sigma}g_{\sigma}(\lambda\epsilon_{\sigma}(\mathbf{t}-\tau_{0}(\mathbf{t})))-2\lambda\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}E_{\vartheta}^{*}(\mathbf{t})d_{\vartheta\sigma}g_{\sigma}(\epsilon_{\sigma}(\mathbf{t}-\tau_{0}(\mathbf{t})))$$
$$\leq\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{u}_{\vartheta\sigma}^{-1}(d'_{\vartheta\sigma})^{*}d'_{\vartheta\sigma}+\hat{v}_{\vartheta\sigma}^{-1}d_{\vartheta\sigma}^{*}d_{\vartheta\sigma}\Big)E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{u}_{\vartheta\sigma}+\hat{v}_{\vartheta\sigma}\lambda^{2}\Big)(\beta_{\sigma})^{2}$$
$$2x_{1}\sum_{\sigma=1}^{n}E_{\vartheta}^{*}(\mathbf{t})w'_{\vartheta\sigma}h_{\sigma}(\epsilon_{\sigma}(\mathbf{t}))-2\lambda x_{1}\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}E_{\vartheta}^{*}(\mathbf{t})w_{\vartheta\sigma}h_{\sigma}(\lambda\epsilon_{\sigma}(\mathbf{t}))\leq x_{1}\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{i}_{\vartheta\sigma}^{-1}$$
$$\times(w'_{\vartheta\sigma})^{*}w'_{\vartheta\sigma}+\hat{j}_{\vartheta\sigma}^{-1}w_{\vartheta\sigma}^{*}w_{\vartheta\sigma}\Big)E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})+x_{1}\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{i}_{\vartheta\sigma}+\hat{j}_{\vartheta\sigma}\lambda^{2}\Big)(\gamma_{\sigma})^{2}, \quad (3.11)$$

and

$$2\sum_{\vartheta=1}^{n}E_{\vartheta}^{*}(\mathbf{t})r'_{\vartheta}(\mathbf{t})-2\lambda\sum_{\vartheta=1}^{n}E_{\vartheta}^{*}(\mathbf{t})r_{\vartheta}(\mathbf{t})\leq\sum_{\vartheta=1}^{n}(\hat{o}_{\vartheta}^{-1}+\tilde{o}_{\vartheta}^{-1})E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})+\sum_{\vartheta=1}^{n}\hat{o}_{\vartheta}r_{\vartheta}'^{*}(\mathbf{t})r'_{\vartheta}(\mathbf{t})$$
$$+\lambda^{2}\sum_{\vartheta=1}^{n}\tilde{o}_{\vartheta\sigma}r_{\vartheta}^{*}(\mathbf{t})r_{\vartheta}(\mathbf{t}). \quad (3.12)$$

Putting the inequalities (3.9)-(3.12) in the inequality (3.8), and applying the result (7.3.2), we obtain

$$\dot{Z}(\mathbf{t})\leq-2\sum_{\vartheta=1}^{n}(a'_{\vartheta}+\hat{k}_{\vartheta})E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}m_{\vartheta\sigma}^{-1}b'_{\vartheta\sigma}(b'_{\vartheta\sigma})^{*}E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})$$
$$+2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}m_{\vartheta\sigma}\kappa_{\sigma}^{2}E_{\sigma}^{*}(\mathbf{t})E_{\sigma}(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}n_{\vartheta\sigma}^{-1}d'_{\vartheta\sigma}(d'_{\vartheta\sigma})^{*}E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t})$$

$$
\begin{aligned}
&+2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} n_{\vartheta\sigma}\rho_\sigma^2 E_\sigma^*(\mathbf{t}-\tau_0(\mathbf{t}))E_\sigma(\mathbf{t}-\tau_0(\mathbf{t})) - \lambda x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} o_{\vartheta\sigma}^{-1} w_{\vartheta\sigma}(w_{\vartheta\sigma})^* \\
&\times E_\vartheta^*(\mathbf{t})E_\vartheta(\mathbf{t}) - 2\lambda x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} o_{\vartheta\sigma}\delta_\sigma^2 E_\sigma^*(\mathbf{t})E_\sigma(\mathbf{t}) + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{m}_{\vartheta\sigma}^{-1}(b'_{\vartheta\sigma})^* b'_{\vartheta\sigma} \\
&+\hat{n}_{\vartheta\sigma}^{-1} b_{\vartheta\sigma}^* b_{\vartheta\sigma}\Big)E_\vartheta^*(\mathbf{t})E_\vartheta(\mathbf{t}) + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{m}_{\vartheta\sigma}+\hat{n}_{\vartheta\sigma}\lambda^2\Big)(\alpha_\sigma)^2 + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{u}_{\vartheta\sigma}^{-1} \\
&\times (d'_{\vartheta\sigma})^* d'_{\vartheta\sigma} + \hat{v}_{\vartheta\sigma}^{-1} d_{\vartheta\sigma}^* d_{\vartheta\sigma}\Big)E_\vartheta^*(\mathbf{t})E_\vartheta(\mathbf{t}) + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{u}_{\vartheta\sigma}+\hat{v}_{\vartheta\sigma}\lambda^2\Big)(\beta_\sigma)^2 \\
&+ x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{i}_{\vartheta\sigma}^{-1}(w'_{\vartheta\sigma})^* w'_{\vartheta\sigma} + \hat{j}_{\vartheta\sigma}^{-1} w_{\vartheta\sigma}^* w_{\vartheta\sigma}\Big)E_\vartheta^*(\mathbf{t})E_\vartheta(\mathbf{t}) + x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\hat{i}_{\vartheta\sigma} \\
&+\hat{j}_{\vartheta\sigma}\lambda^2\Big)(\gamma_\sigma)^2 + \sum_{\vartheta=1}^{n}(\hat{o}_\vartheta^{-1}+\tilde{o}_\vartheta^{-1})E_\vartheta^*(\mathbf{t})E_\vartheta(\mathbf{t}) + \sum_{\vartheta=1}^{n}\hat{o}_\vartheta r_\vartheta'^*(\mathbf{t})r'_\vartheta(\mathbf{t}) \\
&+\lambda^2\sum_{\vartheta=1}^{n}\tilde{o}_{\vartheta\sigma} r_\vartheta^*(\mathbf{t})r_\vartheta(\mathbf{t}) \\
\le&\sum_{\vartheta=1}^{n}\Big(-2(a'_\vartheta+\hat{k}_\vartheta) + \sum_{\sigma=1}^{n}\Big(m_{\vartheta\sigma}^{-1} b'_{\vartheta\sigma}(b'_{\vartheta\sigma})^* + 2m_{\sigma\vartheta}\kappa_\vartheta^2 + n_{\vartheta\sigma}^{-1} d'_{\vartheta\sigma}(d'_{\vartheta\sigma})^* \\
&-\lambda x_1 o_{\vartheta\sigma}^{-1} w_{\vartheta\sigma}(w_{\vartheta\sigma})^* - 2\lambda x_1 o_{\sigma\vartheta}\delta_\vartheta^2 + \hat{m}_{\vartheta\sigma}^{-1}(b'_{\vartheta\sigma})^* b'_{\vartheta\sigma} + \hat{n}_{\vartheta\sigma}^{-1} b_{\vartheta\sigma}^* b_{\vartheta\sigma} \\
&+\hat{u}_{\vartheta\sigma}^{-1}(d'_{\vartheta\sigma})^* d'_{\vartheta\sigma} + \hat{v}_{\vartheta\sigma}^{-1} d_{\vartheta\sigma}^* d_{\vartheta\sigma} + x_1\hat{i}_{\vartheta\sigma}^{-1}(w'_{\vartheta\sigma})^* w'_{\vartheta\sigma} + x_1\hat{j}_{\vartheta\sigma}^{-1} w_{\vartheta\sigma}^* w_{\vartheta\sigma}\Big) \\
&+\hat{o}_\vartheta^{-1}+\tilde{o}_\vartheta^{-1}+\tilde{y}_1\Big)E_\vartheta^*(\mathbf{t})E_\vartheta(\mathbf{t}) - \sum_{\vartheta=1}^{n}\tilde{y}_1 E_\vartheta^*(\mathbf{t})E_\vartheta(\mathbf{t}) + \sum_{\vartheta=1}^{n}\Big(\sum_{\sigma=1}^{n} 2n_{\sigma\vartheta}\rho_\vartheta^2 - \tilde{y}_2\Big)E_\vartheta^*(\mathbf{t} \\
&-\tau_0(\mathbf{t}))E_\vartheta(\mathbf{t}-\tau_0(\mathbf{t})) + \sum_{\vartheta=1}^{n}\tilde{y}_2 E_\vartheta^*(\mathbf{t}-\tau_0(\mathbf{t}))E_\vartheta(\mathbf{t}-\tau_0(\mathbf{t})) + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big((\hat{m}_{\vartheta\sigma} \\
&+\hat{n}_{\vartheta\sigma}\lambda^2)\alpha_\sigma^2 + (\hat{u}_{\vartheta\sigma}+\hat{v}_{\vartheta\sigma}\lambda^2)\beta_\sigma^2 + x_1(\hat{i}_{\vartheta\sigma}+\hat{j}_{\vartheta\sigma}\lambda^2)\gamma_\sigma^2\Big) + \sum_{\vartheta=1}^{n}\Big(\hat{o}_\vartheta r_\vartheta'^*(\mathbf{t})r'_\vartheta(\mathbf{t}) \\
&+\lambda^2\sum_{\vartheta=1}^{n}\tilde{o}_\vartheta r_\vartheta^*(\mathbf{t})r_\vartheta(\mathbf{t})\Big) \\
\le& -\tilde{y}_1\hat{Z}(\mathbf{t}) + \tilde{y}_2 Z(\mathbf{t}-\tau_0(\mathbf{t})) + C. \qquad (3.13)
\end{aligned}
$$

Since $\tilde{y}_1 - \tilde{y}_2 > 0$, it follows from Lemma 3.2.1,

$$
\hat{Z}(\mathbf{t}) \le \sup_{-\xi\le s\le 0}(Z(\epsilon))E^{-\chi t} + \frac{C}{\chi},
$$

where $\chi > 0$ is the unique solution of $\tilde{y}_1 - \tilde{y}_2 E^{\chi\xi} - \chi = 0$, which implies

$$\|E(\mathbf{t})\|^2 \leq \sup_{-\xi \leq s \leq 0} (Z(\epsilon))E^{-\chi t} + \frac{C}{\chi}.$$

Therefore error $E(\mathbf{t})$ converges exponentially to the region

$$\mathcal{B} = \left\{ E(\mathbf{t}) : \|E(\mathbf{t})\|^2 \leq \frac{C}{\chi} \right\}.$$

Therefore, using Definition 2.1, it is simple to determine the QPS between QVNNs (3.2) and (3.1). □

Remark 3.3.1. In particular, if we assume $\lambda = 1$ for QS, then the synchronization error between (3.1) and (3.2) is $E_\vartheta(\mathbf{t}) = \tilde{\epsilon}_\vartheta(\mathbf{t}) - \epsilon_\vartheta(\mathbf{t})$. Now we get the error system as

$$\begin{aligned}
\dot{E}_\vartheta(\mathbf{t}) =& - a'_\vartheta \tilde{\epsilon}_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b'_{\vartheta\sigma} f_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma} g_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 \sum_{\sigma=1}^{n} w'_{\vartheta\sigma} \\
& \times h_\sigma(\epsilon_\sigma(\mathbf{t})) + r'_\vartheta(\mathbf{t}) + l_\vartheta(\mathbf{t}) - \Big(- a_\vartheta \epsilon_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b_{\vartheta\sigma} f_\sigma(\epsilon_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d_{\vartheta\sigma} \\
& \times g_\sigma(\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} h_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) + r_\vartheta(\mathbf{t}) \Big) \\
=& - a'_\vartheta E_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} b'_{\vartheta\sigma} f_\sigma(E_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma} g_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - x_1 \sum_{\sigma=1}^{n} w_{\vartheta\sigma} \\
& \times h_\sigma(E_\sigma(\mathbf{t})) + \Delta a_\vartheta \epsilon_\vartheta(\mathbf{t}) + \sum_{\sigma=1}^{n} \Delta b_{\vartheta\sigma} f_\sigma(\epsilon_\sigma(\mathbf{t})) + \sum_{\sigma=1}^{n} \Delta d_{\vartheta\sigma} g_\sigma(\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) \\
& + x_1 \sum_{\sigma=1}^{n} \Delta w_{\vartheta\sigma} h_\sigma(\epsilon_\sigma(\mathbf{t})) + r'_\vartheta(\mathbf{t}) - r_\vartheta(\mathbf{t}) + l_\vartheta(\mathbf{t}),
\end{aligned} \tag{3.14}$$

where $\Delta a_\vartheta = a_\vartheta - a'_\vartheta$, $f_\sigma(E_\sigma(\mathbf{t}) = f_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) - f_\sigma(\epsilon_\sigma(\mathbf{t}))$, $g_\sigma(E_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) = g_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t} - \tau_0(\mathbf{t}))) - g_\sigma(\epsilon_\sigma(\mathbf{t} - \tau_0(\mathbf{t})))$, $h_\sigma(E_\sigma(\mathbf{t})) = h_\sigma(\tilde{\epsilon}_\sigma(\mathbf{t})) - h_\sigma(\epsilon_\sigma(\mathbf{t}))$, $\Delta b_{\vartheta\sigma} = b'_{\vartheta\sigma} - b_{\vartheta\sigma}$, $\Delta d_{\vartheta\sigma} = d'_{\vartheta\sigma} - d_{\vartheta\sigma}$, $\Delta w_{\vartheta\sigma} = w'_{\vartheta\sigma} - w_{\vartheta\sigma}$.

Corollary 3.3.1. Under Assumptions 3.2.1-3.2.2, if there exist positive constants $\pi_1, \pi_2, m'_{\vartheta\sigma}, n'_{\vartheta\sigma}, o'_{\vartheta\sigma}$ such that

$$
\begin{aligned}
T'_\vartheta =& 2(a'_\vartheta + \hat{k}_\vartheta) - \sum_{\sigma=1}^{n} \Big(m'^{-1}_{\vartheta\sigma} b'_{\vartheta\sigma}(b'_{\vartheta\sigma})^* + 2m'_{\sigma\vartheta}\kappa_\vartheta^2 + n'^{-1}_{\vartheta\sigma} d'_{\vartheta\sigma}(d'_{\vartheta\sigma})^* - x_1 o'^{-1}_{\vartheta\sigma} w_{\vartheta\sigma}(w_{\vartheta\sigma})^* \\
& - 2x_1 o'_{\sigma\vartheta}\delta_\vartheta^2 + \tilde{m}^{-1}_{\vartheta\sigma}(\Delta b_{\vartheta\sigma})^*\Delta b_{\vartheta\sigma} + \tilde{n}^{-1}_{\vartheta\sigma}(\Delta d_{\vartheta\sigma})^*\Delta d_{\vartheta\sigma} + x_1\tilde{u}^{-1}_{\vartheta\sigma}(\Delta w_{\vartheta\sigma})^*\Delta w_{\vartheta\sigma} \Big) \\
& - \tilde{v}_\vartheta^{-1} - \tilde{k}_\vartheta^{-1} - \pi_1 > 0, \\
\hat{T}'_\vartheta =& \sum_{\sigma=1}^{n} 2n'_{\sigma\vartheta}\rho_\vartheta^2 - \pi_2 < 0, \\
& \text{and } \pi_1 - \pi_2 > 0,
\end{aligned} \tag{3.15}
$$

under the controller

$$
l_\vartheta(\mathbf{t}) = -\hat{k}_\vartheta E_\vartheta(\mathbf{t}) - \Delta a_\vartheta \epsilon_\vartheta(\mathbf{t}), \tag{3.16}
$$

then QVNNs (3.1) and (3.2) are QS. Moreover, QVNNs (3.1) and (3.2) are globally exponential converge to the region

$$
\mathcal{B}' = \left\{ E(\mathbf{t}) \in \mathbb{Q}^n : \|E(\mathbf{t})\|^2 \leq \frac{C'}{\chi} \right\}, \tag{3.17}
$$

where $C' = \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\tilde{m}_{\vartheta\sigma}\alpha_\vartheta^2 + \tilde{n}_{\vartheta\sigma}\beta_\vartheta^2 + x_1\tilde{u}_{\vartheta\sigma}\gamma_\vartheta^2\Big) + \sum_{\vartheta=1}^{n}\Big(\tilde{v}_\vartheta r'^*_\vartheta(\mathbf{t})r'_\vartheta(\mathbf{t}) + \tilde{k}_\vartheta r^*_\vartheta(\mathbf{t})r_\vartheta(\mathbf{t})\Big)$, and χ is the unique solution of $\pi_1 - \pi_2 E^{\chi\delta} - \chi = 0$.

Proof. Consider the Lyapunov functional as

$$
\hat{Z}(\mathbf{t}) = \sum_{\vartheta=1}^{n} E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t}). \tag{3.18}
$$

Calculating the derivative of (3.18) and from equations (3.14) and (3.16), we get

$$
\begin{aligned}
\dot{\hat{Z}}(\mathbf{t}) =& 2\sum_{\vartheta=1}^{n} E_{\vartheta}^{*}(\mathbf{t})\dot{E}_{\vartheta}(\mathbf{t}) \\
=& 2\sum_{\vartheta=1}^{n} E_{\vartheta}^{*}(\mathbf{t})\Big\{-a'_{\vartheta}E_{\vartheta}(\mathbf{t}) + \sum_{\sigma=1}^{n} b'_{\vartheta\sigma}f_{\sigma}(E_{\sigma}(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma}g_{\sigma}(E_{\sigma}(\mathbf{t}-\tau_0(\mathbf{t}))) \\
& - x_1\sum_{\sigma=1}^{n} w_{\vartheta\sigma}h_{\sigma}(E_{\sigma}(\mathbf{t})) + \Delta a_{\vartheta}\epsilon_{\vartheta}(\mathbf{t}) + \sum_{\sigma=1}^{n}\Delta b_{\vartheta\sigma}f_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) + \sum_{\sigma=1}^{n}\Delta d_{\vartheta\sigma}g_{\sigma}(\epsilon_{\sigma}(\mathbf{t} \\
& - \tau_0(\mathbf{t}))) + x_1\sum_{\sigma=1}^{n}\Delta w_{\vartheta\sigma}h_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) + r'_{\vartheta}(\mathbf{t}) - r_{\vartheta}(\mathbf{t}) + l_{\vartheta}(\mathbf{t})\Big\} \\
=& -2\sum_{\vartheta=1}^{n}(a'_{\vartheta}+\hat{k}_{\vartheta})E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) + 2E_{\vartheta}^{*}(\mathbf{t})\Big\{\sum_{\sigma=1}^{n} b'_{\vartheta\sigma}f_{\sigma}(E_{\sigma}(\mathbf{t})) + \sum_{\sigma=1}^{n} d'_{\vartheta\sigma}g_{\sigma}(E_{\sigma}(\mathbf{t} \\
& - \tau_0(\mathbf{t}))) - x_1\sum_{\sigma=1}^{n} w_{\vartheta\sigma}h_{\sigma}(E_{\sigma}(\mathbf{t})) + \sum_{\sigma=1}^{n}\Delta b_{\vartheta\sigma}f_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) + \sum_{\sigma=1}^{n}\Delta d_{\vartheta\sigma}g_{\sigma}(\epsilon_{\sigma}(\mathbf{t} \\
& - \tau_0(\mathbf{t}))) + x_1\sum_{\sigma=1}^{n}\Delta w_{\vartheta\sigma}h_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) + r'_{\vartheta}(\mathbf{t}) - r_{\vartheta}(\mathbf{t})\Big\} \\
\leq& -2\sum_{\vartheta=1}^{n}(a'_{\vartheta}+\hat{k}_{\vartheta})E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} m'^{-1}_{\vartheta\sigma}b'_{\vartheta\sigma}(b'_{\vartheta\sigma})^{*}E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) \\
& + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} m'_{\vartheta\sigma}f_{\sigma}^{*}(E_{\sigma}(\mathbf{t}))f_{\sigma}(E_{\sigma}(\mathbf{t})) + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} n'^{-1}_{\vartheta\sigma}d'_{\vartheta\sigma}(b'_{\vartheta\sigma})^{*}E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) \\
& + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} n'_{\vartheta\sigma}g_{\sigma}^{*}(E_{\sigma}(\mathbf{t}-\tau_0(\mathbf{t})))g_{\sigma}(E_{\sigma}(\mathbf{t}-\tau_0(\mathbf{t})))x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} o'^{-1}_{\vartheta\sigma}w_{\vartheta\sigma}(w_{\vartheta\sigma})^{*} \\
& \times E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) - x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} o'_{\vartheta\sigma}h_{\sigma}^{*}(E_{\sigma}(\mathbf{t}))h_{\sigma}(E_{\sigma}(\mathbf{t})) + 2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E_{\vartheta}^{*}(\mathbf{t})\Delta b_{\vartheta\sigma} \\
& \times f_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) + 2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E_{\vartheta}^{*}(\mathbf{t})\Delta d_{\vartheta\sigma}g_{\sigma}(\lambda\epsilon_{\sigma}(\mathbf{t}-\tau_0(\mathbf{t}))) + 2x_1\sum_{\sigma=1}^{n} E_{\vartheta}^{*}(\mathbf{t})\Delta w_{\vartheta\sigma} \\
& \times h_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) + 2\sum_{\vartheta=1}^{n} E_{\vartheta}^{*}(\mathbf{t})r'_{\vartheta}(\mathbf{t}) - 2\sum_{\vartheta=1}^{n} E_{\vartheta}^{*}(\mathbf{t})r_{\vartheta}(\mathbf{t})\Big\}.
\end{aligned} \tag{3.19}
$$

By Assumptions 3.2.2-3.2.3 and Lemmas 3.2.2-3.2.3 and for real constants $\tilde{m}_{\vartheta\sigma}, \tilde{n}_{\vartheta\sigma}, \tilde{u}_{\vartheta\sigma}, \tilde{v}_{\vartheta}, \tilde{k}_{\vartheta} > 0$, we can deduce the following inequality

$$\begin{aligned} 2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E_{\vartheta}^{*}(\mathbf{t})\Delta b_{\vartheta\sigma} f_{\sigma}(\lambda\epsilon_{\sigma}(\mathbf{t})) \leq & \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} \tilde{m}_{\vartheta\sigma}^{-1}(\Delta b_{\vartheta\sigma})^{*}\Delta b_{\vartheta\sigma} E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) \\ & + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} \tilde{m}_{\vartheta\sigma} f_{\sigma}^{*}(\epsilon_{\sigma}(\mathbf{t}))f_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) \\ \leq & \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} \tilde{m}_{\vartheta\sigma}^{-1}(\Delta b_{\vartheta\sigma})^{*}\Delta b_{\vartheta\sigma} E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) \\ & + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} \tilde{m}_{\vartheta\sigma}(\alpha_{\sigma})^{2}. \end{aligned} \tag{3.20}$$

Similarly,

$$\begin{aligned} 2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} E_{\vartheta}^{*}(\mathbf{t})d'_{\vartheta\sigma} g_{\sigma}(\lambda\epsilon_{\sigma}(\mathbf{t}-\tau_0(\mathbf{t}))) \leq & \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} \tilde{n}_{\vartheta\sigma}^{-1}(\Delta d_{\vartheta\sigma})^{*}\Delta d_{\vartheta\sigma} E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) \\ & + \sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} \tilde{n}_{\vartheta\sigma}(\beta_{\sigma})^{2} \\ 2x_1\sum_{\sigma=1}^{n} E_{\vartheta}^{*}(\mathbf{t})w'_{\vartheta\sigma} h_{\sigma}(\epsilon_{\sigma}(\mathbf{t})) \leq & \; x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} \tilde{u}_{\vartheta\sigma}^{-1}(\Delta w_{\vartheta\sigma})^{*}\Delta w_{\vartheta\sigma} E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) \\ & + x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n} \tilde{u}_{\vartheta\sigma}(\gamma_{\sigma})^{2}, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} 2\sum_{\vartheta=1}^{n} E_{\vartheta}^{*}(\mathbf{t})r'_{\vartheta}(\mathbf{t}) - 2\sum_{\vartheta=1}^{n} E_{\vartheta}^{*}(\mathbf{t})r_{\vartheta}(\mathbf{t}) \leq & \sum_{\vartheta=1}^{n}(\tilde{v}_{\vartheta}^{-1}+\tilde{k}_{\vartheta}^{-1})E_{\vartheta}^{*}(\mathbf{t})E_{\vartheta}(\mathbf{t}) + \sum_{\vartheta=1}^{n}\tilde{v}_{\vartheta} r_{\vartheta}'^{*}(\mathbf{t})r'_{\vartheta}(\mathbf{t}) \\ & + \sum_{\vartheta=1}^{n}\tilde{k}_{\vartheta} r_{\vartheta}^{*}(\mathbf{t})r_{\vartheta}(\mathbf{t}). \end{aligned} \tag{3.22}$$

Based on relations (3.9) and putting the equations (3.20)-(3.22) in the equation (3.19), and applying the result (3.15), we obtain

$$
\begin{aligned}
\dot{\hat{Z}}(\mathbf{t}) \leq & -2\sum_{\vartheta=1}^{n}(a'_\vartheta+\hat{k}_\vartheta)E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}m'^{-1}_{\vartheta\sigma}b'_{\vartheta\sigma}(b'_{\vartheta\sigma})^*E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})\\
&+2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}m'_{\vartheta\sigma}\kappa^2_\sigma E^*_\sigma(\mathbf{t})E_\sigma(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}n'^{-1}_{\vartheta\sigma}d'_{\vartheta\sigma}(d'_{\vartheta\sigma})^*E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})\\
&+2\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}n'_{\vartheta\sigma}\rho^2_\sigma E^*_\sigma(\mathbf{t}-\tau_0(\mathbf{t}))E_\sigma(\mathbf{t}-\tau_0(\mathbf{t}))-x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}o'^{-1}_{\vartheta\sigma}w_{\vartheta\sigma}(w_{\vartheta\sigma})^*\\
&\times E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})-2x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}o'_{\vartheta\sigma}\delta^2_\sigma E^*_\sigma(\mathbf{t})E_\sigma(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\tilde{m}^{-1}_{\vartheta\sigma}(\Delta b_{\vartheta\sigma})^*\Delta b_{\vartheta\sigma}\\
&\times E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\tilde{m}_{\vartheta\sigma}(\alpha_\sigma)^2+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\tilde{n}^{-1}_{\vartheta\sigma}(\Delta d_{\vartheta\sigma})^*\Delta d_{\vartheta\sigma}E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})\\
&+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\tilde{n}_{\vartheta\sigma}(\beta_\sigma)^2+x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\tilde{u}^{-1}_{\vartheta\sigma}(\Delta w_{\vartheta\sigma})^*\Delta w_{\vartheta\sigma}E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})\\
&+x_1\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\tilde{u}_{\vartheta\sigma}(\gamma_\sigma)^2+\sum_{\vartheta=1}^{n}(\tilde{v}^{-1}_\vartheta+\tilde{k}^{-1}_\vartheta)E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})+\sum_{\vartheta=1}^{n}\tilde{v}_\vartheta r'^*_\vartheta(\mathbf{t})r'_\vartheta(\mathbf{t})\\
&+\sum_{\vartheta=1}^{n}\tilde{k}_\vartheta r^*_\vartheta(\mathbf{t})r_\vartheta(\mathbf{t})\\
\leq & \sum_{\vartheta=1}^{n}\Big(-2(a'_\vartheta+\hat{k}_\vartheta)+\sum_{\sigma=1}^{n}\Big(m'^{-1}_{\vartheta\sigma}b'_{\vartheta\sigma}(b'_{\vartheta\sigma})^*+2m'_{\sigma\vartheta}\kappa^2_\vartheta+n'^{-1}_{\vartheta\sigma}d'_{\vartheta\sigma}(d'_{\vartheta\sigma})^*-x_1\\
&\times o'^{-1}_{\vartheta\sigma}w_{\vartheta\sigma}(w_{\vartheta\sigma})^*-2x_1o'_{\sigma\vartheta}\delta^2_\vartheta+\tilde{m}^{-1}_{\vartheta\sigma}(\Delta b_{\vartheta\sigma})^*\Delta b_{\vartheta\sigma}+\tilde{n}^{-1}_{\vartheta\sigma}(\Delta d_{\vartheta\sigma})^*\Delta d_{\vartheta\sigma}\\
&+x_1\tilde{u}^{-1}_{\vartheta\sigma}(\Delta w_{\vartheta\sigma})^*\Delta w_{\vartheta\sigma}\Big)+\tilde{v}^{-1}_\vartheta+\tilde{k}^{-1}_\vartheta+\pi_1\Big)E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})-\sum_{\vartheta=1}^{n}\pi_1E^*_\vartheta(\mathbf{t})E_\vartheta(\mathbf{t})\\
&+\sum_{\vartheta=1}^{n}\Big(\sum_{\sigma=1}^{n}2n_{\sigma\vartheta}\rho^2_\vartheta-\pi_2\Big)E^*_\vartheta(\mathbf{t}-\tau_0(\mathbf{t}))E_\vartheta(\mathbf{t}-\tau_0(\mathbf{t}))+\sum_{\vartheta=1}^{n}\pi_2E^*_\vartheta(\mathbf{t}-\tau_0(\mathbf{t}))\\
&\times E_\vartheta(t-\tau_0(\mathbf{t}))+\sum_{\vartheta=1}^{n}\sum_{\sigma=1}^{n}\Big(\tilde{m}_{\vartheta\sigma}\alpha^2_\vartheta+\tilde{n}_{\vartheta\sigma}\beta^2_\vartheta+x_1\tilde{u}_{\vartheta\sigma}\gamma^2_\vartheta\Big)+\sum_{\vartheta=1}^{n}\Big(\tilde{v}_\vartheta r'^*_\vartheta(\mathbf{t})r'_\vartheta(\mathbf{t})\\
&+\tilde{k}_\vartheta r^*_\vartheta(\mathbf{t})r_\vartheta(\mathbf{t})\Big)\\
\leq & -\pi_1\hat{Z}(\mathbf{t})+\pi_2Z(\mathbf{t}-\tau_0(\mathbf{t}))+C'.
\end{aligned}
\tag{3.23}
$$

Since $\pi_1 - \pi_2 > 0$, it follows from Lemma 3.2.1 that

$$\hat{Z}(\mathbf{t}) \leq \sup_{-\xi \leq s \leq 0} (Z(\epsilon))E^{-\chi t} + \frac{C'}{\chi},$$

where $\chi > 0$ is the unique solution of $\pi_1 - \pi_2 E^{\chi\xi} - \chi = 0$, which implies

$$\|E(\mathbf{t})\|^2 \leq \sup_{-\xi \leq s \leq 0} (Z(\epsilon))E^{-\chi t} + \frac{C'}{\chi}.$$

Therefore error $E(\mathbf{t})$ converges exponentially to the region

$$\mathcal{B}' = \left\{ E(\mathbf{t}) : \|E(\mathbf{t})\|^2 \leq \frac{C'}{\chi} \right\}.$$

As a result, the QS between QVNNs (3.2) and (3.1) can be simply determined from Definition 2.1. □

Remark 3.3.2. If two identical systems are taken with $A' = A$, $B' = B$, $D' = D$, $W' = W$, and $R' = R$, and $\lambda = 1$, the problem becomes complete synchronization between the systems (3.1) and (3.2). In this case the global exponential synchronization of two identical systems (3.1) and (3.2) will be obtained. Additionally, if $\lambda = 0$ then the problem becomes stability of the QVNN system (3.1).

Corollary 3.3.2. Suppose that the Assumptions (3.2.1)-(3.2.2) hold, and if there exist positive constants $\pi_1, \pi_2, m'_{\vartheta\sigma}, n'_{\vartheta\sigma}, o'_{\vartheta\sigma}$ such that

$$\begin{aligned} T'_\vartheta =& 2(a'_\vartheta + \hat{k}_\vartheta) - \sum_{\sigma=1}^{n} \Big(m'^{-1}_{\vartheta\sigma} b'_{\vartheta\sigma} (b'_{\vartheta\sigma})^* + 2m'_{\sigma\vartheta}\kappa^2_\vartheta + n'^{-1}_{\vartheta\sigma} d'_{\vartheta\sigma}(d'_{\vartheta\sigma})^* \\ & - x_1 o'^{-1}_{\vartheta\sigma} w_{\vartheta\sigma}(w_{\vartheta\sigma})^* - 2x_1 o'_{\sigma\vartheta}\delta^2_\vartheta \Big) - \pi_1 > 0, \\ \hat{T}'_\vartheta =& \sum_{\sigma=1}^{n} 2n'_{\sigma\vartheta}\rho^2_\vartheta - \pi_2 < 0, \end{aligned}$$

under the controller

$$l_\vartheta(\mathbf{t}) = -\,\hat{k}_\vartheta E_\vartheta(\mathbf{t}), \tag{3.24}$$

then QVNNs (3.1) and (3.2) are global exponentially synchronized.

Remark 3.3.3. Notably, the suggested method may be expanded to function PS of time-varying delayed QVNNs with mismatched parametric values and interaction terms if projective constant λ is replaced by a function in the study of QPS. If the projective constant λ is replaced by -1, the issue becomes quasi anti-synchronization of non-identical QVNNs with time-varying delays and interaction terms.

Remark 3.3.4. To guarantee QPS between two different systems (3.1) and (3.2), with non-identical parameters, a sufficient condition is established via direct method. Theorem 3.3.1 focuses on the activation functions that cannot be directly divided into four real-imaginary components. New results on QPS are produced without decomposition with different assumptions on activation functions and inequality techniques. The whole work of the present chapter is carried out in the quaternion domain. Also, the upper bound of the synchronization error is calculated.

Remark 3.3.5. In comparison to RVNNs and CVNNs, the QVNNs have a better calculation efficacy ([92], [93]) which opens up a wide range of potential real-world applications like colour night vision, picture compression, and 3-D wind forecasting. Furthermore, the proposed results on QPS are broader in comparison to the Lyapunov stability of QVNNs explored in earlier works [94] [95] [96]. Unlike some existing literature that may focus solely on either time-varying delays or interaction terms, this article comprehensively addresses both aspects. Time-varying delays are common in many real-world systems and can significantly impact synchronization dynamics. Likewise, interaction terms capture the complex interplay between NN components, which is crucial for understanding and controlling synchronization behavior. The factors have been used here are non-identical parameters, interaction terms, time-varying delay, a controller and projective coefficient, which have helped us to achieve QPS between systems (3.1) and (3.2). As a result, the present study advances the earlier studies on QVNNs.

3.4 Simulation examples

In this section, the effectiveness and efficiency of the theoretical findings mentioned above are validated using Example 3.4.1. The application of the QVNN model through retrieving a color image is illustrated in Example 3.4.2.

Example 3.4.1. Let as assume the two-dimensional non-identical time varying delayed QVNNs with interaction terms as the drive and response systems as

$$\dot{\epsilon}(\mathbf{t}) = -A\epsilon(\mathbf{t}) + Bf(\epsilon(\mathbf{t})) + Dg(\epsilon(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 W h(\tilde{\epsilon}(\mathbf{t})) + R(\mathbf{t}), \tag{3.25}$$

$$\dot{\tilde{\epsilon}}(\mathbf{t}) = -A'\tilde{\epsilon}(\mathbf{t}) + B'f(\tilde{\epsilon}(\mathbf{t})) + D'g(\tilde{\epsilon}(\mathbf{t} - \tau_0(\mathbf{t}))) + x_1 W' h(\epsilon(\mathbf{t})) + R'(\mathbf{t}) + L(\mathbf{t}), \tag{3.26}$$

where

$$A = \begin{bmatrix} 24 & 0 \\ 0 & 15 \end{bmatrix}, \quad B = \begin{bmatrix} 2 + 1.2\tilde{\imath} + 1.5\tilde{\jmath} + 1.3\tilde{k} & -1 - 0.5\tilde{\imath} - 2.3\tilde{\jmath} - 0.2\tilde{k} \\ 1 + 1.7\tilde{\imath} + 1.4\tilde{\jmath} + 1.6\tilde{k} & 0.8 + 0.9\tilde{\imath} + 0.6\tilde{\jmath} + 0.5\tilde{k} \end{bmatrix},$$

$$D = \begin{bmatrix} 1 - 0.9\tilde{\imath} + 2.1\tilde{\jmath} + 1\tilde{k} & 2 + 2\tilde{\imath} - 0.9\tilde{\jmath} + 1.2\tilde{k} \\ 3 - 0.9\tilde{\imath} + 0.5\tilde{\jmath} + 0.6\tilde{k} & 0.2 - 1.1\tilde{\imath} + 2\tilde{\jmath} + 1.2\tilde{k} \end{bmatrix},$$

$$W = \begin{bmatrix} 1.3 - 0.4\tilde{i} + 1\tilde{j} + 1\tilde{k} & 1 + 1.2\tilde{i} - 1.3\tilde{j} + 0.5\tilde{k} \\ 0.2 - 0.4\tilde{i} + 0.7\tilde{j} + 1.2\tilde{k} & 0.3 - 0.9\tilde{i} + 0.4\tilde{j} + 0.8\tilde{k} \end{bmatrix},$$

$$A' = \begin{bmatrix} 31 & 0 \\ 0 & 15 \end{bmatrix}, \quad B' = \begin{bmatrix} 1.9 + 1.2\tilde{i} + 1.5\tilde{j} + 1.3\tilde{k} & -1.2 - 0.4\tilde{i} - 2.3\tilde{j} - 0.3\tilde{k} \\ 1 + 1.3\tilde{i} + 1.2\tilde{j} + 1.5\tilde{k} & 0.5 + 0.6\tilde{i} + 0.7\tilde{j} + 0.5\tilde{k} \end{bmatrix},$$

$$D' = \begin{bmatrix} 1 - 0.8\tilde{i} + 2\tilde{j} + 0.9\tilde{k} & 1.5 + 1.2\tilde{i} - 0.5\tilde{j} + 1\tilde{k} \\ 1.2 - 0.5\tilde{i} + 0.6\tilde{j} + 0.5\tilde{k} & 0.3 - 0.2\tilde{i} + 1.5\tilde{j} + 1\tilde{k} \end{bmatrix},$$

$$W' = \begin{bmatrix} 0.2 - 0.6\tilde{i} + 1.9\tilde{j} + 0.7\tilde{k} & 1.6 + 0.4\tilde{i} - 0.6\tilde{j} + 0.7\tilde{k} \\ 0.5 - 1.2\tilde{i} + 0.7\tilde{j} + 0.3\tilde{k} & 0.4 - 1\tilde{i} + 1.2\tilde{j} + 0.3\tilde{k} \end{bmatrix},$$

$$R = \begin{bmatrix} 0.2sin(\mathbf{t}) + 0.5cos(\mathbf{t})\tilde{i} + 1sin(\mathbf{t})\tilde{j} + 0.4cos(\mathbf{t})\tilde{k} \\ 0.4sin(\mathbf{t}) + 1cos(\mathbf{t})\tilde{i} + 0.3sin(\mathbf{t})\tilde{j} + 0.6cos(\mathbf{t})\tilde{k} \end{bmatrix},$$

$$R' = \begin{bmatrix} 1cos(\mathbf{t}) + 0.2sin(\mathbf{t})\tilde{i} + 0.3cos(\mathbf{t})\tilde{j} + 0.4sin(\mathbf{t})\tilde{k} \\ 0.3cos(\mathbf{t}) + 0.5sin(\mathbf{t})\tilde{i} + 0.6cos(\mathbf{t})\tilde{j} + 0.4sin(\mathbf{t})\tilde{k} \end{bmatrix},$$

and activation functions are $f(\epsilon^{(\rho)}(\mathbf{t})) = g(\epsilon^{(\rho)}(\mathbf{t})) = h(\epsilon^{(\rho)}(\mathbf{t})) = tanh(\epsilon^{(\rho)}(\mathbf{t}))$, $f(\tilde{\epsilon}^{(\rho)}(\mathbf{t})) = g(\tilde{\epsilon}^{(\rho)}(\mathbf{t})) = h(\tilde{\epsilon}^{(\rho)}(\mathbf{t})) = tanh(\tilde{\epsilon}^{(\rho)}(\mathbf{t}))$, where $\rho = R, I, J, K$ with the initial conditions $\epsilon_1(0) = 1 - 0.3\tilde{i} - 1.2\tilde{j} + 2.2\tilde{k}$, $\epsilon_2(0) = -0.1 + 0.2\tilde{i} + 0.3\tilde{j} - 1.2\tilde{k}$, $\tilde{\epsilon}_1(0) = 0.9 - 0.5\tilde{i} + 1.1\tilde{j} + 3.2\tilde{k}$, $\tilde{\epsilon}_2(0) = 0.9 + 2.2\tilde{i} + 1.3\tilde{j} + 0.6\tilde{k}$, and $\tau_0(\mathbf{t}) = 0.1sin^2(\mathbf{t})$, $\xi = 0.1$, and interaction strength $x_1 = 0.2$. Also, we can calculate the other parameters $\kappa_\sigma = \rho_\sigma = \delta_\sigma = \alpha_\sigma = \beta_\sigma = \gamma_\sigma$, $\sigma = 1, 2$. The real and imaginary components of the trajectories of the systems (3.25) and (3.26) for $\vartheta = 1, 2$ are shown in Figures 3.1-3.2.

Case 1. By choosing projective coefficient $\lambda = 0.3$, and other constants $\tilde{y}_1 = 36$, $\tilde{y}_2 = 4.5$, $m_{\vartheta\sigma} = n_{\vartheta\sigma} = o_{\vartheta\sigma} = \hat{m}_{\vartheta\sigma} = \hat{n}_{\vartheta\sigma} = \hat{u}_{\vartheta\sigma} = \hat{v}_{\vartheta\sigma} = \hat{i}_{\vartheta\sigma} = \hat{j}_{\vartheta\sigma} = \tilde{o}_{\vartheta} = \hat{o}_{\vartheta} = 1$ for $\vartheta, \sigma = 1, 2$. Now, we can calculate $T_1 = 1.3938 > 0$, $T_2 = 5.2425 > 0$, $\tilde{T}_\vartheta = -0.5 < 0$ $(\vartheta = 1, 2)$, also $\tilde{y}_1 - \tilde{y}_2 = 31.5 > 0$. Hence, the conditions stated in (7.3.2) hold under the controllers $l_1(\mathbf{t}) = -31E_1(\mathbf{t}) + 2.1\epsilon_1(\mathbf{t})$, $l_2(\mathbf{t}) = -20E_1(\mathbf{t}) + 0.3\epsilon_1(\mathbf{t})$. In

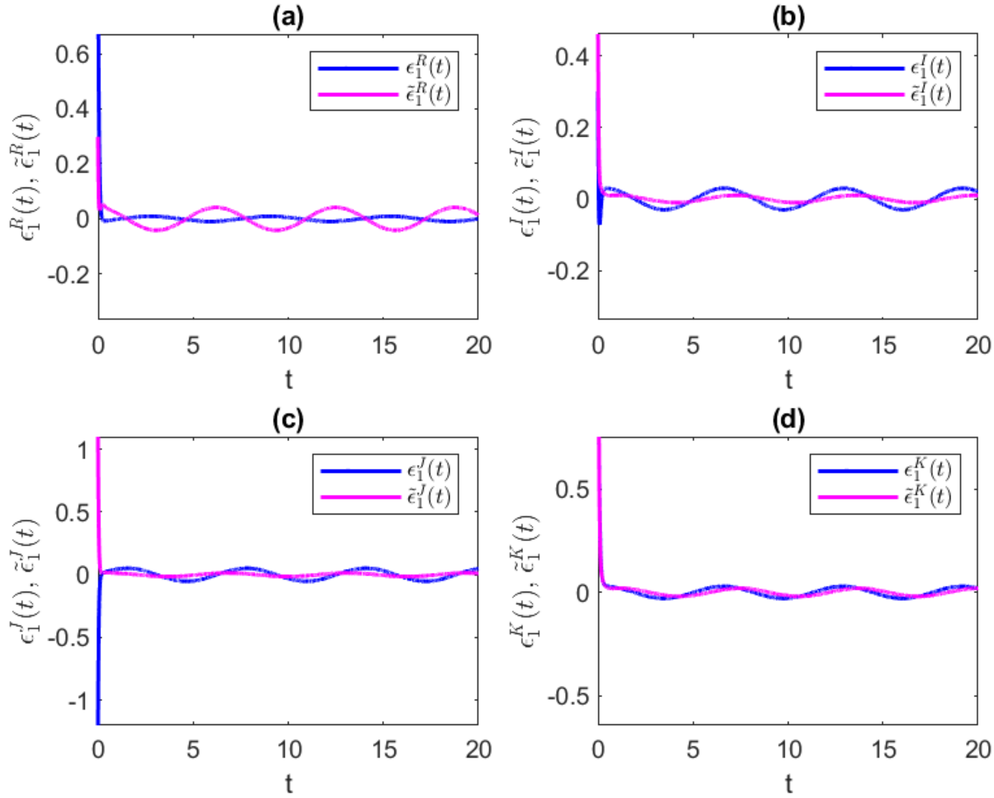

FIGURE 3.1: Trajectories of real and imaginary components of (3.1) and (3.2) for $\vartheta = 1$.

addition, one gets $C = 14.1435$ and $\xi = 15.2734$. So based on Theorem 3.3.1, the convergence ball $\mathcal{B} = \left\{E(\mathbf{t}) \in \mathbb{Q}^n : \|E(\mathbf{t})\|^2 \leq \frac{14.1435}{15.2734} = 0.9260\right\}$ is obtained, which is shown in Figure 3.3. As a result, QPS between the systems (3.25) and (3.26) is accomplished.

Case 2. For QS, let us choose projective coefficient $\lambda = 1$, and take the other constants $\pi_1 = 42$, $\pi_2 = 4.6$, and $m'_{\vartheta\sigma} = n'_{\vartheta\sigma} = o'_{\vartheta\sigma} = \tilde{m}_{\vartheta\sigma} = \tilde{n}_{\vartheta\sigma} = \tilde{u}_{\vartheta\sigma} = \tilde{v}_{\vartheta} = \tilde{k}_{\vartheta} = 1$ $(\vartheta, \sigma = 1, 2)$. Now, we can calculate $T'_1 = 8.686 > 0$, $T'_2 = 37.112 > 0$, $\tilde{T}_{\vartheta} = -0.6 < 0$ $(\vartheta = 1, 2)$, also $\tilde{y}_1 - \tilde{y}_2 = 37.4 > 0$. Hence, the conditions stated in (3.15) hold under the controllers $l_1(\mathbf{t}) = -2E_1(\mathbf{t}) - 7\epsilon_1(\mathbf{t})$, $l_2(\mathbf{t}) = -4E_1(\mathbf{t}) - 1\epsilon_1(\mathbf{t})$. Also we get, $C = 15.1$ and $\xi = 16.9487$. Thus, by Corollary 3.3.1, $\mathcal{B} = \left\{E(\mathbf{t}) \in \mathbb{Q}^n : \|E(\mathbf{t})\|^2 \leq \frac{15.1}{16.9487} = 0.8909\right\}$ is obtained as error bound, which is

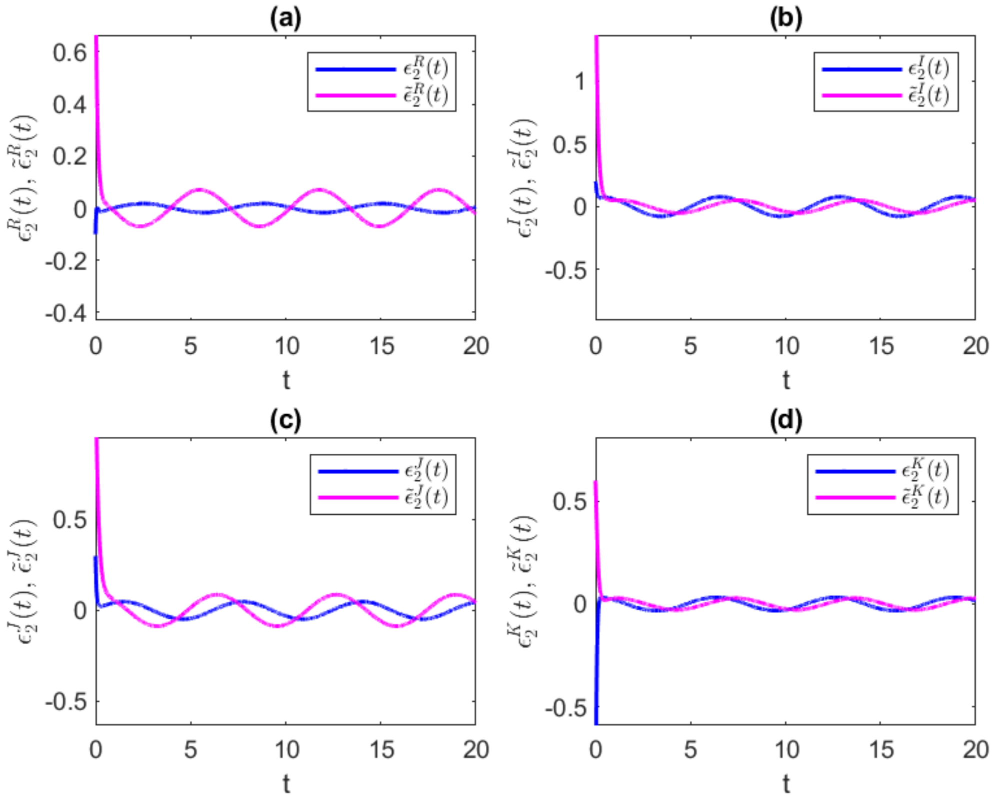

FIGURE 3.2: Trajectories of real and imaginary components of (3.1) and (3.2) for $\vartheta = 2$.

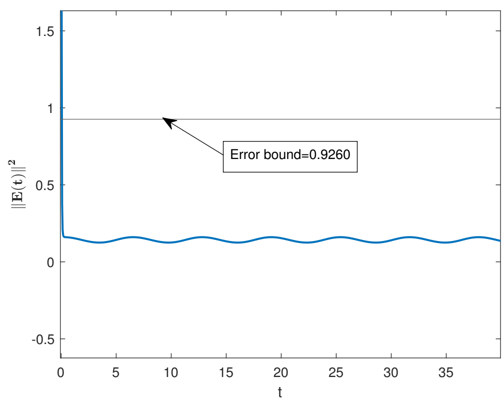

FIGURE 3.3: QPS of error system (3.3).

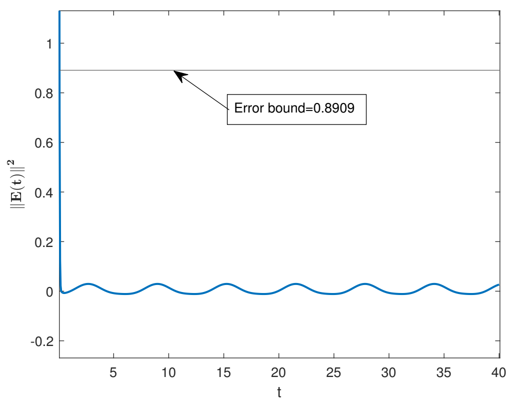

FIGURE 3.4: QS of error system (3.14).

depicted in Figure 3.4. Hence, QS between the systems (3.25) and (3.26) is achieved. Also, Figures 3.5 and 3.6 shows the error system (3.3) with and without interaction

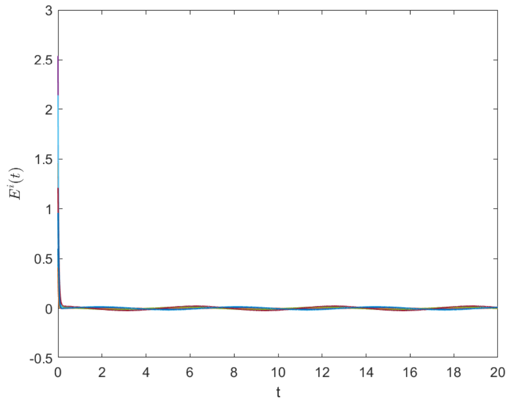

FIGURE 3.5: The error of drive (3.25) and response (3.26) systems with interaction terms.

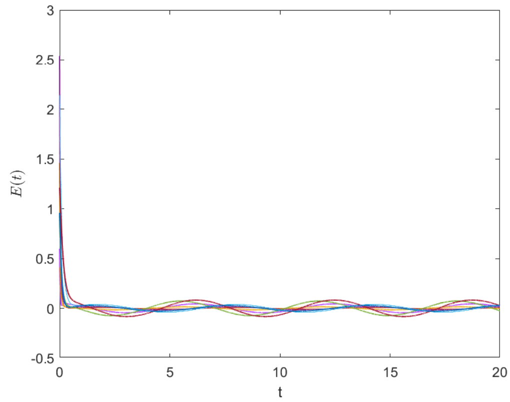

FIGURE 3.6: The error of drive (3.25) and response (3.26) systems without interaction terms.

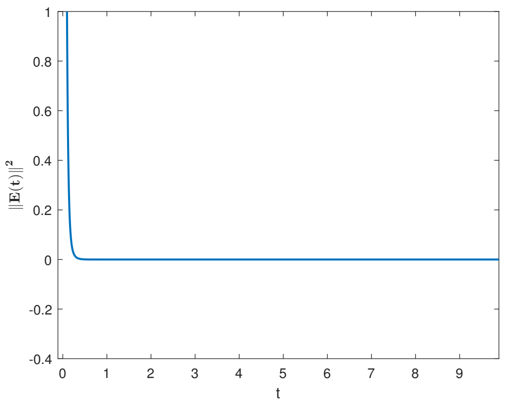

FIGURE 3.7: Plot of synchronization error for two identical QVNNs.

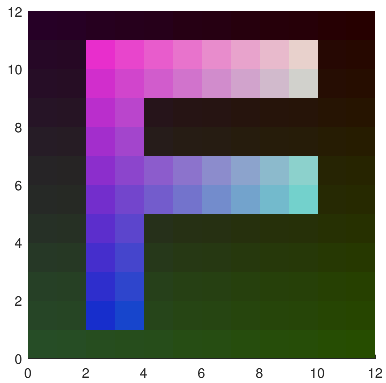

FIGURE 3.8: Original colour image of pattern "*F*".

terms.

Case 3. For Complete synchronization, we take $A' = A = \begin{bmatrix} 24 & 0 \\ 0 & 15 \end{bmatrix}$,

$$B' = B = \begin{bmatrix} 2 + 1.2\tilde{i} + 1.5\tilde{j} + 1.3\tilde{k} & -1 - 0.5\tilde{i} - 2.3\tilde{j} - 0.2\tilde{k} \\ 1 + 1.7\tilde{i} + 1.4\tilde{j} + 1.6\tilde{k} & 0.8 + 0.9\tilde{i} + 0.6\tilde{j} + 0.5\tilde{k} \end{bmatrix},$$

$$D' = D = \begin{bmatrix} 1 - 0.9\tilde{i} + 2.1\tilde{j} + 1\tilde{k} & 2 + 2\tilde{i} - 1\tilde{j} + 1.1\tilde{k} \\ 3 - 1\tilde{i} + 0.4\tilde{j} + 0.6\tilde{k} & 0.2 - 1\tilde{i} + 2\tilde{j} + 1\tilde{k} \end{bmatrix},$$

$$W' = W = \begin{bmatrix} 1.3 - 0.4\tilde{i} + 1\tilde{j} + 1\tilde{k} & 1 + 1.2\tilde{i} - 1.3\tilde{j} + 0.5\tilde{k} \\ 0.2 - 0.3\tilde{i} + 0.6\tilde{j} + 1.2\tilde{k} & 0.3 - 1\tilde{i} + 0.4\tilde{j} + 0.6\tilde{k} \end{bmatrix},$$

and $R' = R = \begin{bmatrix} 0.2sin(\mathbf{t}) + 0.5cos(\mathbf{t})\tilde{i} + 1sin(\mathbf{t})\tilde{j} + 0.4cos(\mathbf{t})\tilde{k} \\ 0.4sin(\mathbf{t}) + 1cos(\mathbf{t})\tilde{i} + 0.3sin(\mathbf{t})\tilde{j} + 0.6cos(\mathbf{t})\tilde{k} \end{bmatrix}$. Thus Conditions (3.24) are satisfied, according to the estimated findings. The error curve is shown in Figure 3.7 and in accordance with Corollary 3.3.2, the systems (3.1) and (3.2) are globally exponentially synchronized.

Example 3.4.2. To associatively memorise the colour image, let us create QVNNs in the shape of (3.25) using the colour image pattern "F" from Figure 3.8. The size of the image "F" is 12×12 pixels, as displayed in Figure 3.8. The QVNNs are created with an equilibrium point of 144-D, which is made up with 144 neurons and store the pattern's colours. Set the parameters of the QVNNs (3.25) as follows:

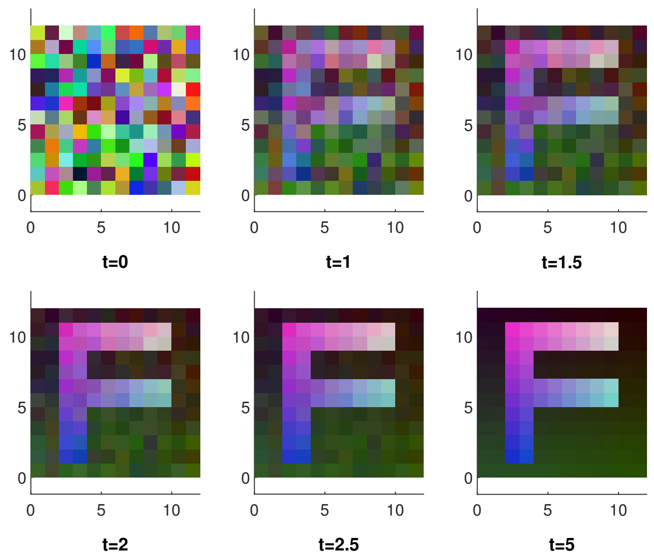

FIGURE 3.9: Simulation of retrieving image "F" with random initials values of time t.

$$a_\vartheta = 1.0 \tag{3.27}$$

$$b_{\vartheta\sigma} = \begin{cases} 4.0 + 0.40\tilde{i} - 0.30\tilde{j} + 0.50\tilde{k}, \vartheta = \sigma \\ 0.40 - 0.50\tilde{i} + 0.50\tilde{j} - 0.30\tilde{k}, \vartheta \neq \sigma \end{cases} \tag{3.28}$$

$$d_{\vartheta\sigma} = \begin{cases} -0.20 + 0.20\tilde{i} - 0.50\tilde{j} + 0.40\tilde{k}, \vartheta < \sigma \\ 0.20 + 0.30\tilde{i} - 0.20\tilde{j} - 0.30\tilde{k}, \vartheta = \sigma \\ -0.10 + 0.20\tilde{i} + 0.30\tilde{j} - 0.50\tilde{k}, \vartheta > \sigma, \end{cases} \tag{3.29}$$

$$h^{(y)}(\epsilon(\mathbf{t})) = tanh(\epsilon(\mathbf{t})), \tau_{\vartheta\sigma} = 1 \ \forall \vartheta, \sigma, \tag{3.30}$$

where $\vartheta, \sigma = 1, 2, \ldots, 144$ and $y = 1, 2, 3$. The equilibrium point of the proposed QVNNs must be $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{144})^T \in \mathbb{Q}^{144}$ in order to recall the colour image pattern "F", where $\epsilon_1 = 0 + 0.15\tilde{i} + 0.30\tilde{j} + 0.1500\tilde{k}$, $\epsilon_2 = 0 + 0.1500\tilde{i} + 0.30\tilde{j} + 0.1400\tilde{k}, \ldots,$ $\epsilon_{144} = 0 + 0.150\tilde{i} + 0\tilde{j} + 0\tilde{k}$, which correspond to the colour $(0.1500, 0.30, 0.250), (0.150, 0.30, 0.140), \ldots, (0.150, 0, 0)$ of the pixels in the image pattern "F". The external input R can be determined using the equilibrium point ϵ as follows:

$$R = (r_1, r_2, \ldots, r_{144})^T \in \mathbb{Q}^{144} \tag{3.31}$$

where $r_1 = -62.80 + 22.70\tilde{i} - 90.50\tilde{j} - 7.250\tilde{k}, r_2 = -630 + 20.950\tilde{i} - 90.50\tilde{j} - 5.660\tilde{k}, \ldots, r_{144} = -91.40 - 234.60\tilde{i} - 90.80\tilde{j} + 221.40\tilde{k}$. Due to space considerations, one can able to list three of the components of R and ϵ here. The developed QVNNs with parameters (3.27)–(3.31) have the capacity to dependably recall the aforementioned pattern "F", according to a simulation with random initial values shown in Figure 3.9.

Remark 3.4.1. Quaternions have a four-dimensional structure and offer a richer representation of color information than complex numbers with only two dimensions. This richness is advantageous for encoding color information in RGB images, as three color channels represent each pixel. Additionally, quaternions naturally handle spatial relationships and rotations, which can benefit image tasks where orientation is essential. Compared to complex numbers, the quaternions provide a more comprehensive representation of spatial transformations. Moreover, QVNNs require fewer neurons than CVNNs to store the same image, suggesting more efficient information representation or compression. As noted in [97], three CVNNs need to build to remember the red, green, and blue intensity matrices in order to remember and retrieve accurate colour pictures. As a result, according to the method suggested in [97], the developed CVNNs require 432 neurons in order to retain a colour figure of 12×12. But in contrast to the CVNNs developed in [97], the designed QVNNs in Example 3.4.2 have 144 neurons, which is a significantly smaller number.

3.5 Conclusions

This chapter investigated the QPS problem of non-identical QVNNs with time-varying delays and interaction terms using the direct rather than the separative method. The QPS criterion has been derived using the Lyapunov functional, a suitable controller, and a few inequality techniques within the context of the quaternion-valued domain. An estimation of the upper bound of the synchronization error has been provided. The QS of time-varying delayed non-identical QVNNs with interaction terms has been discussed by Lyapunov stability. The CS of QVNNs with time-varying delays and interaction terms has also been analyzed in the case of two identical systems. For some special cases, the suitable conditions have also been provided. Finally, the efficacy of the theoretical results is illustrated numerically through Example 3.4.1. Additionally, Example 3.4.2 highlights the practical application of QVNNs combined with associative memory. This example demonstrates that QVNNs accurately restore original color image patterns, showcasing their effectiveness in real-world scenarios.

****